1 Introduction

Preliminary

- Course home page: `http://math.kaist.ac.kr/~schoi/GT2010.html`
  Old `http://mathsci.kaist.ac.kr/~schoi/dgorb.htm`

Some advanced references for the course

- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977. (This is the principal source)
- W. Thurston, Three-dimensional geometry and topolgy, PUP, 1997
- R.W. Sharp, Differential geometry: Cartan's generalization of Klein's Erlangen program.
- T. Ivey and J.M. Landsberg, Cartan For Beginners: Differential geometry via moving frames and exterior differential systems, GSM, AMS
- G. Bredon, Introduction to compact transformation groups, Academic Press, 1972.
- M. Berger, Geometry I, Springer
- S. Kobayashi and Nomizu, Foundations of differential geometry, Springer.

Outline of the orbifolds part

- See the book introduction also.
- Manifolds and differentiable structures: Background materials..
- Lie groups and geometry: Geometry and discrete groups
- Topology of orbifolds: topology and covering spaces.
- The topology of 2-orbifolds: cutting and pasting, classification (not complete yet)
- The geometry of orbifolds
- The deformation space of hyperbolic structures on 2-orbifolds.
- Note that the notes are incomplete... I will try to correct as we go along. The orders may change...

Helpful preliminary knowledge for this chapter:

- Hatcher's "Algebraic topology" Chapters 0,1. (better with Chapter 2...) http://www.math.cornell.edu/~hatcher/AT/ATpage.html
- "Introduction to differentiable manifolds" by Munkres
- "Foundations of differentiable manifolds and Lie groups," by F. Warner.
- "Riemannian manfolds" by Do Carmo.
- S. Kobayashi and Nomizu, Foundations of differential geometry, Springer.
- R. Bishop and R. Crittendon, Geometry of manifolds.
- W. Thurston, Three-dimensional geometry and topology, Princeton Univ. press.
- W. Thurston, Geometry and Topology of 3-manifolds http://www.msri.org/publications/books/gt3m
- Section 1: Manifolds and differentiable structures (Intuitive account)
  - Manifolds
  - Simplicial manifolds
  - Lie groups.
  - Pseudo-groups and $\mathcal{G}$-structures.
  - Differential geometry and $\mathcal{G}$-structures.
  - Principal bundles and connections, flat connections
- Section 2: Lie groups and geometry
  - Projective geometry and conformally flat geometry
  - Euclidean geometry
  - Spherical geometry
  - Hyperbolic geometry and three models
  - Discrete groups: examples

2 Manifolds and differentiable structures (Intuitive account)

2.1 Aim

- The following theories for manifolds will be transfered to the orbifolds. We will briefly mention them here as a "review" and will develop them for orbifolds later (mostly for 2-dimensional orbifolds).

- We follow coordinate-free approach to differential geometry. We do not need to actually compute curvatures and so on.
  - $\mathcal{G}$-structures
  - Covering spaces
  - Riemanian manifolds and constant curvature manifolds
  - Lie groups and group actions
  - Principal bundles and connections, flat connections

2.2 Manifolds

Topological spaces.

- Quotient topology
- We will mostly use cell-complexes: Hatcher's AT P. 5-7 (Consider finite ones for now.)
- Operations: products, quotients, suspension, joins; AT P.8-10

Manifolds.

- A topological n-dimensional manifold (n-manifold) is a Hausdorff space with countable basis and charts to Euclidean spaces E^n; e.g curves, surfaces, 3-manifolds.
- The charts could also go to a positive half-space H^n. Then the set of points mapping to R^{n-1} under charts is well-defined is said to be the boundary of the manifold. (By the invariance of domain theorem)
- $\mathbb{R}^n$, H^n themselves or open subsets of $\mathbb{R}^n$ or H^n.
- $\mathbf{S}^n$ the unit sphere in $\mathbb{R}^{n+1}$. (use `http://en.wikipedia.org/wiki/Stereographic_projection`
- $\mathbb{R}P^n$ the real projective space. (use affine patches)

Manifolds.

- An n-ball is a manifold with boundary. The boundary is the unit sphere $\mathbf{S}^{n-1}$.
- Given two manifolds M_1 and M_2 of dimensions m and n respectively. The product space $M_1 \times M_2$ is a manifold of dimension $m+n$.
- An annulus is a disk removed with the interior of a smaller disk. It is also homeomorphic to a circle times a closed interval.
- The n-dimensional torus T^n is homeomorphic to the product of n circles $\mathbf{S}^1$.
- 2-torus: `http://en.wikipedia.org/wiki/Torus`

More examples

- Let T_n be a group of translations generated by $T_i : x \mapsto x + e_i$ for each $i = 1, 2, .., n$. Then $\mathbb{R}^n/T_n$ is homeomorphic to T^n.
- A connected sum of two n-manifolds M_1 and M_2. Remove the interiors of two closed balls from M_i for each i. Then each M_i has a boundary component homeomorphic to $\mathbf{S}^{n-1}$. We identify the spheres.
- Take many 2-dimensional tori or projective plane and do connected sums. Also remove the interiors of some disks. We can obtain all compact surfaces in this way. `http://en.wikipedia.org/wiki/Surface`

2.3 Discrete group actions

Some homotopy theory (from Hatchers AT)

- X, Y topological spaes. A homotopy is a $f : X \times I \to Y$.
- Maps f and $g : X \to Y$ are *homotopic* if $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all x. The homotopic property is an equivalence relation.
- Homotopy equivalences of two spaces X and Y is a map $f : X \to Y$ with a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to I_X and I_Y respectively.

Fundamental group (from Hatchers AT)

- A path is a map $f : I \to X$.
- A linear homotopy in $\mathbb{R}^n$ for any two paths.
- A *homotopy class* is an equivalence class of homotopic map relative to endpoints.
- The fundamental group $\pi(X, x_0)$ is the set of homotopy class of path with endpoints x_0.
- The product exists by joining. The product gives us a group.
- A change of base-points gives us an isomorphism (not canonical)
- The fundamental group of a circle is $\mathbb{Z}$. Brouwer fixed point theorem
- Induced homomorphisms. $f : X \to Y$ with $f(x_0) = y_0$ induces $f_* : \pi(X, x_0) \to \pi(Y, y_0)$.

Van Kampen Theorem (AT page 43–50)

- Given a space X covered by open subsets A_i such that any two or three of them meet at a path-connected set, $\pi(X, *)$ is a quotient group of the free product $*\pi(A_i, *)$.
- The kernel is generated by $i_j^*(a)i_k^*(a)$ for any a in $\pi(A_i \cap A_j, *)$.
- For cell-complexes, these are useful for computing the fundamental group.
- If a space Y is obtained from X by attaching the boundary of 2-cells. Then $\pi(Y, *) = \pi(X, *)/N$ where N is the normal subgroup generated by "boundary curves" of the attaching maps.
- Bouquet of circles, surfaces,...

Covering spaces and discrete group actions

- Given a manifold M, a covering map $p : \tilde{M} \to M$ from another manifold $\tilde{M}$ is an onto map such that each point of M has a neighborhood O s.t. $p|p^{-1}(O) : p^{-1}(O) \to O$ is a homeomorphism for each component of $p^{-1}(O)$.
- The coverings of a circle.
- Consider a disk with interiors of disjoint smaller disks removed. Cut remove edges and consider...
- The join of two circles example: See Hatcher AT P.56–58
- Homotopy lifting: Given two homotopic maps to M, if one lifts to $\tilde{M}$ and so does the other.
- Given a map $f : Y \to M$ with $f(y_0) = x_0$, f lifts to $\tilde{f} : Y \to \tilde{M}$ so that $\tilde{f}(y_0) = \tilde{x}_0$ if $f_*(\pi(Y, y_0)) \subset p_*(\pi_*(\tilde{M}, \tilde{x}_0))$.

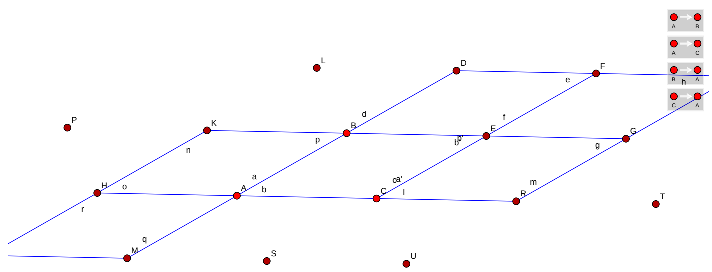
Covering spaces and discrete group actions

- The automorphism group of a covering map $p : M' \to M$ is a group of homeomorphisms $f : M' \to M'$ so that $p \circ f = f$. (also called deck transformation group.)
- $\pi_1(M)$ acts on $\tilde{M}$ on the right by path-liftings.
- A covering is *regular* if the covering map $p : M' \to M$ is a quotient map under the action of a discrete group Γ acting properly discontinuously and freely. Here M is homeomorphic to M'/Γ.
- One can classify covering spaces of M by the subgroups of $\pi(M, x_0)$. That is, two coverings of M are equal iff the subgroups are the same.
- Covering spaces are ordered by subgroup inclusion relations.
- If the subgroup is normal, the corresponding covering is regular.

- A manifold has a *universal covering*, i.e., a covering whose space has a trivial fundamental group. A universal cover covers every other coverings of a given manifold.

- $\tilde{M}$ has the covering automorphism group Γ isomorphic to $\pi_1(M)$. A manifold M equals $\tilde{M}/\Gamma$ for its universal cover $\tilde{M}$. Γ is a subgroup of the deck transformation group.

  - Let $\tilde{M}$ be $\mathbb{R}^2$ and T^2 be a torus. Then there is a map $p : \mathbb{R}^2 \to T^2$ sending (x, y) to $([x], [y])$ where $[x] = x \mod 2\pi$ and $[y] = y \mod 2\pi$.
  - Let M be a surface of genus 2. $\tilde{M}$ is homeomorphic to a disk. The deck transformation group can be realized as isometries of a hyperbolic plane.



2.4 Simplicial manifolds

Simplicial manifolds

- An n-simplex is a convex hull of $n + 1$-points (affinely independent). An n-simplex is homeomorphic to B^n.

- A simplicial complex is a locally finite collection S of simplices so that any face of a simplex is a simplex in S and the intersection of two elements of S is an element of S. The union is a topological set, a *polyhedron*.

- We can define barycentric subdivisions and so on.

- A link of a simplex σ is the simplicial complex made up of simplicies opposite σ in a simplex containing σ.

- An n-manifold X can be constructed by gluing n-simplices by face-identifications. Suppose X is an n-dimensional triangulated space. If the link of every p-simplex is homeomorphic to a sphere of $(n-p-1)$-dimension, then X is an n-manifold.
- If X is a simplicial n-manifold, we say X is orientable if we can give an orientations on each simplex so that over the common faces they extend each other.

2.5 Surfaces

Surfaces

Canonical construction
Given a polygon with even number of sides, we assign identification by labeling by alphabets $a_1, a_2, .., a_1^{-1}, a_2^{-1}, , , ,$ so that a_i means an edge labelled by a_i oriented counter-clockwise and a_i^{-1} means an edge labelled by a_i oriented clockwise. If a pair a_i and a_i or a_i^{-1} occur, then we identify them respecting the orientations.

- A bigon: We divide the boundary into two edges and identify by labels a, a^{-1}.
- A bigon: We divide the boundary into two edges and identify by labels a, a.
- A square: We identify the top segment with the bottom one and the right side with the left side. The result is a 2-torus.
- Any closed surface can be represented in this manner.
- A $4n$-gon. We label edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, ...a_n, b_n, a_n^{-1}, b_n^{-1}.$$

  The result is a connected sum of n tori and is orientable. The genus of such a surface is n.
- A $2n$-gon. We label edges $a_1a_1a_2a_2....a_nb_n$. The result is a connected sum of n projective planes and is not orientable. The genus of such a surface is n.
- The results are topological surfaces and a 2-dimensional simplicial manifold.
- We can remove the interiors of disjoint closed balls from the surfaces. The results are surfaces with boundary.

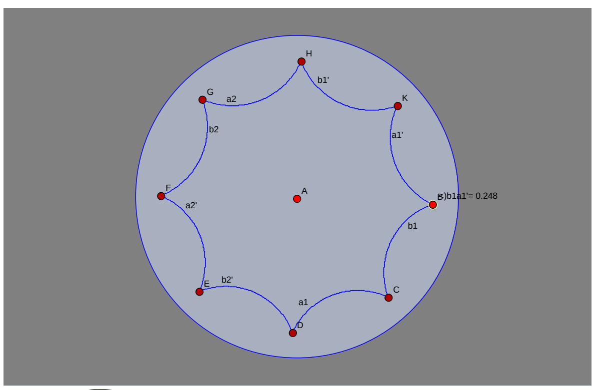
- The fundamental group of a surface can now be computed. A surface is a cell complex starting from a 1-complex which is a bouquet of circles and attached with a cell. (See AT P.51)

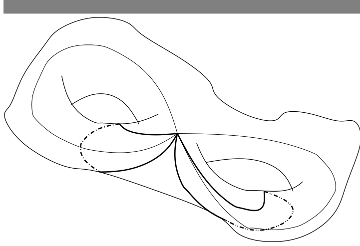
$$\pi(S) = \{a_1, b_1, ..., a_g, b_g | [a_1, b_1][a_2, b_2]...[a_g, b_g]\}$$

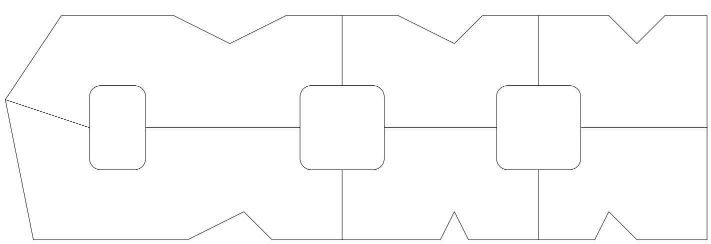
  for orientable S of genus g.

- An Euler characteristic of a simplicial complex is given by $E - F + V$. This is a topological invariant. We can show that the Euler characteristic of an orientable compact surface of genus g with n boundary components is $2 - 2g - n$.

- In fact, a closed orientable surface contains $3g - 3$ disjoint simple closed curves so that the complement of its union is a disjoint union of pairs of pants, i.e., spheres with three holes. Thus, a pair of pants is an "elementary" surface.







2.6 Lie groups

Section 1: Lie groups

- A Lie group is a space of symmetries of some space. More formally, a Lie group is a manifold with a group operation $\circ : G \times G \to G$ that satisfies
  - $\circ$ is smooth.
  - the inverse $\iota : G \to G$ is smooth also.
- Examples:
  - The permutation group of a finite set form a 0-dimensional manifold, which is a finite set.
  - $\mathbb{R}, \mathbb{C}$ with $+$ as an operation. ($\mathbb{R}^+$ with $+$ is merely a Lie semigroup.)
  - $\mathbb{R} - \{O\}, \mathbb{C} - \{O\}$ with $*$ as an operation.
  - $T^n = \mathbb{R}^n/\Gamma$ with $+$ as an operation and O as the equivalence class of $(0, 0, ..., 0)$. (The three are abelian ones.)
- - $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$: the general linear group.
  - $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | det(A) = 1\}$: the special linear group.
  - $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | A^T A = \mathrm{I}\}$: the orthogonal group.
  - $Isom(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n | T(x) = Ax + b \text{ for } A \in O(n-1, \mathbb{R}), b \in \mathbb{R}^n\}$.
  - Proofs: One can express the operations as polynomials or rational functions.
- Products of Lie groups are Lie groups.
- A covering space of a connected Lie group form a Lie group.
- A *Lie subgroup* of a Lie group is a subgroup that is a Lie group with the induced operation and is a submanifold.
  - $O(n) \subset SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$.
  - $O(n-1) \subset Isom(\mathbb{R}^n)$.
- A homomorphism $f : G \to H$ of two Lie groups G, H is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms.
- The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also.
- If G, H are simply connected, f induces a unique homomorphism of Lie algebra of G to that of H which is Df and conversely.

2.7 Lie algebras

Lie algebras

- A Lie algebra is a vector space V with an operation $[,] : V \times V \to V$ that satisfies:
  - $[x, x] = 0$ for $x \in L$. (Thus, $[x, y] = -[y, x]$.)
  - Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.
- Examples:
  - Sending $V \times V$ to O is a Lie algebra (abelian ones.)
  - Direct sums of Lie algebras is a Lie algebra.
  - A subalgebra is a subspace closed under $[,]$.
  - An ideal K of L is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in L$.
- A homomorphism of a Lie algebra is a linear map preserving $[,]$.
- The kernel of a homomorphism is an ideal.

Lie groups and Lie algebras

- Let G be a Lie group. A left translation $L_g : G \to G$ is given by $x \mapsto g(x)$.
- A left-invariant vector field of G is a vector field so that the left translation leaves it invariant, i.e., $dL_g(X(h)) = X(gh)$ for $g, h \in G$.
- The set of left-invariant vector fields form a vector space under addition and scalar multiplication and is vector-space isomorphic to the tangent space at I. Moreover, $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.
- The Lie algebra of G is the the Lie algebra of the left-invariant vector fields on G.
- Example: The Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to $gl(n, \mathbb{R})$:
  - For X in the Lie algebra of $GL(n, \mathbb{R})$, we can define a flow generated by X and a path $X(t)$ along it where $X(0) = \mathrm{I}$.
  - We obtain an element of $gl(n, \mathbb{R})$ by taking the derivative of $X(t)$ at 0 seen as a matrix.
  - The brackets are preserved.
  - A Lie algebra of an abelian Lie group is abelian.

Lie group actions

- A Lie group G-action on a smooth manifold X is given by a smooth map $G \times X \to X$ so that $(gh)(x) = (g(h(x))$ and $I(x) = x$. (left action)
- A right action satisfies $(x)(gh) = ((x)g)h$.
- Each Lie algebra element correspond to a vector field on X by using a vector field.
- The action is faithful if $g(x) = x$ for all x, then g is the identity element of G.
- The action is transitive if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.
- Example:
  - $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$.
  - $PGL(n+1, \mathbb{R})$ acting on $\mathbb{R}P^n$.

Lie algebras

- Given X in the Lie algebra $\mathfrak{g}$ of G, there is an integral curve $X(t)$ through I. We define the exponential map $\exp : \mathfrak{g} \to G$ by sending X to $X(1)$.
- The exponential map is defined everywhere, smooth, and is a diffeomorphism near O.
- The matrix exponential defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{1}{k!} A^k$$

  is the exponential map $gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$.

3 Pseudo-group and $\mathcal{G}$-structures

Pseudo-groups

- In this section, we introduce pseudo-groups.
- However, we are mainly interested in classical geometries, Clifford-Klein geometries. We will be concerned with Lie group G acting on a manifold M.
- Most obvious ones are euclidean geometry where G is the group of rigid motions acting on the euclidean space $\mathbb{R}^n$. The spherical geometry is one where G is the group $O(n+1)$ of orthogonal transformations acting on the unit sphere $\mathbf{S}^n$.

Pseudo-groups

- Topological manifolds form too large a category to handle.
- To restrict the local property more, we introduce *pseudo-groups*. A *pseudo-group* $\mathcal{G}$ on a topological space X is the set of homeomorphisms between open sets of X so that
  - The domains of $g \in \mathcal{G}$ cover X.
  - The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
  - The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
  - The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
  - If $g : U \to V$ is a homeomorphism for U, V open subsets of X. If U is a union of open sets U_α for $\alpha \in I$ for some index set I such that $g|U_\alpha$ is in $\mathcal{G}$ for each α, then g is in $\mathcal{G}$.

- The trivial pseudo-group is one where every element is a restriction of the identity $X \to X$.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is TOP, the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group C^r, $r \geq 1$, of the set of C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group PL of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- (G, X)-pseudo group. Let G be a Lie group acting on a manifold X. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ where U is the set of all open subsets of X.
- The group $\mathrm{isom}(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or orthogonal group $O(n+1, \mathbb{R})$ acting on $\mathbf{S}^n$ give examples.

3.1 $\mathcal{G}$-manifold

$\mathcal{G}$-manifold

$\mathcal{G}$-manifold is obtained as a manifold glued with special type of gluings only in $\mathcal{G}$.

- Let $\mathcal{G}$ be a pseudo-group on $\mathbb{R}^n$. A $\mathcal{G}$-manifold is a n-manifold M with a maximal $\mathcal{G}$-atlas.

- A $\mathcal{G}$-atlas is a collection of charts (imbeddings) $\phi : U \to \mathbb{R}^n$ where U is an open subset of M such that whose domains cover M and any two charts are $\mathcal{G}$-compatible.
  - Two charts $(U, \phi), (V, \psi)$ are $\mathcal{G}$-*compatible* if the transition map
$$\gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}.$$

- One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely.
- A $\mathcal{G}$-map $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism so that if f sends a domain of a chart ϕ into a domain of a chart ψ, then
$$\psi \circ f \circ \phi^{-1} \in \mathcal{G}.$$
That is, f is an element of $\mathcal{G}$ locally up to charts.

3.2 Examples

Examples

- $\mathbb{R}^n$ is a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on $\mathbb{R}^n$.
- $f : M \to N$ be a local homeomorphism. If N has a $\mathcal{G}$-structure, then so does M so that the map in a $\mathcal{G}$-map. (A class of examples such as θ-annuli and π-annuli.)
- Let Γ be a discrete group of $\mathcal{G}$-homeomorphisms of M acting properly and freely. Then M/Γ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets.
- T^n has a C^r-structure and a PL-structure.
- Given (G, X) as above, a (G, X)-manifold is a $\mathcal{G}$-manifold where $\mathcal{G}$ is the restricted pseudo-group.
- A euclidean manifold is a $(\mathrm{isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.
- A spherical manifold is a $(O(n+1), \mathbf{S}^n)$-manifold.

4 Differential geometry and $\mathcal{G}$-structures

Differential geometry and $\mathcal{G}$-structures

- We wish to understand geometric structures in terms of differential geometric setting; i.e., using bundles, connections, and so on.
- Such an understanding help us to see the issues in different ways.
- Actually, this is not central to the lectures. However, we should try to relate to the traditional fields where our subject can be studied in another way.
- We will say more details later on.

4.1 Riemannian manifolds

Riemanian manifolds and constant curvature manifolds.

- A differentiable manifold has a Riemannian metric, i.e., inner-product at each tangent space smooth with respect coordinate charts. Such a manifold is said to be a Riemannian manifold.
- An isometric immersion (imbedding) of a Riemannian manifold to another one is a (one-to-one) map that preserve the Riemannian metric.
- We will be concerned with isometric imbedding of M into itself usually.
- A length of an arc is the value of an integral of the norm of tangent vectors to the arc. This gives us a metric on a manifold. An isometric imbedding of M into itself is an isometry always.
- A geodesic is an arc minimizing length locally.

- A constant curvature manifold is one where the sectional curvature is identical to a constant in every planar direction at every point.
- Examples:
  - A euclidean space E^n with the standard norm metric has a constant curvature $= 0$.
  - A sphere $\mathbf{S}^n$ with the standard induced metric from $\mathbb{R}^{n+1}$ has a constant curature $= 1$.
  - Find a discrete torsion-free subgroup Γ of the isometry group of E^n (resp. $\mathbf{S}^n$). Then E^n/Γ (resp. $\mathbf{S}^n/\Gamma$) has constant curvature $= 0$ (resp. 1).

4.2 Lie groups and group actions

Lie groups and group actions.

- A Lie group is a smooth manifold G with an associative smooth product map $G \times G \to G$ with identity and a smooth inverse map $\iota : G \to G$. (A Lie group is often the set of symmetries of certain types of mathematical objects.)
- For example, the set of isometries of $\mathbf{S}^n$ form a Lie group $O(n+1)$, which is a classical group called an orthogonal group.
- The set of isometries of the euclidean space $\mathbb{R}^n$ form a Lie group $\mathbb{R}^n \cdot O(n)$, i.e., an extension of $O(n)$ by a translation group in $\mathbb{R}^n$.

- Simple Lie groups are classified. Examples $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $O(n,\mathbb{R})$, $O(n,m)$, $GL(n,\mathbb{C})$, $U(n)$, $SU(n)$, $SP(2n,\mathbb{R})$, $Spin(n)$,....
- An action of a Lie group G on a space X is a map $G\times X \to X$ so that $(gh)(x) = g(h(x))$.
- For each $g \in G$, g gives us a map $g : X \to X$ where the identity element correspond to the identity map of X.
- Examples: $\mathbb{R}^n \cdot O(n)$ on $\mathbb{R}^n$ and $O(n)$ on $\mathbf{S}^n$.

4.3 Principal bundles and connections, flat connections

Principal bundles and connections, flat connections

- Let M be a manifold and G a Lie group. A principal fiber bundle P over M with a group G:
  - P is a manifold.
  - G acts freely on P on the right. $P \times G \to P$.
  - $M = P/G$. $\pi : P \to M$ is differentiable.
  - P is locally trivial. $\phi : \pi^{-1}(U) \to U \times G$.
- Example 1: $L(M)$ the set of frames of $T(M)$. $GL(n,\mathbb{R})$ acts freely on $L(M)$. $\pi : L(M) \to M$ is a principal bundle.
- P a bundle space, M the base space. $\pi^{-1}(x)$ a fiber.
- $\pi^{-1}(x) = \{ug|g \in G\}$.

- A bundle can be constructed by mappings
$$\{\phi_{\beta,\alpha} : U_\alpha \cap U_\beta \to G | U_\alpha, U_\beta \text{"trivial" open subsets of} M\}$$
so that
$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}$$
for any triple $U_\alpha, U_\beta, U_\gamma$.
- G', G Lie groups. $f : G' \to G$ a monomorphism. $P(G', M) \to P(G, M)$ inducing identity $M \to M$ is called a reduction of the structure group G to G'. There maybe many reductions for given G' and G.
- $P(G, M)$ is reducible to $P(G', M)$ if and only if $\phi_{\alpha,\beta}$ can be taken to be in G'. (See Kobayashi-Nomizu, Bishop-Crittendon for details.)

Associated bundles

- Associated bundle: Let F be a manifold with a left-action of G.
- G acts on $P \times F$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in M, x \in F.$$

- The quotient space $E = P \times_G F$.
- π_E is induced and $\pi_E^{-1}(U) = U \times F$. The structure group is the same.
- Example: Tangent bundle $T(M)$. $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$. Let $F = \mathbb{R}^n$. Obtain $L(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$.
- Example: Tensor bundltes $T_s^r(M)$. $GL(n, \mathbb{R})$ acts on $T_s^r(\mathbb{R})$. Let $F = T_s^r(\mathbb{R})$.

Connections

- $P(M, G)$ a principal bundle.
- A connection decomposes each $T_u(P)$ for each $u \in P$ into
  - $T_u(P) = G_u \oplus Q_u$ where G_u is a subspace tangent to the fiber. (G_u the vertical space, Q_u the horizontal space.)
  - $Q_{ug} = (R_g)_* Q_u$ for each $g \in G$ and $u \in P$.
  - Q_u depend smoothly on u.
- A *horizontal* lift of a piecewise-smooth path τ on M is a piecewise-smooth path τ' lifting τ so that the tangent vectors are all horizontal.
- A horizontal lift is determined once the initial point is given.

- Given a curve on M with two endpoints, the lifts defines a parallel displacement between fibers above the two endpoints. (commuting with G-actions).
- Fixing a point x_0 on M, this defines a holonomy group.
- The curvature of a connection is a measure of how much a horizontal lift of small loop in M is a loop in P.
- The flat connection: In this case, we can lift homotopically trivial loops in M^n to loops in P. Thus, the flatness is equivalent to local lifting of coordinate chart of M to horizontal sections in P.
- A flat connection on P gives us a smooth foliation of dimension n transversal to the fibers.

- The associated bundle E also inherits a connection and hence horizontal lifings.
- The flatness is also equivalent to the local lifting property.
- The flat connection on E gives us a smooth foliation of dimension n transversal to the fibers.
- Summary: A connection gives us a way to identify fibers given paths on X-bundles over M. The flatness gives us locally consistent identifications.

The principal bundles and G-structures.

- Given a manifold M of dimension n, a Lie group G acting on a manifold X of dimension n.
- We form a principal bundle P and then the associated bundle E fibered by X with a flat connection.
- A section $f : M \to E$ which is transverse everywhere to the foliation given by the flat connection.
- This gives us a (G, X)-structure and coversely a (G, X)-structure gives us P, E, f and the flat connection.