1 Introduction

About this lecture

- Notions of Inference
- Inference Rules
- Hypothetical Rules
- Derived Rules
- The Propositional Rules
- Equivalences
- The soundness and the completeness of deductions.
- We go over the last three Hypothetical Rules in Lecture 6.
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer
- A mathematical introduction to logic, H. Enderton, Academic Press.
- Whitehead, Russel, Principia Mathematica (our library). (This could be a project idea.)
- `http://plato.stanford.edu/contents.html` has much resource. See “classical logic”.
- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Derivations in Sentential Calculus". (or SC Derivations.) and “The completeness of the SC rules.”
- `http://jvrosset.free.fr/Goedel-Proof-Truth.pdf` “Does Godels incompleteness prove that truth transcends proof?”

Some helpful references

- `http://en.wikipedia.org/wiki/Truth_table`,
- `http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html`, complete (i.e. has all the steps)
- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

2 Derived Rules

Derived Rules

- Suppose that one proved a logical formula, which are not in the ten elementary rules. Then we can substitute the symbols with wffs and still obtain valid logical formula.
- Example: $P \to Q, \neg Q \vdash \neg P$. (Modus Tollens (MT)).
- Substitution instance: P to $(R \vee S)$ and Q to $\neg C$. Then obtain $(R \vee S) \to \neg C$, $\neg\neg C. \vdash \neg(R \vee S)$.

Examples

- Prove MT:
- 1. $P \to Q$ A
- 2. $\neg Q$ A.
- 3.: $\neg\neg P$. for $\neg I$.
- 4.: P. $\neg E$.
- 5.: Q 1,4 $\leftarrow E$.
- 6.: $Q \wedge \neg Q$. 2.5. $\wedge I$.
- 7. $\neg P$.

Derived Rules

- Modus Tollens (MT): $P \to Q, \neg Q \vdash \neg P$.
- Hypothetical syllogism (HS): $P \to Q, Q \to R \vdash P \to R$.
- Absorption (ABS): $P \to Q \vdash P \to (P \wedge Q)$.
- Constructive Dilemma (CD): $P \vee Q$, $P \to R$, $Q \to S \vdash R \vee S$.
- Repeat or Reiteration (RE): $P \vdash Q \to P$.
- Contradiction (CON): $P, \neg P \vdash Q$.
- Disjunctive syllogism (DS): $P \vee Q, \neg P \vdash Q$.

Examples

- Prove CD: $P \vee Q, P \to R, Q \to S \vdash R \vee S$.
- 1. $P \vee Q$ A
- 2. $P \to R$ A.
- 3. $Q \to S$. A.
- 4.: P for $\to I$.
- 5.: R from $\to E$.
- 6.: $R \vee S$ $\vee I$.
- 7. $P \to (R \vee S)$.
- 8.: Q for $\to I$.
- 9.: S.
- 10.: $R \vee S$.
- 11. $Q \to (R \vee S)$.
- 12. $R \vee S$.

Examples

- Prove DS: $P \vee Q, \neg P \vdash Q$.
- 1. $P \vee Q$ A
- 2. $\neg P$ A.
- 3.: P for $\to I$.
- 4.: Q. 2.3. (CON)
- 5. $P \to Q$.
- 6.: Q for $\to I$.
- 7: $Q \to Q$.
- 8. Q.

3 Theorems

Theorems

- Theorems are wff deduced from no assumptions. They are just tautologies. (At least in this book)
- $\neg(P \land \neg P)$, or $\neg P \lor P$.
- $P \to ((P \to Q) \to Q)$.
- $P \to (Q \to P)$.
- $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$.
- $(((\neg P \to \neg Q) \to (Q \to P))$.

Example

- Deduce (Prove) $\vdash ((\neg P \to \neg Q) \to (Q \to P))$.
- 1. :$\neg P \to \neg Q$ H. for $\to I$.
- 2. :: Q. H for $\to I$.
- 3. ::: $\neg P$
- 4. ::: $\neg Q$. 1.3.
- 5. ::: $Q \land \neg Q$.
- 6. :: P
- 7. : $Q \to P$. 2-5
- 8. $\neg P \to \neg Q \to (Q \to P)$.

Example

- Deduce $\vdash (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$.
- 1. : $(P \to (Q \to R))$ for $\to I$.
- 2. :: $(P \to Q)$ for $\to I$.
- 3. ::: P for $\to I$.
- 4. ::: $(Q \to R)$ 1.3.
- 5. ::: $P \to R$. 2.4.
- 6. ::: R.
- 7. :: $P \to R$. 3-6
- 8. : $(P \to Q) \to (P \to R)$. 2-7
- 9. $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ 1-8.

4 Equivalences

Equivalences

- Equivalences $\phi \leftrightarrow \psi$ for two wff ϕ and ψ. We prove by $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$.
- Clearly, equivalence is exactly a tautology for the form $\phi \leftrightarrow \psi$.
- The equivalences can be used to replace some subwffs with equivalent subwffs.
- $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$. DeMorgan's law. (DM)
- $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$. (DM)
- $P \vee Q \leftrightarrow Q \vee P$. Commutation (COM)
- $P \wedge Q \leftrightarrow Q \wedge P$. (COM)
- $P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$. Association (ASSOC).
- $P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$. Association (ASSOC).

More equivalences

- $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$. Distribution (DIST)
- $P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$. (DIST)
- $P \leftrightarrow \neg\neg P$. Double negation (DN)
- $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$. Transposition (TRANS)
- $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$. Material Implication (MI)
- $(P \wedge Q) \rightarrow R \leftrightarrow (P \rightarrow (Q \rightarrow R))$. Exportation (EXP)
- $P \leftrightarrow (P \wedge P)$. Tautology (TAUT)
- $P \leftrightarrow (P \vee P)$. (TAUT)
- The equivalences can be verified by the truth table method or by deduction.

More derived rules

- Theorem introduction (TI): Any substituted version of a theorem may be introduced with at any line of the proof.
- Equivalence introduction (using above notations): Given χ with subwff ϕ and an equivalence $\phi \leftrightarrow \psi$, we deduce χ' with some subwffs of form ϕ replaced with subwffs of form ψ.

Example

- We use the equivalence $\neg P \vee Q \leftrightarrow \neg(P \wedge \neg Q)$. DM.
- We shall prove that $\neg P \vee Q, \vdash P \rightarrow Q$
- 1. $\neg P \vee Q$. A
- 2. : P H. for $\rightarrow I$.
- 3. ::$\neg Q$ H for $\neg I$.
- 4. :: $P \wedge \neg Q$.
- 5. :: $\neg(P \wedge \neg Q)$. 1. (DM)
- 6. :: $(P \wedge \neg Q) \wedge \neg(P \wedge \neg Q)$.
- 7. : Q. 3-6
- 8. $P \rightarrow Q$. 2-7.

4.1 Soundness and completeness of deductions

Soundness

- A logical system is a formal system with
  - An alphabet, a set of statement symbols with logical connectives.
  - well-formed formulas
  - A set of axioms.
  - Rules of inference.
- See also `http://plato.stanford.edu/entries/logic-classical/`
- A logical system is consistent if not all wff can be deduced. (Equivalently, exactly one of ϕ and $\neg\phi$ can be deduced.)
- Given any truth-false assigment to atomic formula so that the axioms are all true, the soundness means that by applying rules of inference you obtain true statements only.
- That is, we cannot deduce a falsehood.

Completeness

- The completeness means that if a formula is true from logical truth assignment from a set of assumptions Σ, then the formula can be deduced from Σ.
- This is true for the first order theories but not true for higher-order theories. Also, true if there are finitely or countably many statement symbols.
- See Chapters 3 and 4 of Cameron.