# 1 Introduction

### Outline

- Section 3: Topology of 2-orbifolds
  - Topology of orbifolds
  - Smooth 2-orbifolds and triangulations
- Covering spaces
  - Fiber-product approach
  - Path-approach by Haefliger
- Topological operations on 2-orbifolds: constructions and decompositions

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## 2 Definition

**$2$-orbifolds**

- We now wish to concentrate on 2-orbifolds.
- Singularities
  - We simply have to classify finite groups in $O(2)$: $\mathbb{Z}_2$ acting as a reflection group or a rotation group of angle $\pi/2$, a cyclic groups $C_n$ of order $\geq 3$ and dihedral groups $D_n$ of order $\geq 4$.
  - According to this the singularities are of form:
    - A silvered point
    - A cone-point of order $\geq 2$.
    - A corner-reflector of order $\geq 2$.

**$2$-orbifolds**

- On the boundary of a surface with a corner, one can take mutually disjoint open arcs ending at corners. If two arcs meet at a corner-point, then the corner-point is a *distinguished one*. If not, the corner-point is *ordinary*. The choice of arcs will be called the *boundary pattern*.
- As noted above, given a surface with corner and a collection of discrete points in its interior and the boundary pattern, it is possible to put an orbifold structure on it so that the interior points become cone-points and the distinguished corner-points the corner-reflectors and boundary points in the arcs the silvered points of any given orders.

**The triangulations of $2$-orbifolds and classification**

- One can put a Riemannian metric on a 2-orbifold so that the boundary is a union of geodesic arcs and each corner-reflector have angles $\pi/n$ for its order $n$ and the cone-points have angles $2\pi/n$.
- Proof: First construct such a metric on the boundary by putting such metrics on the boundary by using a broken geodesic in the euclidean plane and around the cone points and then using partition of unity.
- By removing open balls around cone-points and corner-reflectors, we obtain a smooth surface with corners.
- Find a smooth triangulation of so that the interior of each side is either completely inside the boundary with the corners removed.
- Extend the triangulations by cone-construction to the interiors of the removed balls.

**The triangulations of $2$-orbifolds and classification**

- Theorem: Any 2-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors.
- A 2-orbifold is classified by the underlying smooth topology of the surface with corner and the number and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.
- proof: basically, strata-preserving isotopies.
- In general, a smooth orbifold has a smooth topological stratification and a triangulation so that each open cell is contained in a single strata.
- Smooth topological triangulations satisfying certain weak conditions have a triangulation.
- One should show that the stratification of orbifolds by orbit types satisfies this condition.

**Existence of locally finite good covering**

- Let $X$ be an orbifold. Give it a Riemannian metric.
- There exists a good covering: each open set is connected and charts have cells as cover and the intersection of any finite collection again has such properties.
- Each point has an open neighborhood with an orthogonal action.
- Now choose sufficiently small ball centered at the origin so that it has a convexity property. (That is, any path can be homotoped into a geodesic.)
- Find a locally finite subcollection.
- Then intersection of any finite collection is still convex and hence has cells as cover.

# 3 Covering spaces of orbifolds

**Covering spaces of orbifold**

- Let $X'$ be an orbifold with a smooth map $p : X' \to X$ so that for each point $x$ of $X$, there is a connected model $(U, G, \phi)$ and the inverse image of $p(\psi(U))$ is a union of open sets with models isomorphic to $(U, G', \pi)$ where $\pi : U \to U/G'$ is a quotient map and $G'$ is a subgroup of $G$. Then $p : X' \to X$ is a *covering* and $X'$ is a *covering orbifold* of $X$.
- Abstract definition: If $X'$ is a $(X_1, X_0)$-space and $p_0 : X'_0 \to X_0$ is a covering map, then $X'$ is a *covering orbifold*.

- We can see it as an orbifold bundle over $X$ with discrete fibers. We can choose the fibers to be acted upon by a discrete group $G$, and hence a principal $G$-bundle. This gives us a regular (Galois) covering.

**Examples (Thurston)**

- $Y$ a manifold. $\tilde{Y}$ a regular covering map $\tilde{p}$ with the automorphism group $\Gamma$. Let $\Gamma_i, i \in I$ be a sequence of subgroups of $\Gamma$.
  - The projection $\tilde{p}_i : \tilde{Y} \times \Gamma_i \backslash \Gamma \to \tilde{Y}$ induces a covering $p_i : (\tilde{Y} \times \Gamma_i \backslash \Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y$ where $\Gamma$ acts by
  $$\gamma(\tilde{x}, \Gamma_i \gamma_i) = (\gamma(\tilde{x}), \Gamma_i \gamma_i \gamma^{-1})$$
  - This is same as $\tilde{Y}/\Gamma_i \to Y$ since $\Gamma$ acts transitively on both spaces.
  - Fiber-products $\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma \to \tilde{Y}$. Define left-action of $\Gamma$ by
  $$\gamma(\tilde{x}, (\Gamma_i \gamma_i)_{i \in I}) = (\gamma(\tilde{x}), (\Gamma_i \gamma_i \gamma^{-1})), \gamma \in \Gamma.$$
  We obtain the fiber-product
  $$(\tilde{Y} \times \prod_{i \in I} \Gamma_i \backslash \Gamma)/\Gamma \to \tilde{Y}/\Gamma = Y.$$

**Developable orbifold**

- We can let $\Gamma$ be a discrete group acting on a manifold $\tilde{Y}$ properly discontinuously but maybe not freely.
- One can find a collection $X_i$ of coverings so that
  - $\Gamma_i = \{\gamma \in \Gamma | \gamma(X_i) = X_i\}$ is finite and if $\gamma(X_i) \cap X_i \neq \emptyset$, then $\gamma$ is in $\Gamma_i$.
  - The images of $X_i$ cover $\tilde{Y}/\Gamma$.
- $Y = \tilde{Y}/\Gamma$ has an *orbifold quotient* of $\tilde{Y}$ and $Y$ is said to be *developable*.
- In the above example, we can let $\Gamma$ be a discrete group acting on a manifold $\tilde{Y}$ properly discontinuously but maybe not freely. $Y^f$ is then the fiber product of orbifold maps $\tilde{Y}/\Gamma_i \to Y$.

**Doubling an orbifold with mirror points**

- A *mirror point* is a singular point with the stablizer group $\mathbb{Z}_2$ acting as a reflection group.
- One can double an orbifold $M$ with mirror points so that mirror-points disappear. (The double covering orbifold is orientable.)
  - Let $V_i$ be the neighborhoods of $M$ with charts $(U_i, G_i, \phi_i)$.

  - Define new charts $(U_i \times \{-1, 1\}, G_i, \phi_i^*)$ where $G_i$ acts by $(g(x, l) = (g(x), s(g)l)$ where $s(g)$ is 1 if $g$ is orientation-preserving and $-1$ if not and $\phi_i^*$ is the quotient map.
  - For each embedding, $i : (W, H, \psi) \to (U_i, G_i, \phi_i)$ we define a lift $(W \times \{-1, 1\}, H, \psi^*) \to (U_i \times \{-1, 1\}, G_i, \phi_i^*$. This defines the gluing.
  - The result is the doubled orbifold and the local group actions are orientation preserving.
  - The double covers the original orbifold with Galois group $\mathbb{Z}_2$.

**Doubling an orbifold with mirror points**

- In the abstract definition, we simply let $X_0'$ be the orientation double cover of $X_0$ where $G$-acts on $X'$ preserving the orientation.
- For example, if we double a corner-reflector, it becomes a cone-point.

**Some Examples**

- Clearly, manifolds are orbifolds. Manifold coverings provide examples.
- Let $Y$ be a tear-drop orbifold with a cone-point of order $n$. Then this cannot be covered by any other type of an orbifold and hence is a universal cover of itself.
- A sphere $Y$ with two cone-points of order $p$ and $q$ which are relatively prime.
- Choose a cyclic action of $Y$ of order $m$ fixing the cone-point. Then $Y/Z_m$ is an orbifold with two cone-points of order $pm$ and $qm$.

**Universal covering by fiber-product**

- A universal cover of an orbifold $Y$ is an orbifold $\tilde{Y}$ covering any covering orbifold of $Y$.
- We will now show that the universal covering orbifold exists by using fiber-product constructions. For this we need to discuss elementary neighborhoods. An *elementary* neighborhood is an open subset with a chart components of whose inverse image are open subsets with charts.
- We can take the model open set in the chart to be simply connected.
- Then such an open set is elementary.

**Fiber-product for $D^n/G_i$**

- If $V$ is an orbifold $D^n/G$ for a finite group $G$.
  - Any covering is $D^n/G_1$ for a subgroup $G_1$ of $G$.
  - Given two covering orbifolds $D^n/G_1$ and $V/G_2$, a covering morphism is induced by $g \in G$ so that $gG_1g^{-1} \subset G_2$.
  - The covering morphism is in one-to-one correspondence with the double cosets of form $G_2gG_1$ for $g$ such that $gG_1g^{-1} \subset G_2$.
  - The covering automorphism group of $D^n/G'$ is given by $N(G_1)/G_1$.

**Fiber-product for $D^n/G_i$**

- Given coverings $p_i : V/G_i \to V/G$ for $G_i \subset G$ for $V$ homeomorphic to a cell, we form a fiber-product.

$$V^f = (V \times \prod_{i \in I} G_i \backslash G)/G \to V/G$$

- If we choose all subgroups $G_i$ of $G$, then any covering of $V/G$ is covered by $V^f$ induced by projection to $G_i$-factor (universal property)

**The construction of the fiber-product of a sequence of orbifolds**

- Let $Y_i, i \in I$ be a collection of the orbifold-coverings of $Y$.
- We cover $Y$ by elementary neighborhoods $V_j$ for $j \in J$ forming a good cover.
- We take inverse images $p_i^{-1}(V_j)$ which is a disjoint union of $V/G_k$ for some finite group $G_k$.
- Fix $j$ and we form one fiber product by $V/G_k$ by taking one from $p_i^{-1}(V_j)$ for each $i$.
- Fix $j$ and we form a fiber-product of $p_i^{-1}(V_j)$, which will essentially be the disjoint union of the above fiber products indiced by the product of the component indices for each $i$.
- Over regular points of $V_j$, this is the ordinary fiber-product.

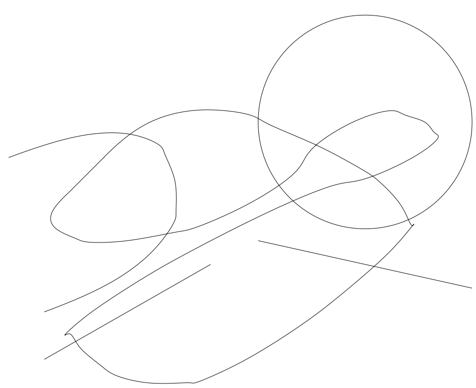
**The construction of the fiber-product of a sequence of orbifolds**

- Now, we wish to patch these up using imbeddings. Let $U \to V_j \cap V_k$. We can assume $U = V_j \cap V_k$ which has a convex cell as a cover.
  - We form the fiber products of $p_i^{-1}(U)$ as before which can be realized in $V_j$ and $V_k$.

- Over the regular points in $V_j$ and $V_k$, they are isomorphic. Then they are isomorphic.
- Thus, each component of the fiber-product can be identified.

- By patching, we obtain a covering $Y^f$ of $Y$ with the covering map $p^f$.

**Thurston's example of fiber product**

- Let $I$ be the unit interval. Make two endpoints into silvered points.
- Then $I_1 = I$ is double covered by $\mathbf{S}^1$ with the deck transformation group $\mathbb{Z}_2$. Let $p_1$ denote the covering map.
- $I_2 = I$ is also covered by $I$ by a map $x \mapsto 2x$ for $x \in [0, 1/2]$ and $x \mapsto 2 - 2x$ for $x \in [1/2, 1]$. Let $p_2$ denote this covering map.
- Then the fiber product of $p_1$ and $p_2$ is what?
- Cover $I$ by $A_1 = [0, \epsilon), A_2 = (\epsilon/2, 1 - \epsilon/2), A_3 = (\epsilon, 1]$.
  - Over $A_1$, $I_1$ has an open interval and $I_2$ has two half-open intervals. The fiber-product is a union of two copies of open intervals.
  - Over $A_2$, the fiber product is a union of four copies of open intervals.
  - Over $A_3$, the fiber product is a union of two copies of open intervals.
- By pasting considerations, we obtain a circle mapping 4-1 almost everywhere to $I$.
- 

**The construction of the universal cover**

- The collection of cover of an orbifold is countable upto isomorphisms preserving base points. (Cover by a countable good cover and for each elementary neighborhood, there is a countable choice.)

- Take a fiber product of $Y_i$, $i = 1, 2, 3, ...$. The fiber-product $\tilde{Y}$ with a base point $*$. We take a connected component.
- The for any cover $Y_i$, there is a morphism $\tilde{Y} \to Y_i$.
- The universal cover is unique up to covering orbifold-isomorphisms by the universality property.

**Properties of the universal cover**

- The group of automorphisms of $\tilde{Y}$ is called the fundamental group and is denoted by $\pi_1(Y)$.
- $\pi_1(Y)$ acts transitively on $\tilde{Y}$ on fibers of $\tilde{p}^{-1}(x)$ for each $x$ in $Y$. (To prove this, we choose one covering of $Y$ from a class of base-point preserving isomorphism classes of coverings of $Y$. Then the universal cover with any base-point occurs will occur in the list and hence a map from $\tilde{Y}$ to it preserving base-points.)
- $\tilde{Y}/\pi_1(Y) = Y$.
- Any covering of $Y$ is of form $\tilde{Y}/\Gamma$ for a subgroup $\Gamma$ of $\pi_1(Y)$.
- The isomorphism classes of coverings of $Y$ is the set of conjugacy classes of subgroups of $\pi_1(Y)$.

**Properties of the universal cover**

- The group of automorphism is $N(\Gamma)/\Gamma$.
- A covering is regular if and only if $\Gamma$ is normal.
- A *good orbifold* is an orbifold with a cover that is a manifold.
- An *very good orbifold* is an orbifold with a finite cover that is a manifold.
- A good orbifold has a simply-connected manifold as a universal covering space.

**Induced homomorphism of the fundamental group**

- Given two orbifolds $Y_1$ and $Y_2$ and an orbifold-diffeomorphism $g : Y_1 \to Y_2$. Then the lift to the universal covers $\tilde{Y}_1$ and $\tilde{Y}_2$ is also an orbifold-diffeomorphism. Furthermore, once the lift value is determined at a point, then the lift is unique.
- Also, homotopies $f_t : Y_1 \to Y_2$ of orbifold-maps lift to homotopies in the universal covering orbifolds $\tilde{f}_t : \tilde{Y}_1 \to \tilde{Y}_2$. Proof: we consider regular parts and model neighborhoods where the lift clearly exists uniquely.
- Given orbifold-diffeomorphism $f : Y \to Z$ which lift to a diffeomorphism $\tilde{f} : \tilde{Y} \to \tilde{Z}$, we obtain $f_* : \pi_1(Y) \to \pi_1(Z)$.
- If $g$ is homotopic to $f$, then $g_* = f_*$.

# 4 Path-approach to the universal covering spaces

**Path-approach to the universal covering spaces.**

- $G$-paths. Given an etale groupoid $X$. A $G$-path $c = (g_0, c_1, g_1, ..., c_k, g_k)$ over a subdivision $a = t_0 \leq t_1 \leq ... \leq t_k = b$ of interval $[a, b]$ consists of
  - continuous maps $c_i : [t_{i-1}, t_i] \to X_0$
  - elements $g_i \in X_1$ so that $s(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, .., k-1$ and $t(g_i) = c_i(t_i)$ for $i = 1, .., k$.
- The initial point is $t(g_0)$ and the terminal point is $s(g_k)$.
- The two operations define an equivalence relation:
  - Subdivision. Add new division point $t'_i$ in $[t_i, t_{i+1}]$ and $g'_i = 1_{c_i(t'_i)}$ and replacing $c_i$ with $c'_i, g'_i, c''_i$ where $c'_i, c''_i$ are restrictions to $[t_i, t'_i]$ and $[t'_i, t_{i+1}]$.
  - Replacement: replace $c$ with $c' = (g'_0, c'_1, g'_1, .., c'_k, g'_k)$ as follows. For each $i$ choose continuous map $h_i : [t_{i-1}, t_i] \to X_1$ so that $s(h_i(t)) = c_i(t)$ and define $c'_i(t) = t(h_i(t))$ and $g'_i = h_i(t_i) g_i h_{i+1}^{-1}(t_i)$ for $i = 1, .., k-1$ and $g'_0 = g_0 h_1^{-1}(t_0)$ and $g'_k = h_k(t_k) g_k$.

**Compositions of $G$-paths**

- All paths are defined on $[0, 1]$ from now on.
- Given two paths $c = (g_0, c_1, .., c_k, g_k)$ over $0 = t_0 \leq t_1 \leq ... \leq t_k = 1$ and $c' = (g'_0, c'_1, .., c'_{k'}, g'_{k'})$ such that the terminal point of $c$ equals the initial point of $c'$, the composition $c * c'$ is the $G$-path $c'' = (g''_0, c''_1, .., g''_{k+k'})$ so that
  - $t''_i = t_i/2$ for $i = 0, .., k$ and $t''_i = 1/2 + t'_{i-k}/2$ and
  - $c''_i(t) = c_i(2t)$ for $i = 1, .., k$ and $c''_i(t) = c'_{i-k}(2t-1)$ for $i = k+1, ..., k+k'$.
  - $g''_i = g_i$ for$i = 1, .., k-1$ and $g''_k = g_k g'_0, g''_i = g'_{i-k}$ for $i = k+1, .., k+k'$.
- The inverse $c^{-1}$ is $(g'_0, c'_1, ..., c'_k, g'_k)$ over the subdivision where $t'_i = 1 - t_i$ so that $g'_i = g_{k-i}^{-1}$ and $c'_i(t) = c_{k-i+1}(1-t)$.

**Homotopies of $G$-paths**

- There are two types
  - equivalences
  - An elementary homotopy is a family of $G$-paths $c^s = (g_0^s, c_1^s, ..., g_k^s)$ over the subdivision $0 = t_0^s \leq t_1^s \leq ... \leq t_k^s = 1$ so that $t_k^s, g_i^s, c_i^s$ depends continously on $s$.
  - A homotopy class of $c$ is denoted $[c]$.

- $[c*c']$ is well-defined in the homotopy classes $[c]$ and $[c']$. Hence, we define $[c]*[c']$.
- $[c*(c'*c'')] = [(c*c')*c'']$.
- The constant path $e_x = (1_x, x, 1_x)$. Then $[c*c^{-1}] = [e_x]$ if the initial point of $c$ is $x$ and $[c^{-1}*c] = [e_y]$ if the terminal point of $c$ is $y$. Thus, $[c]^{-1} = [c^{-1}]$.

**Fundamental group $\pi_1(X, x_0)$**

- The fundamental group $\pi_1(X, x_0)$ based at $x_0 \in X_0$ is the group of loops based at $x_0$.
- A continuous homomorphism $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.
- This is well-defined up to conjuations.
- An equivalence induces an isomorphism.
- Seifert-Van Kampen theorem: $X$ an orifold. $X_0 = U \cup V$ where $U$ and $V$ are open and $U \cap V = W$. Assume that the groupoid restrictions $G_U$, $G_V$, $G_W$ to $U, V, W$ are connected. And let $x_0 \in W$. Then $\pi_1(X, x_0)$ is the quotient group of the free product $\pi_1(G_U, x_0) * \pi_1(G_V, x_0)$ by the normal subgroup generated by $j_U(\gamma)j_W(\gamma^{-1})$ for $\gamma \in \pi_1(G_W, x_0)$ for $j_U$ the induced homomorphism $\pi_1(G_W, x_0) \to \pi_1(G_U, x_0)$ and $j_V$ the induced homomorphism $\pi_1(G_W, x_0) \to \pi_1(G_V, x_0)$ .

**Examples**

- Let a discrete group $\Gamma$ act on a connected manifold $X_0$ properly discontinuously. Then $(\Gamma, X_0)$ has an orbifold structure. Any loop can be made into a $G$-path $(1_x, c, \gamma)$ so that $\gamma(x) = c(1)$. and $c(0) = x$. Thus, there is an exact sequence

$$1 \to \pi_1(X_0, x_0) \to \pi_1((\Gamma, X_0), x_0) \to \Gamma \to 1$$

- A two-orbifold that is a disk with an arc silvered has the fundamental group isomorphic to $Z_2$.
- A two-dimensional orbifold with cone-points which is boundariless and with no silvered point.
- A tear drop: A sphere with one cone-point of order $n$ has the trivial fundamental group

**Examples**

- An annulus with one boundary component silvered has a fundamental group isomorphic to $Z \times Z_2$.

  The fundamental group can be computed by removing open-ball neighborhoods of the cone-points and using Van-Kampen theorem.

- Suppose that a two-dimensional orbifold has boundary and silvered points. Then remove open-ball neighborhoods of the cone-points and corner-reflector points. Then the fundamental group of remaining part can be computed by Van-Kampen theorem by taking open neighborhoods of silvered boundary arcs. Finally, adding the open-ball neighborhoods, we compute the fundamental group.

- The fundamental group of a three-dimensional orbifold can be computed similarly.

**Seifert fibered $3$-manifold Examples**

- We can obtain a 2-orbifold from a Seifert fibered 3-manifold $M$.
- $X_0$ will be the union of patches transversal to the fibers.
- $X_1$ will be the arrows obtained by the flow.
- The orbifold $X$ will be a 2-dimensional one with cone-points whose orders are obtained as the numerators of the fiber-order.
- The fundamental group of $X$ is then the quotient of the ordinary fundamental group $\pi_1(M)$ by the central cyclic group $\mathbb{Z}$ generated by the generic fiber.

**Covering spaces and the fundamental group**

- One can build the theory of covering spaces using the fundamental group.
- Given a covering $X' \to X$:
  - For every $G$-path $c$ in $X$, there is a lift $G$-path in $X'$. If we assign the initial point, the lift is unique.
  - If $c'$ is homotopic to $c$, then the lift of $c'$ is also homotopic to the lift of $c$ provided the initial points are the same.
  - $\pi_1(X', x_0') \to \pi_1(X, x_0)$ is injective.
  - A map from a simply connected orbifold to an orbifold lifts to a cover. The lift is unique if the base-point lift is assigned. Thus, a simply connected cover of an orbifold covers any cover of the given orbifold.
  - From this, we can show that the fiber-product construction is simply-connected and hence is a universal cover.
  - Two simply-connected coverings of an orbifold are isomorphic and if base-points are given, we can find an isomorphism preserving the base-points.

**Covering spaces and the fundamental group**

- A simply-connected covering of an orbifold $X$ is a Galois-covering with the Galois-group isomorphic to $\pi_1(X, x_0)$.
- Proof: Consider $p^{-1}(x_0)$. Choose a base-point $\tilde{x}_0$ in it. Given a point of $p^{-1}(x_0)$, connected it with $\tilde{x}_0$ by a path. The paths map to the fundamental group. The Galois-group acts transitively on $p^{-1}(x)$. Hence the Galois-group is isomorphic to the fundamental group.

**The existence of the universal cover using path-approach**

- The construction follows that of the ordinary covering space theory.
  - Let $\hat{X}$ be the set of homotopy classes $[c]$ of $G$-paths in $X$ with a fixed starting point $x_0$.
  - We define a topology on $\hat{X}$ by open set $U_{[c]}$ that is the set of paths ending at a simply-connected open subset $U$ of $X$ with homotopy class $c * d$ for a path $d$ in $U$.
  - Define a map $\hat{X} \to X$ sending $[c]$ to its endpoint other than $x_0$.
  - Define a map $\hat{X} \times X_1 \to \hat{X}$ given by $([c], g) \to [c * g]$. This defines a right $G$-action on $\hat{X}$. This makes $\hat{X}$ into a bundle.
  - Define a left action of $\pi_1(X, x_0)$ on $\hat{X}$ given by $[c] * [c'] = [c * c']$ for $[c'] \in \pi_1(X, x_0)$. This is transitive on fibers.
  - We show that $\hat{X}$ is a simply connected orbifold.