Logic and the set theory

Lecture 20: The set theory (NS)

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About this lecture

- Further set theory

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  - Ordered pairs

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

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- Sets for mathematics, F.W. Lawvere, R. Rosebrugh, Cambridge

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- To use this freely without much difficulty, the set theory was invented.
- In this lecture, we will be more rigorous than in HTP and use axioms to establish facts.

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- $\rightarrow$: $\{\{a\}, \{a, b\}\} = \{\{a\}\}$. $\{a\} = \{a, b\}$. $b \in \{a\}$. $b = a$.

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- $\leftarrow$: clear
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- (i) If $a = b$, then (a, b) is a singleton, and so is (x, y). we obtain $x = y$. $x \in \{a\}$. Thus $x = y = a = b$.
- (ii) If $a \neq b$, then since both (a, b) and (x, y) contain exactly one singletons, it follows that $a = x$. $\{a, b\} = \{x, y\}$ also. $b \in \{x, y\}$. Since $b \neq x$, $b = y$.

Relations

- covered in HTP

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- Define $X^Y = \{f : Y \to X\}$. The set of functions. This is a set!

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- $\emptyset^{Y} = \{f : Y \to \emptyset\} = \emptyset$. $f(y) =?$ is undefined. No existence.

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- When I if finite, $\bigcup_{i\in I} A_i = A_{i_1} \cup \cdots \cup A_{i_n}$.

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- The function $f : Z \to X \times Y$ given by $f(z) = (z(a), z(b))$ is one-to-one and onto.
- $\prod_{i \in \{a,b\}} X_i$ where $X_1 = X$ and $X_2 = Y$.

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- Odered triples, quadruples, and ... n-tuples...
- Given a subset J of I, we form a function $\prod_{i \in I} X_i \to \prod_{i \in J} X_i$ given by sending $f : I \to \bigcup_{i \in I} X_i$ to the restriction $f|J : J \to \bigcup_{i \in J} X_i$ where domains and range are restricted.

Inverses and composites

- Covered in HTP.