1 Introduction

Outline

- Lie groups
  - Lie algebras
  - Lie group actions
- Geometries
  - Euclidean geometry
  - Spherical geometry
  - Affine geometry
  - Projective geometry
  - Conformal geometry: Poincare extensions
  - Hyperbolic geometry
    - Lorentz group
    - Geometry of hyperbolic space
    - Beltrami-Klein model
    - Conformal ball model
    - The upper-half space model
  - Discrete groups: examples (In the next handout.)

Some helpful references

- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977. (This is the principal source)
- W. Thurston, Three-dimensional geometry and topolgy, PUP, 1997
- M. Berger, Geometry I, Springer
- J. Ratcliffe, Foundations of hyperbolic manifolds, Springer
- M. Kapovich, Hyperbolic Manifolds and Discrete Groups, Birkhauser.
- My talk `http://math.kaist.ac.kr/~schoi/Titechtalk.pdf`

2 Lie groups

2.1 Lie groups

Section 1: Lie groups

- A Lie group is a space of symmetries of some space. More formally, a Lie group is a manifold with a group operation $\circ : G \times G \to G$ that satisfies
  - $\circ$ is smooth.
  - the inverse $\iota : G \to G$ is smooth also.
- Examples:
  - The permutation group of a finite set form a 0-dimensional manifold, which is a finite set.
  - $\mathbb{R}, \mathbb{C}$ with $+$ as an operation. ($\mathbb{R}^+$ with $+$ is merely a Lie semigroup.)
  - $\mathbb{R} - \{O\}, \mathbb{C} - \{O\}$ with $*$ as an operation.
  - $T^n = \mathbb{R}^n/\Gamma$ with $+$ as an operation and O as the equivalence class of $(0, 0, ..., 0)$. (The three are abelian ones.)
- - $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$: the general linear group.
  - $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | det(A) = 1\}$: the special linear group.
  - $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) | A^T A = \mathrm{I}\}$: the orthogonal group.
  - $Isom(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n | T(x) = Ax + b \text{ for } A \in O(n-1, \mathbb{R}), b \in \mathbb{R}^n\}$.
  - Proofs: One can express the operations as polynomials or rational functions.
- Products of Lie groups are Lie groups.
- A covering space of a connected Lie group form a Lie group.
- A *Lie subgroup* of a Lie group is a subgroup that is a Lie group with the induced operation and is a submanifold.
  - $O(n) \subset SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$.
  - $O(n-1) \subset Isom(\mathbb{R}^n)$.
- A homomorphism $f : G \to H$ of two Lie groups G, H is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms.
- The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also.
- If G, H are simply connected, f induces a unique homomorphism of Lie algebra of G to that of H which is Df and conversely.

2.2 Lie algebras

Lie algebras

- A Lie algebra is a vector space V with an operation $[,] : V \times V \to V$ that satisfies:
  - $[x, x] = 0$ for $x \in L$. (Thus, $[x, y] = -[y, x]$.)
  - Jacobi identity $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.
- Examples:
  - Sending $V \times V$ to O is a Lie algebra (abelian ones.)
  - Direct sums of Lie algebras is a Lie algebra.
  - A subalgebra is a subspace closed under $[,]$.
  - An ideal K of L is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in L$.
- A homomorphism of a Lie algebra is a linear map preserving $[,]$.
- The kernel of a homomorphism is an ideal.

Lie groups and Lie algebras

- Let G be a Lie group. A left translation $L_g : G \to G$ is given by $x \mapsto g(x)$.
- A left-invariant vector field of G is a vector field so that the left translation leaves it invariant, i.e., $dL_g(X(h)) = X(gh)$ for $g, h \in G$.
- The set of left-invariant vector fields form a vector space under addition and scalar multiplication and is vector-space isomorphic to the tangent space at I. Moreover, $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.
- The Lie algebra of G is the the Lie algebra of the left-invariant vector fields on G.
- Example: The Lie algebra of $GL(n, \mathbb{R})$ is isomorphic to $gl(n, \mathbb{R})$:
  - For X in the Lie algebra of $GL(n, \mathbb{R})$, we can define a flow generated by X and a path $X(t)$ along it where $X(0) = \mathrm{I}$.
  - We obtain an element of $gl(n, \mathbb{R})$ by taking the derivative of $X(t)$ at 0 seen as a matrix.
  - The brackets are preserved.
  - A Lie algebra of an abelian Lie group is abelian.

Lie algebras

- Given X in the Lie algebra $\mathfrak{g}$ of G, there is an integral curve $X(t)$ through I. We define the exponential map $\exp : \mathfrak{g} \to G$ by sending X to $X(1)$.
- The exponential map is defined everywhere, smooth, and is a diffeomorphism near O.
- The matrix exponential defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{1}{k!} A^k$$

  is the exponential map $gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$.

Lie group actions

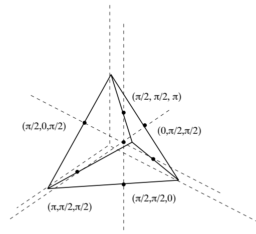
- A Lie group G-action on a smooth manifold X is given by a smooth map $G \times X \to X$ so that $(gh)(x) = (g(h(x))$ and $I(x) = x$. (left action)
- A right action satisfies $(x)(gh) = ((x)g)h$.
- Each Lie algebra element correspond to a vector field on X by using a vector field.
- The action is faithful if $g(x) = x$ for all x, then g is the identity element of G.
- The action is transitive if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.
- Example:
  - $GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$.
  - $PGL(n+1, \mathbb{R})$ acting on $\mathbb{R}P^n$.

3 Geometries

3.1 Euclidean geometry

Euclidean geometry

- The Euclidean space is $\mathbb{R}^n$ and the group $Isom(\mathbb{R}^n)$ of rigid motions is generated by $O(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.
- A system of linear equations gives us a subspace (affine or linear)
- This gives us the model for Euclidean axioms....



3.2 Spherical geometry

Spherical geometry

- Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$.
- Many great sphere exists and they are subspaces... (They are given by homogeneous system of linear equations in $\mathbb{R}^{n+1}$.)
- The lines are replaced by great circles and lengths and angles are also replaced.
- The transformation group is $O(n+1)$.

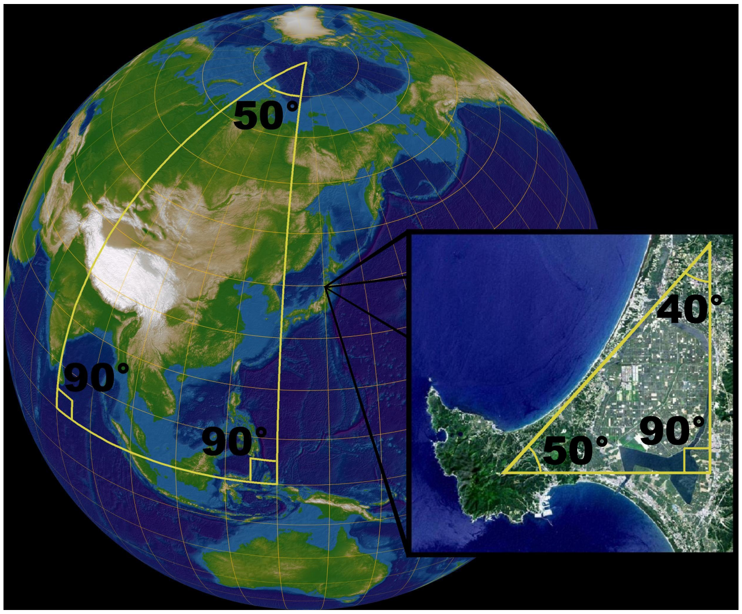
Spherical trigonometry

- Many spherical triangle theorems exist... `http://mathworld.wolfram.com/SphericalTrigonometry.html`
- Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \quad (1)$$

$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \quad (2)$$

-

- 

3.3 Affine geometry

Affine geometry

- A vector space $\mathbb{R}^n$ becomes an affine space by forgetting the origin.
- An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.
- The Euclidean space $\mathbb{R}^n$ with the group $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations form the affine geometry.
- Of course, angles and lengths do not make sense. But the notion of lines exists.
- The set of three points in a line has an invariant based on ratios of lengths.

3.4 Projective geometry

Projective geometry

- Projective geometry was first considered from fine art.
- Desargues (and Kepler) first considered points at infinity.
- Poncelet first added infinite points to the euclidean plane.
- Projective transformations are compositions of perspectivities. Often, they send finite points to infinite points and vice versa. (i.e., two planes that are not parallel).

- The added points are same as ordinary points up to projective transformations.
- Lines have well defined infinite points and are really circles topologically.
- Some notions lose meanings. However, many interesting theorems can be proved. Duality of theorems plays an interesting role.
- See for an interactive course: `http://www.math.poly.edu/courses/projective_geometry/`
- and `http://demonstrations.wolfram.com/TheoremeDePappusFrench/`, `http://demonstrations.wolfram.com/TheoremeDePascalFrench/`, `http://www.math.umd.edu/~wphooper/pappus9/pappus.html`, `http://www.math.umd.edu/~wphooper/pappus/`

- Formal definition with topology is given by Felix Klein using homogeneous coordinates.
- The projective space $\mathbb{R}P^n$ is $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R}$.
- Each point is given a homogeneous coordinates: $[v] = [x_0, x_1, ..., x_n]$.
- The projective transformation group $\mathrm{PGL}(n+1, \mathbb{R}) = GL(n+1, \mathbb{R})/\sim$ acts on $\mathbb{R}P^n$ by each element sending each ray to a ray using the corresponding general linear maps.
- Here, each element of g of $\mathrm{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $GL(n+1, \mathbb{R})$ of g. Also any coordinate change can be viewed this way.

- The affine geometry can be "imbedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{R}P^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, ..., x_n]\}$. This is called an affine patch. The subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$.
- The subspace of points $\{[0, x_1, x_2, ..., x_n]\}$ is the complement homeomorphic to $\mathbb{R}P^{n-1}$. This is the set of infinities, i.e., directions in $\mathbb{R}P^n$.
- From affine geometry, one can construct a unique projective geometry and conversely using this idea. (See Berger for the complete abstract approach.)

- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponding to a projective subspace in $\mathbb{R}P^n$ in a one-to-one manner while the dimension drops by 1.
- The independence of points are defined. The dimension of a subspace is the maximal number of independent set minus 1.
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Acutally a line is $\mathbb{R}P^1$ or a line $\mathbb{R}^1$ with a unique infinity.

- Cross ratios of four points on a line (x, y, z, t). There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. Thus $b = b(x, y, z, t)$ is the cross-ratio.
- If the four points are on $\mathbb{R}^1$, the cross ratio is given as

$$(x, y; z, t) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

  if we can write

$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$

- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2.
- For us $n = 2$ is important. Here we have a familiar projective plane as topological type of $\mathbb{R}P^2$, which is a Mobius band with a disk filled in at the boundary. `http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/`

3.5 Conformal geometry

Conformal geometry

- Reflections of $\mathbb{R}^n$. The hyperplane $P(a, t)$ given by $a \cot x = b$. Then $\rho(x) = x + 2(t - a \cdot x)a$.
- Inversions. The hypersphere $S(a, r)$ given by $|x - a| = r$. Then $\sigma(x) = a + (\frac{r}{|x-a|})^2(x - a)$.

- We can compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This can be accomplished by a stereographic projection from the unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ from the northpole $(0, 0, ..., 1)$. Then these reflections and inversions induce conformal homeomorphisms.
- The group of transformations generated by these homeomorphisms is called the Mobius transformation group.
- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.
- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobius transformation is a either a fractional linear transformation of form

$$z \mapsto \frac{az + b}{cz + d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

  or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.
- In higher-dimensions, a description as a sphere of null-lines and the special Lorentzian group exists.

Poincare extensions

- We can identify E^{n-1} with $E^{n-1} \times \{O\}$ in E^n.
- We can extend each Mobius transformation of $\hat{E}^{n-1}$ to $\hat{E}^n$ that preserves the upper half space U: We extend reflections and inversions in the obvious way.
- The Mobius transformation of $\hat{E}^n$ that preserves the open upper half spaces are exactly the extensions of the Mobius transformations of $\hat{E}^{n-1}$.
- $M(U^n) = M(\hat{E}^{n-1})$.
- We can put the pair $(U^n, \hat{E}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobius transformation.
- Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere.

3.6 Hyperbolic geometry

Lorentzian geometry

- A hyperbolic space H^n is defined as a complex Riemannian manifold of constant curvature equal to -1.
- Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions.
- But it is realized as a "sphere" in a Lorentzian space.

- A Lorentzian space is $\mathbb{R}^{1,n}$ with an inner product

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n.$$

- A Lorentzian norm $||x|| = (x \cdot y)^{1/2}$, a positive, zero, or positive imaginary number.
- One can define Lorentzian angles.
- The null vectors form a light cone divide into positive, negative cone, and 0.
- Space like vectors and time like vectors and null vectors.
- Ordinary notions such as orthogonality, orthonormality,...

Lorentz group

- A Lorentzian transformation is a linear map preserving the inner-product.
- For J the diagonal matrix with entries $-1, 1, ..., 1$, $A^t J A = J$ iff A is a Lorentzian matrix.
- A Lorentzian transformation sends time-like vectors to time-like vectors. It is either positive or negative.
- The set of Lorentzian transformations form a Lie group $O(1, n)$.
- The set of positive Lorentzian transformations form a Lie subgroup $PO(1, n)$.

Hyperbolic space

- Given two positive time-like vectors, there is a time-like angle

$$x \cdot y = ||x||||y||cosh\eta(x, y)$$

- A hyperbolic space is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone.
- Topologically, it is homeomorphic to $\mathbb{R}^n$. Minkowsky model
- One induces a metric from the Lorentzian space which is positive definite.
- This gives us a Riemannian metric of constant curvature -1. (The computation is very similar to the computations for the sphere.)
- $PO(1, n)$ is the isometry group of H^n which is homogeneous and directionless.
- A hyperbolic line is an intersection of H^n with a time-like two-dimensional vector subspace.
- The hyperbolic sine law, The first law of cosines, The second law of cosines...

- One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π.
- One can assign any lengths to a hyperbolic triangle.
- The triangle formula can be generalized to formula for quadrilateral, pentagon, hexagon.
- Basic philosophy here is that one can push the vertex outside and the angle becomes distances between lines. (See Ratcliffe, `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)
- hyperbolic law of sines:

$$\sin A / \sinh a = \sin B / \sinh b = \sin C / \sinh c$$

- hyperbolic law of cosines:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$$

$$\cosh c = (\cosh A \cosh B + \cos C) / \sinh A \sinh B$$

Beltrami-Klein models of hyperbolic geometry

- Beltrami-Klein model is directly obtained from the hyperboloid model.
- $d_k(P, Q) = 1/2 \log |(AB, PQ)|$ where A, P, Q, B are on a segment with endpoints A, B and
$$(AB, PQ) = \left| \frac{AP}{BP} \frac{BQ}{AQ} \right|.$$
- There is an imbedding from H^n onto an open ball B in the affine patch $\mathbb{R}^n$ of $\mathbb{R}P^n$. This is standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$.
- B can be described as a ball of radius 1 with center at O.
- The isometry group $PO(1, n)$ also maps injectively to a subgroup of $PGL(n + 1, \mathbb{R})$ that preserves B.
- The projective automorphism group of B is precisely this group.
- The metric is induced on B. This is precisely the metric given by the log of the cross ratio. Note that $\lambda(t) = (\cosh t, \sinh t, 0, ..., 0)$ define a unit speed geodesic in H^n. Under the Riemannian metric, we have $d(e_1, (\cosh t, \sinh t, 0, ..., 0)) = t$ for t positive.
- Under d_k, we obtain the same. Since any geodesic segment of same length is congruent under the isometry, we see that the two metrics coincide. Betrami-Klein model

- Beltrami-Klein model is nice because you can see outside. The outside is the anti-de Sitter space http://en.wikipedia.org/wiki/Anti_de_Sitter_space
- Also, we can treat points outside and inside together.
- Each line (hyperplane) in the model is dual to a point outside. (i.e., orthogonal by the Lorentzian inner-product) A point in the model is dual to a hyperplane outside. Infact any subspace of dimension i is dual to a subspace of dimension $n-i-1$ by orthogonality.
- For $n=2$, the duality of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
- The distance between two hyperplanes can be obtained by two dual points. The two dual points span an orthogonal plane to the both hyperperplanes and hence provide a shortest geodesic.

The conformal ball model (Poincare ball model)

- The stereo-graphic projection H^n to the plane P given by $x_0 = 0$ from the point $(-1, 0, ..., 0)$.
- The formula for the map $\kappa : H^n \to P$ is given by
$$\kappa(x) = \left(\frac{y_1}{1+y_0}, ..., \frac{y_n}{1+y_0}\right),$$
where the image lies in an open ball of radius 1 with center O in P. The inverse is given by
$$\zeta(x) = \left(\frac{1+|x|^2}{1-|x|^2}, \frac{2x_1}{1-|x|^2}, ..., \frac{2x_n}{1-|x|^2},\right).$$
- Since this is a diffeomorphism, B has an induced Riemannian metric of constant curvature -1.

- We show
$$\cosh d_B(x,y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)},$$
and inversions acting on B preserves the metric. Thus, the group of Mobius transformations of B preserve metric.
- The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1-|x|^2)^2$.
- It follows that the group of Mobius transformations acting on B is precisely the isometry group of B. Thus, $Isom(B) = M(\mathbf{S}^{n-1})$.
- Geodesics would be lines through O and arcs on circles perpendicular to the sphere of radius 1.

The upper-half space model.

- Now we put B to U by a Mobius transformation. This gives a Riemannian metric constant curvature -1.
- We have by computations $\cosh d_U(x,y) = 1 + |x-y|^2/2x_n y_n$ and the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(E^{n-1})$.
- Geodesics would be arcs on lines or circles perpendicular to E^{n-1}.
- Since $\hat{E}^1$ is a circle and $\hat{E}^2$ is the complex sphere, we obtain $Isom^+(B^2) = PSL(2,\mathbb{R})$ and $Isom^+(B^3) = PSL(2,\mathbb{C})$.
- Orientation-preserving isometries of hyperbolic plane can have at most one fixed point. elliptic, hyperbolic, parabolic.

$$z \mapsto e^{i\theta}, z \mapsto az, a \neq 1, a \in \mathbb{R}^+, z \mapsto z+1$$

- Isometries of a hyperbolic space: loxodromic, hyperbolic, elliptic, parabolic.
- Up to conjugations, they are represented as Mobius transformations which has forms
  - $z \mapsto \alpha z, Im\alpha \neq 0, |\alpha| \neq 1$.
  - $z \mapsto az, a \neq 1, a \in \mathbb{R}^+$.
  - $z \mapsto e^{i\theta} z, \theta \neq 0$.
  - $z \mapsto z+1$.