Logic and the set theory

Lecture 9: Predicate Calculus

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About this lecture

- Reasoning in predicate calculus

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- Inference rules for the universal quantifiers

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- Theorems and quantifier equivalence rules

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.

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- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Monadic Predicate Calculus" and MPC completeness, Predicate Calculus, Predicate Calculus Deriviations

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- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.

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- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.
- `http://logic.philosophy.ox.ac.uk/`. See "Predicate Calculus" in Tutorial.

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- http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html, complete (i.e. has all the steps)

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- http://svn.oriontransfer.org/TruthTable/index.rhtml, has xor, complete.

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- 1. $\forall x(M_1 x \rightarrow M_2 x)$. A, 2. $M_1 S$ A, 3. $M_1 S \rightarrow M_2 S$. 4. $M_2 S$.

Example

- $\neg Fa \vdash \neg \forall x Fx$.

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- 1. $\neg Fa$ A.

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- 4.: $Fa \wedge \neg Fa$. 1.3

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- 4.: $Fa \wedge \neg Fa$. 1.3
- 5. $\neg\forall x Fx$. 1-4

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  - We can introduce one quantifier at a time.

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- 3. Fa.

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- 5. $\forall x Fx$.
- 6. $\forall x Gx$.
- 7. $\forall x Fx \wedge \forall x Gx$.
- Is the converse true also. How does one prove it?

Examples

- $\forall x(Fx \to (Gx \lor Hx)), \forall x \neg Gx \vdash \forall x Fx \to \forall x Hx$.

Examples

- $\forall x(Fx \to (Gx \vee Hx)), \forall x \neg Gx \vdash \forall x Fx \to \forall x Hx$.
- 1. $\forall x(Fx \to (Gx \vee Hx))$ A.

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- 3. $Fa \to (Ga \vee Ha)$.

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- 10. $\forall x Fx \rightarrow \forall x Hx$.

Interchangiblility proof:

- $\forall x \forall y$ interchangible to $\forall y \forall x$.

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- $\forall x \forall y \phi(x, y) \leftrightarrow \forall y \forall x \phi(x, y)$.

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- $\forall x \forall y \phi(x, y) \leftrightarrow \forall y \forall x \phi(x, y)$.
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- The second one is similar and proved after the inference rules for existential quantifiers.

Inference rules for existential quantifiers

- Existential introduction $\exists I$.

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- ϕ contains α. Replace with $\exists\beta\phi^{\beta/\alpha}$. Here we replace some occurrences of α with β.

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Inference rules for existential quantifiers

- Example: $\neg\exists x F x \vdash \forall x \neg F x$.

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- Example: $\neg\exists xFx \vdash \forall x\neg Fx$.
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Inference rules for existential quantifiers

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- 4.: $\exists xFx \wedge \neg\exists xFx$.
- 5. $\neg Fa$.
- 6. $\forall x\neg Fx$.
- The converse can be proven also.

Example

- Example $\neg\forall x Fx \vdash \exists x \neg Fx$.

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- Example $\neg\forall x Fx \vdash \exists x \neg Fx$.
- 1. $\neg\forall x Fx$.

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- Example $\neg\forall x Fx \vdash \exists x \neg Fx$.
- 1. $\neg\forall x Fx$.
- 2.: $\neg\exists x \neg Fx$. (H)

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- 3.:: $\neg Fa$ (H) (note this also.)

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- 4.:: $\exists x \neg Fx$. 3 ($\exists I$).

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- 1. $\neg\forall x Fx$.
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- 5.:: $\exists x \neg Fx \wedge \neg\exists x \neg Fx$ 2.4.

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- 2.: $\neg\exists x \neg Fx$. (H)
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- 6.: Fa. 3-5.

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- 6.: Fa. 3-5.
- 7.: $\forall x Fx$. 6. ($\forall I$).

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- 6.: Fa. 3-5.
- 7.: $\forall x Fx$. 6. ($\forall I$).
- 8.: $\forall x Fx \wedge \neg\forall x Fx$. 1. 7.

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- Example $\neg\forall xFx \vdash \exists x\neg Fx$.
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- 7.: $\forall xFx$. 6. ($\forall I$).
- 8.: $\forall xFx \wedge \neg\forall xFx$. 1. 7.
- 9. $\exists x\neg Fx$. 2-8.

Existential Elimination

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  - α may not occur in hypothesis in effect.

Example

- Example: $\forall x(Fx \to Gx), \exists x Fx \vdash \exists x Gx$.

Example

- Example: $\forall x(Fx \rightarrow Gx), \exists xFx \vdash \exists xGx$.
- 1. $\forall x(Fx \rightarrow Gx)$ A

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- 1. $\forall x(Fx \to Gx)$ A
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- 3.: Fa. for $\exists E$.
- 4.: $Fa \to Ga$. 1,3.

Example

- Example: $\forall x(Fx \to Gx), \exists xFx \vdash \exists xGx$.
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- 5.: Ga.
- 6.: $\exists xGx$.
- 7. $\exists xGx$.

Example

- Example: $\forall x \neg Fx \vdash \neg \exists x Fx$.

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- Example: $\forall x \neg Fx \vdash \neg \exists x Fx$.
- 1. $\forall x \neg Fx$. A.

Example

- Example: $\forall x \neg Fx \vdash \neg \exists x Fx$.
- 1. $\forall x \neg Fx$. A.
- 2.: $\exists x Fx$. H for $\neg I$.

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- 1. $\forall x \neg Fx$. A.
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- 1. $\forall x \neg Fx$. A.
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- 5.:: $Fa \wedge \neg Fa$.

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- 3.:: Fa. by $\exists E$.
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- Example: $\forall x \neg Fx \vdash \neg \exists x Fx$.
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- 4.:: $\neg Fa$. 1,3.
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- 7. $\exists x Fx \rightarrow \neg \exists x Fx$. 2-7.

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- 1. $\forall x \neg Fx$. A.
- 2.: $\exists x Fx$. H for $\neg I$.
- 3.:: Fa. by $\exists E$.
- 4.:: $\neg Fa$. 1,3.
- 5.:: $Fa \wedge \neg Fa$.
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- 8. $\neg \exists x Fx$. (Use. $\neg P \rightarrow P, \vdash P$ or $P \rightarrow \neg P, \vdash \neg P$)

Example

- $\exists x \neg Fx \vdash \neg \forall x Fx$.

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- 2.: $\forall x Fx$. (H)

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Example

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- 5.:: $Fa \wedge \neg Fa$. 3.4.

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Some strategies

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  - $\exists y Fay$: We prove Fab.
  - $\exists x Fxx$: We prove Faa.

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  - $\forall x \neg Fx$: We prove $\neg Fa$.
  - $\forall x \exists y Fxy$: We prove $\exists y Fay$.
  - $\exists y Fay$: We prove Fab.
  - $\exists x Fxx$: We prove Faa.
- 3. To prove the forms in negation, conjunction, disjunction, conditional, or biconditional, then use propositional calculus methods.... (Similar to $(\forall x P) \to (\forall x Q)$).

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- 3.: $\exists x \neg Fx$.
- 4.:: $\neg Fa$. (for $\exists E$).
- 5.:: Fa 2.
- 6.:: $\neg Fa \wedge Fa$.
- 7.: $\neg Fa \wedge Fa$.
- 8. $\neg(\forall x Fx \wedge \exists x \neg Fx)$

Examples

- $\vdash \forall x F x \vee \exists x \neg F x$.

Examples

- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.

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- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
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- $\vdash \forall x Fx \vee \exists x \neg Fx$.
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- 4.::: $\exists x \neg Fx$

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- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.

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- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
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- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg \neg Fa$.

Examples

- $\vdash \forall xFx \vee \exists x\neg Fx$.
- 1.: $\neg\forall xFx$. H for $\rightarrow I$.
- 2.:: $\neg\exists x\neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
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- 7.:: Fa.

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- 7.:: Fa.
- 8.:: $\forall x Fx$.

Examples

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- 6.:: $\neg\neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.

Examples

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- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
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- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg \neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.
- 10.: $\neg \neg \exists x \neg Fx$.

Examples

- $\vdash \forall x Fx \lor \exists x \neg Fx$.
- 1.: $\neg\forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg\exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \land \neg\exists x \neg Fx$.
- 6.:: $\neg\neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \land \neg\forall x Fx$.
- 10.: $\neg\neg\exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.

Examples

- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg \exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg \neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.
- 10.: $\neg \neg \exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.
- 12. $\neg \forall x Fx \rightarrow \exists x \neg Fx$.

Examples

- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg \exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg\neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.
- 10.: $\neg\neg \exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.
- 12. $\neg \forall x Fx \rightarrow \exists x \neg Fx$.
- 13. $\neg\neg \forall x Fx \vee \exists x \neg Fx$. MI

Examples

- $\vdash \forall x Fx \vee \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg \exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \wedge \neg \exists x \neg Fx$.
- 6.:: $\neg\neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \wedge \neg \forall x Fx$.
- 10.: $\neg\neg \exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.
- 12. $\neg \forall x Fx \rightarrow \exists x \neg Fx$.
- 13. $\neg\neg \forall x Fx \vee \exists x \neg Fx$. MI
- 14. $\forall x Fx \vee \exists x \neg Fx$. DN.

Examples

- $\vdash \forall x Fx \lor \exists x \neg Fx$.
- 1.: $\neg \forall x Fx$. H for $\rightarrow I$.
- 2.:: $\neg \exists x \neg Fx$. H for $\neg I$.
- 3.::: $\neg Fa$ H. for $\neg I$.
- 4.::: $\exists x \neg Fx$
- 5.::: $\exists x \neg Fx \land \neg \exists x \neg Fx$.
- 6.:: $\neg\neg Fa$.
- 7.:: Fa.
- 8.:: $\forall x Fx$.
- 9.:: $\forall x Fx \land \neg \forall x Fx$.
- 10.: $\neg\neg \exists x \neg Fx$.
- 11.: $\exists x \neg Fx$.
- 12. $\neg \forall x Fx \rightarrow \exists x \neg Fx$.
- 13. $\neg\neg \forall x Fx \lor \exists x \neg Fx$. MI
- 14. $\forall x Fx \lor \exists x \neg Fx$. DN.
- There is a way using equivalences.

Examples

- The rule $\vdash \forall x P \rightarrow \exists x P$.

Examples

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- $\vdash \forall x \forall y P \rightarrow \exists x \exists y P$.
- $\vdash \forall x \exists y P \rightarrow \exists x \exists y P$.
- In fact, we can use a theorem to generate many more theorems....

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- We studied equivalences called interchanges: $\exists x \exists y \leftrightarrow \exists y \exists x$ and $\forall x \forall y \leftrightarrow \forall y \forall x$.

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- $\vdash \neg \forall x Fx \leftrightarrow \exists x \neg Fx$. This was proved above.

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- $\vdash \neg\forall x Fx \leftrightarrow \exists x \neg Fx$. This was proved above.
- $\vdash \forall x \neg Fx \leftrightarrow \neg\exists x Fx$. This was proved above.

Equivalences

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