Spherical triangles and the two components of the $\mathrm{SO}(3)$-character space of the fundamental group of a closed surface of genus 2.

Suh-Young Choi

Department of Mathematical Science
KAIST, Daejeon, South Korea
mathsci.kaist.ac.kr/ schoi (the copy of the lecture)

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Abstract

- We use geometric techniques to explicitly find the topological structure of the space of $\mathrm{SO}(3)$-representations of the fundamental group of a closed surface of genus 2 quotient by the conjugation action of $\mathrm{SO}(3)$.
- There are two components of the space. We will describe the topology of each of the two components and describe the corresponding $\mathrm{SU}(2)$-character spaces.
- For each component, there is a sixteen to one branch-covering and the branch locus is a union of 2-spheres and 2-tori.
- The main purpose is to find the explicit cell-decompositions.

Defintion of *G*-character spaces

- G a compact Lie group (algebraic)
- π a fundamental group of a compact surface.
- $\mathrm{Hom}(\pi, G)$ is an algebraic set in G^n for which G acts by conjugation.
- $\mathrm{Hom}(\pi, G)/G$ is a semi-algebraic set, called the *G-character space* of π.
- For $G = \mathrm{SU}(2)$, this is a well-known space.

Main motivation

- $\pi = \pi_1(\Sigma)$ for a real 2-dimensional closed surface.
- $G = \mathrm{SO}(3)$ or $G = \mathrm{SL}(3, \mathbb{R})$.
- The inclusion $Hom(\pi, \mathrm{SO}(3))/\mathrm{SO}(3) \to Hom(\pi, \mathrm{SL}(3, \mathbb{R}))/\mathrm{SL}(3, \mathbb{R})$.
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- $\mathcal{C}_0, \mathcal{C}_1$ into two components but not to the Teichmuller component.
- Question: what are the topology of the two components? (Goldman 1990)
- We are interested in non-Teichmuller components.
- We hope to understand from the imbedded subspaces.

History

- The classical work of Narashimhan, Ramanan, Seshadri, Newstead and so on [NR], [NS],[Ne2] show that the space of $\mathrm{SU}(2)$-characters for a genus-two closed surface is diffeomorphic to $\mathbb{CP}^3$.

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- See Goldman [G,1985] for a part of the beginning of the topological approach to the subject. Goldman found the symplectic structures on the character spaces.

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- Huebshmann [Hu, 1998] also showed that this space branch-covers the $\mathrm{SO}(3)$-character spaces. (See also Florentino-Lawton [FL].)
- Higgs bundle techniques as initiated by Donaldson [Do], Corlette [Cor], Hitchin [Hit], and Simpson [Sim]. There are now extensive accomplishments in this area using these techniques.
- See Bradlow, Garcia-Prada, and P. Gothen [BGG1] and [BGG2].

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- The main point of our method seems to be that we have more direct way to relate the $\mathrm{SO}(3)$-character space with the $\mathrm{SU}(2)$-character space with cell-structures preserved under the branching map.

The main objects

- Let Σ be a closed surface of genus 2 and $\pi_1(\Sigma)$ its fundamental group and let $\mathrm{SO}(3)$ denote the group of special orthogonal matrices with real entries.
- The space of homomorphisms $\pi_1(\Sigma) \to \mathrm{SO}(3)$ admits an action by $\mathrm{SO}(3)$ given by
$$h(\cdot) \mapsto g \circ h(\cdot) \circ g^{-1}, \text{ for } g \in \mathrm{SO}(3).$$
- $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SO}(3))$ as an algebraic subset of $\mathrm{SO}(3)^4$.
- We denote by $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ the Hausdorff quotient space of the space under the action of $\mathrm{SO}(3)$: i.e., the *space of $\mathrm{SO}(3)$-characters* of $\pi_1(\Sigma)$.

The main objects

- We define a solid tetrahedron G in the positive octant of $\mathbb{R}^3$ by the equation $x + y + z \geq \pi$, $x \leq y + z - \pi$, $y \leq x + z - \pi$, and $z \leq x + z - \pi$.
- There is a natural action of the Klein four-group on G by isometries generated by three involutions each fixing a maximal segment in G (See Figure 1.)
- We will denote the Klein four-group by V, isomorphic to $\mathbb{Z}_2^2$.
- A *double Klein four-group* isomorphic to $\mathbb{Z}_2^4$.

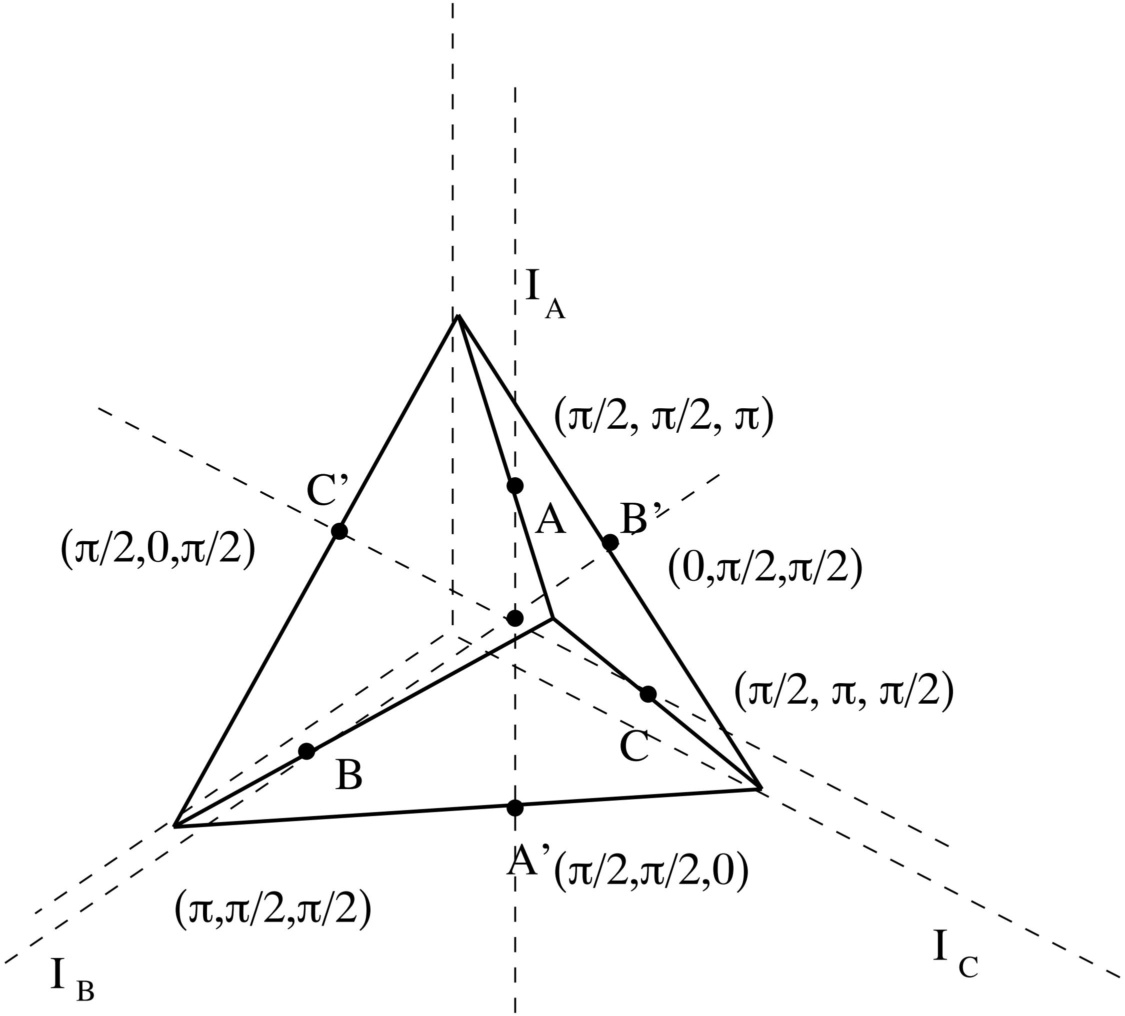

Figure: 1. The tetrahedron and the Klein four-group-symmetries. The three edges in front are labeled A, B, and C in front and the three opposite edges are labeled A', B', and C'.

The main result A

Theorem A

Let $\pi_1(\Sigma)$ the fundamental group of a closed surface Σ of genus 2.

(i) *The component $\mathcal{C}_0$ of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ is homeomorphic to the quotient space of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ by a double Klein four-group action.*

(ii) *$\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ is homeomorphic to $\mathbb{CP}^3$.*

(iii) *The quotient by the double Klein four-group induces a 16-to-1 branch-covering of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ onto $\mathcal{C}_0$.*

(iv) *$\mathcal{C}_0$ has an orbifold structure with singularities in a union of six 2-spheres meeting transversally.*

The main result A

- $\mathbb{CP}^3$ is a T^3-fibration over the tetrahedron where fibers over the interior are T^3, the fibers over the interiors of faces are T^2, the fibers over the interiors of the edges are circles, and the fiber over each of the vertices is a point.
- We will see $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ as $\mathbb{CP}^3$ by inserting into $\mathbb{CP}^3$ the four 3-balls corresponding to the vertices and inserting solid tori at the circles over the interior of edges.

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- The parameters of solid tori over the open edges will converge to 3-balls as they approach the fibers above the vertices. (clasping)
- The subspace of abelian characters consist of 2-tori over the interior of faces and the boundary 2-tori of the solid tori over edges and the boundary sphere of the vertex 3-balls. (crossing over the edges and identified to a sphere over the vertices.)

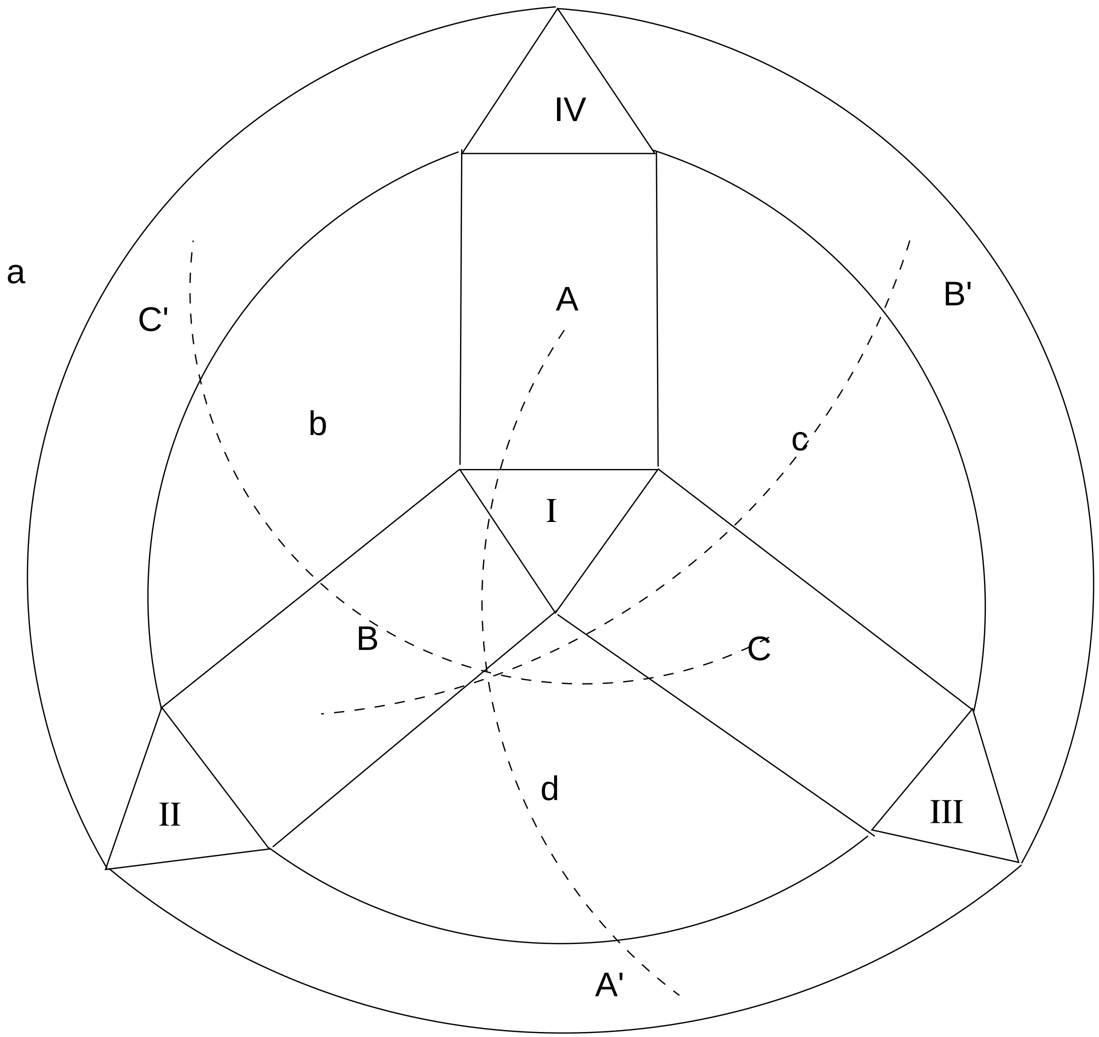

Figure: The face diagram of blown-up solid tetrahedron and regions to be explained later.

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- The octahedral manifold is a torus fibration over an octahedron so that over the interior of the octahedron the fibers are 3-dimensional tori and over the interior of faces the fibers are 2-dimensional tori and over the interior of the edges the fibers are circles and the over the vertex the fibers are 3-spheres.

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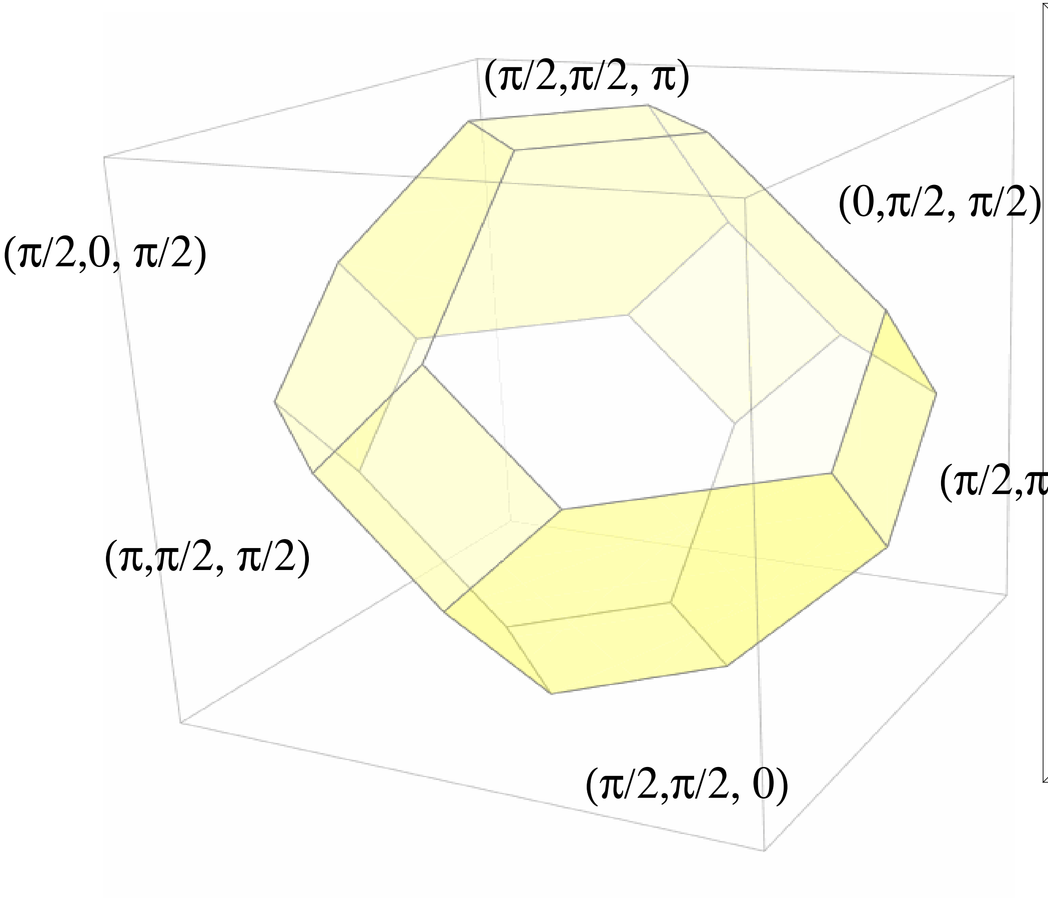
- Let Σ_1 denote a surface of genus two with one puncture, and $\mathrm{rep\_}_I(\pi_1(\Sigma), \mathrm{SU}(2))$ be the quotient space of the subspace of $\mathrm{Hom}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ determined by the condition that the holonomy of the boundary curve $-\mathrm{I}$ under the conjugation action.


(π/2,π/2, π)
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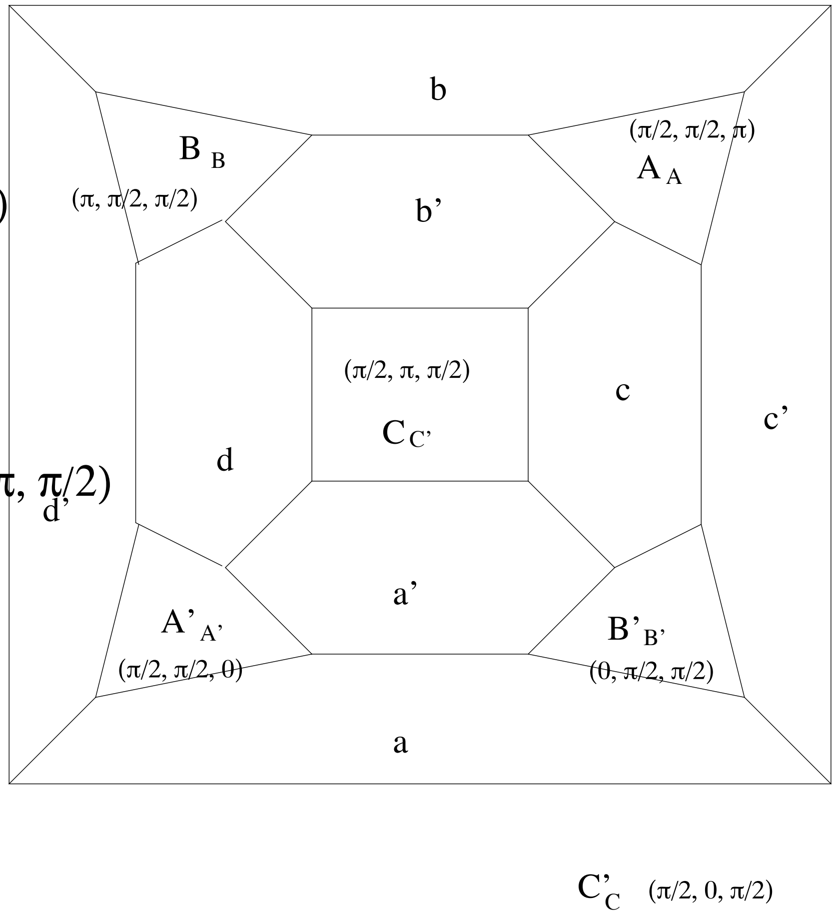
b
B B
(π, π/2, π/2)
b'
(π/2, π/2, π)
A A
(π/2, π, π/2)
C C'
c
c'
d
d'
a'
A' A'
(π/2, π/2, 0)
B' B'
(0, π/2, π/2)
a
C' C (π/2, 0, π/2)

The main result B

Theorem B

(i) $\mathcal{C}_1$ *is homeomorphic to the double Klein four-group quotient of an octahedral manifold.*

(ii) $\mathrm{rep}_{-I}(\pi_1(\Sigma_1, \mathrm{SU}(2))$ *is homeomorphic to an octahedral manifold seen as a torus fibration over an octahedron except at the vertices.*

(iii) $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ *branch-covers* $\mathcal{C}_1$ *in a* 16 *to* 1 *manner by an action of* $\mathbb{Z}_2^4$ *and has a cell structure.*

(iv) *There is a* $\mathbb{Z}_2^4$*-action preserving the torus fibers. The branch locus is a union of six* 2*-tori meeting transversally.*

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- The relationship of $\mathrm{SU}(2)$ with $\mathrm{SO}(3)$.
- The $\mathrm{SO}(3)$-character space for Σ, which has two components $\mathcal{C}_0$, containing the identity representation, and the other component $\mathcal{C}_1$.

Outline: $\mathcal{C}_0$

- $\mathcal{C}_0$ as a quotient space of the T^3-bundle over the blown-up tetrahedron above, and the explicit quotient relations for $\mathcal{C}_0$ by going over each of the faces of the blown-up tetrahedron.

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- The $\mathrm{SU}(2)$-character space of the fundamental group of Σ and the geometric representations of such characters using the spherical triangles. The character space is $\mathbb{CP}^3$.
- The topology of $\mathcal{C}_0$ and the $\mathbb{Z}_2^4$-action on the $\mathrm{SU}(2)$-character space of the fundamental group of Σ to branch-cover $\mathcal{C}_0$.

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- Describe the $\mathbb{Z}_2^4$-action on the above manifold to branch-cover $\mathcal{C}_1$.

Spherical triangles

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  - A *lune* is the closed domain in $\mathbf{S}^2$ bounded by two segments connecting two antipodal points forming an angle $< \pi$.
  - A *hemisphere* is the closed domain bounded by a great circle in $\mathbf{S}^2$.

Generalized triangles

We say that ordinary triangles to be *nondegenerate triangles*. We define *degenerate triangles*:

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- A *pointed-hemisphere* is a hemisphere with three ordered points on the boundary great circle where a segment between any two not containing the other is of length $\leq \pi$.

Generalized triangles

- A *pointed-segment* is a segment of length $\leq \pi$ with three ordered points where two are the endpoints and one is on the segment. Here again, the third point could be identical with one of the endpoint, and the *pointed-segment* is *degenerate*.

Generalized triangles

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- A *pointed-point* is a point with three identical vertices.

Angles

- The notion of angles for nondegenerate triangles is the same as in geometry.
- We now associate angles to each of the three vertices of degenerate triangles by the following rules. The angles are numbers in $[0, \pi]$. Let us use indices in $\mathbb{Z}_3$:
  - If a vertex v_i has two nonzero length edges l_{i-1} and l_{i+1} ending at v_i, then we define the angle θ_i at v_i to be the interior angle between the edge vectors oriented away from v_i.

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  - If a vertex v_i is such that exactly one of l_{i-1} or l_{i+1} has a zero length, say l_{i-1} without loss of generality, ... then we choose an arbitrary great circle $\mathbf{S}^1_{i-1}$ containing v_i .. We take the counter-clockwise unit tangent vector for $\mathbf{S}^1_{i-1}$, to be called the *direction vector* at v_i for l_{i-1}, and we take the inward unit tangent vector for l_{i+1} at v_i... (an *infinitesimal edge*.)

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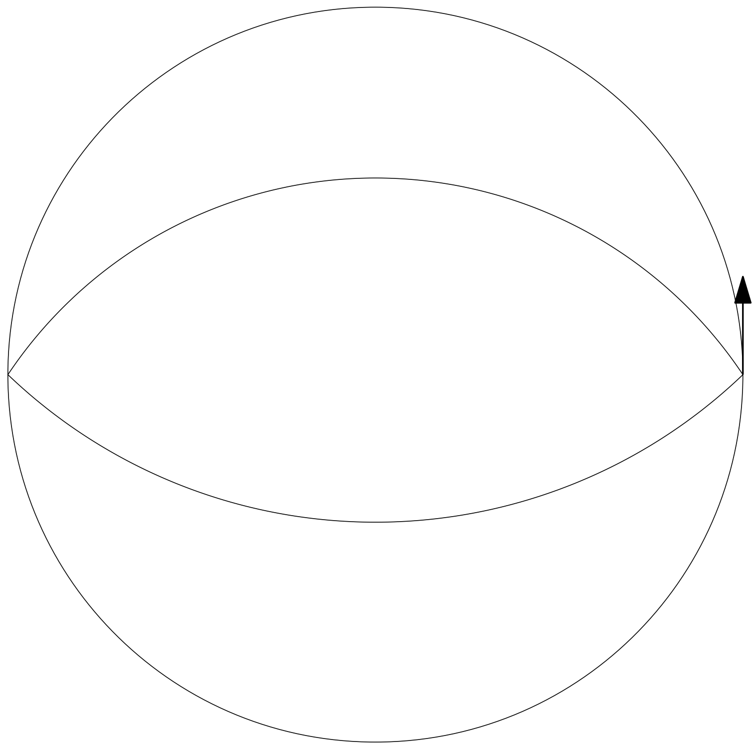
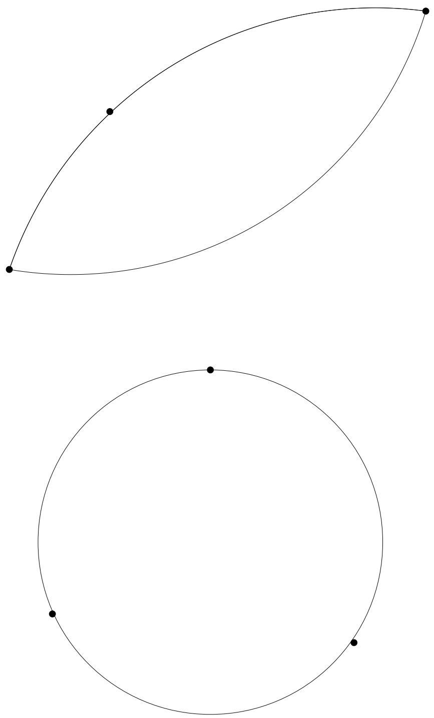
  - If a vertex v_i is such that both of l_{i-1} or l_{i+1} are zero lengths, then we have a pointed-point, and the angles to the three vertices are given arbitrarily so that they sum up to π.

Examples of generalized triangles and angles





The space of generalized triangles and angles

- Let $\hat{G}$ denote the space of generalized triangles with angles assigned.
- Let $\hat{G}$ be given a metric defined by letting $D(L, M)$ to be maximum of
  - the Hausdorff distance between regions L and M of $\mathbf{S}^2$
  - and the Hausdorff distances between corresponding points and segments of L and M
  - and the absolute values of the differences between the corresponding angles respectively.

Proposition (1.1)

$\hat{G}$ is compact under the metric, and the subspace of nondegenerate triangles are dense in $\hat{G}$.

The space of generalized triangles and angles

- The isometry group $\mathrm{SO}(3)$ acts properly on $\hat{G}$.
- The quotient topological space is denoted by $\tilde{G}$. This is a compact metric space with metric induced from $\hat{G}$ by taking the Hausdorff distances between the orbits.
- We will denote by G^o the quotient space of the space of nondegenerate triangles by the $\mathrm{SO}(3)$-action.

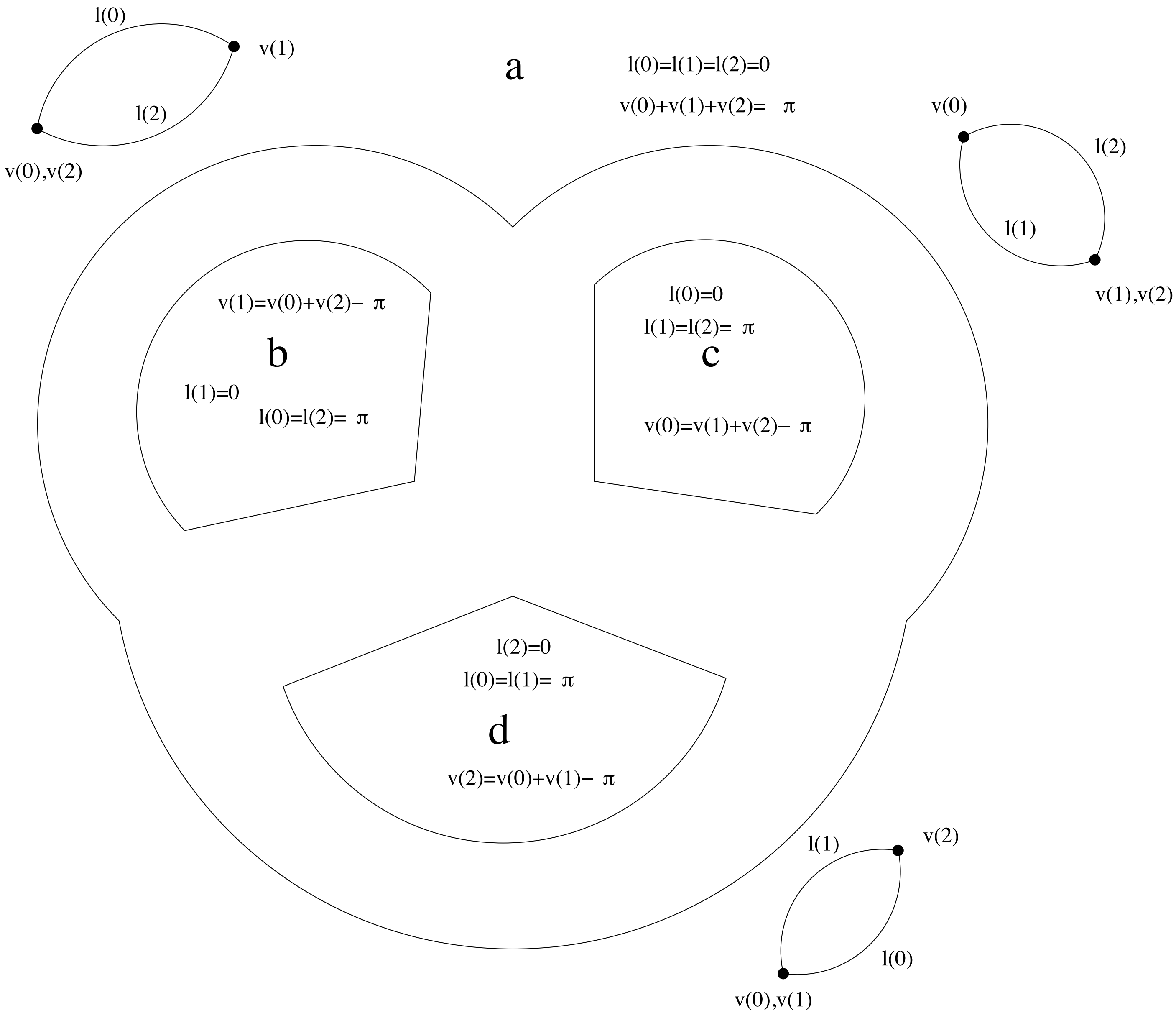
Theorem (1.5)

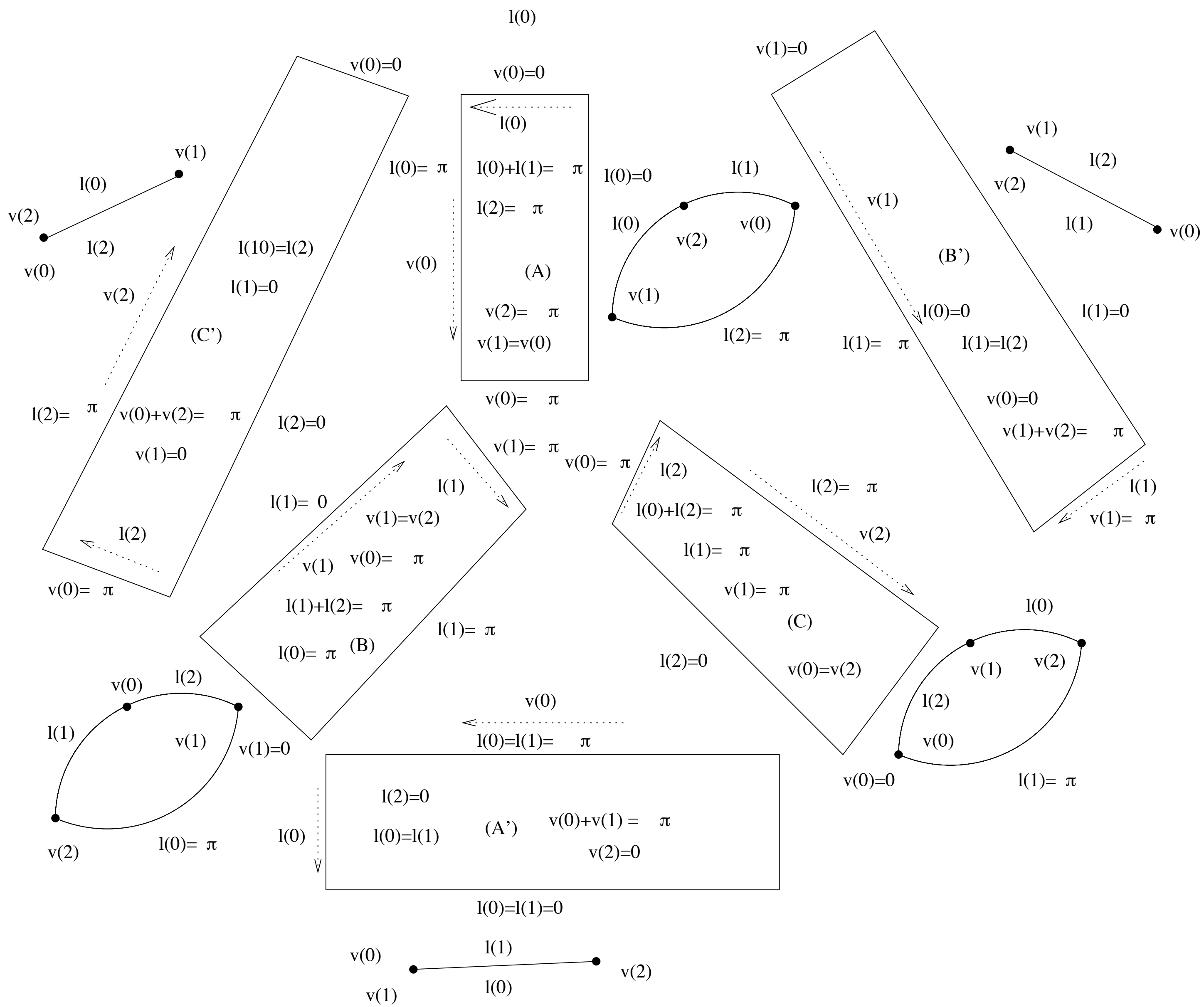
The geometric-limit configuration space $\tilde{G}$ is homeomorphic to a blown-up solid tetrahedron with G^o as the interior.

Proof: embed by $(\theta_1, \theta_2, \theta_3, l_1, l_2, l_3)$.

Parameterizing the degenerate triangles

- We will classify the degenerate triangles according to their types and show that the collection form nice topology of triangles and rectangles, i.e., 2-cells.
- Let us denote by $l(i)$ the coordinate function measuring length of l_i for $i = 0, 1, 2$, and $v(j)$ the coordinate function measuring the angle of v_i for $i = 0, 1, 2$.


l(0)
v(1)
l(2)
v(0),v(2)
a
l(0)=l(1)=l(2)=0
v(0)+v(1)+v(2)= π
v(0)
l(2)
l(1)
v(1),v(2)
v(1)=v(0)+v(2)− π
b
l(1)=0
l(0)=l(2)= π
l(0)=0
l(1)=l(2)= π
c
v(0)=v(1)+v(2)− π
l(2)=0
l(0)=l(1)= π
d
v(2)=v(0)+v(1)− π
l(1)
v(2)
l(0)
v(0),v(1)


l(0)
v(0)=0
v(0)=0
l(0)
l(0)= π
l(0)+l(1)= π
l(2)= π
v(0)
(A)
v(2)= π
v(1)=v(0)
v(0)= π
v(1)= π
v(1)=0
v(1)
l(2)
v(2)
l(1)
v(0)
(B')
l(0)=0
l(1)=0
l(1)= π
l(1)=l(2)
v(0)=0
v(1)+v(2)= π
l(1)
v(1)= π
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l(0)
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v(0)
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l(10)=l(2)
l(1)=0
(C')
l(2)= π
v(0)+v(2)= π
l(2)=0
v(1)=0
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v(0)= π
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l(0)= π
(B)
l(1)= π
v(0)= π
l(2)
l(0)+l(2)= π
l(1)= π
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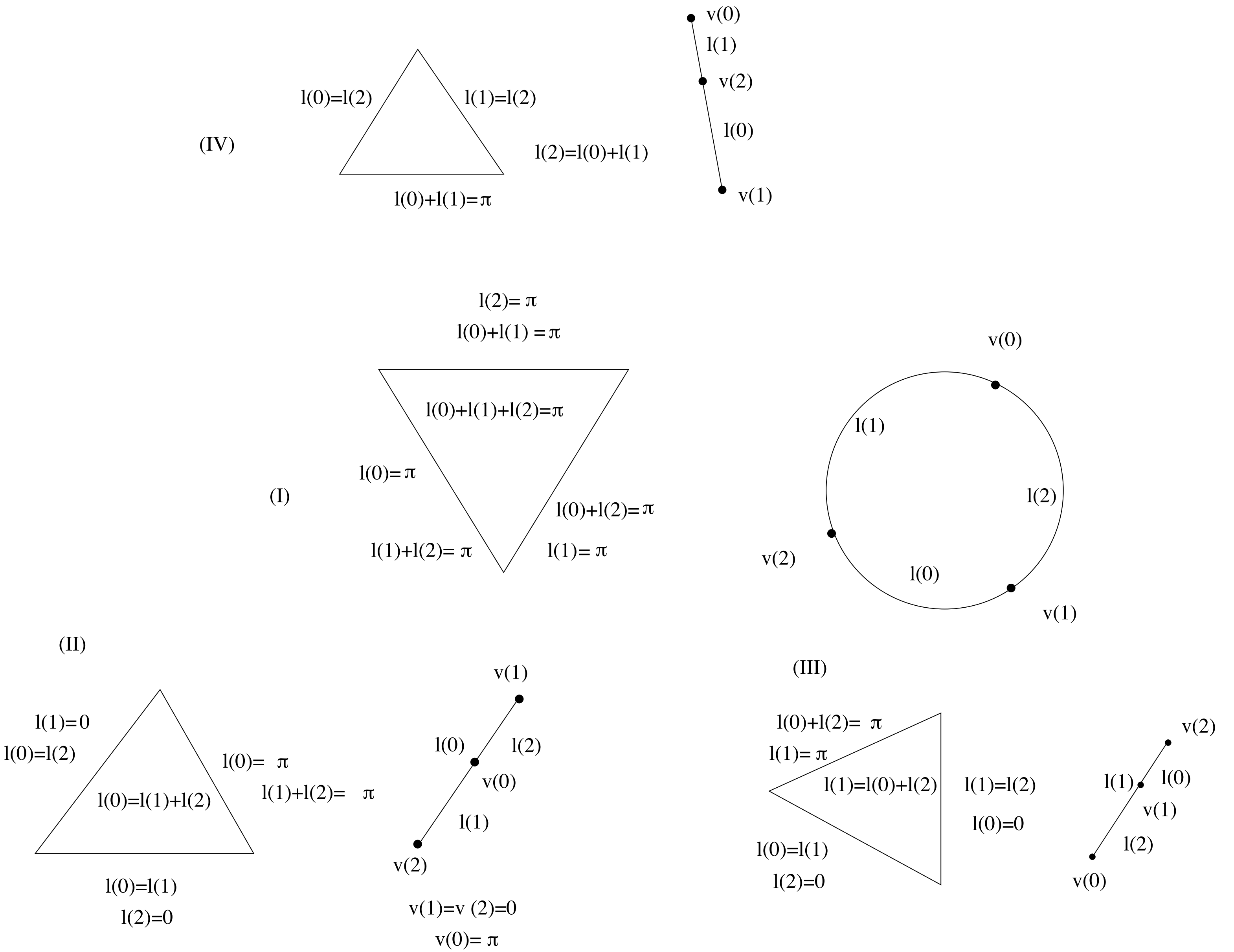
v(1)
v(2)
l(2)
v(0)
v(0)=0
l(1)= π
v(0)
l(2)
l(1)
v(1)
v(1)=0
v(2)
l(0)= π
v(0)
l(0)=l(1)= π
l(2)=0
l(0)
l(0)=l(1)
(A')
v(0)+v(1) = π
v(2)=0
l(0)=l(1)=0
v(0)
l(1)
v(2)
v(1)
l(0)


(IV)
l(0)=l(2)
l(1)=l(2)
l(2)=l(0)+l(1)
l(0)+l(1)=π
v(0)
l(1)
v(2)
l(0)
v(1)
l(2)= π
l(0)+l(1) =π
(I)
l(0)+l(1)+l(2)=π
l(0)=π
l(0)+l(2)=π
l(1)+l(2)= π
l(1)= π
v(0)
l(1)
l(2)
v(2)
l(0)
v(1)
(II)
l(1)=0
l(0)=l(2)
l(0)= π
l(1)+l(2)= π
l(0)=l(1)+l(2)
l(0)=l(1)
l(2)=0
v(1)
l(0)
l(2)
v(0)
l(1)
v(2)
v(1)=v (2)=0
v(0)= π
(III)
l(0)+l(2)= π
l(1)= π
l(1)=l(0)+l(2)
l(1)=l(2)
l(0)=0
l(0)=l(1)
l(2)=0
v(2)
l(1)
l(0)
v(1)
l(2)
v(0)

The Klein four-group action

- The map I_A in $\tilde{G}^o$ can be described as first find an element μ in $\tilde{G}^o$ and representing it as a triangle with vertices v_0, v_1, v_2 and taking a triangle with vertices $v_0' = -v_0$ and $v_1' = -v_1$ and $v_2' = v_2$.

The Klein four-group action

- $$I_A : (v(0), v(1), v(2), l(0), l(1), l(2)) \mapsto (\pi - v(0), \pi - v(1), v(2), \pi - l(0), \pi - l(1), l(2)). \quad (1)$$
- Similarly, the map I_B changes the triangle with vertices v_0, v_1, and v_2 to one with $v_0, -v_1$, and $-v_2$:
  $$(v(0), v(1), v(2), l(0), l(1), l(2)) \mapsto (v(0), \pi - v(1), \pi - v(2), l(0), \pi - l(1), \pi - l(2)). \quad (2)$$
- Similarly, the map I_C changes the triangle with vertices v_0, v_1, and v_2 to one with $-v_0, v_1$, and $-v_2$:
  $$(v(0), v(1), v(2), l(0), l(1), l(2)) \mapsto (\pi - v(0), v(1), \pi - v(2), \pi - l(0), l(1), \pi - l(2)). \quad (3)$$
- For our geometric degenerate triangles, we do the same. For regions a, b, c, and d, the transformations are merely the linear extensions or equivalently extensions with respect to the metrics.

Matrix-multiplication by geometry

- An element of $\mathrm{SO}(3)$ can be written as $R_{x,\theta}$ where x is a fixed point and an angle θ, $0 \leq \theta \leq 2\pi \mod 2\pi$.
- For the identity element, x is not determined but $\theta = 0$.
- For any nonidentity element, x is determined up to antipodes: $R_{x,\theta} = R_{-x,2\pi-\theta}$.

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- Let w_0, w_1, and w_2 be vertices of a triangle oriented in the clockwise direction. Then

$$R_{w_2,2\theta_2} \circ R_{w_1,2\theta_1} \circ R_{w_0,2\theta_0} = \mathrm{I}:$$

- Denoting the rotation at w_0, w_1, w_2 by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively, we obtain

$$\mathcal{C}\mathcal{B}\mathcal{A} = \mathrm{I}, \mathcal{C}^{-1} = \mathcal{B}\mathcal{A}, \mathcal{C} = \mathcal{A}^{-1}\mathcal{B}^{-1}. \qquad (4)$$

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- These work even for the degenerate triangles.

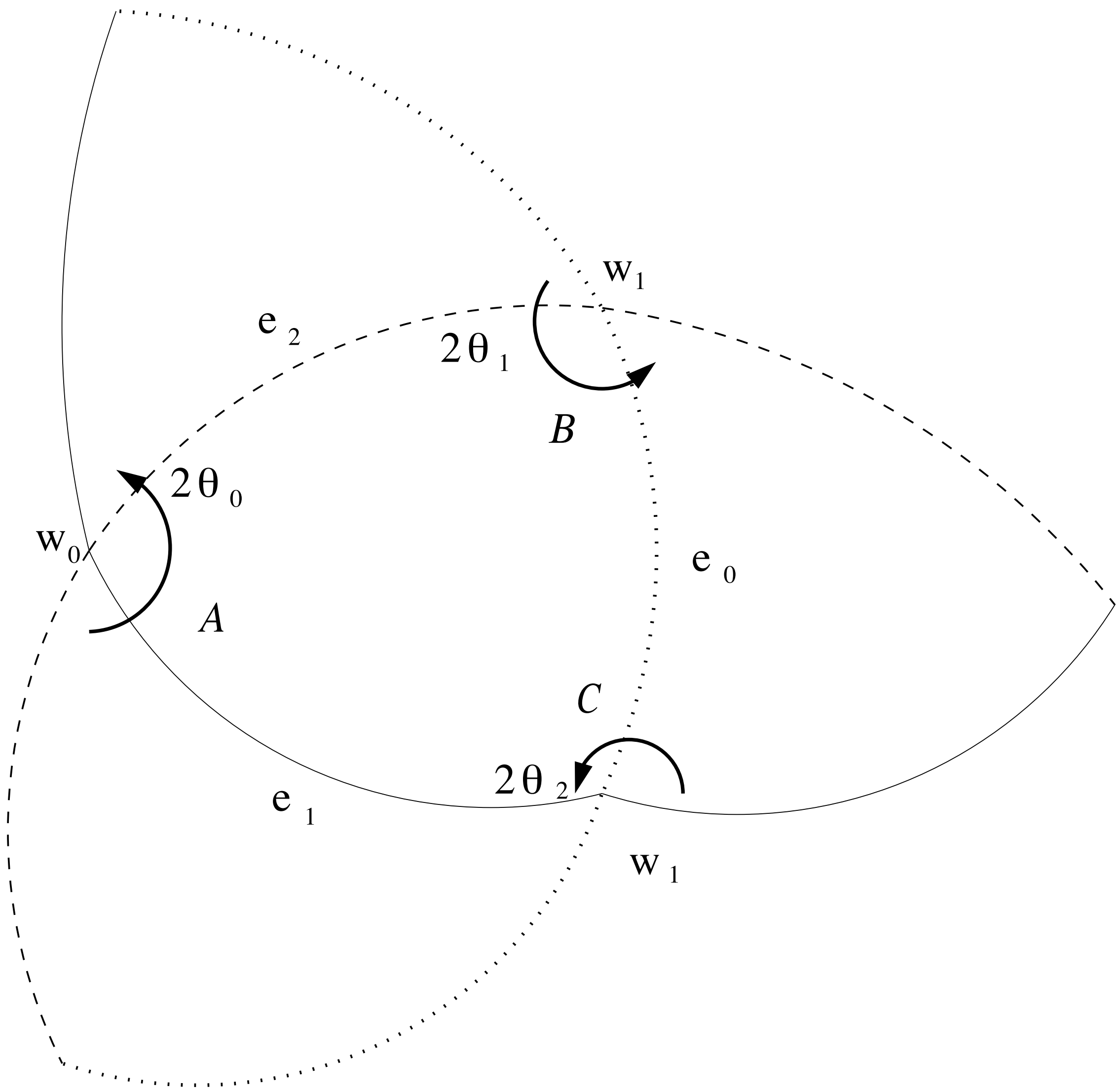

Figure: Multiplication by geometry. Triangular representations

The SO(3)-character space of the fundamental group of a pair of pants

- Let P be a pair of pants and let $\tilde{P}$ be the universal cover.
- Let c_0, c_1, and c_2 denote three boundary components of P oriented using the boundary orientation.
- Let $\pi_1(P)$ denote the fundamental group of P seen as a group of deck transformations generated by three elements $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ parallel to the boundary components of P satisfying $\mathcal{C}\mathcal{B}\mathcal{A} = I$.

- Take a triangle on the sphere $\mathbf{S}^2$ with geodesic edges so that each edge has length $< \pi$ so that the vertices are ordered in a clockwise manner in the boundary of the triangle.

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- Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \tag{5}$$

$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \tag{6}$$

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$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \quad (6)$$

- The region gives us an open tetrahedron in the positive octant of $\mathbb{R}^3$ with vertices

$$(\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, \pi, \pi)$$

and thus we have $0 < \theta_i < \pi$. This is a regular tetrahedron with edge lengths all equal to $\sqrt{2}\pi$.

Lemma (2.2)

$\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *contains a dense open set where each character is a triangular.*

Proposition (2.3)

$\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *is homeomorphic to the quotient of the tetrahedron G by a* $\{I, I_A, I_B, I_C\}$*-action.* (*See Figure 1.*)

$\mathrm{SO}(3)$ and $\mathrm{SU}(2)$: geometric relationships

- $\mathrm{SO}(3)$ can be identified with $\mathbb{R}P^3$ in the following way: Take B^3 of radius π in $\mathbb{R}^3$. Then for each $g \in \mathrm{SO}(3)$ we choose the fixed point with angle $\theta < \pi$ and take the point in the ray to the point in B^3 of distance θ from the origin. If $\theta = \pi$, then we take both points in the boundary $\mathbf{S}^1$ of B^3 in the direction and identify them.

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- Since $\mathrm{SU}(2)$ double-covers $\mathrm{SO}(3)$, the Lie group $\mathrm{SU}(2)$ is diffeomorphic to $\mathbf{S}^3$. Take the ball B^3_2 of radius 2π so that the boundary is identified with a point. Hence, we obtain $\mathbf{S}^3$. Let $||v||$ denote the norm of a vector v in B^3_2. Take the map from $B^3_2 \to B^3$ given by sending a vector v to v if $||v|| \leq \pi$ or to $(\pi - ||v||)v$ if $||v|| > \pi$. This is a double-covering map clearly.

SO(3) and SU(2): geometric relationships

- Since we have $R_{x,\theta} = R_{-x,4\pi-\theta}$, an element of SU(2) can be considered as a fixed point of $\mathbf{S}^2$ with angles in $[-2\pi, 2\pi]$ where -2π and 2π are identified or with angles in $[0, 4\pi]$ where 0 and 4π are identified.
- $$\begin{aligned} -\mathrm{I}R_{w,\theta} &= R_{w,2\pi} \circ R_{w,\theta} = R_{w,2\pi+\theta} \\ &= R_{-w,4\pi-2\pi-\theta} = R_{-w,2\pi-\theta}. \end{aligned} \tag{7}$$

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- ## Definition (3.3)

By choosing θ to be in $(0, 2\pi)$, $R_{x,\theta}$ is now a point in $B_2^{3,o} - \{O\}$. Thus, each point of $\mathbf{S}^3 - \{I, -I\}$, we obtain a unique rotation $R_{x,\theta}$ for $\theta \in (0, 2\pi)$, $x \in \mathbf{S}^2$ and conversely. (

normal representation)

- The "multiplication by geometry" also works in $\mathrm{SU}(2)$: Let w_0, w_1, and w_2 be vertices of a triangle, possibly degenerate, oriented in the clockwise direction.
- Let e_0, e_1, and e_2 denote the opposite edges. Let θ_0, θ_1, and θ_2 be the respective angles for $0 \leq \theta \leq \pi$. Then

$$R_{w_2,2\theta_2} \circ R_{w_1,2\theta_1} \circ R_{w_0,2\theta_0} = -\mathrm{I}.$$

  Here the minus sign is needed.
- We can even do this for immersed triangles with angles $> \pi$.

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  Here the minus sign is needed.
- We can even do this for immersed triangles with angles $> \pi$.
- By lifting the representations, we obtain (See Proposition 3.2)

Proposition

$\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ *is homeomorphic to the tetrahedron, and map to* $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *as a* 4 *to* 1 *branched covering map induced by the Klein four-group* V*-action.* □

The character space of a closed surface of genus 2

- First, we discuss the two-components of the character space.
- Next, we discuss how to view a representation as two related representations of the fundamental groups of two pairs of pants glued by three pasting maps.

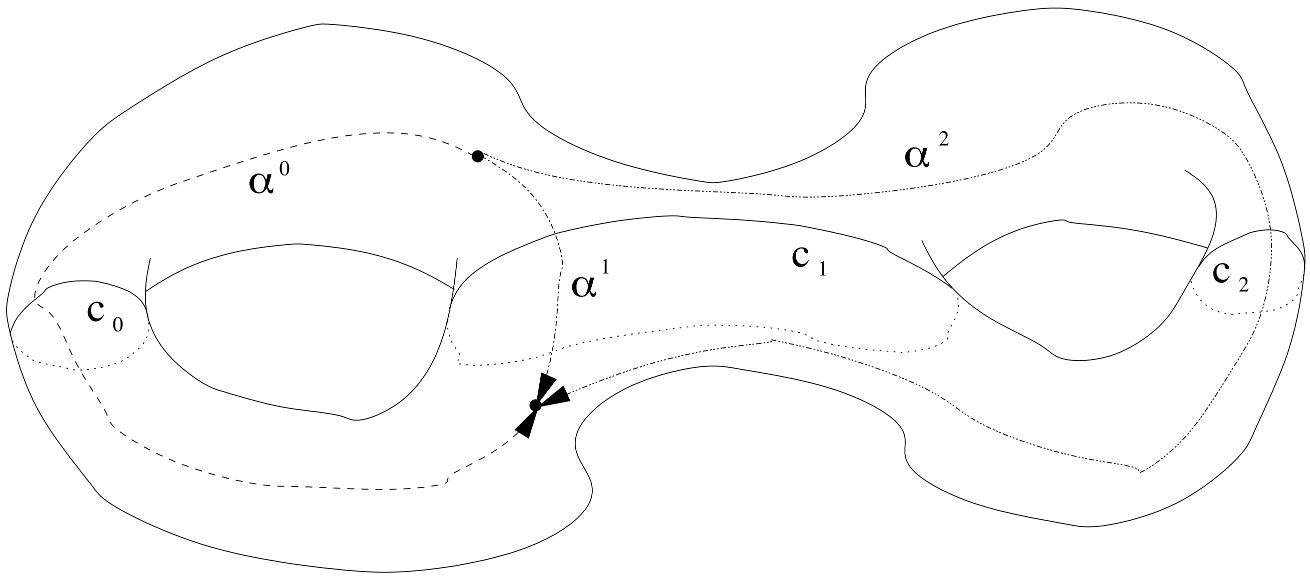

Figure: Σ and closed curves.

Two components

- Three sccs c_0, c_1, and c_2 on Σ so that we have two pairs of pants S_0 and S_1 so that $S_0 \cap S_1 = c_0 \cup c_1 \cup c_2$.
- Let scc d_1 and d_2 dual to c_1 and c_2 respectively.
- There are two components of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ as shown by Goldman [G, 1988]. The Stiefel-Whitney class in $H^2(\Sigma, \pi_1(\mathrm{SO}(3))) = \mathbb{Z}_2$ of the flat bundle classifies the component.

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- $\mathcal{C}_0$ the identity component.
- $\mathcal{C}_1$ the other component. This contains a representation sending c_1 and d_1 to

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } B_1 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

and c_2 and d_2 to the identity matrix.

Two components

- The base point x^* of Σ in the interior of S_0. Given a representation $h : \pi_1(\Sigma) \to \mathrm{SO}(3)$, we obtain a representations $h_0 : \pi_1(S_0) \to \mathrm{SO}(3)$ and $h_1 : \pi_1(S_1) \to \mathrm{SO}(3)$.
- Let c_0^0, c_1^0, and c_2^0 denote the sccs on S_0 with base point x_0^* that are freely homotopic to c_0, c_1, and c_2 respectively, Let us choose a base point x_1^* in S_1 and oriented sccs c_0^1, c_1^1, and c_2^1 homotopic to c_0, c_1, and c_2.

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- Relation
$$[c_1^0, d_1][c_2^0, d_2] = 1.$$
- $h_0(c_i^0)$ are conjugate to $h_1(c_i^1)$ by $P_i \in \mathrm{SO}(3)$, i.e.,
$$P_i h_0(c_i^0) P_i^{-1} = h_1(c_i^1) \text{ for } i = 0, 1, 2.$$
We call P_i the pasting map for c_i for $i = 0, 1, 2$.

Proposition (4.2)

We have

$$h(d_1) = P_0^{-1} \circ P_1, h(d_2) = P_0^{-1} \circ P_2.$$

Proposition (4.3)

Let h_0 and h_1 are representations of the fundamental groups of pairs of pants S_0 and S_1 from a $\mathrm{SO}(3)$-representation h. The angles $(\theta_0, \theta_1, \theta_2)$ of h_0 and $(\theta'_0, \theta'_1, \theta'_2)$ of h_1 satisfy the equation

$$\theta'_i = \theta_i \text{ or} \quad (9)$$
$$\theta'_i = \pi - \theta_i \text{ for } i = 0, 1, 2. \quad (10)$$

- We consider the identity component $\mathcal{C}_0$.

Proposition (5.3)

If h is in the identity component $\mathcal{C}_0$ of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$, and h_0 and h_1 be obtained as above by restrictions to S_0 and S_1. then

(a) *We can conjugate h_1 so that $h_0 = h_1$ and corresponding angles are equal.*

(b) *For each representation h in $\mathcal{C}_0$, we can associate a pair of identical degenerate or nondegenerate triangles and an element of $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, i.e., the parameter space of the pasting angles. (not normally a unique association for the degenerate triangle cases.)*

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- This gives a surjective map

$$\mathcal{T} : \tilde{G} \times T^3 \to \mathcal{C}_0 \subset \mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3)).$$

We need to find the equivalence relation $\sim$ on $\tilde{G} \times T^3$ to make the above map induce a homeomorphism.

- We describe the action below where I_A, I_B, I_C inside are the transformations on $\tilde{G}$ described above:

$$\begin{aligned} I_A &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_A(x), \phi_0, 2\pi - \phi_1, 2\pi - \phi_2) \\ I_B &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_B(x), 2\pi - \theta_0, \phi_1, 2\pi - \phi_2) \\ I_C &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_C(x), 2\pi - \phi_0, 2\pi - \phi_1, \phi_2). \end{aligned} \tag{11}$$

  Since the action correspond to changing the fixed points of c_i and hence does not change the associated representations, we have $\mathcal{T} \circ I_A = \mathcal{T} \circ I_B = \mathcal{T} \circ I_C = \mathcal{T}$.
- By above, the set of triangular characters and $\tilde{G}^o \times T^3/\{I, I_A, I_B, I_C\}$ are in one-to-one correspondence.

The equivalence relation

- The equivalence relation $\sim$ on the union of these are very complicated and we obmit these.
- The following is the main result:

Theorem (5.27)

The identity component $\mathcal{C}_0$ of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ is homeomorphic to $\tilde{G} \times T^3/\sim$. Thus $\mathcal{C}_0$ is a topological complex consisting of a 3-dimensional copy of $H(F_2)$, and 4-dimensional C^A, C^B, and C^C, and the space of abelian representations, coming from the boundary of the 3-ball $\tilde{G}$ and the 6-dimensional complex from the interior of $\mathcal{C}_0$.

The $\mathrm{SU}(2)$-character space of $\pi(\Sigma)$.

- We will first find the topological type of the $\mathrm{SU}(2)$-character space of π and then in the next section do the same for the $\mathrm{SO}(3)$-character space.

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- We first study $\mathbb{CP}^3$ as a quotient space of a tetrahedron times a 3-torus. Then we represent each $\mathrm{SU}(2)$-character by a generalized triangle and pasting angles as in the $\mathrm{SO}(3)$-case.
- An $\mathrm{SU}(2)$-character of a pair of pants corresponds to a generalized triangle in a one-to-one manner except for the degenerate ones. The space of pasting maps in $\mathrm{SU}(2)$ is now $\mathbf{S}^1_2$.

The SU(2)-character space of $\pi(\Sigma)$

- Let $\mathbb{CP}^3$ denote the complex projective space. According to the toric manifold theory, $\mathbb{CP}^3$ admits a T^3-action given by

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [e^{i\theta_1} z_0, e^{i\theta_2} z_1, e^{i\theta_3} z_2, z_3] \tag{12}$$

and the quotient map is given by

$$[z_0, z_1, z_2, z_3] \mapsto \pi(|z_0|^2, |z_1|^2, |z_2|^2) / \sum_{i=0}^{3} |z_i|^2, z_i \in \mathbb{C} \tag{13}$$

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- The image is a standard 3-simplex $\triangle^*$ in the positive quadrant of $\mathbb{R}^3$ given by the plane given by $x_0 + x_1 + x_2 \leq \pi$ and the fibers are the orbits of T^3-action. The fibers are given by $\mathbb{R}^3$ quotient out by the standard lattice L^* with generators $(2\pi, 0, 0), (0, 2\pi, 0), (0, 0, 2\pi)$.

On T_2^3, an equivalence relation is given by the $\mathbb{Z}_2$-action sending (ϕ_0, ϕ_1, ϕ_2) to $(\phi_0 + 2\pi, \phi_1 + 2\pi, \phi_2 + 2\pi)$: We obtain $T^3/\mathbb{Z}_2$.

Proposition (6.1)

By considering fibers of faces of G, we can realize $\mathbb{CP}^3$ as the quotient space $G \times T_2^3/\mathbb{Z}_2$ of under an equivalence relation given as follows:

- *In the interior, the equivalence is trivially given.*
- *For the face a, the equivalence relation on $a \times T_2^3/\mathbb{Z}_2$ is given by*

$$(v, \phi_0, \phi_1, \phi_2) \sim (v', \phi'_0, \phi'_1, \phi'_2)$$

if and only if $v = v'$ and two vectors (ϕ_0, ϕ_1, ϕ_2) and $(\phi'_0, \phi'_1, \phi'_2)$ are the same up to the $\mathbf{S}^1$-action generated by vectors parallel to $(2\pi, 2\pi, 2\pi)$ normal to a.

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- *For faces b, c, and d, the equivalence relation is similarly defined.*
- *In the edges and the vertices, the equivalence relation is induced from the facial ones.*

Theorem (6.2)

$\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ is diffeomorphic to $\mathbb{CP}^3$ considered as a $T^3/\mathbb{Z}_2$-fibration over G with the following properties:

- *Each edge of G corresponding to the region A, A', B, B', C, C' of $\tilde{G}$ correspond a solid torus fibration over the interior of edges of $\tilde{G}$. Here, the solid torus end is identified to a 3-ball.*

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- *Three of them meet in a 3-ball over each vertex of $\tilde{G}$ according to the pattern of the edges of $\tilde{G}$.*
- *The set of abelian characters $\chi_2(\Sigma)$ forms a subspace with an orbifold structure with 16 singularities. It consists of the two-torus fibrations over faces of G which meet at the boundary components of the above solid torus fibration.*

Triangular characters

- We find a description of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ as a quotient space of $\tilde{G} \times T_2^3/\mathbb{Z}_2$: For the open domain of triangular characters, a representation of $\pi_1(\Sigma)$ gives us a unique triangle on $\mathbf{S}^2$ by Proposition 4.2 and hence unique pasting map. Thus, the space of triangular characters is homeomorphic to $\tilde{G}^o \times T_2^3/\mathbb{Z}_2$.
- By density, the map
$$\tilde{G} \times T_2^3/\mathbb{Z}_2 \to \mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$$
is onto.

- For the face a, the equivalence relation on $a \times T_2^3/\mathbb{Z}_2$ is given by $(v, \phi_0, \phi_1, \phi_2) \sim (v', \phi_0', \phi_1', \phi_2')$ if and only if $v = v'$ and two vectors (ϕ_0, ϕ_1, ϕ_2) and $(\phi_0', \phi_1', \phi_2')$ are the same up to the $\mathbf{S}^1$-action generated by vectors parallel to $(2\pi, 2\pi, 2\pi)$ normal to a.
- For faces b, c, and d, the equivalence relation is defined again using the respective $\mathbf{S}^1$-action generated by vectors parallel to $(-2\pi, 2\pi, 2\pi), (2\pi, -2\pi, -2\pi), (2\pi, 2\pi, -2\pi)$ perpendicular to b, c, d respectively.
- The quotient space $T_{2,a}^2$ is homeomorphic to a 2-torus. Thus, the character space here is in one-to-one correspondence with $a \times T_{2,a}^2$. Similarly, we obtain $T_{2,b}^2$, $T_{2,c}^2$, and $T_{2,d}^2$ for respective faces b, c, and d.

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- We take a union of $a \times T_{2,a}^2$, $b \times T_{2,b}^2$, $c \times T_{2,c}^2$, and $d \times T_{2,d}^2$. Note that as we cross an edge through a tie from a face to another face, we change one of the vertex of a lune triangle to its antipode.
- Hence, we can consider as a fibration over $\partial \tilde{G}$ with fibers homeomorphic to T^2 except at vertices where the fibers are homeomorphic to a 2-sphere.

Lemma (6.3)

- *The subspace over a tie in one of the regions A, B, C, A', B', and C' but not in U is homeomorphic to $\mathbf{S}^1 \times B^2$. Thus, over the interior of each of A, A', B, B', C, C', there is a bundle over an open interval with fibers homeomorphic to the solid tori.*

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- *Hence, the region above each of* A, A', B, B', C, C' *is homeomorphic to the quotient space of a solid torus times an interval with the solid torus over each end identified with a* 3*-ball.*

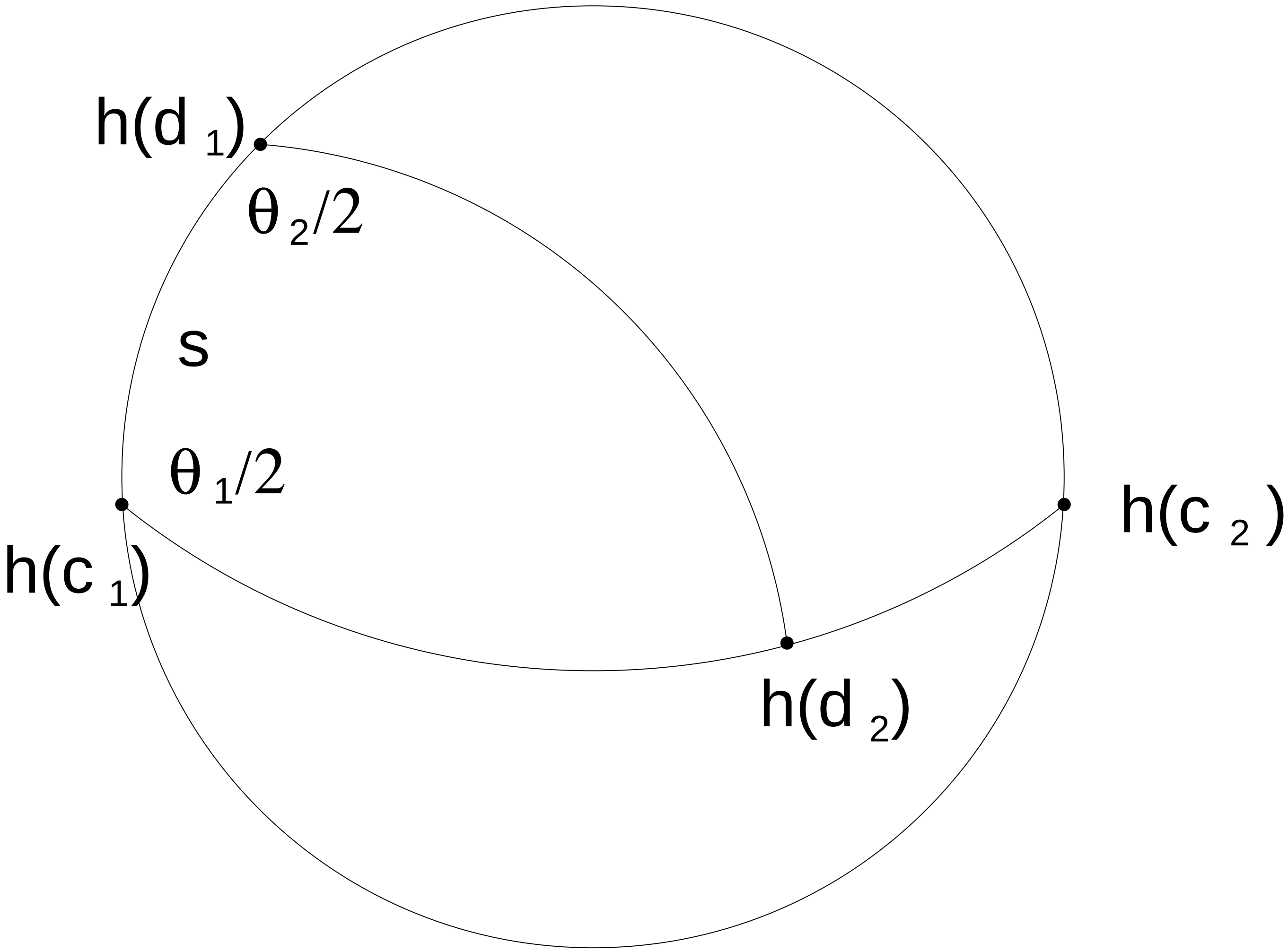

Figure: Finding topology of space over regions A, A', B, B', C, and C'

The topology of the quotient space $\tilde{G} \times T^3/\sim$ or $\mathcal{C}_0$

- Clearly, there is a group V' of order 16 action on $\tilde{G} \times T_2^3/\mathbb{Z}_2/\sim$ generated by the $\{\mathrm{I}, I_A, I_B, I_C\}$-action similar to equations 11
- and the Klein four-group acting on each of the fibers $\mathbf{S}_2^1 \times \mathbf{S}_2^1 \times \mathbf{S}_2^1/\mathbb{Z}_2$:
  by i_a sending $(\theta_0, \theta_1, \theta_2) \to (\theta_0 + 2\pi, \theta_1 + 2\pi, \theta_2)$ and i_b sending $(\theta_0, \theta_1, \theta_2) \to (\theta_0, \theta_1 + 2\pi, \theta_2 + 2\pi)$ and i_c sending $(\theta_0, \theta_1, \theta_2) \to (\theta_0 + 2\pi, \theta_1, \theta_2 + 2\pi)$.

- **Theorem (7.5)**

  $\tilde{G} \times T^3/\sim$ is homeomorphic to a quotient of $\mathbb{CP}^3$ under the product of the two Klein four-group actions generated by fiberwise and axial action:

  - *The branch loci of I_A, I_B, I_C are given as follows: six 2-spheres corresponding to the axes of I_A, I_B, and I_C. There are two 2-spheres over each axis, and over each axis, the two 2-spheres are disjoint. All three 2-spheres over different axis meet at the same point as above.*
  - *The branched loci of i_a, i_b, i_c are 2-spheres also over A, A', B, B', C, C'.*

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- Then we introduce equivalence relation so that $\tilde{O} \times T^3/\sim$ becomes homeomorphic to $\mathcal{C}_1$. This will be done by considering the interior and each of the boundary regions as in the previous sections.
- Finally, we will show that the quotient space is homeomorphic to an octahedral manifold.

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- *Every character in the component* $\mathcal{C}_1$ *is associated with generalized triangles* $(\triangle_0, \triangle_1)$ *whose associated angles are* $(\theta_0, \theta_1, \theta_2)$ *and* $(\theta_0, \pi - \theta_1, \theta_2)$*.*

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For our convention, the pasting map P_0 sends v_0 to v_0', P_1 sends v_1 to $-v_1'$ and P_2 sends v_2 to v_2'. Note we do not have a canonical choices for P_i which we need to get a coordinate system as of yet.

- The set of possible nondegenerate triangles for $\triangle_0$ and $\triangle_1$ is then described as the intersection of $\tilde{G}^o \cap \kappa(\tilde{G}^o)$ where κ is the map sending $(\theta_0, \theta_1, \theta_2)$ to $(\theta_0, \pi - \theta_1, \theta_2)$.

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- Since $\tilde{G}^o$ is given by

$$\begin{aligned} \theta_0 + \theta_1 + \theta_2 &> \pi \\ \theta_0 &< \theta_1 + \theta_2 - \pi, \\ \theta_1 &< \theta_2 + \theta_0 - \pi \\ \theta_2 &< \theta_0 + \theta_1 - \pi \end{aligned} \tag{14}$$

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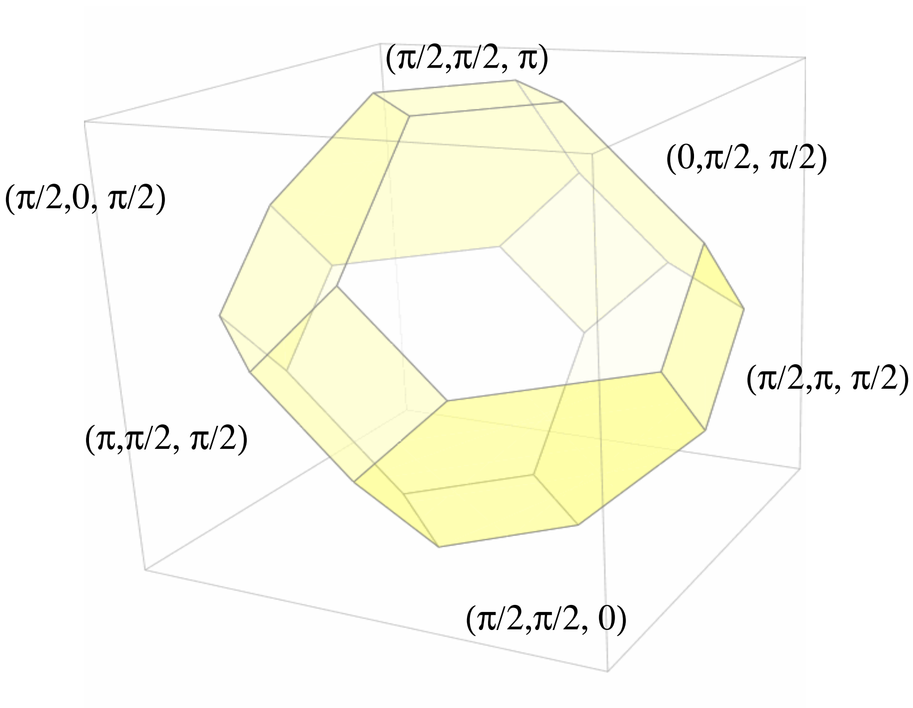
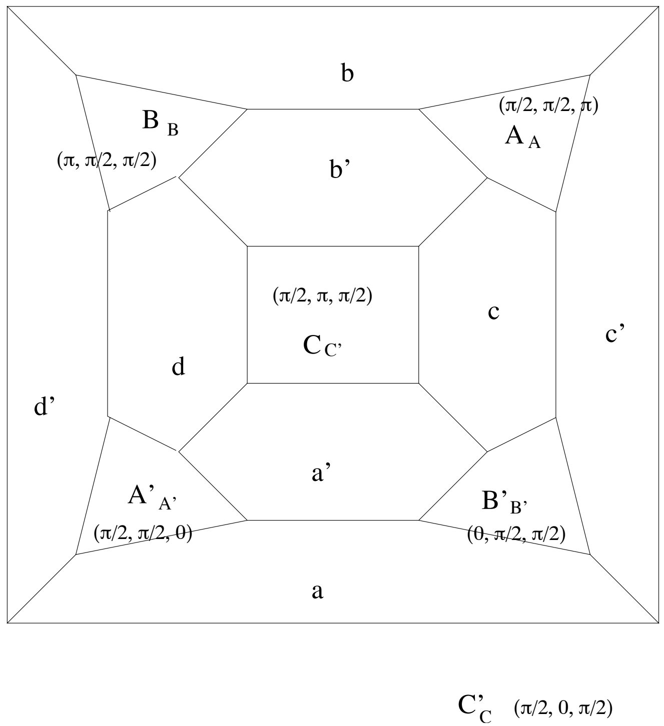
- it follows that our domain is an octahedron O given by eight equations

$$\begin{aligned} \theta_0+\theta_1+\theta_2 &> \pi : (a) \\ \theta_0+\theta_2 &> \theta_1 : (a') \\ \theta_0 &< \theta_1+\theta_2-\pi, : (c) \\ \theta_0+\theta_1 &< \theta_2 : (c') \\ \theta_1 &< \theta_2+\theta_0-\pi : (b) \\ 2\pi &< \theta_0+\theta_1+\theta_2 : (b') \\ \theta_2 &< \theta_0+\theta_1-\pi : (d) \\ \theta_1+\theta_2 &< \theta_0 : (d') \end{aligned} \tag{15}$$


(π/2,π/2, π)
(0,π/2, π/2)
(π/2,0, π/2)
(π/2,π, π/2)
(π,π/2, π/2)
(π/2,π/2, 0)


b
B B
(π, π/2, π/2)
b'
(π/2, π/2, π)
A A
(π/2, π, π/2)
C C'
c
c'
d
d'
a'
A' A'
(π/2, π/2, 0)
B' B'
(0, π/2, π/2)
a
C' C (π/2, 0, π/2)

- The Klein four-group $\{I, I_A, I_B, I_C\}$ acts on the resulting polyhedron $\tilde{O}$ as isometric actions. They are obtained by replacing v_i to $-v_i$ for some $i = 0, 1, 2$, and they have the same formula as in the $\mathcal{C}_0$ case.

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- They are as follows in terms of coordinates

$$I_A : (v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l'(0), l'(1), l'(2)) \mapsto$$
$$(\pi - v(0), \pi - v(1), v(2), \pi - l(0), \pi - l(1), l(2), \pi - v(0)', \pi - v(1)', v(2)', \pi - l(0)', \pi - l(1)', l(2)') \quad (16)$$

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- The map I_B changes the triangle with vertices v_0, v_1, and v_2 to one with $v_0, -v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l(0)', l(1)', l(2)') \mapsto$$
$$(v(0), \pi - v(1), \pi - v(2), l(0), \pi - l(1), \pi - l(2), v(0)', \pi - v(1)', \pi - v(2)', l(0)', \pi - l(1)', \pi - l(2)'). \quad (17)$$

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- They are as follows in terms of coordinates

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- The map I_B changes the triangle with vertices v_0, v_1, and v_2 to one with $v_0, -v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l(0)', l(1)', l(2)') \mapsto$$
$$(v(0), \pi - v(1), \pi - v(2), l(0), \pi - l(1), \pi - l(2), v(0)', \pi - v(1)', \pi - v(2)', l(0)', \pi - l(1)', \pi - l(2)'). \quad (17)$$

- The map I_C changes the triangle with vertices v_0, v_1, and v_2 to one with $-v_0, v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l(0)', l(1)', l(2)') \mapsto$$
$$(\pi - v(0), v(1), \pi - v(2), \pi - l(0), l(1), \pi - l(2), \pi - v(0)', v(1)', \pi - v(2)', \pi - l(0)', l(1)', \pi - l(2)'). \quad (18)$$

Proposition (8.2)

Using pasting map construction as above, we have a continuous onto map

$$\mathcal{T} : \tilde{O} \times T^3 \to \mathcal{C}_1.$$

Furthermore, we have

$$\mathcal{T} \circ I_A = \mathcal{T} \circ I_B = \mathcal{T} \circ I_C = \mathcal{T}.$$

The upper and lower triangles are related by the relation in O:

$$(\theta_0, \theta_1, \theta_2) \leftrightarrow (\theta_0, \pi - \theta_1, \theta_2).$$

Theorem

The map $\mathcal{T} : \tilde{O} \times T^3 \to \mathcal{C}_1$ induces a homeomorphism $\tilde{O} \times T^3/\sim \to \mathcal{C}_1$ where $\sim$ is an appropriate equivalence relation.

The SU(2)-pseudo-characters for the above component

- We define the SU(2)-character space $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ of a punctured genus 2 surface Σ_1 with the puncture holonomy $-I$ as the quotient space of the subspace of $\mathrm{Hom}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ where $h(c) = -I$ by conjugations where c is a simple closed curve around the puncture.

Theorem (9.1)

$\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ *is homeomorphic to the filled octahedral manifold.*

- We have a surjective map from $\tilde{O} \times T^3_2/\mathbb{Z}_2$ to $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$.

- We will use the same equivalence relation on regions above a, b, c, d, a', b', c', and d' as in the $\mathrm{SO}(3)$-case except that now the fibers are $T^3_2/\mathbb{Z}_2$.
- The equivalence relation is given by identifying all elements proportional to $(2\pi, 2\pi, 2\pi)$-vector in $T^3_2/\mathbb{Z}_2$ to O for face $a \times T^3_2/\mathbb{Z}_2$, by identifying all elements proportional to $(2\pi, -2\pi, 2\pi)$-vector in T^3_2 to O for face $a' \times T^3_2/\mathbb{Z}_2$, and so on.
- Thus, the character space here is in one-to-one correspondence with $a^o \times T^2$.
- Similar statements are true for the other regions.

- For regions, A_A,B_B, $C_{C'}$,$A'_{A'}$, $B'_{B'}$, and C'_C, we will use a different but similar identification to the $\mathrm{SO}(3)$-cases.
- Let us take the case A_A first. Here the pasting angles are in $\mathbf{S}^1_2$. By an extended multiplication by geometry for $\mathrm{SU}(2)$, we obtain the fixed point u_1 of $h(d_1)$ and $h(d_2)$ uniquely determined in this case.
- Therefore, by reading the coordinates of $h(d_2)$ in terms of u_1, we obtain a map $A_A \times T^3_2/\mathbb{Z}_2 \to \mathcal{C}_1$ whose image is an imbedded 3-sphere since given all points of $\mathbf{S}^2$ arise as a point and all rotation angles occur by our constructions, where again we used our control of P_2 with pasting angles fixed at v_0 and v_2.
- Moreover, $A_A \times T^3_2/\mathbb{Z}_2/\sim \to \mathcal{C}_1$ is an embedding since $\sim$ is chosen for this to hold.

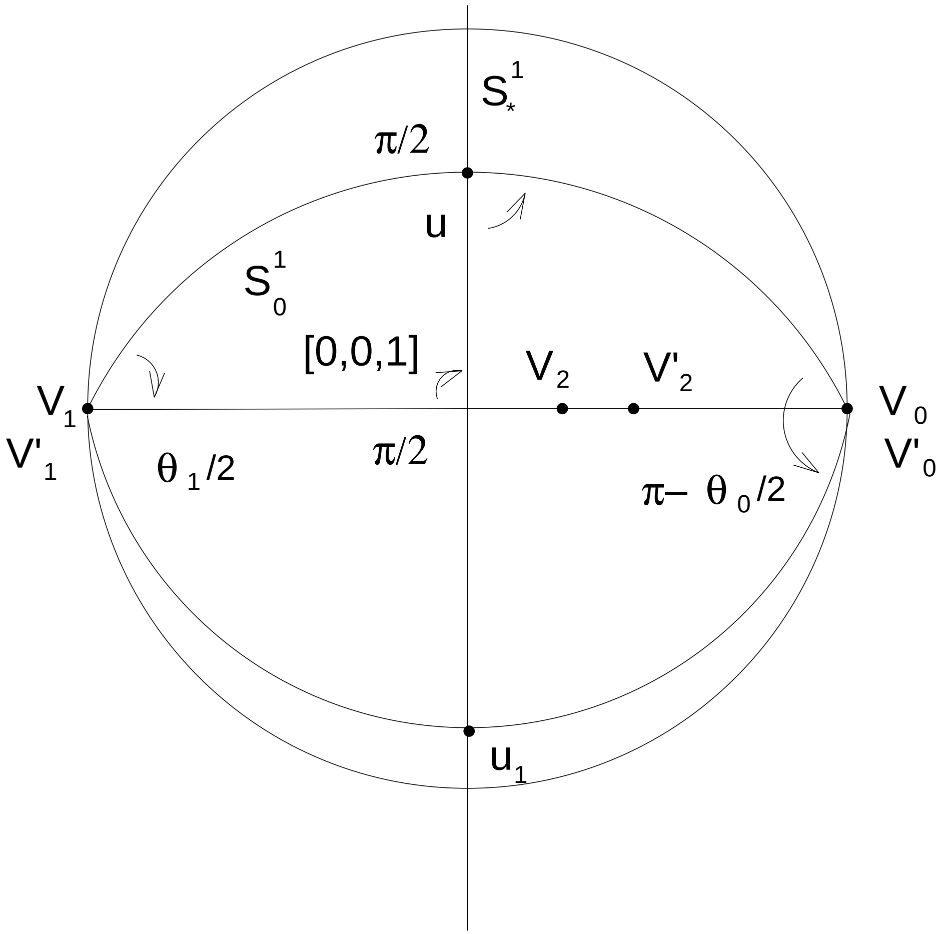

Figure: How to find a fixed point of $h(d_1)$ for region A_A.

- Since $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ is topologically a manifold, the tubular neighborhood of the region above A_A is homeomorphic to $\mathbf{S}^3 \times B^3$ and covers the tubular neighborhood of the 3-sphere in $\mathrm{SO}(3)$ case 4 to 1.

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- In all other cases B_B, $C_{C'}$,$A'_{A'}$, $B'_{B'}$, and C'_C, similar arguments show that the rectangle times T_2^3 maps to a 3-sphere in $\mathcal{C}_1$.
- Since the boundary of the closure M is a union of six five-dimensional manifolds homeomorphic to $\mathbf{S}^3 \times \mathbf{S}^2$. Thus, we can glue six neighborhoods of the three spheres over A_A,B_B,$C_{C'}$,$A'_{A'}$, $B'_{B'}$, and C'_C, homeomorphic to $\mathbf{S}^3 \times B^3$ to obtain the octahedral manifold. Hence, it follows that $\tilde{O} \times T_2^3/\mathbb{Z}_2/\sim$ is homeomorphic to the octahedral manifold.

The topology of the other component $\mathcal{C}_1$

- Again, we define i_a, i_b, and i_c on $T_2^3/\mathbb{Z}_2$: i_a is defined by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1 + 2\pi, \phi_2)$ and i_b by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0, \phi_1 + 2\pi, \phi_2 + 2\pi)$ and i_c by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1, \phi_2 + 2\pi)$. Again, this amounts to changing some of the pasting maps by multiplications by $-\mathrm{I}$.

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- Let us study what are the branch loci of I_A, I_B, I_C. This is defined by equations 11.

The topology of the other component $\mathcal{C}_1$

- Again, we define i_a, i_b, and i_c on $T_2^3/\mathbb{Z}_2$: i_a is defined by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1 + 2\pi, \phi_2)$ and i_b by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0, \phi_1 + 2\pi, \phi_2 + 2\pi)$ and i_c by $(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1, \phi_2 + 2\pi)$. Again, this amounts to changing some of the pasting maps by multiplications by $-\mathrm{I}$.
- Let us study what are the branch loci of I_A, I_B, I_C. This is defined by equations 11.

Theorem (10.1)

The other component $\mathcal{C}_1$ is homeomorphic to the quotient of filled octahedral manifold with the product Klein four-group action given by the action of I_A, I_B, I_C, i_a, i_b, and i_c.

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