1 Introduction

Outline

- Definition of geometric structures on 2-orbifolds
  - Using charts
  - Goodness of geometric 2-orbifolds.
  - Using development pair.
  - Flat X-bundles and transversal sections.
- The deformation spaces of geometric structures on 2-orbifolds
- The local homeomorphism theorem from the deformation space to the representation space.

Some helpful references

- S. Choi, Geometric structures on orbifolds and holonomy representations, Geometriae Dedicata 104: 161 – 199, 2004.
- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977.
- M. Kapovich, Hyperbolic Manifolds and Discrete Groups: Lectures on Thurston's Hyperbolization, Birkhauser's series "Progress in Mathematics", 2000.
- Hubbard, Teichmuller Theory and Applications to Geometry, Topology, and Dynamics Volume 2: Four Theorems by William Thurston `http://matrixeditions.com/TeichmullerVol2.html`.

2 Definition

Definition of geometric structures on orbifolds

- Let (X, G) be a pair defining a geometry. That is, G is a Lie group acting on a manifold effectively and transitively.
- Given an orbifold M, there is at least three ways to define (X, G)-geometric structure on M.
  - Using atlas of charts.
  - A developing map from the universal covering space.
  - A cross-section of the flat orbifold X-bundle.

Atlas of charts approach

- Given an atlas of charts for M, for each chart (U, K, ϕ) we find an X-chart $\rho : U \to X$ and an injective homomorphism $h : K \to G$ so that ρ is an equivariant map.
- For each imbedding $i : (V, H, \psi) \to (U, K, \phi)$ where V has an X-chart $\rho' : V \to X$ and equivariant with respect to an injective homomorphism $h' : H \to G$, we have
$$\rho \circ i = g \circ \rho', h'(\cdot) = gh(i^*(\cdot))g^{-1}$$
- If we simply identify with open subsets of X, the above simplifies greatly and i is a restriction of an element of g and i^* is a conjugation by g also.
- This gives us a way to build an orbifold from pieces of X.
- A maximal such atlas of X-charts is called *an* (X, G)-*structure* on M.

Atlas of charts approach

- An (X, G)-*map* $M \to N$ is a smooth map f so that for each x and $y = f(x)$, there are charts (U, K, ϕ) and (V, H, ψ) so that f sends $\phi(U)$ into $\psi(V)$ and lifts to $\tilde{f} : U \to V$ so that $\rho' \circ \tilde{f} = g \circ \rho$ and $h'(i^*(\cdot)) = gh(\cdot)g^{-1}$.
- In otherwards, f is a restriction of an element g of G up to charts with a homomorphism $K \to H$ induced by a conjugation by an element of G.

Atlas of charts approach

- (X, G)-orbifold is always good.
- Proof:
  - Basically build a germ of local (X, G)-maps from M to X which is a principal bundle and is a manifold: For each (U, K, ϕ), we build $G(U) = G \times U/K$ and a projection $G(U) \to U$. We paste these together to find $G(M)$.
  - $G(M)$ is a manifold since K acts on $G \times U$ freely.
  - The foliation given by pasting $g_0 \times U$ is a foliation by open manifolds with the same dimension as M. Each leave of the foliation is covers M.
- If G is a subrgroup of a linear group, then M is very good by Selberg's lemma.
- Thus M is a quotient $\tilde{M}/\Gamma$ where Γ contains copies of all of the local group.

The developing maps and holonomy homomorphisms

- Let $\tilde{M}$ denote the universal cover of M with a deck transformation group π.
- Then we obtain a *developing map* $D : \tilde{M} \to X$ by first finding an initial chart $\rho : U \to X$ and continuing by extending maps by patches.
- One uses a nice cover of $\tilde{M}$ and extend. The map is well-defined independently of which path of charts one took to arrive at a given chart. To show this, we need to homotopy and consider three charts simultaneously.
- This gives an (X, G)-structure on $\tilde{M}$ as well and the cover map is an (X, G)-map.

The developing maps and holonomy homomorphisms

- Since we can change the initial chart to $k \circ \rho$ for any $k \in G$, we see that $k \circ D$ is another developing map and conversely any developing map is of such form.
- Given a deck transformation $\gamma : \tilde{M} \to \tilde{M}$, we see that $D \circ \gamma$ is a developing map also and hence equals $h(\gamma) \circ D$ for some $h(\gamma) \in G$.
- The map $h : \pi \to G$ is a homomorphism, so-called the holonomy homomorphism.
- The pair (D, h) is said to be the *development pair*.
- The development pair is determined up to an action of G given by $(D, h(\cdot)) \to (g \circ D, gh(\cdot)g^{-1})$.

The developing maps and holonomy homomorphisms

- Conversely, a developing map (D, h) gives us X-charts:
- For each open chart (U, K, ψ), we lift to a component of $p^{-1}(U)$ in $\tilde{M}$ and obtain a restriction of D to the component. This gives us X-charts.
- A different choice of components gives us the compatible charts.
- Local group actions and imbeddings satisfy the desired properties.
- Thus, a development pair completely determines the (X, G)-structure on M.

Definition as flat bundles with sections

- Given an (X, G)-manifold with X-charts, form a G-bundle $G(M)$ as above. This is a principal G-bundle. We form an associated an X-bundle $X(M)$ using the G-action on X.

- $X(G) = G(X) \times X/G$ where G acts on the right on $G(X)$ and left on X. and G acts on $G(M) \times X$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in G(M), x \in X.$$

- A flat G-bundle is an object obtained by patching open sets $G \times U$ by the left action of G, and so is a flat X-bundle

Flat X-bundles

- A foliation in $G(M)$ induces a foliation in $G(M) \times X$ and hence a foliation in $X(M)$ transversal to fibers. This corresponds to a flat G-connection.

- A flat G-connection on $X(M)$ is a way to identify each fibers of $X(M)$ with X locally-consistently.

- A flat G-connection on $X(M)$ gives us a flat G-connection on $X(\tilde{M})$. Since $\tilde{M}$ is a simply-connected manifold, $X(\tilde{M})$ equals $X \times \tilde{M}$ as an X-bundle. $X(\tilde{M})$ covers $X(M)$ and hence $X(M) = X \times \tilde{M}/\pi_1(M)$ where the connection corresponds to foliations with leaves of type $x \times \tilde{M}$.

- Hence this gives us a representation $h : \pi_1(M) \to G$ so that for any $\gamma \in \pi_1(M)$, the corresponding action in $X \times \tilde{M}$ is given by $(x, m) \to (h(\gamma)x, \gamma(x))$.

- Conversely, given a representation h, we can build $X \times \tilde{M}$ and act by $\gamma(x, m) = (h(\gamma), \gamma(m))$ to obtain a flat X-bundle $X(M)$.

Flat X-bundles with sections

- Conversely, an atlas of (X, G)-charts gives us a flat X-bundle $X(M)$ with a section $s; M \to X(M)$.

- An atlas gives us a development pair (D, h). We obtain a section $D' : \tilde{M} \to X \times \tilde{M}$ transversal to the foliation. The left-action of $\pi_1(M)$ gives us a section $s : M \to X(M)$ transversal to the foliation.

- On the other hand, given a transversal section $s : M \to X(M)$, we obtain a transversal section $s' : \tilde{M} \to X \times \tilde{M}$. By a projection to X, we obtain an immersion $D : \tilde{M} \to X$ so that $D \circ \gamma = h(\gamma) \circ D$ for some $h(\gamma)$ in G. The map $h : \pi_1(M) \to G$ is a homomorphism. Hence we obtain a development pair.

The equivalences of three notions.

- Hence, given an atlas of X-charts, i.e., a (X, G)-structure, we determine a development pair (D, h).
- Given a development pair (D, h), we determine an atlas of X-charts, i.e., an (X, G)-structure.
- Given a development pair (D, h), we determine a flat X-bundle $X(M)$ with a transversal section $M \to X(M)$.
- Given a section $s : M \to X(M)$ to a flat X-bundle, we determine a development pair (D, h).
- Thus, these three class of defintions are equivalent.