1 Introduction

Outline

- Section 3: Topology of 2-orbifolds: Compact group actions
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  - Tubes and slices.
  - Path-lifting, covering homotopy
  - Locally smooth actions
  - Smooth actions
  - Equivariant triangulations
  - Newman's theorem

Some helpful references

- Bredon, Introduction to compact transformation groups, Academic Press
- Hsiang, Cohomology theory of topological transformation group, Springer, 1975
- Soren Illman, Smooth equivariant triangulations of G-manifolds for G a finite group, Math. Ann. 233, 199–220 (1978)

2 Compact group actions

- A group action $G \times X \to X$ with $e(x) = x$ for all x and $gh(x) = g(h(x))$. That is, $G \to Homeo(X)$ so that the product operation becomes compositions.
- We only need the result for finite group actions.
- An *equivariant* map $\phi : X \to Y$ between G-spaces is a map so that $\phi(g(x)) = g(\phi(x))$.
- An *isotropy subgroup* $G_x = \{g \in G | g(x) = x\}$.
- $G_{g(x)} = gG_xg^{-1}$. $G_x \subset G_{\phi(x)}$ for an equivariant map ϕ.
- Tietze-Gleason Theorem: G a compact group acting on X with a closed invariant set A. Let G also act linearly on $\mathbb{R}^n$. Then any equivariant $\phi : A \to \mathbb{R}^n$ extends to $\phi : X \to \mathbb{R}^n$.

Orbit spaces

- An orbit of x is $G(x) = \{g(x)|g \in G\}$.
- $G/G_x \to G(x)$ is one-to-one onto continuous function.
- An orbit type is given by the conjugacy class of G_x in G. The orbit types form a partially ordered set.
- Denote by X/G the space of orbits with quotient topology.
- For $A \subset X$, $G(A) = \bigcup_{g \in G} g(A)$ is the *saturation* of A.
- Properties:
  - $\pi : X \to X/G$ is an open, closed, and proper map.
  - X/G is Hausdorff.
  - X is compact iff X/G is compact.
  - X is locally compact iff X/G is locally compact.

Orbit spaces: Examples

- Let $X = G \times Y$ and G acts as a product.
- For k, q relatively prime, the action of Z_k on S^3 in C^2 generated by a matrix

  $$\begin{bmatrix} e^{2\pi i/k} & 0 \\ 0 & e^{2\pi qi/k} \end{bmatrix}$$

  giving us a Lens space.
- We can also consider S^1-actions given by

  $$\begin{bmatrix} e^{2\pi ki\theta} & 0 \\ 0 & e^{2\pi qi\theta} \end{bmatrix}$$

  Then it has three orbit types.
- Consider in general the action of torus T^n-action on C^n given by

  $$(c_1, ..., c_n)(y_1, ..., y_n) = (c_1 y_1, ..., c_n y_n), |c_i| = 1, y_i \in C.$$

Orbit spaces: Examples

- Then there is a homeomorphism $h : C^n/T^n \to (R^+)^n$ given by sending

  $$(y_1, ..., y_n) \mapsto (|y_1|^2, ..., |y_n|^2).$$

  The interiors of sides represent different orbit types.

- H a closed subgroup of Lie group G. The left-coset space G/H where G acts on the right also.
- $G/G_x \to G(x)$ is given by $gG_x \mapsto g(x)$ is a homeomorphism if G is compact.
- Twisted product: X a right G-space, Y a left G-space. A left action is given by $g(x, y) = (xg^{-1}, gy)$. The twisted product $X \times_G Y$ is the quotient space.
- $p : X \to B$ is a principal bundle with G acting on the left. F a right G-space. Then $F \times_G X$ is the associated bundle.

Orbit spaces: Bad examples

- The Conner-Floyd example: There is an action of Z_r for $r = pq$,p, q relatively prime, on an Euclidean space of large dimensions without stationary points.
- Proof:
  - Find a simplicial action Z_{pq} on $S^3 = S^1 * S^1$ without stationary points obtained by joining action of Z_p on S^1 and Z_q on the second S^1.
  - Find an equivariant simplicial map $h : S^3 \to S^3$ which is homotopically trivial.
  - Build the infinite mapping cylinder which is contactible and imbed it in an Euclidean space of high-dimensions where Z_{pq} acts orthogonally.
  - Find the contractible neighborhood. Taking the product with the real line makes it into a Euclidean space.

Orbit spaces: Bad examples

- Hsiang-Hsiang: If G is any compact, connected, nonabelian Lie group, then there is an action of G on any euclidean space of sufficiently high dimension for which the fixed point set F has any given homotopy type. (F could be empty.)

Twisted product

- G a compact subgroup, X right G-space and Y left G-space. $X \times_G Y$ is the quotient space of $X \times Y$ where $[xg, y] \sim [x, gy]$ for $g \in G$.
- H a closed subgroup of G $G \times_H Y$ is a left G-space by the action $g[g', a] = [gg', a]$. This sends equivalence classes to themselves.
- The inclusion $A \to G \times_H A$ induces a homeomorphism $A/H \to (G \times_H A)/G$.
- The isotropy subgroup at $[e, a]$: $[e, a] = g[e, a] = [g, a] = [h^{-1}, h(a)]$. Thus, $G_{[e,a]} = H_a$.
- Example: Let $G = S^1$ and A be the unit-disk and $H = \mathbb{Z}_3$ generated by $e^{2\pi/3}$. G and H acts in a standard way in A. Then consider $G \times_H A$.

Tubes and slices

- X a G-space. P an orbit of type G/H. A *tube about* P is a G-equivariant imbedding $G \times_H A \to X$ onto an open neighborhood of P where A is a some space where H acts on.
- Every orbit passes the image of $e \times A$.
- $P = G(x)$ for $x = [e, a]$ where a is the stationary point of H in A.
- In general $G_x = H_a \subset H$ for $x = [e, a]$.
- A slice: $x \in X$, A set $x \in S$ such that $G_x(S) = S$. Then S is a *slice* if $G \times_{G_x} S \to X$ so that $[g, s] \to g(s)$ is a tube about G_x.
- S is a slice iff S is the image of $e \times A$ for some tube.

Tubes and slices

- Let $x \in S$ and $H = G_x$. Then the following are equivalent:
  - There is a tube $\phi : G \times_H A \to X$ about $G(x)$ such that $\phi([e, A]) = S$.
  - S is a slice at x.
  - $G(S)$ is an open neighborhood of $G(x)$ and there is an equivariant retraction $f : G(S) \to G(x)$ with $f^{-1}(x) = S$.
- It would be a good exercise to apply these theories to above examples....

The existence of tubes

- Let X be a completely regular G-space. There is a tube about any orbit of a complete regular G-space with G compact. (Mostow)
- Proof:
  - Let x_0 have an isotropy group H in G.
  - Find an orthogonal representation of G in $\mathbb{R}^n$ with a point v_0 whose isotropy group is H.
  - There is an equivalence $G(x_0)$ and $G(v_0)$. Extend this to a neighborhood.
  - For $\mathbb{R}^n$, we can find the equivariant retraction. Transfer this on X.
- If G is a finite group acting on a manifold, then a tube is a union of disjoint open sets and a slice is an open subset where G_x acts on.

Path-lifting and covering homotopy theorem.

- Let X be a G-space, G a compact Lie group, and $f : I \to X/G$ any path. Then there exists a lifting $f' : I \to X$ so that $\pi \circ f' = f$.
- Let $f : X \to Y$ be an equivariant map. Let $f' : X/G \to Y/G$ be an induced map. Let $F' : X/G \times I \to Y/G$ be a homotopy preserving orbit types that starts at f'. Then there is an equivariant $F : X \times I \to Y$ lifting F' starting at f.
- If G is finite and X a smooth manifold with a smooth G-action and if the functions are smooth, then the lifts can be chosen to be also smooth.

Locally smooth actions

- M a G-space, G a compact Lie group, P an orbit of type G/H. V a vector space where H acts orthogonally. Then a *linear* tube in M is a tube of the form $\phi : G \times_H V \to M$.
- Let S be a slice. S is a *linear slice* if $G \times_{G_x} S \to M$ given by $[g, s] \to g(s)$ is equivalent to a linear slice. (If G_x-space S is equivalent to the orthogonal G_x-space.)
- If there is a linear tube about each orbit, then M is said to be *locally smooth*.

Locally smooth actions

- There exists a maximum orbit type G/H for G. (That is, H is conjugate to a subgroup of each isotropy group.)
- Proof:
  - Near each tube, we find the maximal orbit types has to be dense and open.
- The maximal orbits so obtained are called *principal orbits*.
- If M is a smooth manifold and compact Lie G acts smoothly, this is true.

Manifold

- M a smooth manifold, G a compact Lie group acting smoothly on M.
- If G is finite, then this is equivalent to the fact that each $i_g : M \to M$ given by $x \mapsto g(x)$ is a diffeomorphism.
- Let n be the dimension of M and d the dimension of the maximal orbit.
- $M^* = M/G$ is a manifold with boundary if $n - d \leq 2$.

Manifold

- Proof:
  - Let $k = n - d$, the codimension of the principal orbits.
  - Consider a linear tube $G \times_K V$. The orbit space $(G \times_K V)^* \cong V^*$.
  - Let S be the unit sphere in V. Then V^* is a cone over S^*.
  - $\dim M^* = \dim V^* = \dim S^* + 1$.
  - If $k = 0$, then M^* is discrete. If M is a sphere, then M^* is one or two points.
  - If $k = 1$, then M^* is locally a cone over one or two points. Hence M^* is a 1-manifold.
  - If $k = 2$, then M^* is locally a cone over an arc or a circle. (as S^* is a 1-manifold by the previous step.)
- Example: $\mathbb{Z}_2$ action on $\mathbb{R}^3$ generated by the antipodal map. The result is not a manifold.

Smooth actions

- Recall smooth actions.
- G-compact Lie group acting smoothly on M. Then there exists an invariant Riemannian metric on M.
- $G(x)$ is a smooth manifold. $G/G_x \to G(x)$ is a diffeomorphism.
- Exponential map: For $X \in T_pM$, there is a unique geodesic γ_X with tangent vector at p equal to X. The exponential map $\exp : T_pM \to M$ is defined by $X \mapsto \gamma_X(1)$.
- If A is an invariant smooth submanifold, then A has an open invariant tubular neighborhood.
- The smooth action of a compact Lie group is locally smooth.
- Proof:
  - Use the fact that orbits are smooth submanifolds and the above item.

Some facts needed later

- The subspace $M_{(H)}$ of same orbit type G/H is a smooth locally-closed submanifold of M.
- A a closed invariant submanifold. Then any two open (resp. closed) invariant tubular neighborhoods are equivariantly isotopic.

Newman's theorem

- Let M be a connected topological n-manifold. Then there is a finite open covering $\mathcal{U}$ of the one-point compactification of M such that there is no effective action of a compact Lie group with each orbit contained in some member of $\mathcal{U}$. (Proof: algebraic topology)
- If G is a compact Lie group acting effectively on M, then M^G is nowhere dense.

Equivariant triangulations

- Sören Illman proved:
- Let G be a finite group. Let M be a smooth G-manifold with or without boundary. Then we have:
  - There exists an equivariant simplicial complex K and a smooth equivariant triangulation $h : K \to M$.
  - If $h : K \to M$ and $h_1 : L \to M$ are smooth triangulations of M, there exist equivariant subdivisions K' and L' of K and L, respectively, such that K' and L' are G-isomorphic.
- This result was widely used once a proof by Yang (1963) was given. But an error was discovered by Siebenmann (1970) and proved in 1977.