Topology of orbifolds II

S. Choi

[1] Department of Mathematical Science
KAIST, Daejeon, South Korea

Lectures at KAIST

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Outline

- Section 3: Topology of orbifolds
  - Topology of orbifolds
    - Definitions,
    - Orbifold maps, singular set,
    - Examples
    - Abstract definitions using groupoid.
    - Smooth structures, fiber bundles, and Riemannian metrics
    - Gauss-Bonnet theorem (due to Satake)
    - Smooth 2-orbifolds and triangulations
  - Covering spaces
    - Fiber-product approach
    - Path-approach by Haefliger

Some helpful references

- I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA. 42, 359–363 (1956)
- I. Satake, The Gauss-Bonnet theorm for V-manifolds, J. Math. Soc. Japan, 9, 464–492 (1957)
- W. Thuston, Orbifolds and Seifert space, Chapter 5, notes
- A. Adem, J. Leida, and Y. Ruan, Orbifolds and stringy topology, Cambridge, 2007.
- J. Ratcliffe, Chapter 13 in Foundations of hyperbolic manifolds, Springer]

Some helpful references

- I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA. 42, 359–363 (1956)
- I. Satake, The Gauss-Bonnet theorm for V-manifolds, J. Math. Soc. Japan, 9, 464–492 (1957)
- W. Thuston, Orbifolds and Seifert space, Chapter 5, notes
- A. Adem, J. Leida, and Y. Ruan, Orbifolds and stringy topology, Cambridge, 2007.
- J. Ratcliffe, Chapter 13 in Foundations of hyperbolic manifolds, Springer]

Some helpful references

- I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA. 42, 359–363 (1956)
- I. Satake, The Gauss-Bonnet theorm for V-manifolds, J. Math. Soc. Japan, 9, 464–492 (1957)
- W. Thuston, Orbifolds and Seifert space, Chapter 5, notes
- A. Adem, J. Leida, and Y. Ruan, Orbifolds and stringy topology, Cambridge, 2007.
- J. Ratcliffe, Chapter 13 in Foundations of hyperbolic manifolds, Springer]

Some helpful references

- I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA. 42, 359–363 (1956)
- I. Satake, The Gauss-Bonnet theorm for V-manifolds, J. Math. Soc. Japan, 9, 464–492 (1957)
- W. Thuston, Orbifolds and Seifert space, Chapter 5, notes
- A. Adem, J. Leida, and Y. Ruan, Orbifolds and stringy topology, Cambridge, 2007.
- J. Ratcliffe, Chapter 13 in Foundations of hyperbolic manifolds, Springer]

Some helpful references

- I. Satake, On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA. 42, 359–363 (1956)
- I. Satake, The Gauss-Bonnet theorm for V-manifolds, J. Math. Soc. Japan, 9, 464–492 (1957)
- W. Thuston, Orbifolds and Seifert space, Chapter 5, notes
- A. Adem, J. Leida, and Y. Ruan, Orbifolds and stringy topology, Cambridge, 2007.
- J. Ratcliffe, Chapter 13 in Foundations of hyperbolic manifolds, Springer]

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- M. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grad. Texts in Math. 319, Springer-Verlag, New York, 1999.
- A. Haefliger, Orbi-espaces, In: Progr. Math. 83, Birkhauser, Boston, MA, 1990, pp. 203–212.
- M. Kato, On Uniformization of orbifolds, Adv. Studies in Pure Math. 9, 149–172 (1986)
- Y. Matsumoto and J. Montesinos-Amilibia, A proof of Thurston's uniformization theorem of geometric orbifolds, Tokyo J. Mathematics 14, 181–196 (1991)
- I. Moerdijk, Orbifolds as groupoids: an introduction. math.DG./0203100v1
- S. Choi, Geometric Structures on Orbifolds and Holonomy Representations, Geometriae Dedicata 104: 161–199, 2004.

Some helpful references

- A. Verona, Stratified mappings—structure and triangulability. Lecture Notes in Mathematics, 1102. Springer-Verlag, 1984. ix+160 pp.
- S. Weinberger, The topological classification of stratified spaces. Chicago Lectures in Mathematics. University of Chicago Press, 1994. xiv+283 pp.

Some helpful references

- A. Verona, Stratified mappings—structure and triangulability. Lecture Notes in Mathematics, 1102. Springer-Verlag, 1984. ix+160 pp.
- S. Weinberger, The topological classification of stratified spaces. Chicago Lectures in Mathematics. University of Chicago Press, 1994. xiv+283 pp.

Definitions

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- X a Hausdorff second countable topological space. Let n be fixed.
- An open subset $\tilde{U}$ in $\mathbb{R}^n$ with a finite group G acting smoothly on it. A G-invariant map $\tilde{U} \to O$ for an open subset O of X inducing a homeomorphism $\tilde{U}/G \to O$. An orbifold chart is such a triple $(\tilde{U}, G, \phi)$.
- An embedding $i : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth imbedding $i : \tilde{U} \to \tilde{V}$ with $\phi = \psi \circ i$ which induces the inclusion map $U \to V$ for $U = \phi(\tilde{U})$ and $V = \phi(\tilde{V})$.
  - Equivalently, i is an imbedding inducing the inclusion map $U \to V$ and inducing an injective homomorphism $i^* : G \to H$ so that $i \circ g = i^*(g) \circ i$ for every $g \in G$. $i^*(G)$ will act on the open set that is the image of i.
  - Note here i can be changed to $h \circ i$ for any $h \in H$. The images of $h \circ i$ will be disjoint for representatives h for $H/i^*(G)$.

Definitions

- Two charts $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$ are *compatible* if for every $x \in U \cap V$, there is an open neighborhood W of x in $U \cap V$ and a chart $(\tilde{W}, K, \mu)$ such that there are embeddings to $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$. (One can assume W is a component of $U \cap V$.)
- If we allow $\tilde{U}$ to be an open subset of the closed upper half space, then the orbifold has boundary.

Definitions

- Two charts $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$ are *compatible* if for every $x \in U \cap V$, there is an open neighborhood W of x in $U \cap V$ and a chart $(\tilde{W}, K, \mu)$ such that there are embeddings to $(\tilde{U}, \phi)$ and $(\tilde{V}, \psi)$. (One can assume W is a component of $U \cap V$.)
- If we allow $\tilde{U}$ to be an open subset of the closed upper half space, then the orbifold has boundary.

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definition of orbifold

- Since G acts smoothly, G acts freely on an open dense subset of $\tilde{U}$.
- An *orbifold atlas* on X is a family of compatible charts $\{(\tilde{U}, \phi)\}$ covering X.
- Two orbifold atlases are *compatible* if charts in one atlas are compatible with charts in the other atlas.
- Atlases form a partially ordered set. It has a maximal element.
- Given an atlas, there is a unique maximal atlas containing it.
- An orbifold is X with a maximal orbifold atlas.
- One can obtain an atlas of linear charts only: that is, charts where $\tilde{U}$ is $\mathbb{R}^n$ and $G \subset O(n)$. That is, for each point, one can find a subgroup G_x stablizing the point and suitable G_x-invariant neighborhood in $\tilde{U}$. Then G_x acts linearly up to a choice of coordinate charts since smooth action is locally smooth (linear).

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
  - This is independently defined up to conjugacy for any choice of y.
  - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
  - This is independently defined up to conjugacy for any choice of y.
  - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
  - This is independently defined up to conjugacy for any choice of y.
  - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
  - This is independently defined up to conjugacy for any choice of y.
  - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
    - This is independently defined up to conjugacy for any choice of y.
    - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- If we have $\tilde{U}$ with G acting freely, we can drop this from the atlas and replace with many charts with trivial group.
- A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is smooth if for each point $x \in X$, there is a chart $(\tilde{U}, G, \phi)$ with $x \in U$ and a chart $(\tilde{V}, H, \psi)$ with $f(x) \in V$ so that $f(V) \subset U$ and f lifts to $\tilde{f} : \tilde{U} \to \tilde{V}$ as a smooth map.
- Two orbifolds are diffeomorphic if there is a smooth orbifold-map with a smooth inverse orbifold-map.
- $x \in X$. A local group G_x of x is obtained by taking a chart $(\tilde{U}, G, \phi)$ around x and finding the stabilizer G_y of y for an inverse image point y of x.
  - This is independently defined up to conjugacy for any choice of y.
  - Smaller charts will give you the same conjugacy class. Thus, one can take a linear chart. Once a linear chart is achieved, G_x is well-defined up to conjugacy (Thus, as an abstract group with an action.)

Definitions

- A singular set is a set of points where G_x is not trivial.
- The subset of the singular set where G_x is constant is a relatively closed submanifold.
- Thus X becomes a stratified smooth topological space where the strata is given by the conjugacy classes of G_x.
- A *suborbifold* Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P. (Compare with P. 35 of Adem.)

Definitions

- A singular set is a set of points where G_x is not trivial.
- The subset of the singular set where G_x is constant is a relatively closed submanifold.
- Thus X becomes a stratified smooth topological space where the strata is given by the conjugacy classes of G_x.
- A *suborbifold* Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P. (Compare with P. 35 of Adem.)

Definitions

- A singular set is a set of points where G_x is not trivial.
- The subset of the singular set where G_x is constant is a relatively closed submanifold.
- Thus X becomes a stratified smooth topological space where the strata is given by the conjugacy classes of G_x.
- A *suborbifold* Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P. (Compare with P. 35 of Adem.)

Definitions

- A singular set is a set of points where G_x is not trivial.
- The subset of the singular set where G_x is constant is a relatively closed submanifold.
- Thus X becomes a stratified smooth topological space where the strata is given by the conjugacy classes of G_x.
- A *suborbifold* Y of an orbifold X is an imbedded subset such that for each point y in Y and and a chart $(\tilde{V}, G, \phi)$ of X for a neighborhood V of y there is a chart for y given by $(P, G|P, \phi)$ where P is a closed submanifold of $\tilde{V}$ where G acts on and $G|P$ is the image of the restriction homomorphism of G to P. (Compare with P. 35 of Adem.)

Examples

- Clearly, manifolds are orbifolds.
- Let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure.
- Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.
- Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

Examples

- Clearly, manifolds are orbifolds.
- Let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure.
- Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.
- Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

Examples

- Clearly, manifolds are orbifolds.
- Let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure.
- Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.
- Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

Examples

- Clearly, manifolds are orbifolds.
- Let G be a finite group acting on a manifold M smoothly. Then M/G is a topological space with an orbifold structure.
- Let $M = T^n$ and $\mathbb{Z}_2$ act on it with generator acting by $-I$. For $n = 2$, $M/\mathbb{Z}_2$ is topologically a sphere and has four singular points. For $n = 4$, we obtain a Kummer surface with sixteen singular points.
- Let X be a smooth surface. Take a discrete subset. For each point, take a disk neighborhood D with a chart (D', Z_n, q) where D' is a disk and Z_n acts as a rotation with O as a fixed point and $q : D' \to D$ as a cyclic branched covering.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - The interior is given charts with trivial groups.
  - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
  - The corner point is given charts with a dihedral group as a group.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - The interior is given charts with trivial groups.
  - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
  - The corner point is given charts with a dihedral group as a group.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - The interior is given charts with trivial groups.
  - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
  - The corner point is given charts with a dihedral group as a group.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
    - The interior is given charts with trivial groups.
    - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
    - The corner point is given charts with a dihedral group as a group.

Examples

- Given a manifold M with boundary. We can double it as a manifold and obtain $\mathbb{Z}_2$-action. Then M has an orbifold structure.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - The interior is given charts with trivial groups.
  - The interior of a boundary curve is given charts with $\mathbb{Z}_2$ as a group. (silvering)
  - The corner point is given charts with a dihedral group as a group.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
    - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
    - If two such arcs meet, then the vertex is given a dihedral group as a group.
    - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
    - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
  - If two such arcs meet, then the vertex is given a dihedral group as a group.
  - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
  - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
  - If two such arcs meet, then the vertex is given a dihedral group as a group.
  - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
  - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
    - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
    - If two such arcs meet, then the vertex is given a dihedral group as a group.
    - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
    - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
  - If two such arcs meet, then the vertex is given a dihedral group as a group.
  - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
  - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

Examples

- An embedded arc in the surface orbifold as above ending at two silvered boundary points is a one-dimensional suborbifold.
- Take a surface and make the boundary be a union of piecewise smooth curves with corners.
  - Some arcs are given $\mathbb{Z}_2$ as groups but not all.
  - If two such arcs meet, then the vertex is given a dihedral group as a group.
  - Then the union of the interiors of the remaining arcs is the boundary of the orbifold.
  - A nicely imbedded arc ending at a corner may not be a suborbifold unless it is in the boundary of the surface.

An abstract definition using Lie groupoid

- We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach.
- A topological groupoid consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps: the source map $s : G_1 \to G_0$, target map $t : G_1 \to G_0$, an associative composition map $m : G_1\,{}_s\times_t G_1 \to G_1$ a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(y)$ and $gu(x) = g = u(x)g$ and an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.
- A Lie groupoid is one where G_0 and G_1 are smooth manifolds.
- M a smooth manifold. Let $G_0 = G_1 = M$ and all maps identity, then this is a *unit groupoid*.

An abstract definition using Lie groupoid

- We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach.
- A topological groupoid consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps: the source map $s : G_1 \to G_0$, target map $t : G_1 \to G_0$, an associative composition map $m : G_1 {}_s\times_t G_1 \to G_1$ a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(y)$ and $gu(x) = g = u(x)g$ and an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.
- A Lie groupoid is one where G_0 and G_1 are smooth manifolds.
- M a smooth manifold. Let $G_0 = G_1 = M$ and all maps identity, then this is a *unit groupoid*.

An abstract definition using Lie groupoid

- We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach.
- A topological groupoid consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps: the source map $s : G_1 \to G_0$, target map $t : G_1 \to G_0$, an associative composition map $m : G_1\,{}_s\times_t\, G_1 \to G_1$ a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(y)$ and $gu(x) = g = u(x)g$ and an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.
- A Lie groupoid is one where G_0 and G_1 are smooth manifolds.
- M a smooth manifold. Let $G_0 = G_1 = M$ and all maps identity, then this is a *unit groupoid*.

An abstract definition using Lie groupoid

- We will try to avoid the definitions using the category theory as related to the theory of stacks in algebraic geometry as much as possible and use the more concrete set theoretic approach.
- A topological groupoid consists of a space G_0 of objects and a space G_1 of arrows with five continuous maps: the source map $s : G_1 \to G_0$, target map $t : G_1 \to G_0$, an associative composition map $m : G_1 {}_s\times_t G_1 \to G_1$ a unit map $u : G_0 \to G_1$ so that $su(x) = x = tu(y)$ and $gu(x) = g = u(x)g$ and an inverse map $i : G_1 \to G_1$ so that if $g : x \to y$, then $i(g) : y \to x$ and $i(g)g = u(x)$ and $gi(g) = u(y)$.
- A Lie groupoid is one where G_0 and G_1 are smooth manifolds.
- M a smooth manifold. Let $G_0 = G_1 = M$ and all maps identity, then this is a *unit groupoid.*

More on Lie groupoid

- isotropy group at x is the set of all arrows from x to itself.
- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: $\phi : H \to G, \psi : K \to G$ the fiber product $H \times_G K$ is the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $\phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(k)g$.

More on Lie groupoid

- isotropy group at x is the set of all arrows from x to itself.
- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: $\phi : H \to G, \psi : K \to G$ the fiber product $H \times_G K$ is the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $\phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(k)g$.

More on Lie groupoid

- isotropy group at x is the set of all arrows from x to itself.
- A *homomorphism* of Lie groupoids $\phi : H \to G$ is a pair of smooth maps $\phi_0 : H_0 \to G_0$ and $\phi_1 : H_1 \to G_1$ commuting with all structure maps.
- The fiber-product: $\phi : H \to G, \psi : K \to G$ the fiber product $H \times_G K$ is the Lie groupoid whose objects are (y, g, z) for $y \in H_0, z \in K_0$, and arrow $\phi(y) \to \psi(z)$ and whose arrows $(y, g, z) \to (y', g', z')$ are pairs (h, k) of arrows $h : y \to y', k : z \to z'$ so that $g'\phi(h) = \psi(k)g$.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- ϕ is an equivalence if it is an etale map and
  - If ϕ_0 induces an isomorphism of stablizer group from x to $\phi_0(x)$.
  - If ϕ induces a bijection of orbit spaces.
- If G and G' are differentiable etale groupoid, then $\phi : G \to G'$ is a differentiable equivalence if ϕ_0 is an equivalence and is a local diffeomorphism.
- This generates an equivalence relation on groupoids.
- Two groupoids are equivalent iff they are Morita equivalent: i.e., there exists another pseudogroup and an equivalence map from it to the two groupoids.

More on Lie groupoid

- The nerve of a groupoid: Let G be a Lie groupoid. Define

$$G_n = \{(g_1, ..., g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

  as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ by sending $(g_1, ..., g_n)$ to $(g_1, ..., g_i g_{i+1}, ..., g_n)$. This forms a simplicial manifold.
- The classifying space BG is the geometric realization as a simplicial complex.
- An orbifold X with G as the Lie groupoid has π_n defined as $\pi_n(BG)$.

More on Lie groupoid

- The nerve of a groupoid: Let G be a Lie groupoid. Define

$$G_n = \{(g_1, ..., g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

  as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ by sending $(g_1, ..., g_n)$ to $(g_1, ..., g_i g_{i+1}, ..., g_n)$. This forms a simplicial manifold.
- The classifying space BG is the geometric realization as a simplicial complex.
- An orbifold X with G as the Lie groupoid has π_n defined as $\pi_n(BG)$.

More on Lie groupoid

- The nerve of a groupoid: Let G be a Lie groupoid. Define

$$G_n = \{(g_1, ..., g_n) | g_i \in G_1, s(g_i) = t(g_{i+1})\}$$

  as a fiber product. The face operator $d_i : G_n \to G_{n-1}$ by sending $(g_1, ..., g_n)$ to $(g_1, ..., g_i g_{i+1}, ..., g_n)$. This forms a simplicial manifold.
- The classifying space BG is the geometric realization as a simplicial complex.
- An orbifold X with G as the Lie groupoid has π_n defined as $\pi_n(BG)$.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

An abstract definition

- A Lie group K acting smoothly on M. The action Lie groupoid L is given by $L_0 = M$ and $L_1 = K \times M$ with s projection and t the action.
- An *orbifold groupoid* is a proper etale Lie groupoid.
- A groupoid is *proper* if $s \times t : G_1 \to G_0 \times G_0$ is proper.
- A groupoid is *etale* if s and t are local diffeomorphisms.
- Theorem: Let $\mathcal{G}$ be a proper effective etale groupoid. Then its orbit space $|\mathcal{G}|$ can be given the structure of an effective orbifold.
- Example: M a smooth manifold with an atlas $\mathcal{U}$. Let M_0 be the disjoint union $\coprod_{U \in \mathcal{U}} U$ and M_1 be $\coprod_{U,V \in \mathcal{U}} U \times_X V$. Then the space of orbits is M.

Action of a Lie groupoid

- Let G be an orbifold groupoid. A left G-space is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.
  - This map is defined on (g, e) with $\pi(e) = s(g)$ and written $\mu(g, e) = g.e$.
  - It satisfies the action identity: $\pi(g.e) = t(g)$, $1_x.e = e$, and $g.(h.e) = (gh).e$ for $h : x \to y$ and $g : y \to z$ and $e \in E$ with $\pi(e) = x$.
- A right G-space is left G^{op}-space obtained by switching the source and target map.

Action of a Lie groupoid

- Let G be an orbifold groupoid. A left G-space is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.
  - This map is defined on (g, e) with $\pi(e) = s(g)$ and written $\mu(g, e) = g.e$.
  - It satisfies the action identity: $\pi(g.e) = t(g)$, $1_x.e = e$, and $g.(h.e) = (gh).e$ for $h : x \to y$ and $g : y \to z$ and $e \in E$ with $\pi(e) = x$.
- A right G-space is left G^{op}-space obtained by switching the source and target map.

Action of a Lie groupoid

- Let G be an orbifold groupoid. A left G-space is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.
  - This map is defined on (g, e) with $\pi(e) = s(g)$ and written $\mu(g, e) = g.e$.
  - It satisfies the action identity: $\pi(g.e) = t(g)$, $1_x.e = e$, and $g.(h.e) = (gh).e$ for $h : x \to y$ and $g : y \to z$ and $e \in E$ with $\pi(e) = x$.
- A right G-space is left G^{op}-space obtained by switching the source and target map.

Action of a Lie groupoid

- Let G be an orbifold groupoid. A left G-space is a manifold E equipped with an action by G: Such an action is given by two maps: an anchor $\pi : E \to G_0$ and an action $\mu : G_1 \times_{G_0} E \to E$.
  - This map is defined on (g, e) with $\pi(e) = s(g)$ and written $\mu(g, e) = g.e$.
  - It satisfies the action identity: $\pi(g.e) = t(g)$, $1_x.e = e$, and $g.(h.e) = (gh).e$ for $h : x \to y$ and $g : y \to z$ and $e \in E$ with $\pi(e) = x$.
- A right G-space is left G^{op}-space obtained by switching the source and target map.

Differentiable structures on orbifolds

- Suppose we are given smooth structures on each $(\tilde{U}, G, \phi)$, i.e., $\tilde{U}$ is given a smooth structure and G is a smooth action on it. We assume that all embeddings in the atlas is smooth. Then M is given a smooth structure.
- Given a chart $(\tilde{U}, G, \phi)$, the space of smooth forms is the space of smooth forms in $\tilde{U}$ invariant under the G-action. A smooth form on the orbifold is the collection of smooth forms on each of the charts so that under embeddings they correspond.
- One can define an integral of smooth singular simplices into charts. This can be extended to any smooth simplex using partition of unity and barycentric subdivisions of the simplex.

Differentiable structures on orbifolds

- Suppose we are given smooth structures on each $(\tilde{U}, G, \phi)$, i.e., $\tilde{U}$ is given a smooth structure and G is a smooth action on it. We assume that all embeddings in the atlas is smooth. Then M is given a smooth structure.
- Given a chart $(\tilde{U}, G, \phi)$, the space of smooth forms is the space of smooth forms in $\tilde{U}$ invariant under the G-action. A smooth form on the orbifold is the collection of smooth forms on each of the charts so that under embeddings they correspond.
- One can define an integral of smooth singular simplices into charts. This can be extended to any smooth simplex using partition of unity and barycentric subdivisions of the simplex.

Differentiable structures on orbifolds

- Suppose we are given smooth structures on each $(\tilde{U}, G, \phi)$, i.e., $\tilde{U}$ is given a smooth structure and G is a smooth action on it. We assume that all embeddings in the atlas is smooth. Then M is given a smooth structure.
- Given a chart $(\tilde{U}, G, \phi)$, the space of smooth forms is the space of smooth forms in $\tilde{U}$ invariant under the G-action. A smooth form on the orbifold is the collection of smooth forms on each of the charts so that under embeddings they correspond.
- One can define an integral of smooth singular simplices into charts. This can be extended to any smooth simplex using partition of unity and barycentric subdivisions of the simplex.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds

- Given a locally finite covering of X, then we can define a smooth partition of unity (in the same way as in the manifold case. See Munkres.)
  - We refine to obtain a cover whose closures are invariant compact subsets.
  - The idea is to find smooth functions on each chart which vanishes outside the invariant compact subsets.
  - The images of compact subsets can be chosen to cover X.
  - Thus, these functions become functions on X which sums to a positive valued function.
  - We divide by the sum.

Differentiable structures on orbifolds: Integration

- An orbifold X is orientable if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each imbedding of charts to another charts is orientation-preserving.
- An n-form can be integrated on an orientable orbifold.

$$\int_{\tilde{U}} \omega = \frac{1}{|G|} \int_U \omega'$$

  where $(\tilde{U}_i, G, \phi)$ is the chart for U. (Otherwise, one can define n-density to integrate.)
- Then any n-form can be integrated by using a partition of unity.

Differentiable structures on orbifolds: Integration

- An orbifold X is orientable if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each imbedding of charts to another charts is orientation-preserving.
- An n-form can be integrated on an orientable orbifold.

$$\int_{\tilde{U}} \omega = \frac{1}{|G|} \int_{U} \omega'$$

where $(\tilde{U}_i, G, \phi)$ is the chart for U. (Otherwise, one can define n-density to integrate.)
- Then any n-form can be integrated by using a partition of unity.

Differentiable structures on orbifolds: Integration

- An orbifold X is orientable if one can choose an atlas of charts where $\tilde{U}$ is given an orientation with G acting in an orientation-preserving manner and each imbedding of charts to another charts is orientation-preserving.
- An n-form can be integrated on an orientable orbifold.

$$\int_{\tilde{U}} \omega = \frac{1}{|G|} \int_{U} \omega'$$

  where $(\tilde{U}_i, G, \phi)$ is the chart for U. (Otherwise, one can define n-density to integrate.)
- Then any n-form can be integrated by using a partition of unity.

Differentiable structures on orbifolds: Integration

- Poincare duality pairing: For a compact orbifold X

$$\int : H^p(X) \otimes H_c^{n-q}(X) \to \mathbb{R}.$$

  This is nondegenerate if X has a finite good cover.

- A cover of an orbifold is *good* if each U is of form $\mathbb{R}^n/G$ and all of its intersections is of the form. In this case, the standard differentiable form arguments work (See Bott-Tu). A compact orbifold has a finite good cover. (Note the confusing terminology here.)

Differentiable structures on orbifolds: Integration

- Poincare duality pairing: For a compact orbifold X

$$\int : H^p(X) \otimes H_c^{n-q}(X) \to \mathbb{R}.$$

  This is nondegenerate if X has a finite good cover.
- A cover of an orbifold is *good* if each U is of form $\mathbb{R}^n/G$ and all of its intersections is of the form. In this case, the standard differentiable form arguments work (See Bott-Tu). A compact orbifold has a finite good cover. (Note the confusing terminology here.)

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), (U'^*, G'^*, \phi'^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to (U^{*.'}, G*.', \phi*.')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
  $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
  for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), (U'^*, G'^*, \phi'^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to (U^{*\cdot'}, G{*.'}, \phi{*.'})$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
    $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
    for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), ({U'}^*, {G'}^*, {\phi'}^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to (U^{*.'}, G*.', \phi*.')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
  $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
  for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), (U'^*, G'^*, \phi'^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to (U^{*\cdot'}, G*.', \phi*.')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
  $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
  for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group $\mathbf{G}$ acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), (U'^*, G'^*, \phi'^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to (U^{*.'}, G*.', \phi*.')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
  $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
  for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Bundles over orbifolds

- An orbifold-bundle (or V-bundle) E over an orbifold X is given by a smooth orbifold E and a smooth map $\pi : E \to X$ so that
  - Let F be a smooth manifold with a Lie group **G** acting on it smoothly.
  - Pair of defining families $\mathcal{F}$ for X and $\mathcal{F}'$ for E so that (U, G, ϕ) of X corresponds to (U^*, G^*, ϕ^*) so that $U^* = U \times F$ and $\pi \circ \phi^* = \phi \circ \pi$.
  - Given $(U, G, \phi), (U^*, G^*, \phi^*)$, and $(U', G', \phi), ({U'}^*, {G'}^*, {\phi'}^*)$ there is a correspondence of embeddings $\lambda : (U, G, \phi) \to (U', G', \phi)$ and $\lambda^* : (U^*, G^*, \phi^*) \to ({U^{*\cdot}}', G{*.}', \phi{*.}')$ where $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ for $(p, q) \in U^* = U \times F$ with $g_\lambda(p) \in \mathbf{G}$.
  - We have
    $$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$
    for embeddings $(U, G, \phi) \xrightarrow{\lambda} (U', G', \phi') \xrightarrow{\mu} (U'', G'', \phi'')$.
  - If $F = \mathbf{G}$, then this is a principle orbifold bundle.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Tangent bundles, Tensor bundles

- Given an orbifold, we can build a tangent orbifold-bundle by taking $F = \mathbb{R}^n$ $\mathbf{G} = GL(n, \mathbb{R})$ and $g_\lambda(p)$ be the Jacobian of λ at p.
- We can build any tensor bundles in this way.
- Frame bundles also.
- A Riemannian metric on an orbifold is given by equivariant Riemannian metric on each chart which matches up under imbeddings.
- Such can be built using partition of unity again from any given Riemannian metrics on charts.
- Orthogonal frame bundles can be build in this way.
- Connections, cuvature, geodesics, and exponential maps can be defined.

Gauss-Bonnet theorem

- Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be
$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$
where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.
- Such a triangulation always seem to exist always. (Proved in Verona.)
- Theorem (Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then
$$(2/O_m) \int_M K dw = \chi(M),$$
where O_m is the volume of the m-sphere.
- The proof essentially follows that of Chern for manifolds.

Gauss-Bonnet theorem

- Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be
$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$
where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.
- Such a triangulation always seem to exist always. (Proved in Verona.)
- Theorem (Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then
$$(2/O_m)\int_M K dw = \chi(M),$$
where O_m is the volume of the m-sphere.
- The proof essentially follows that of Chern for manifolds.

Gauss-Bonnet theorem

- Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be
$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$
where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.
- Such a triangulation always seem to exist always. (Proved in Verona.)
- Theorem (Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then
$$(2/O_m) \int_M K dw = \chi(M),$$
where O_m is the volume of the m-sphere.
- The proof essentially follows that of Chern for manifolds.

Gauss-Bonnet theorem

- Assuming that X admits a finite smooth triangulation so that interior of each cell lies in singularity with locally constant isotopy groups, then we define the Euler characteristic to be
$$\chi(X) = \sum_k (-1)^{\dim s_k} 1/N_{s_k}$$
where s_k denotes the kth-cell and N_{s_k} the order of the isotropy group.
- Such a triangulation always seem to exist always. (Proved in Verona.)
- Theorem (Allendoerfer-Weil, Hopf) Let M be a compact Riemannian orbifold of even dimension m. Then
$$(2/O_m)\int_M K dw = \chi(M),$$
where O_m is the volume of the m-sphere.
- The proof essentially follows that of Chern for manifolds.