The topological and geometrical finiteness of complete flat Lorentzian 3-manifolds with free fundamental groups (Preliminary)

Suhyoung Choi

Department of Mathematical Science
KAIST, Daejeon, South Korea
mathsci.kaist.ac.kr/∼schoi (Copies of my lectures are posted)
email: schoi@math.kaist.ac.kr

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Centre de recherches mathématiques, Université de Montréal

Abstract

- We prove the topological tameness of a 3-manifold with a free fundamental group admitting a complete flat Lorentzian metric; i.e., a Margulis space-time isomorphic to the quotient of the complete flat Lorentzian space by the free and properly discontinuous isometric action of the free group of rank ≥ 2.

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- We will use our particular point of view that a Margulis space-time is a real projective manifold in an essential way.
- The basic tools are a bordification by a closed $\mathbb{R}P^2$-surface with a free holonomy group, the important work of Goldman, Labourie, and Margulis on geodesics in the Margulis space-times and the 3-manifold topology.

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- The tameness and many other results are also obtained indepedently by Jeff Danciger, Fanny Kassel and François Guéritaud.

Content

Tame manifolds

- An open n-manifold can sometimes be compactified to a compact n-manifold with boundary. Then the open manifold is said to be *tame*.
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A nontame 3-manifold

essentially can be “simply” thought of as a union of an increasing sequence of compression bodies M_i so that each $M_i \to M_{i+1}$ is an imbedding by homotopy equivalence not isotopic to a homeomorphism. (Ohshika's observation.)

- Hyperbolic 3-manifolds with finitely generated fundamental groups are shown to be tame by Bonahon, Agol and Calegari-Gabai. See Bowditch [7] for details.
- Earlier, geometrically finite hyperbolic 3-manifolds are shown to be tame by Marden (and Thurston). This is relevant to us.

- Let $V^{2,1}$ denote the vector space $\mathbb{R}^3$ with a Lorentzian norm of sign $1, 1, -1$, and
- the Lorentzian space-time $E^{2,1}$ can be thought of as the vector space with translation by any vector allowed.
- We will concern ourselves with only the subgroup $\mathrm{Isom}^+(E^{2,1})$ of orientation-preserving isometries, isomorphic to $\mathbb{R}^3 \rtimes \mathrm{SO}(2,1)$ or

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- $P(V^{2,1})$ is defined as the quotient space

$$V^{2,1} - \{O\}/\sim \text{ where v} \sim \text{w if and only if v} = sw \text{ for } s \in \mathbb{R} - \{0\}.$$

The sphere of directions $\mathbb{S} := \mathbb{S}(V^{2,1})$ is defined as the quotient space

$$V^{2,1} - \{O\}/\sim \text{ where v} \sim \text{w if and only if v} = sw \text{ for } s > 0,$$

and equals the double cover $\widehat{\mathbb{R}P^2}$ of $\mathbb{R}P^2$.

Our spherical view of $E^{2,1}$ and homogeneous coordinates

- The projective sphere $\mathbf{S}^3 := \mathbb{S}(\mathbb{R}^4 - \{O\})$ with coordinates t, x, y, z with projective automorphism group $\mathrm{Aut}(\mathbf{S}^3)$ isomorphic to $\mathrm{SL}_\pm(4, \mathbb{R})$.
- $\mathbf{S}^3$ double-covers the real projective space.
- The upper hemisphere given by $t > 0$ is identical with $[1, x, y, z]$ and is identified with $E^{2,1}$ with boundary $\mathbb{S}$.
- $$\mathrm{Isom}^+(E^{2,1}) \subset \mathrm{Aut}(\mathbf{S}^3).$$
- $\mathrm{Isom}^+(E^{2,1})$ acts on $\mathbb{S}$ by sending it by $\mathcal{L}$ to $\mathrm{Aut}(\mathbb{S})$.
- We map $E^{2,1}$ to a unit 3-ball in $\mathbb{R}^3$ by the map
$$[1, x, y, z] \to \frac{(x, y, z)}{\sqrt{1 + x^2 + y^2 + z^2}}.$$
- $\mathbb{S}$ goes to the unit sphere $x^2 + y^2 + z^2 = 1$.

- The Lorentzian structure divides $\mathbb{S}$ into three open domains $\mathbb{S}_+, \mathbb{S}_0, \mathbb{S}_-$ separated by two conics $\mathrm{bd}\mathbb{S}_+$ and $\mathrm{bd}\mathbb{S}_-$.
- Recall that $\mathbb{S}_+$ of the space of future time-like vectors is the Beltrami-Klein model of the hyperbolic plane $\mathbb{H}^2$ where $SO(2,1)$ acts as the orientation-preserving isometry group. Here the metric geodesics are precisely the projective geodesics and vice versa.

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- The geodesics in $\mathbb{S}_+$ are straight arcs and $\mathrm{bd}\mathbb{S}_+$ forms the ideal boundary of $\mathbb{S}_+$.
- For a finitely generated discrete, non-elementary, subgroup Γ in SO(2, 1), $\mathbb{S}_+/\Gamma$ has a complete hyperbolic structure as well as a real projective structure with the compatible geodesic structure.
- Nonelementary Γ has no parabolics if and only if $\mathbb{S}_+/\Gamma$ is a geometrically finite hyperbolic surface.

- Suppose that Γ is a finitely generated Lorentzian isometry group acting freely and properly on $E^{2,1}$. We assume that Γ is not amenable (i.e., not solvable). Then $E^{2,1}/\Gamma$ is said to be a *Margulis space-time*.
- Γ injects under $\mathcal{L}$ to $\mathcal{L}(\Gamma)$ acting properly discontinuously and freely on $\mathbb{S}_+$. By Mess [34], Γ must be a free group of rank ≥ 2.

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- Γ injects under $\mathcal{L}$ to $\mathcal{L}(\Gamma)$ acting properly discontinuously and freely on $\mathbb{S}_+$. By Mess [34], Γ must be a free group of rank ≥ 2.
- Then $\mathbb{S}_+/\Gamma$ is a complete genus $\tilde{g}$ hyperbolic surface with b ideal boundary components.

Theorem A (Bordification by an $\mathbb{R}P^2$-surface)

Let $\Gamma \subset \mathrm{Isom}_+(E^{2,1})$ be a fg. free group of rank $g \geq 2$ acting on the hyperbolic 2-space $\mathbb{H}^2$ properly discontinuously and freely without any parabolic holonomy.

Then there exists a Γ-invariant open domain $\mathcal{D} \subset \mathbb{S}(V^{2,1})$ such that $\mathcal{D}/\Gamma$ is a closed surface Σ with a real projective structure induced from $\mathbb{S}$ unique up to the antipodal map $\mathcal{A}$. (The genus equals g.)

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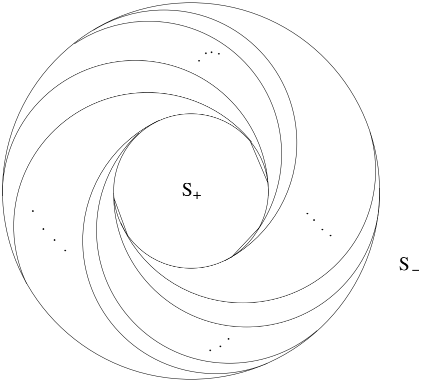

Figure : The domain $\mathcal{D}$ covering Σ.

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- The surface is a quotient of a domain in $\mathbb{S}$ by a group of projective automorphisms.
- This is an $\mathbb{R}P^2$-analog of the standard Schottky uniformization of a Riemann surface as a $\mathbb{C}P^1$-manifold as observed by Goldman. There is an equivariant map shrinking all complementary intervals to points.

Handlebody

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Theorem B (Compactification)

Let M be a Margulis space-time $\mathsf{E}^{2,1}/\Gamma$ *and* $\mathcal{L}(\Gamma)$ *has no parabolic element. Then M is homeomorphic to the interior of a solid handlebody of genus equal to the rank of* Γ*.*

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Simplifying Assumption

$\mathcal{L}(\Gamma) \subset \mathrm{SO}^+(2,1)$. Up to double covering, always true.

Convex decomposition of real projective surfaces

- A *properly convex* domain in $\mathbb{R}P^2$ is a bounded convex domain of an affine subspace in $\mathbb{R}P^2$. A real projective surface is *properly convex* if it is a quotient of a properly convex domain in $\mathbb{R}P^2$ by a properly disc. and free action of a subgroup of $\mathrm{PGL}(3, \mathbb{R})$.
- A disjoint collection of simple closed geodesics $c_1, \ldots, c_m$ *decomposes* a real projective surface S into subsurfaces $S_1, .., S_n$ if each S_i is the closure of a component of $S - \bigcup_{i=1,..,m} c_i$. We do not allow a curve c_j to have two one-sided neighborhoods in only one S_i for some i.

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Theorem 3.1 ([13])

Let Σ be a closed orientable real projective surface with principal geodesic or empty boundary and $\chi(\Sigma) < 0$.
Then Σ has a collection of disjoint simple closed principal geodesics decomposing Σ into properly convex real projective surfaces with principal geodesic boundary and of negative Euler characteristic and/or π-annuli with principal geodesic boundary.

Null half-planes

- Let $\mathcal{N}$ denote the *nullcone* in $V^{2,1}$.
- If $v \in \mathcal{N} - \{O\}$, then its orthogonal complement $v^{\perp}$ is a *null plane* which contains $\mathbb{R}v$, which separates $v^{\perp}$ into two half-planes.
- Since $v \in \mathcal{N}$, its direction lies in either $\mathrm{bd}\mathbb{S}_+$ or $\mathrm{bd}\mathbb{S}_-$. Choose an arbitrary element u of $\mathbb{S}_+$ or $\mathbb{S}_-$ respectively, so that the directions of v and u both lie in the same $\mathrm{Cl}(\mathbb{S}_+)$ or $\mathrm{Cl}(\mathbb{S}_-)$ respectively.

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- Define the *null half-plane* $\mathscr{W}(v)$ (or the *wing*) associated to v as:

$$\mathscr{W}(v) := \{w \in v^{\perp} \mid \mathrm{Det}(v, w, u) > 0\}.$$

We will now let $\varepsilon([v]) := [\mathscr{W}(v)]$ for convenience.

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- The map $[v] \longmapsto \varepsilon(v)$ is an $SO(2,1)$-equivariant map

$$\mathrm{bd}\mathbb{S}_+ \to \mathcal{S}$$

  for the space $\mathcal{S}$ of half-arcs of form $\varepsilon(v)$ for $v \in \mathrm{bd}\mathbb{S}_+$.

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- Hence $\mathbb{S}_0$ has a SO(2, 1)-equivariant quotient map

$$\Pi : \mathbb{S}_0 \to P(\mathcal{N} - \{O\}) \cong \mathbf{S}^1$$

where $\varepsilon([v]) = \Pi^{-1}([v])$ for each $v \in \mathcal{N} - \{O\}$.

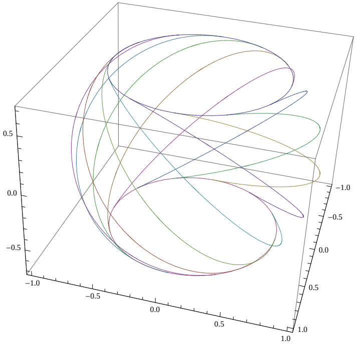
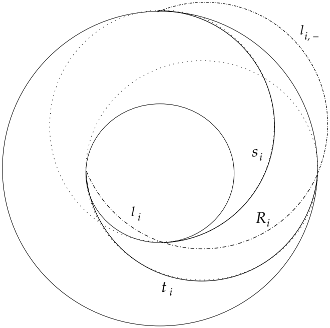


Figure : The tangent geodesics to disks $\mathbb{S}_+$ and $\mathbb{S}_-$ in the unit sphere $\mathbb{S}$ imbedded in $\mathbb{R}^3$.

- $\mathbb{S}_+/\Gamma$ is an open hyperboic surface, compactified to Σ' by adding number of ideal boundary components.
- Σ' is covered by $\mathbb{S}_+ \cup \bigcup_{i\in\mathcal{J}} \mathbf{b}_i$ where $\mathbf{b}_i$ are ideal open arcs in $\mathrm{bd}\mathbb{S}_+$.
- Let $s_i = \varepsilon(p_i)$ and $t_i = \varepsilon(q_i)$. Then $\mathbf{l}_i, s_i, t_i, \mathbf{l}_{i,-}$ bound a *strip* invariant under $\langle \mathbf{g}_i \rangle$. We denote by $\mathcal{R}_i$ the open strips union with $\mathbf{l}_i$ and $\mathbf{l}_{i,-}$.



Proof of Theorem A

- We define $\mathcal{A}_i = \mathcal{R}_i \cap \mathbb{S}_0$ for $i \in \mathcal{J}$, which equals $\bigcup_{x \in \mathbf{b}_i} \varepsilon(x)$.
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- We finally define

$$\begin{aligned} \tilde{\Sigma} &= \tilde{\Sigma}'_+ \cup \coprod_{i \in \mathcal{J}} \mathcal{R}_i \cup \tilde{\Sigma}'_- \\ &= \tilde{\Sigma}'_+ \cup \coprod_{i \in \mathcal{J}} \mathcal{A}_i \cup \tilde{\Sigma}'_- \\ &= \Omega_+ \cup \coprod_{i \in \mathcal{J}} \mathcal{R}_i \cup \Omega_- \qquad (1) \\ &= \mathbb{S} - \bigcup_{x \in \Lambda} \mathrm{Cl}(\varepsilon(x)). \qquad (2) \end{aligned}$$

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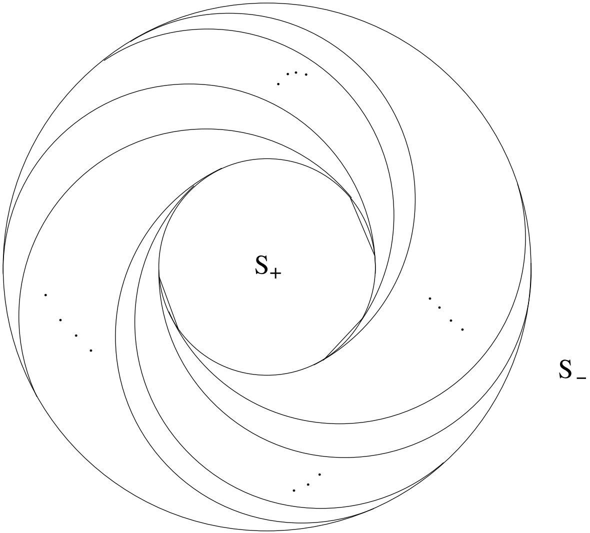
  an open domain in $\mathbb{S}$ where Λ is the limit set.
- Since the collection whose elements are of form $\mathcal{R}_i$ mapped to itself by Γ, we showed that Γ acts on this open domain.


S_+
S_-

Margulis invariants

- Given an element $g \in \Gamma - \{\mathrm{I}\}$, let us denote by $\mathsf{v}_+(g)$, $\mathsf{v}_0(g)$, and $\mathsf{v}_-(g)$ the eigenvectors of the linear part $\mathcal{L}(g)$ of g corresponding to eigenvalues > 1, $= 1$, and < 1 respectively.
- $\mathsf{v}_+(g)$ and $\mathsf{v}_-(g)$ are null vectors and $\mathsf{v}_0(g)$ is space-like and of unit norm. We choose so that $\mathsf{v}_-(g) \times \mathsf{v}_+(g) = \mathsf{v}_0(g)$.

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- $\mathsf{v}_+(g)$ and $\mathsf{v}_-(g)$ are null vectors and $\mathsf{v}_0(g)$ is space-like and of unit norm. We choose so that $\mathsf{v}_-(g) \times \mathsf{v}_+(g) = \mathsf{v}_0(g)$.
- We recall the Margulis invariant $\alpha : \Gamma - \{\mathrm{I}\} \to \mathbb{R}$

$$\alpha(g) := \mathbf{B}(gx - x, \mathsf{v}_0(g)) \text{ for } g \in \Gamma - \{\mathrm{I}\}, x \in \mathsf{E}^{2,1},$$

which is independent of the choice of x in $\mathsf{E}^{2,1}$. (See [20] for details.)

- If Γ acts freely on $\mathsf{E}^{2,1}$, then Margulis invariants of nonidentity elements are all positive or all negative by the Opposite sign-lemma of Margulis.

Diffused Margulis invariants of Labourie

- By following the geodesics in Σ_+, we obtain a so-called geodesic flow

$$\Phi : \mathbb{U}\Sigma_+ \times \mathbb{R} \to \mathbb{U}\Sigma_+.$$

  A *geodesic current* is a Borel probability measure on $\mathbb{U}(\mathbb{S}_+/\Gamma)$ invariant under the geodesic flow, supported on a union of weakly recurrent geodesics.

- Let $[u]$ denote the element of $H^1(\Gamma_0, \mathsf{V}^{2,1})$ given by Γ for the linear part Γ_0 of Γ.

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- We extend the function

$$\mathcal{C}_{\text{per}}(\Sigma_+) \to \mathbb{R} \text{ by } \mu_\gamma \mapsto \frac{\alpha(\gamma)}{l_{\mathbb{S}_+}(\gamma)}.$$

  to the diffused one $\Phi_{[u]} : \mathcal{C}(\mathbb{S}_+/\Gamma) \to \mathbb{R}_{\geq 0}$.

- $\Gamma = \Gamma_{0,[u]}$ acts properly if and only if $\Phi_{[u]}(\mu) > 0$ for all $\mu \in \mathcal{C}(\Sigma) - \{O\}$ (or $\Phi_{[u]}(\mu) < 0$) [30]

Neutralized sections

- They in [30] (following Fried) constructed a flat affine bundle **E** over the unit tangent bundle $\mathbb{U}\Sigma_+$ of Σ_+ by forming $E^{2,1} \times \mathbb{U}\mathbb{S}_+$ and taking the quotient by the diagonal action $\gamma(x, v) = (h(\gamma)(x), \gamma(v))$ for a deck transformation γ of the cover $\mathbb{U}\mathbb{S}_+$ of $\mathbb{U}\Sigma_+$ where

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- The cover of **E** is denoted by $\hat{\mathbf{E}}$ and is identical with $E^{2,1} \times \mathbb{US}_+$. We denote by

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- We define **V** as the quotient of $V^{2,1} \times \mathbb{US}_+$ by the linear action of Γ and the action of $\mathbb{US}_+$.

Neutralized sections

- A *neutral section* of **V** is an SO(2, 1)-invariant section which is parallel along geodesic flow of $\mathbb{U}\Sigma_+$.
- A neutral section $\boldsymbol{\nu} : \mathbb{U}\Sigma_+ \to \mathbf{V}$ arises from a graph of the SO(2, 1)-invariant map

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  with the image in the space of unit space-like vectors in $\mathsf{V}^{2,1}$:
- $\tilde{\boldsymbol{\nu}}$ is defined by sending a unit vector u in $\mathbb{U}\mathbb{S}_+$ to the normalization of $\rho(\mathrm{u}) \times \alpha(\mathrm{u})$ of the null vectors $\rho(\mathrm{u})$ and $\alpha(\mathrm{u})$ with directions the the start point and the end point in $\mathrm{bd}\mathbb{S}_+$ of the geodesic tangent to u in $\mathbb{S}_+$.

Let $\mathbb{U}_{\rm rec}\Sigma_+ \subset \mathbb{U}\Sigma_+$ denote the unit vectors tangent to weakly recurrent geodesics of Σ.

Lemma 4.1 ([30])

Let Σ_+ be as above. Then

- $\mathbb{U}_{\rm rec}\Sigma_+ \subset \mathbb{U}\Sigma_+$ *is a connected compact geodesic flow invariant set and is a subset of the compact set* $\mathbb{U}\Sigma''_+$.
- *The inverse image* $\mathbb{U}_{\rm rec}\mathbb{S}_+$ *of* $\mathbb{U}_{\rm rec}\Sigma_+$ *in* $\mathbb{U}_{\rm rec}\mathbb{S}_+$ *is precisely the set of unit vectors tangent to geodesics with both endpoints in* Λ.

- The above conjugates the geodesic flow ϕ_t on Σ_+ with one Φ_t in $E^{2,1}$ where each geodesic with direction $\vec{u}$ at p goes to a geodesic in the direction of $\boldsymbol{\nu}(\vec{u})$.
- We find the section $\tilde{\mathcal{N}} : \mathbb{U}_{\mathrm{rec}}\mathbb{S}_+ \rightarrow \hat{\mathbf{E}}$ lifting $\mathcal{N}$ satisfying

$$\tilde{\mathcal{N}} \circ \phi_t = \Phi_{t'} \circ \tilde{\mathcal{N}} \text{ and } \tilde{\mathcal{N}} \circ \gamma = \gamma \circ \tilde{\mathcal{N}} \qquad (3)$$

for each deck transformation γ of $\mathbb{U}\mathbb{S}_+ \rightarrow \mathbb{U}\boldsymbol{\Sigma}_+$.

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for each deck transformation γ of $\mathbb{U}\mathbb{S}_+ \to \mathbb{U}\Sigma_+$.

Proposition 4.2

The lift of the neutralized section $\tilde{\mathcal{N}}$ induces a continuous function $\mathcal{N} : \mathcal{G}_{rec}\mathbb{S}_+ \to \mathcal{G}_{rec}\mathsf{E}^{2,1}$ where

- *if the oriented geodesic l in $\mathbb{S}_+$ is g-invariant for $g \in \Gamma$, then g acts on the space-like geodesic L_g the image under $\mathcal{N}$ as a translation.*
- *the convergent set of elements of $\mathcal{G}_{rec}\mathbb{S}_+$ maps to a convergent set in $\mathcal{G}_{rec}\mathsf{E}^{2,1}$.*
- *Finally, the map is surjective.*

Repeat: Our view of $E^{2,1}$ and coordinates

- The projective sphere $\mathbf{S}^3 = \mathbb{S}(\mathbb{R}^4 - \{O\})$ with coordinates t, x, y, z with projective automorphism group $\mathrm{Aut}(\mathbf{S}^3)$ isomorphic to $\mathrm{SL}_{\pm}(4, \mathbb{R})$.
- The upper hemisphere given by $t > 0$ is identical with $[1, x, y, z]$ and is identified with $\mathrm{E}^{2,1}$ with boundary $\mathbb{S}$.
- $\mathrm{Isom}^+(\mathrm{E}^{2,1}) \subset \mathrm{Aut}(\mathbf{S}^3)$.
- $\mathrm{Isom}^+(\mathrm{E}^{2,1})$ acts on $\mathbb{S}$ by sending it by $\mathcal{L}$ to $\mathrm{Aut}(\mathbb{S})$.
- We map $\mathrm{E}^{2,1}$ to a unit 3-ball by the map

$$[1, x, y, z] \to \frac{(x, y, z)}{\sqrt{1 + x^2 + y^2 + z^2}}.$$

A Lemma on projective automorphisms

Lemma 5.1

Let v_i^j *for* $j = 1, 2, 3, 4$ *be four sequences points of* $\mathbf{S}^3$. *Suppose that* $\mathrm{v}_i^j \to \mathrm{v}_\infty^j$ *for each* j *and mutually distinct independent points* $\mathrm{v}_\infty^1, \ldots, \mathrm{v}_\infty^4$. *Then we can choose a sequence* h_i *of elements of* $\mathrm{Aut}(\mathbf{S}^3)$ *so that*

- $h_i(\mathrm{v}_i^j) = \mathrm{e}_j$,
- h_i *is represented by uniformly convergent matrices and*

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- $h_i(\mathrm{v}_i^j) = \mathrm{e}_j$,
- h_i *is represented by uniformly convergent matrices and*
- $h_i \to h_\infty$ *uniformly for* $h_\infty \in \mathrm{Aut}(\mathbf{S}^3)$ *under* C^s*-topology for every* $s \geq 0$.

Projective boost automorphism

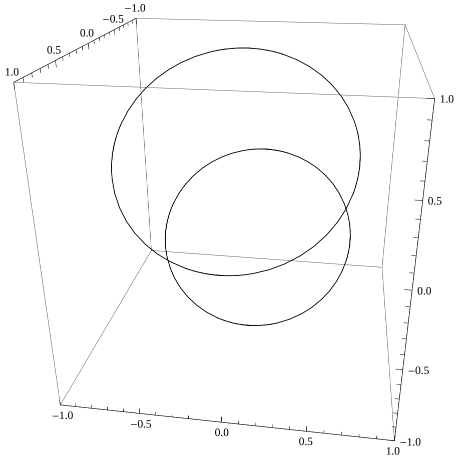
- A projective automorphism g that is of form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \lambda > 1, k \neq 0 \tag{4}$$

under a homogeneous coordinate system of $\mathbf{S}^3$ is said to be a *projective boost automorphism*.

- In affine coordinates,

$$(x, y, z) \mapsto (\lambda x, y + k, \frac{1}{\lambda} z), x, y, z \in \mathbb{R}$$



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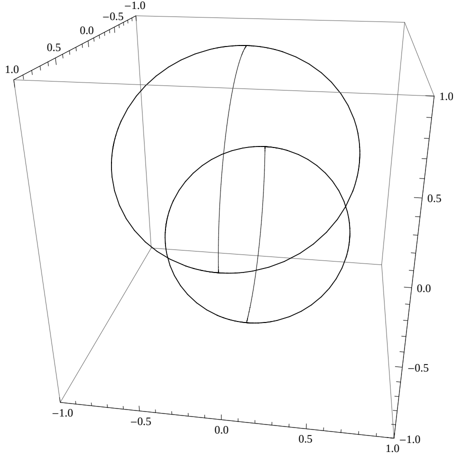
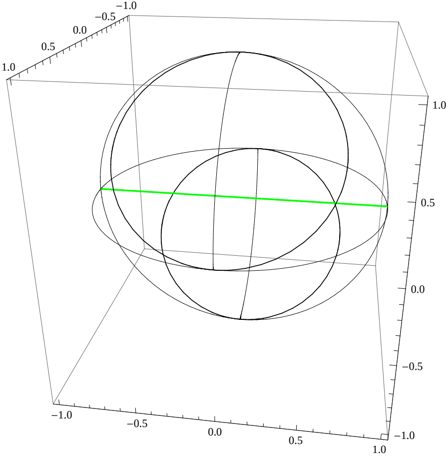


The action of a Lorentzian isometry $\hat{g}$ on the hemisphere $\mathscr{H}$ where the boundary sphere $\mathbb{S}$ is the unit sphere with center $(0, 0, 0)$ here.

- The arc on $\mathbb{S}$ given by $y = 0$ is the invariant geodesic in $\mathbb{S}_+$ and with end points the fixed points of $\hat{g}$.



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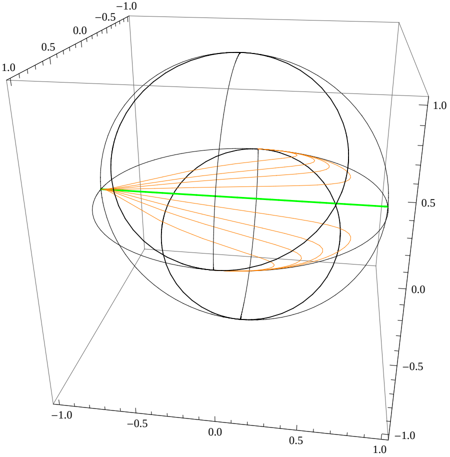
- The arc given by $x = 0$ and $z = 0$ is a line where $\hat{g}$ acts as a translation in the positive y-axis direction for $\hat{g} \neq \mathrm{I}$.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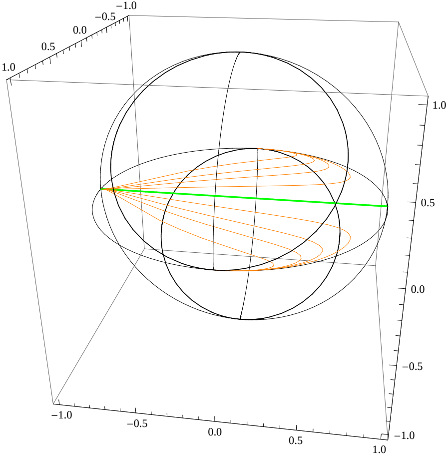
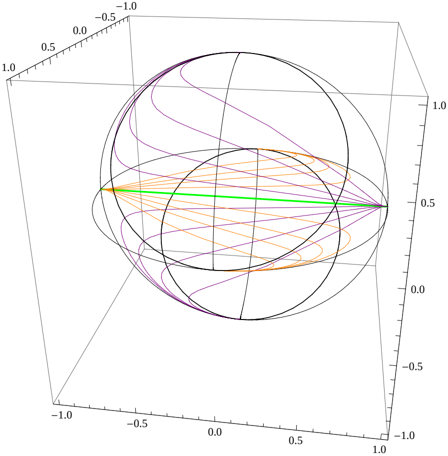


- The plane $z = 0$ is where $\hat{g}$ acts as an expansion-translation (stable disk),



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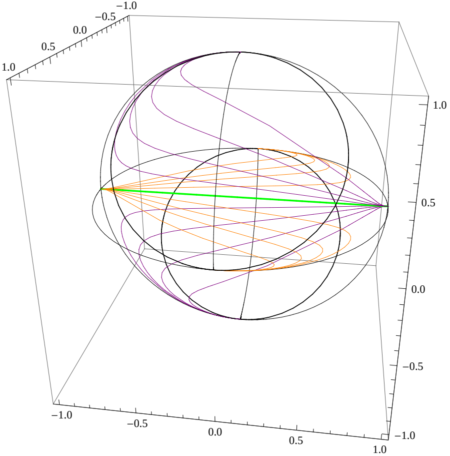

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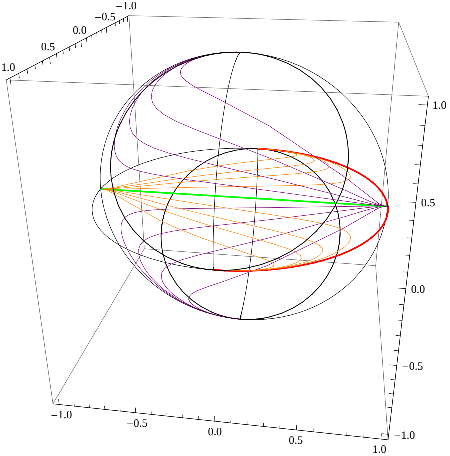

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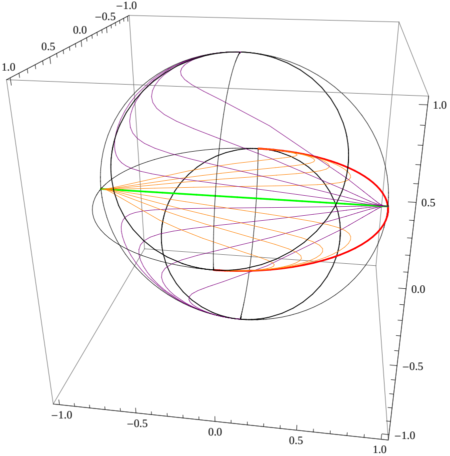

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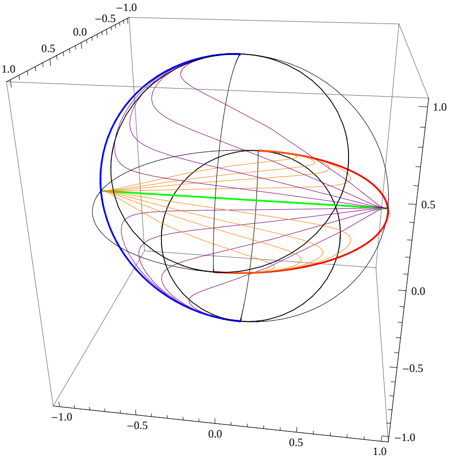

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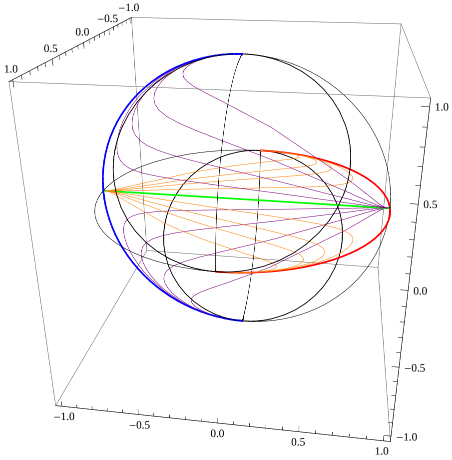

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- The semicircle defined by $y \geq 0$ and $z = 0$ is η^+, “the attracting arc”.
- The semicircle defined by $x = 0$ and $y \leq 0$ is η^-, “the repelling arc”.

Lemma 5.2 (Central)

Let $g_{\lambda,k}$ denote the automorphism on $\mathbf{S}^3$ defined by the equation 4 for a homogeneous coordinate system with functions t, x, y, z in the given order and let $\mathbb{S}$ given by $t = 0$, $\mathbf{S}^2_0$ given by $x = 0$, and $\mathscr{H}$ given by $t > 0$. We assume that $k \geq 0, \lambda > 0$.

Then as $\lambda, k \to +\infty$ where $k/\lambda \to 0$, we obtain

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- *$g_{\lambda,k}|\mathbf{S}^3 - \mathbf{S}^2_0$ converges to a rational map Π_0 given by sending $[t, x, y, z]$ to $[0, \pm 1, 0, 0]$ where the sign depends on the sign of x/t if $t \neq 0$ and the sign of x if $t = 0$.*

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- *$g_{\lambda,k}|(\mathbf{S}^2_0 \cap \mathscr{H}) - \eta_-$ converges in the compact open topology to a rational map Π_1 given by sending $[t, 0, y, z]$ to $[0, 0, 1, 0]$.*

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- *For a properly convex compact set K in $\mathscr{H} - \eta_-$, the geometric limit of a subsequence of $\{g_{\lambda,k}(K)\}$ as $\lambda, k \to \infty$, is either a point $[0, 1, 0, 0]$ or $[0, -1, 0, 0]$ or the segment η_+.*

Proposition 5.3 (Properness of the action on the bordification)

Let Γ be a discrete group of orientation-preserving fg. Lorentzian isometries acting freely and properly discontinuously on $\mathsf{E}^{2,1}$ isomorphic to a free group of finite rank ≥ 2 with $\tilde{\Sigma}$ as determined above. Assuming the positive diffused Margulis invariants:
Then Γ acts freely and properly discontinuously on $\mathsf{E}^{2,1} \cup \tilde{\Sigma}$ as a group of projective automorphisms of $\mathbf{S}^3$.

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- Proof: Suppose that there exists a sequence $\{g_i\}$ of elements of Γ and a compact subset K of $\mathsf{E}^{2,1} \cup \tilde{\Sigma}$ so that

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- Recall that the Fuchsian Γ-action on the boundary $\mathrm{bd}\mathbb{S}_+$ of the standard disk $\mathbb{S}_+$ in $\mathbb{S}$ forms a discrete convergence group:

Choosing the coordinatization of each g_i.

- For every sequence g_j in Γ, there is a subsequence g_{j_k} and two (not necessarily distinct) points a, b in the circle $\mathrm{bd}\mathbb{S}_+$ such that
  - the sequences $g_{j_k}(x) \to a$ locally uniformly in $\mathrm{bd}\mathbb{S}_+ - \{b\}$.
  - $g_{j_k}^{-1}(y) \to b$ locally uniformly on $\mathrm{bd}\mathbb{S}_+ - \{a\}$ respectively as $k \to \infty$. (See [1] for details.) We may assume $a \neq b$.

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- We compute

$$\nu_i := \frac{\rho_i \times \alpha_i}{|||\rho_i \times \alpha_i|||}$$

.

- Since we have $\{a_i\} \to a$, we obtain that the sequence $\overline{a_i[\nu_i]a_{i,-}} = \mathrm{Cl}(\varepsilon(a_i))$ converges to a segment $\overline{a[\nu]a_-} = \mathrm{Cl}(\varepsilon(a))$ where $[\nu]$ is the direction of

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  for nonzero vectors α and β corresponding to a and b respectively.
- Since the geodesics with end points a_i, r_i pass the bounded part of the unit tangent bundle of $\mathbb{S}_+$, it follows that L_{g_i} are convergent as well by Proposition 4.2.
- Each L_{g_i} pass a point p_i, and $\{p_i\}$ forms a convergent sequence in $\mathsf{E}^{2,1}$. By choosing a subsequence, we assume wlg $p_i \to p_\infty$ for $p_\infty \in \mathsf{E}^{2,1}$.

The coordinate changes so that g_i becomes one of form in equation 4 from a converging subsequence

- We now introduce $h_i \in \mathrm{Aut}(\mathbf{S}^3)$ coodinatizing $\mathbf{S}^3$ for each i. We choose h_i so that

$$\begin{aligned} h_i(p_i) &= [1,0,0,0], h_i(a_i) = [0,1,0,0], \\ h_i(b_i) &= [0,0,0,1], \text{ and } h_i([\nu_i]) = [0,0,1,0]. \end{aligned} \tag{6}$$

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Lemma 5.4

By conjugating g_i by h_i as defined above, we have

$$\lambda(g_i) \to +\infty, k(g_i) \to +\infty, \text{ and } \frac{k(g_i)}{\lambda(g_i)} \to 0. \tag{7}$$

The conclusion of the proof of Proposition 5.3.

- Let $\mathbf{S}_i^0$ denote the sphere containing the weak stable plane of g_i, and $\mathbf{S}_i^+$ the sphere containing the stable plane of g_i. The sequences of these both geometrically converge.
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- For the compact set K, we cover it by convex open balls $B_j, j = 1, \dots, K$, of two types: Ones that are at least ϵ away from $\mathbf{S}_i^0$ for $i > I_0$ and ones that are dumbel types with the two parts at least $\epsilon/2$ away from $\mathbf{S}^0$ for $i > I_0$.

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- Then under g_i, the sequences of images of balls will converge to a or a_- and the sequences of images of the dumbels will converge to $\overline{a[\nu]a_-}$.

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- Then under g_i, the sequences of images of balls will converge to a or a_- and the sequences of images of the dumbels will converge to $\overline{a[\nu]a_-}$.
- The coordinate change by h_i will verify this.
- Thus, for every small compact ball B_j, we have $g_i(B_j) \cap B_k = \emptyset$ for $i > J^{j,k}$. For $J = \max\{J^{j,k}\}_{j=1,\dots,K,k=1,\dots,K}$, we have $g_i(K) \cap K = \emptyset$ for $i > J$.

The proof of Tameness

- Thus, $\tilde{\Sigma}/\Gamma$ is a closed surface of genus g and the boundary of the 3-manifold $M := (\mathrm{E}^{2,1} \cup \tilde{\Sigma})/\Gamma$ by Proposition 5.3. We now show that M is compact.

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Proposition 5.5

Each simple closed curve γ in $\tilde{\Sigma}$ bounds a simple disk in $\mathrm{E}^{2,1} \cup \tilde{\Sigma}$. Let c be a simple closed curve in Σ that is homotopically trivial in M. Then c bounds an imbedded disk in M.

Proof.

This is just Dehn's lemma. □

A system of circles

- We can find a collection of disjoint simple curves γ_i, $i \in \mathcal{J}$, on $\tilde{\Sigma}$ for an index set $\mathcal{J}$ so that the following hold:
  - $\bigcup_{i \in \mathcal{J}} \gamma_i$ is invariant under Γ.

A system of circles

- We can find a collection of disjoint simple curves γ_i, $i \in \mathcal{J}$, on $\tilde{\Sigma}$ for an index set $\mathcal{J}$ so that the following hold:
  - $\bigcup_{i\in\mathcal{J}} \gamma_i$ is invariant under Γ.
  - $\bigcup_{i\in\mathcal{J}} \gamma_i$ cuts $\tilde{\Sigma}$ into a union of open pair-of-pants P_k, $k \in K$, for an index set K. The closure of each P_k is a closed pair-of-pants.
  - $\{P_k\}_{k\in K}$ is a Γ-invariant set.

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  - $\{P_k\}_{k\in K}$ is a Γ-invariant set.
  - Under the covering map $\pi : \tilde{\Sigma} \to \tilde{\Sigma}/\Gamma$, each γ_i for $i \in I$ maps to a simple closed curve in a one-to-one manner and each P_k for $k \in K$ maps to an open pair-of-pants as a homeomorphism.

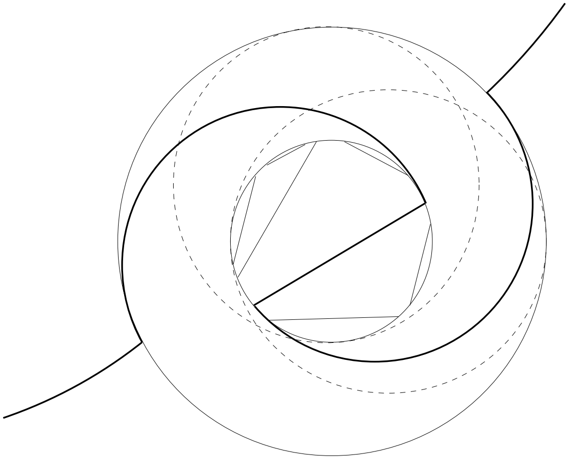

Figure : The arcs in $\mathbb{S}_+$ and an example of $\hat{\gamma}_i$ in the bold arcs.

Corollary 6.1

In $E^{2,1}$, there exists a Γ-invariant nonempty convex open domain $\mathcal{D}$ whose boundary in $E^{2,1}$ is asymptopic to $\mathrm{bd}D(\Lambda)$, homeomorphic to a circle. ($D(\Lambda)$ is the properly convex invariant set in $\mathbb{S}$ containing Λ.) There exists another Γ-invariant convex open domain $\mathcal{D}'$ whose boundary in $E^{2,1}$ is asymptotic to $\mathscr{A}(\mathrm{bd}D(\Lambda))$ so that the closures of $\mathcal{D}$ and $\mathcal{D}'$ are disjoint. Moreover, every weakly recurrent space-like geodesic is contained in a manifold

$$(E^{2,1} - \mathcal{D} - \mathcal{D}')/\Gamma$$

with concave boundary.

Remark: Mess first obtained these invariant domains (see also Barbot [3] for proof).

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Theorem 6.2

There exists a compact core in a Margulis space-time containing all weakly recurrent space-like geodesics.

Gracias!

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