The deformation spaces of convex real projective structures on manifolds or orbifolds with ends: openness and closedness

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Real projective structure

A $\mathbb{R}P^n$-structure on an orbifold is given by having charts from U_is to open subsets of $\mathbb{R}P^n$ with transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$.

Projective, affine, and hyperbolic geometry

- $\mathbb{R}P^n = P(\mathbb{R}^{n+1}) = (\mathbb{R}^{n+1} - \{O\})/\sim$ where $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{O\}$.
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- $\mathbb{R}P^n - \mathbb{R}P^{n-1}_\infty$ is an affine space A^n where the group of projective automorphisms of A^n is exactly $Aff(A^n)$.

$$A^n \hookrightarrow \mathbb{R}P^n, Aff(A^n) \hookrightarrow \mathrm{PGL}(n+1, \mathbb{R}).$$

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- The upper part of $q = -1$ is the model of the hyperbolic n-space H^n.
- The cone $q < 0$ corresponds to the convex open n-ball in $B^n \hookrightarrow A^n \subset \mathbb{R}P^n$ correspond to H^n in a one-to-one manner.

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- $Isom(H^n) = Aut(B^n) = PO(1, n) \hookrightarrow \mathrm{PGL}(n+1, \mathbb{R})$.

Real projective structures on orbifolds

An $\mathbb{R}P^n$-structure on M/Γ with simply connected M is given by an immersion $D: M \to \mathbb{R}P^n$ equivariant with respect to a homomorphism $h: \Gamma \to \mathrm{PGL}(n+1, \mathbb{R})$ where Γ is the fundamental group of M/Γ.

- The pair (D, h) is only determined up to the action by $g \in \mathrm{PGL}(n+1, \mathbb{R})$ given by

$$g(D, h(\cdot)) = (g \circ D, gh(\cdot)g^{-1}).$$

- Conversely, $[(D, h)]$ determines the $\mathbb{R}P^n$-structure.

Deformation spaces of convex $\mathbb{R}P^n$-structures

- Given an orbifold S, a *convex $\mathbb{R}P^n$-structure* is given by a diffeomorphism $f : S \to \Omega/\Gamma$ for a convex domain Ω in $\mathbb{R}P^n$ and Γ a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$.
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is $\{(D, h)\}/\sim$ where $(D, h) \sim (D', h')$ if there is an isotopy $\tilde{f} : \tilde{S} \to \tilde{S}$ such that $D' = D \circ \tilde{f}$ and $h'(g) = h(g)$ for each $g \in \pi_1(S)$ or $D' = k \circ D$ and $h'(\cdot) = kh(\cdot)k^{-1}$ for $k \in \mathrm{PGL}(n+1, \mathbb{R})$.

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- Alternatively, $\mathrm{CDef}(S) = \{f : S \to \Omega/\Gamma\}/\sim$ where $f \sim g$ for $f : S \to \Omega/\Gamma$ and $g : S \to \Omega'/\Gamma'$ if there exists a projective diffeomorphism $k : \Omega/\Gamma \to \Omega'/\Gamma'$ so that $k \circ f$ is isotopic to g.

The hol map: The local homeomorphism property

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Theorem A

Let $\mathcal{O}$ be a closed n-orbifold or noncompact tame with radial or totally geodesic ends. Then the following map is a local homeomorphism:

$$\mathrm{hol} : \mathrm{Def}_{(E)}(\mathcal{O}) \to \mathrm{rep}_{(E)}(\pi_1(O), \mathrm{PGL}(n+1, \mathbb{R}))$$

in the stable subspace. (*note: no convexity condition is needed for this.*)

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Proof.

This follows as in the compact cases using the bump functions. However, we may need the bump functions extending to the ends for radial ends. (comments: this would be hard to generalize for non-R- or T-ends) □

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Proposition (Basic Convexity)

- *An $\mathbb{R}P^n$-orbifold is convex if and only if the developing map D sends the universal cover to a convex open domain in $\mathbb{R}P^n$.*
- *An $\mathbb{R}P^n$-orbifold is properly convex if and only if D sends the universal cover to a properly convex open domain in a compact domain in an affine patch of $\mathbb{R}P^n$.*
- *If a convex $\mathbb{R}P^n$-orbifold is not properly convex, then its holonomy is virtually reducible.*

Benoist's "maximally complete" results

Benoist in his papers "Convexes divisibles I-IV":

Proposition (Benoist)

Suppose that a discrete subgroup Γ of $\mathrm{PGL}(n+1,\mathbb{R})$ acts properly on a properly convex n-dimensional open domain Ω so that Ω/Γ is a compact orbifold. Then the following statements are equivalent.

- *Every FI subgroup of Γ has a trivial center.*
- *Every FI subgroup of Γ is irreducible in $\mathrm{PGL}(n+1,\mathbb{R})$. (or strongly irreducible).*
- *The Zariski closure of Γ is semisimple.*
- *Γ does not contain a normal infinite nilpotent subgroup.*
- *Γ does not contain a normal infinite abelian subgroup.*

Benoist's result continued

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- **Theorem (Benoist's Closedness)**

  Let Γ be a discrete subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ *with a trivial virtual center. Suppose that a discrete subgroup Γ of* $\mathrm{PGL}(n+1, \mathbb{R})$ *acts on a properly convex n-dimensional open domain Ω so that* Ω/Γ *is a compact orbifold. Then every representation of a component of* $\mathrm{Hom}(\Gamma, \mathrm{PGL}(n+1, \mathbb{R}))$ *containing the inclusion representation also acts on a properly convex n-dimensional open domain cocompactly.*

S. Tillman's example

- There is a census of small hyperbolic orbifolds with graph-singularity. (See the paper by D. Heard, C. Hodgson, B. Martelli, and C. Petronio [2])
- There is a complete hyperbolic structure on the orbifold based on $\mathbf{S}^3$ with handcuff singularity with two points removed. The singularity orders are three.

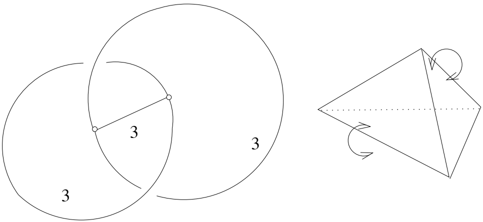

Figure: The handcuff graph

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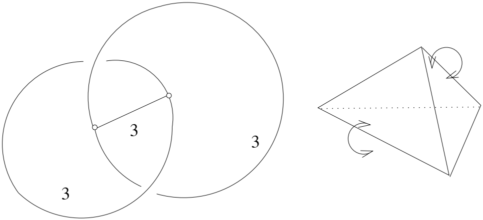
- There is a one-parameter space of deformations of the structures to $\mathbb{R}P^3$-structures as seen by simple matrix computations.
- More examples due to myself, Ballas, Danciger, Gye-Seon Lee, Greene: Some of these are properly and strictly convex and irreducible by our theory to be presented.



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End orbifold

- An $\mathbb{R}P^n$-orbifold has *radial ends* if each end has an end neighborhood foliated by concurrent geodesics for each chart ending at the common point of concurrency.
- Each end has a neighborhood diffeomorphic to a closed orbifold times an open interval.

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- Crampon-Marquis arXiv:1202.5442 and Cooper-Long-Tillman arXiv:1109.0585 also studies finite-covolume cases: i.e.; “cusped cases".

Open and closed properties

Theorem B

Let $\mathcal{O}$ be a noncompact topologically tame n-orbifold with admissible ends satisfying (IE) and (NA). Then

- *In $\mathrm{Def}^{i}_{E,u,ce}(\mathcal{O})$, the subspace $\mathrm{CDef}_{E}(\mathcal{O})$ of SPC-structures is open. (SPC means "stable properly convex")*

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- *Suppose further that $\pi_1(\mathcal{O})$ contains no nontrivial nilpotent normal subgroup. The deformation space $\mathrm{CDef}_{E,u,ce}(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ maps homeomorphic to a union of components of $\mathrm{rep}^{i}_{E,u,ce}(\pi_1(\mathcal{O}), \mathrm{PGL}(n+1,\mathbb{R}))$.*

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Here "u" indicates unique fixed point conditions. However it is not essential here.

(Cooper-Long-Tillman are using "flag" condition.) "ce" means lens or horospherical condition.

Theorem C

Let $\mathcal{O}$ be a strict SPC noncompact topologically tame n-dimensional orbifold with admissible ends satisfying (IE) and (NA). Suppose that $\pi_1(\mathcal{O})$ has no infinite nilpotent subgroup as a virtual normal subgroup. Then

- *$\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups.*

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- *In $\mathrm{Def}^i_{E,u,ce}(\mathcal{O})$, the subspace $\mathrm{SDef}_E(\mathcal{O})$ of strict SPC-structures with respect to the ends is open.*

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$$\mathrm{rep}^i_{E,u,ce}(\pi_1(\mathcal{O}), \mathrm{PGL}(n+1, \mathbb{R})).$$

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Hilbert metrics

- A *Hilbert metric* on an SPC-structure is defined as a distance metric given by cross ratios. (We do not assume strictness here.)
- Let Ω be a properly convex domain. Then $d_\Omega(p, q) = \log(o, s, q, p)$ where o and s are endpoints of the maximal segment in Ω containing p, q.

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- Let Ω be a properly convex domain. Then $d_\Omega(p, q) = \log(o, s, q, p)$ where o and s are endpoints of the maximal segment in Ω containing p, q.
- This gives us a well-defined Finsler metric.
- Given an SPC-structure on $\mathcal{O}$, there is a Hilbert metric d_H on $\tilde{\mathcal{O}}$ and hence on $\tilde{\mathcal{O}}$. This induces a metric on $\mathcal{O}$.

Relatively hyperbolicity and strict SPC-structures

- We will use Bowditch's result to show

Theorem (D)

Let $\mathcal{O}$ be a topologically tame strictly SPC-orbifold with admissible ends satisfying (IE) and (NA). Then $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to the end groups $\pi_1(E_1), ..., \pi_1(E_k)$.

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- Fact: Suppose that $\pi_1(E_l), .., \pi_1(E_k)$ are hyperbolic for some $0 \leq l < k$, $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to $\pi_1(E_1), \ldots, \pi_1(E_{l-1})$ iff so it is with respect to $\pi_1(E_1), \ldots, \pi_1(E_k)$. (Drutu)

- Proof: We denote this quotient space $\mathrm{bd}\tilde{\mathcal{O}}_1/\sim$ by B, a compact metrizable space.
- We will use Theorem 0.1. of Yaman [5]: We show that $\pi_1(\mathcal{O})$ acts on the set B as a geometrically finite convergence group.

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- The group acts properly discontinuously on the set of triples in B.
- An end group Γ_x for end vertex x is a parabolic subgroup fixing x since the elements in Γ_x fixes only the contracted set in B and since there are no essential annuli.

- Proof continued: Let p be a point that is not a horospherical endpoint or a singleton corresponding an lens-shaped end. We show that p is a conical limit point.

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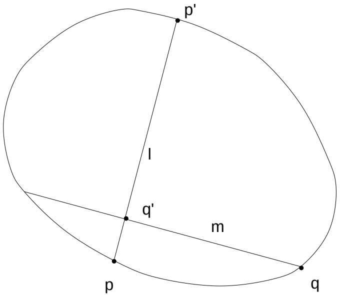

Figure: A shortest geodesic m to a geodesic l.

Converse

We will prove the partial converse to the above Theorem D:

Theorem (E)

Let $\mathcal{O}$ be a topologically tame SPC-orbifold with admissible ends satisfying (IE) and (NA). Suppose that $\pi_1(\mathcal{O})$ is relatively hyperbolic group with respect to the admissible end groups $\pi_1(E_1), ..., \pi_1(E_k)$ where E_i are horospherical for $i = 1, ..., m$ and lens-shaped for $i = m+1, ..., k$ for $0 \leq m \leq k$.

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- *Assume that $\mathcal{O}$ is SPC. Then $\mathcal{O}$ is strictly SPC.*
- *Let $\mathcal{O}_1$ be obtained by removing the concave neighborhoods of hyperbolic ends. Then if $\mathcal{O}$ is SPC, then $\mathcal{O}_1$ is strictly SPC.*

Proof.

Suppose not. We obtain a triangle T with ∂T in $\partial \tilde{\mathcal{O}}_1$. □

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Lemma

Suppose that $\mathcal{O}$ is a topologically tame properly convex n-orbifold with admissible ends and $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its ends. $\mathcal{O}$ has no essential tori or essential annuli. Then every triangle T in $\tilde{\mathcal{O}}$ with $\partial T \subset \partial \tilde{\mathcal{O}}$ is contained in the closure of a convex hull of one of its ends.

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Proof.

Uses asymptotic cones in Drutu-Sapir's work. □

Proofs of Theorem B and C

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- We show that the Koszul-Vinberg function can be perturbed to positive definite functions in the affine suspensions by controlling the ends.– This proves the openness part of Theorem B.
- For theorem C, we use "Strict SPC iff rel. hyperbolic".
- As we deform a strict SPC structure, we do not change the rel. hyperbolicity. Thus, strict SPC property is preserved. The openness part of Theorem C is done.

- We also need to show that the limiting convex real projective structure of a sequence of SPC-structure is also SPC. We show this by showing that the universal covers Ω_i must converge geometrically to a properly convex domain of nonempty interior. (up to duality) (Essentially because we have only horospherical or lens-type ends.)

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- Let $g_1, \ldots, g_m$ denote the set of generator of $\pi_1(\mathcal{O})$.

$$\mathbf{d}(h_i(g_j)(x_0), \mathrm{bd}\Omega_i) \geq C_0 \text{ for a uniform constant } C_0 : \tag{1}$$

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- We make use of Benzecri's estimation that there are two fixed balls B_r and B_R so that

$$B_r \subset \tau_i(\Omega_i) \subset B_R$$

up to projective transformations. Then $\tau_i g_i \tau_i^{-1}$ must be bounded and convergent.

Suhyoung Choi, Craig D. Hodgson, Gye-Seon Lee,
Projective Deformations of Hyperbolic Coxeter 3-Orbifolds
arXiv:1003.4352

D. Cooper, D.D. Long, and M. B. Thistlethwaite,
Flexing closed hyperbolic manifolds,
Geom. Topol. 11 (2007), 2413–2440.

D. Cooper, D.D. Long, and M. B. Thistlethwaite,
Computing varieties of representations of hyperbolic 3-manifolds into $\mathrm{SL}(4, \mathbb{R})$,
Experiment. Math. 15 (2006), no. 3, 291–305.

C. Drutu and M. Sapir,
Tree-graded spaces and asymptotic cones of groups,
Topology 44 (2005), no. 5, 959–1058.

W. Goldman.
Geometric structures on manifolds and varieties of representations.
Contemp. Math., 74:169–198, 1988.

D. Heard, C. Hodgson, B. Martelli, and C. Petronio,
Hyperbolic graphs of small complexity,
Experiment. Math. 19 (2010), no. 2, 211–236

D. Johnson and J.J. Millson.
Deformation spaces associated to compact hyperbolic manifolds.
In *Discrete Groups in Geometry and Analysis*, Proceedings of a conference held at Yale University in honor of G. D. Mostow, Springer-Verlag, Berlin, 1986.

Ludovic Marquis,
Espace des modules de certains polyedres projectifs miroirs,
arXiv:0806.3569

A. Yaman
A topological characterisation of relatively hyperbolic groups,
J. reine ange. Math. 566 (2004), 41 – 89.