THE PL-METHODS FOR HYPERBOLIC 3-MANIFOLDS TO PROVE TAMENESS

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ABSTRACT. Using PL-methods, we prove the Marden's conjecture that a hyperbolic 3-manifold M with finitely generated fundamental group and with no parabolics are topologically tame. Our approach is to form an exhaustion M_i of M and modify the boundary to make them 2-convex. We use the induced path-metric, which makes the submanifold M_i negatively curved and with Margulis constant independent of i. By taking the convex hull in the cover of M_i corresponding to the core, we show that there exists an exiting sequence of surfaces Σ_i. Some of the ideas follow those of Agol. We drill out the covers of M_i by a core $\mathcal{C}$ again to make it negatively curved. Then the boundary of the convex hull of Σ_i is shown to meet the core. By the compactness argument of Souto, we show that infinitely many of Σ_i are homotopic in $M - \mathcal{C}^o$. Our method should generalize to a more wider class of piecewise hyperbolic manifolds.

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Date: January 10, 2006 (version 3).
1991 *Mathematics Subject Classification.* Primary 57M50.
Key words and phrases. hyperbolic 3-manifolds, tameness, Kleinian groups.
The author gratefully acknowledges support from the Korea Reseach Foundation Grant (KRF-2002-070-C00010).

Recently, Agol initiated a really interesting approach to proving Marden's conjecture by drilling out closed geodesics. In this paper, we use truncation of the hyperbolic manifolds and drilling out compact cores of them to prove the tameness conjecture. We do use the Agol's idea of using covering spaces and taking convex hulls of the cores. Our approach is somewhat different in that we do not use end-reductions and pinched Riemannian hyperbolic metrics; however, we use the incomplete hyperbolic metric itself. The hard geometric analysis and geometric convergence techniques can be avoid using the techniques of this paper. By a negatively curved space, we mean a metric space whose universal cover is CAT(-1). Except for developing a somewhat complicated theory of deforming boundary to make the submanifolds of codimension 0 negatively curved, we do not need any other highly developed techniques. Also, we might be able to generalize the techniques to negatively curved polyhedral 3-manifolds and complexes obtained from groups.

Note also that there is a recent paper by Calegari and Gabai [6] using modified least area surfaces and closed geodesics. The work here is independently developed from their line of ideas. Also, we note that there were earlier attempts by Freedman [11], Freedman-McMullen [13], which were very influential for the later success by Agol and Calegari-Gabai, and another earlier unsuccessful attempt by Ohshika, using the least area surfaces.

In this paper, we let M be a hyperbolic 3-manifold with a Scott's core homeomorphic to a compression body. We suppose that M has a finitely generated fundamental group and the holonomy is purely loxodromic and has ends $E, E_1, \ldots, E_n$. Let $F_1, \ldots, F_n$ be the incompressible surfaces in neighborhoods of the ends $E_1, \ldots, E_n$ respectively. Let $N(E)$ be a neighborhood of an end E with no incompressible surface associated.

The Marden's conjecture states that a hyperbolic 3-manifold with a finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold. It will be sufficient to prove for the above M to prove the complete conjecture.

The cases when the group contains parabolic elements are left out, which we will work out on a later occasion.

Theorem A . *Let M be as above with ends $E, E_1, \ldots, E_n$, and $\mathcal{C}$ be a compact core of M. Then E has an exiting sequence of surfaces of genus equal to that of the boundary component of $\partial\mathcal{C}$ corresponding to E.*

Theorem B . *M is tame; that is, M is homeomorphic to the interior of a compact manifold.*

This paper has three parts: In Part 1, let M be a codimension 0 submanifold of a hyperbolic 3-manifold N of infinite volume with certain nice boundary conditions. M is locally finitely triangulated. Suppose that M is 2-*convex* in N in the sense that any tetrahedron T in N with three of its side in M must be inside M. Now let L be a finitely triangulated, compact codimension 0-submanifold M so that ∂L is incompressible in M with a number of closed geodesics $\mathbf{c_1}, \ldots, \mathbf{c_n}$ removed. Given $\epsilon > 0$, we show that ∂L can be isotopied to a hyperbolically triangulated surface so that it bounds in M a 2-convex submanifold whose ϵ-neighborhood contains $\mathbf{c_1}, \ldots, \mathbf{c_n}$. The isotopy techniques will be based on PL-type arguments and deforming by crescents. An important point to be used in the proof is that the crescents avoid closed geodesics and geodesic laminations. Thus, the isotopy does not pass through the closed geodesics by as small amount as we wish. (These will be the contents of Theorem C and Corollary D.)

Part 2 is as follows: A *general hyperbolic manifold* is a manifold with boundary modeled on subdomains in the hyperbolic space. A general hyperbolic manifold is 2-*convex* if every isometry from a tetrahedron with an interior of one of its side removed extends to the tetrahedron itself. We show that a 2-convex general hyperbolic manifold is negatively curved. The proof is based on the analysis of the geometry of the vertices of the boundary required by the 2-convexity. We will also define a hyperbolic surface as a triangulated surface where each triangle gets mapped to a geodesic triangle and the sum of the induced angles at each vertex is always greater than or equal to 2π. We show the area bound of such surfaces. Finally, we show that the boundary of the convex hull of a core in a general hyperbolic manifold with finitely generated fundamental group can be deformed to a nearby hyperbolic-surface, which follows from the local analysis of geometry.

In Part 3, we will give the proof of Theorems A and B using the results of Part 1 and 2. The outline is given in the abstract and in the beginning of Part 3. The proof itself is rather short spanning 9-10 pages only.

We like to thank the valuable comments and encouragements from Professors Francis Bonahon, Richard Canary, Bill Goldman, Misha Kapovich, Sadayoshi Kojima, Kenichi Ohshika, Curt McMullen, Yair Minsky and Bill Thurston.

Part 1. 2-convex hulls of submanifolds of hyperbolic manifolds

1. Introduction to Part 1

The purpose of Part I is to deform a submanifold of codimension 0 of general hyperbolic manifold into a negatively curved one, i.e., Corollary D.

We will be working in a more general setting. A *general hyperbolic manifold* is a Riemannian manifold M with corner and a geodesic metric that admits a geodesic triangulation so that each 3-simplex is isometric with a compact hyperbolic simplex. Let $\tilde M$ be a universal cover of M and $\pi_1(M)$ the group of deck transformations. M admits a local isometry, so-called developing map, $\mathbf{dev} : \tilde M \to \mathbb{H}^3$ for the hyperbolic space $\mathbb{H}^3$ equivariant with respect to a homomorphism $h : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$. The pair $(\mathbf{dev}, h)$ is only determined up to action

$$(\mathbf{dev}, h) \mapsto (g \circ \mathbf{dev}, g \circ h(\dot{)} \circ g^{-1}) \text{ for } g \in \mathrm{PSL}(2, \mathbb{C}).$$

We remark that in Thurston's notes [17] a locally convex general hyperbolic manifold is shown to be covered by a convex domain in $\mathbb{H}^3$ or, equivalently, it can be extended to a complete hyperbolic manifold. However, a general hyperbolic manifold can be much worse although here we would be looking at mostly general hyperbolic manifolds that are covered by coverings of some domains in the hyperbolic spaces.

A general hyperbolic manifold M is 2-*convex* if given a hyperbolic 3-simplex T, a local-isometry $f : T - F^o \to M^o$ for a face F of T extends to an isometry $T \to M^o$. (See [9] for more details. Actually projective version applies here by the Klein model of the hyperbolic 3-space.)

- By a *totally geodesic hypersurface*, we mean the union of components of the inverse image under a developing map of a totally geodesic plane in $\mathbb{H}^3$.
- A *local totally geodesic hypersurface* is an open neighborhood of a point in the hypersurface.
- For a point, a *local half-space* is the closure in the ball of the component of an open ball around it with a totally geodesic hypersurface passing through it.
- The local totally geodesic hypersurface intersected with the local-half space is said to be the *side* of the local half-space.
- A local half-space with its side removed is said to be an *open* local half-space.

A surface $f : S \to M$ is said to be *triangulated* if S is triangulated and each triangle is mapped to a totally geodesic triangle in M. (We will generalize this notion a bit.)

- An interior vertex of f is a *saddle-vertex* if every open local half-space associated with the vertex does not contain the local image of f with the vertex removed.
- A *strict saddle-vertex* is a saddle-vertex where every associated closed local half-space does not contain the local image.
- An interior vertex of f is a *convex vertex* if a local open half-space associated with the vertex contains the local image of f removed with the vertex.

(In these definitions, we take a slightly larger ambient open manifold of M to make sense of open local half-space. Also, we can see these definitions better by looking at the unit tangent bundle U at the vertex: Then the image of f corresponds to a path in U.) If P is a local totally geodesic plane passing through the vertex whose one-sided neighborhood contains the local image of f, then we say P is a *supporting plane*.

An interior vertex of f is an *hyperbolic-vertex* if the sum of angles of the triangles around a vertex is $\geq 2\pi$. A saddle-vertex is a hyperbolic-vertex by Lemma 7.4.

A map $f : S \to M$ is a saddle-map if each vertex is a saddle-vertex, and f is a hyperbolic-map if each vertex is a hyperbolic-one. A saddle-map is a hyperbolic-map but not conversely in general. For an imbedding f and an orientation, a convex vertex is said to be a *concave vertex* if the local half-space is in the exterior direction. Otherwise, the convex vertex is a *convex vertex*. We also know that if the boundary of a general hyperbolic submanifold N of M is saddle-imbedded, then N is 2-convex (see Proposition 2.5).

Theorem C . *Let M be an orientable 2-convex general hyperbolic 3-manifold, and Σ be a closed surface in M. Suppose that each component of Σ is incompressible in M if we remove a finite number of the image $\mathbf{c_1}, \dots, \mathbf{c_n}$ of the closed geodesics in the interior M^o of M. Then for arbitrarily given small $\epsilon > 0$, we can isotopy Σ so that Σ becomes a saddle-imbedded surface. Finally, during the isotopy Σ might pass through some of $\mathbf{c_i}$ but with as small an amount as possible.*

Corollary D . *Let N be an orientable 2-convex general hyperbolic 3-manifold, and M be a compact codimenion-zero submanifold of N with boundary ∂M. Suppose that each component of ∂M is incompressible in M if we remove a finite number of the image $\mathbf{c_1}, \dots, \mathbf{c_n}$ of the closed geodesics in the interior M^o of M. Then for arbitrarily given small $\epsilon > 0$, we can isotopy M to a homeomorphic general hyperbolic 3-manifold M' in N so that M' is 2-convex and an ϵ-neighborhood of M' contains the collection of closed geodesics $\mathbf{c_1}, \dots, \mathbf{c_n}$.*

We say that M' obtained from M by the above process is a 2-*convex hull* of M. Although M' is not necessarily a subset of M, the curves $\mathbf{c_1}, \dots, \mathbf{c_n}$ is a subset of an ϵ-neighborhood of M' and hence we have certain amount of control. (Here the geodesics are allowed to self-intersect.)

In section 2, we review some hyperbolic manifold theory and discuss saddle vertices and relationship with 2-convexity.

In section 3, we introduce so-called crescents: Let M be a 2-convex general hyperbolic manifold and Σ a closed subsurface, possibly with many components. We take the inverse image $\tilde{\Sigma}$ in $\tilde{M}$ of Σ in M, incompressible in the ambient 2-convex general hyperbolic manifold with a number of geodesics $\mathbf{c_1}, \dots, \mathbf{c_n}$ removed. A crescent is a connected domain bounded by a totally geodesic hypersurface and an open surface in $\tilde{\Sigma}$. The portion of boundary in the totally geodesic hypersurface is said to be the I-part and the portion in $\tilde{\Sigma}$ is said to be the α-part. A crescent may contain another crescents and so on, and the folding number of a crescent is the maximum intersection number of the generic path from the outer part in the surface to the innermost component of the crescent with the surface removed. We show that for a given closed surface Σ, the folding number is bounded above.

A *highest-level crescent* is an innermost one that is contained in a crescent with highest folding number which achieves the folding number. We show that a highest-level crescent is always contained in an innermost crescent; i.e., so called the secondary highest-level crescent. In a secondary highest level crescent, the closure of the α-part and the I-part are isotopic. We also show that the secondary highest-level crescents meet nicely extending their α-parts in $\tilde{\Sigma}$, following [9].

In section 4, we introduce the crescent-isotopy theory to isotopy a surface in a general hyperbolic manifold so that all of its vertices become saddle-vertices: We form the union of secondary highest-level crescents and can isotopy the union of their α-parts to the complement I in the boundary of their union. This is essentially the crescent move. (In this paper, by isotopies, we mean the deck-transformation group equivariant isotopies unless we specify otherwise. At least, if the isotopy in each step is not equivariant, we will make it so after the final isotopy is completed.)

However, there might be some parts of $\tilde{\Sigma}$ meeting I tangentially from above. We need to first push these parts upward first using so-called convex truncations.

Also, after the move, there might be pleated parts which are not triangulated. We present a method to perturb these parts to triangulated parts without increasing the levels or the set of crescents by much.

Next, we use the crescent isotopy and perturbations to obtain saddle-imbedded surface isotopic to Σ:

- We take the highest folding number and take all outer secondary highest-level crescents,
  - do some convex truncations,
  - do the crescent isotopies and
  - convex perturbations.
- Next, we take all inner highest-level crescents of the same level as above, do some truncations, and do crescent moves and perturb as we did above. Now the highest folding number decreases by one.
- We do the next step of the induction until we have no crescents any more.

In this case, all the vertices are saddle-vertices. This completes the proof of Theorem C.

Finally, we prove Corollary D by applying our results to a codimension-zero submanifold M with incompressible boundary in the ambient manifold with some geodesics removed.

2. Preliminary

In this section, we review the hyperbolic space and the Kleinian groups briefly. We discuss the relationship between the 2-convexity of general hyperbolic manifolds.

2.1. Hyperbolic manifolds.

The hyperbolic n-space is a complete Riemannian metric space $(\mathbb{H}^n, d)$ of constant curvature equal to -1. We will be concerned about hyperbolic plane and hyperbolic spaces, i.e., $n = 2, 3$, in this paper.

The upper half space model for $\mathbb{H}^2$ is the pair

$$(U^2, \mathrm{PSL}(2,\mathbb{R}) \cup \overline{\mathrm{PSL}}(2,\mathbb{R}))$$

where U^2 is the upper half space.

The Klein model of $\mathbb{H}^2$ is the pair $(B^2, \mathrm{PO}(1,2))$ where B^2 is the unit disk and $\mathrm{PO}(1,2)$ is the group of projective transformations acting on B^2.

A *Fuchsian* group is a discrete subgroup of the group of isometries of $\mathrm{PO}(1,2)$ of the group $\mathrm{Isom}(\mathbb{H}^2)$ of isometries of $\mathbb{H}^2$.

There are many models of the hyperbolic 3-space: The upper half-space model consists of the upper half-space U of $\mathbb{R}^3$ and the group of isometries are identified as the group of similarities of $\mathbb{R}^3$ preserving U, which is identified as the union of $\mathrm{PSL}(2,\mathbb{C})$ and its conjugate $\overline{\mathrm{PSL}}(2,\mathbb{C})$.

We shall use the Klein model mostly: The Klein model consists of the unit ball in $\mathbb{R}^3$ and the group of isometries are identified with the group of projective transformations preserving the unit ball, which is identified as $\mathrm{PO}(1,3)$.

A *Kleinian* group is a discrete subgroup of the group of isometries $\mathrm{PO}(1,3)$, i.e., the group $\mathrm{Isom}(\mathbb{H}^3)$ of isometries of $\mathbb{H}^3$.

A *parabolic* element γ of a Kleinian group is a nonidentity element such that $(\gamma(x), x)$, $x \in \mathbb{H}^3$, has no lower bound other than 0. A *loxodromic* element of a Kleinian group is an isometry with a unique invariant axis. A *hyperbolic* element is a loxodromic one with invariant hyperplanes.

For Fuchsian groups, a similar terminology holds.

In this paper, we will restrict our Kleinian groups to be torsion-free and have no parabolic elements and all elements are orientation-preserving.

2.2. **Saddle-vertices.** Let M be a general hyperbolic manifold. We now classify the vertices of a triangulated map $f : S \to M$ where we do not yet require the general position property of f but identify the vertex with its image.

By a *straight geodesic* in a general hyperbolic manifold, we mean a geodesic that maps to geodesics in $\mathbb{H}^3$ under the developing maps.

Lemma 2.1. *Let $f : S \to M$ be a triangulated map.*

- *An interior vertex of S is either a convex-vertex, a concave vertex, or a saddle-vertex.*
- *A saddle-vertex which is not strict one has to be one of the following:*
  - (i) *a vertex with a totally geodesic local image.*
  - (ii) *a vertex on an edge in the intersection of two totally geodesic planes where f locally maps into one sides of each plane.*
  - (iii) *A vertex which is contained in at least three edges in the image of f in a local totally geodesic plane P and the edges are not contained in any closed half-plane of P. The local half-space bounded by P is the unique one containing the image of f.*
- *If f is a general position map, then a saddle-vertex is a strict saddle-vertex.*

Proof. Suppose v is a saddle-vertex and not a strict one and not of form (i) or (ii). Since v is not strict, there is a supporting plane. If there are more than three supporting planes in general position, then v is a strict convex-vertex. If there are two supporting plane, then an edge in the image of f is in the edge of intersection of the two planes in order that v be a nonstrict saddle vertex, which is absurd.

Hence the supporting plane is unique. If there are no three edges as described in (iii), we can easily find another supporting plane. □

Lemma 2.2. *A saddle-vertex of type* (ii) *and* (iii) *of Lemma 2.1 can be deformed to a strict saddle-vertex by an arbitrarily small amount by pushing if necessary the vertex from the boundary of the closed local half-space containing the local image of f in the direction of the open half-space.*

Proof. If the saddle-vertex is a strict one, then we leave it alone. If the saddle-vertex is not a strict one, a closed local half-space contains the local image of f. Let U be the unit tangent bundle at the vertex. A closed hemisphere H contains the path corresponding to the local image of f.

In case (ii) of Lemma 2.1, there are actually two closed hemispheres H_1 and H_2 whose intersection contains the local image of f in U. Therefore, we choose a direction in the interior of the intersection of H_1 and H_2. Then by Lemma 2.3, the result of a sufficiently small deformation is a strict saddle vertex.

In case (iii) of Lemma 2.1, let w_1, w_2, w_3 be the points on the unit tangent bundle at the vertex corresponding the three edges. Then by moving vertex in the direction, the corresponding directions w_1', w_2', w_3' of the perturbed edges form a strictly convex triangle in an open hemisphere in U and the direction vector v of the movement not in the hemisphere. v, w_1', w_2', w_3' are vertices of a geodesic triangulation of U into a 2-skeleton of the topological tetrahedron and every triple of them form vertices of a strictly convex triangle in an open hemisphere. Therefore, there is no closed hemisphere in U containing all of them. Hence, we obtain a strict saddle vertex. □

We will need the following much later:

Lemma 2.3. *Suppose that $f : \Sigma \to M$ is a triangulated imbedding. Let v be a vertex of f and $f' : U_v \to U_v^M$ be the induced map from the link of v in Σ to that of v in M. Suppose that there exists a segment l of length $> \pi$ in U_v^M with endpoints in the image of f' separating two points in the image of f' so that the minor arc $\overline{xy}$ meets l transversely. Then v is a strict saddle vertex.*

Proof. If a closed hemisphere contains l, then it must contain l in its boundary. Therefore, x or y is not in the hemisphere, and there is no closed hemisphere containing l and x and y. □

Given an oriented surface, a convex vertex is either a *convex vertex* or a *concave vertex* depending on whether the supporting local half-space is in the outer normal direction or in the inner normal direction.

Proposition 2.4. *A vertex of an oriented imbedded triangulated surface is either a saddle-vertex or a convex vertex or a concave vertex.*

Proof. Straightforward. □

In this paper, we consider only metrically complete submanifolds, i.e., locally compact ones.

Proposition 2.5. *A general hyperbolic manifold M is 2-convex if and only if each vertex of ∂M is a convex vertex or a saddle-vertex.*

Proof. Suppose M is 2-convex. If a vertex x of ∂M is a concave, we can find a local half-open space in M with its side passing through x. The side meets ∂M only at x. From this, we can find a 3-simplex inside with a face in the side. This contradicts 2-convexity of M.

Conversely, suppose that ∂M has only convex vertices or saddle-vertices. Let $f : T - F^o \to M^o$ be a local-isometry from a 3-simplex T and a face F of T. We may lift this map to $\tilde{f} : T - F^o \to \tilde{M}^o$ where $\tilde{M}$ is the universal cover of M where $\tilde{f}$ is an imbedding.

Since $\tilde{M}$ is metrically complete, $\tilde{f}$ extends to $\tilde{f}' : T \to \tilde{M}$. Suppose that f does not extend to $f' : T \to M^o$. This implies that $\tilde{f}'(F)$ meets $\partial\tilde{M}$ where $\tilde{f}'(\partial F)$ does not meet $\partial\tilde{M}$. The subset $K = \partial\tilde{M} \cap \tilde{f}'(F)$ has a vertex x of $\partial\tilde{M}$ which is an extreme point of the convex hull of K in the image of F. We can tilt $\tilde{f}'(T)$ by a supporting line l at x a bit and the new 3-simplex meets $\partial\tilde{M}$ at x only. This implies that x is not a saddle-vertex but a concave vertex, a contradiction. □

3. Crescents

Let M be a metrically complete 2-convex general hyperbolic manifold from now on and $\tilde{M}$ its universal cover. Let Γ denote the deck transformation group of $\tilde{M} \to M$.

Let Σ be a properly imbedded compact subsurface of an orientable general hyperbolic manifold M with more than one components in general. We denote by $\tilde{\Sigma}$ the inverse image of Σ in the universal cover $\tilde{M}$ of M. ($\tilde{\Sigma}$ is not connected in general and components may not be universal covers of Σ.) We assume that the triangulated $\tilde{M}$ is in general position and so is $\tilde{\Sigma}$ under the developing maps.

For each component Σ_0 of Σ and a component Σ'_0 of $\tilde{\Sigma}$ mapping to Σ_0, there exists a subgroup $\Gamma_{\Sigma'_0}$ acting on Σ'_0 so that the quotient space is isometric to Σ_0.

Hypothesis 3.1. We will now assume that Σ is incompressible in M with a number of straight closed geodesics $\mathbf{c_1}, \dots, \mathbf{c_n}$ removed.

First, we introduce crescents for $\tilde{\Sigma}$ which is the inverse image of a surface Σ in a 2-convex general hyperbolic manifold. We define the folding number of crescents and show that they are bounded above.

We define the highest level crescents, i.e., the innermost crescents in the crescent with the highest folding number incurring the highest folding number. We show that closed geodesics avoid the interior of crescents. Given a highest level crescent, we show that there is an innermost crescent that has a connected I-part to which the closure of the α-part is isotopic in the crescent by the incompressibility of Σ. These are the *secondary highest-level crescents*. We show that the secondary highest-level crescent is homeomorphic to its I-part times the unit interval.

Next, we show that if two highest-level crescents meet each other in their I-parts tangentially, then they both are included in a bigger secondary highest-level crescent.

Furthermore, if two secondary highest-level crescents meet in their interiors, then they meet nicely extending their α-parts. This is the so-called transversal intersection of two crescents.

3.1. **Definition of crescents.** We list here key notions associated with crescents that are used very often in this paper.

Definition 3.2. Clearly, we require $\tilde{\Sigma}$ to be a properly and tamely imbedded subsurface disjoint from $\tilde{M}$.

- If $\tilde{\Sigma}$ is not necessarily triangulated but each point of it has a convex 3-ball neighborhood B where the closure of one of the component of $B - \tilde{\Sigma}$ is the closure of a component of a convex 3-ball with a closed triangulated disk with boundary in ∂B removed. In this case, $\tilde{\Sigma}$ is said to be *nicely* imbedded.
- If $\tilde{\Sigma}$ is a union of triangulated compact triangles but the vertices are not necessarily in general position, we say that $\tilde{\Sigma}$ is *triangulated.*
- If $\tilde{\Sigma}$ is triangulated by compact triangles whose vertices are in general position, we say that $\tilde{\Sigma}$ is *well-triangulated.*

Definition 3.3. We assume that $\tilde{\Sigma}$ is nicely imbedded at least. A *crescent* $\mathcal{R}$ for $\tilde{\Sigma}$ is

- a connected domain in $\tilde{M}$ which is a closure of a connected open domain in $\tilde{M}$,
- so that its boundary is a disjoint union of a (connected) open domain in $\tilde{\Sigma}$ and the closed subset that is the disjoint union of totally geodesic 2-dimensional domains in $\tilde{M}$ that develops into a **common** totally geodesic hypersurface in $\mathbb{H}^3$ under **dev**.

We denote by $\alpha_{\mathcal{R}}$ the domain in $\tilde{\Sigma}$ and $I_{\mathcal{R}}$ the union of totally geodesic domains. To make the definition canonical, we require $I_{\mathcal{R}}$ to be a maximal totally geodesic set in the boundary of $\mathcal{R}$. We say that $I_{\mathcal{R}}$ and $\alpha_{\mathcal{R}}$ the *I-part* and the *α-part* of $\mathcal{R}$.

As usual Σ is oriented so that there are outer and inner directions to normal vectors.

Definition 3.4. The subset $\tilde{\Sigma} \cap \mathcal{R}$ may have more than one components. For each component of $\mathcal{R} - \tilde{\Sigma}$, we can assign a *folding number* which is the minimal generic intersection number of that a path in the interior of $\mathcal{R}$ from $\alpha_{\mathcal{R}}$ meeting $\tilde{\Sigma}$ to reach to the component. The folding number of $\mathcal{R}$ is the maximum of the folding numbers for all of the components.

Definition 3.5. A *sub-crescent* $\mathcal{S}$ of a crescent $\mathcal{R}$ is the closure of a component of $\mathcal{R} - A$ where A is a union of components of $\tilde{\Sigma} \cap \mathcal{R}$ where we define the I-part to be the union of maximal domains in the intersection of the $I_{\mathcal{R}}$ with $\mathcal{S}$ and $\alpha_{\mathcal{S}}$ to be $\partial\mathcal{S} - I_{\mathcal{S}}$. In these case $\mathcal{R}$ is a *super-crescent* of $\mathcal{S}$.

Given a crescent $\mathcal{S}$, an *ambient folding number* is the maximum of the folding number of super-crescents of $\mathcal{S}$.

A priori, a crescent may have an infinite folding number. However, we will soon show that the folding number is finite.

Note that a proper sub-crescent has a strictly less folding number than the original crescent and a strictly greater ambient folding number than the original one.

Definition 3.6. The *I-part hypersurface* is the inverse image in $\tilde{M}$ under the developing map of the totally geodesic plane P containing the developing image of the I-part of the crescent.

A closed subset K of $\tilde{M}$ is a *geometric limit* or just *limit* of a sequence of closed subsets K_i if for each compact ball B in $\tilde{M}$, $K_i \cap B$ converges to $K \cap B$ in the Hausdorff metric sense.

Definition 3.7. A noncompact domain will be called a *crescent* if it is bounded by a (connected) domain in $\tilde{\Sigma}$ and the union of totally geodesic domains developing into a common totally geodesic plane in $\mathbb{H}^3$ and is a geometric limit of compact crescents. Again the I-part is the maximal totally geodesic subset of the boundary of the crescent. The α-part is the complement in the boundary of the crescent and is a connected open subset of $\tilde{\Sigma}$. Of course they need not be limits of the corresponding subsets of the compact crescents. (See Proposition 3.12 for a related idea.)

Definition 3.8. A crescent is an *outer* one if its interior to the α-part is in the outer normal direction of $\tilde{\Sigma}$. It is an *inner* one otherwise.

Definition 3.9. The boundary $\partial I_{\mathcal{R}}$ of $I_{\mathcal{R}}$ is the set of boundary points in the I-part hypersurface of $\mathcal{R}$. Also, for any subset A of $I_{\mathcal{R}}$, we define ∂A to be the set of boundary points in the I-part hypersurface. We define $I^o_{\mathcal{R}}$ to be the interior, i.e., $I_{\mathcal{R}} - \partial I_{\mathcal{R}}$.

Definition 3.10. A *pinched simple closed curve* is a simple curve pinched at most three points or pinched at a connected arc. The boundary of the I-part is a disjoint union of pinched simple curves.

3.2. **Properties of crescents.** The following is a really important property since this shows we can use crescents in general hyperbolic manifolds without worrying about whether the I-parts meet the boundary of the ambient manifold.

Proposition 3.11. *Let $\mathcal{R}$ be a crescent in a 2-convex ambient general hyperbolic manifold M. Then $\mathcal{R}$ is disjoint from $\partial\tilde{M}$. In fact, if $\mathcal{R}$ is compact, then $\mathcal{R}$ is uniformly bounded away from $\partial\tilde{M}$.*

Proof. Suppose that $\mathcal{R}$ meets $\partial\tilde{M}$. Since the closure of $\alpha_{\mathcal{R}}$ being a subset of $\tilde{\Sigma}$ is disjoint from $\partial\tilde{M}$, it follows that $I_{\mathcal{R}}$ meets $\partial\tilde{M}$ in its interior points and away from the boundary points in the ambient totally geodesic subsurface P in $\tilde{M}$.

We find the extreme point of $I_{\mathcal{R}} \cap \partial M$ and find the supporting line. This point has a local half-space in $\mathcal{R}$. By tilting the I-part a bit by the supporting line, we find a local half-space in M and in it a local totally geodesic hypersurface meeting ∂M at a point. This contradicts the 2-convexity of $\tilde{M}$.

The second part follows from the disjointness of $\mathcal{R}$ to ∂M and the compactness of $\mathcal{R}$. □

The following shows the closedness of set of points of $\tilde{M} - \tilde{\Sigma}$ in crescents.

Proposition 3.12. *Let $\mathcal{R}_i$ be a sequence of crescents. Suppose that x is a point of $\tilde{M} - \tilde{\Sigma}$ which is a limit of a sequence of points in the union of $\mathcal{R}_i$. Then x is contained in a crescent.*

Proof. Let $x_i \in \bigcup_j \mathcal{R}_j$ be a sequence converging to x. We may assume that x is not an element of any $\mathcal{R}_j$.

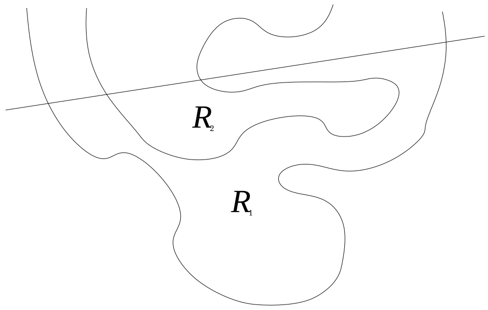

FIGURE 1. A 2-dimensional section of crescents $\mathcal{R}_1$ and $\mathcal{R}_2$ where $\mathcal{R}_2$ is 1-nested and is innermost.

Using geometric convergence, there exists a totally geodesic hypersurface P through x and a geometric limit of a sequence of $I_{\mathcal{R}_j}$ converging to a subset D in P.

Then D is separating in $\tilde{M} - \tilde{\Sigma}$. If not, there exists a simple closed curve γ in $\tilde{M}$ meeting D only once, which means $I_{\mathcal{R}_j}$ for a sufficiently large j meets γ only once as well.

Now, $\tilde{M} - D$ may have more than one components. Since $x \in D$, we take an open domain L bounded by D and a subset of $\tilde{\Sigma}$ and whose closure contains x. The closure of L is a crescent containing x since R_j is a geometric limit of a sequence of compact crescents $R_{j,i}$, $i = 1, 2, 3, \ldots$ and we can choose a subsequence from these converging to L. □

Definition 3.13. Let Σ_i be a sequence of nicely imbedded surfaces isotopic to Σ. By the method of above Proposition 3.12, for any sequence of crescents R_i for surface Σ_i with a geometric limit K with nonempty interior, it follows that there is a maximal crescent R contained in K with I_R in the geometric limit of a subsequence of totally geodesic hypersurfaces P_i containing I_{R_i}. R is said to be *generalized limit* of the sequence R_i.

The definition is needed since we might have some parts of domain degenerating to lower-dimensional objects. The generalized limit might not be unique. In fact, there could be two with disjoint interiors.

Proposition 3.14. *Suppose that a sequence of crescents R_i converges to a crescent R in the generalized sense. Then α_R is a subset of a limit of any subsequence of* $\mathrm{Cl}(\alpha_{R_i})$.

Proof. The union of a totally geodesic 2-dimensional domain and $\mathrm{Cl}(\alpha_{R_i})$ is the boundary of R_i. The limit of a subsequence of the boundary of R_i converges to a subset containing the boundary of R. □

As usual, we assume that the holonomy group of Σ does not consist strictly of parabolic or elliptic or identity elements.

A *size* of a crescent is the supremum of the distances $d(x, \alpha_{\mathcal{R}})$ for $x \in I_{\mathcal{R}}$. We show that this is globally bounded by a constant depending only on Σ.

First, a complementary result is proved:

Lemma 3.15. *Every $x \in \alpha_{\mathcal{R}}$ satisfies*

$$d(x, I_{\mathcal{R}}) \leq N$$

for a uniform constant N depending on M and Σ only.

Proof. If not, since there is a compact fundamental domain in $\tilde{\Sigma}$, using deck transformations acting on $\tilde{\Sigma}$, we obtain a sequence of bigger and bigger compact crescents where the corresponding sequence of the I-parts leave any compact subset of $\tilde{M}$ and the corresponding sequence of α-parts meets a fixed compact subset of $\tilde{M}$. Therefore, we form a subsequence of the developing images of the I-parts converging to a point of the sphere at infinity of $\mathbb{H}^3$.

Let $\mathcal{R}_i$ be the corresponding crescents. Then $\alpha_{\mathcal{R}_i}$ is a subsurface with boundary in the I-parts, and $\alpha_{\mathcal{R}_i}$ contains any compact subset of $\tilde{\Sigma}$ eventually.

Let c be a closed curve in Σ with nonidentity holonomy. Let $\tilde{c}$ be a component of its inverse image in $\tilde{\Sigma}$. Since $\tilde{c}$ must escape any compact subset of $\tilde{\Sigma}$, $\tilde{c}$ escape $\alpha_{\mathcal{R}_i}$. Thus, $\tilde{c}$ must meet all $I_{\mathcal{R}_i}$ for i sufficiently large. Since the developing image of $\tilde{c}$ has two well-defined endpoints, this means that the limit of the sequence of I-parts must contain at least two points, a contradiction. □

Proposition 3.16. *Let M and Σ be as above. Then $d(x, \alpha_{\mathcal{R}})$ for $x \in I_{\mathcal{R}}$ is uniformly bounded above by a constant depending only on M and Σ and, hence, there is an upper bound to the size of a crescent. There is an upper bound to the folding number of crescents depending only on Σ and M.*

Proof. By above Lemma 3.15, $\alpha_{\mathcal{R}}$ is in the N-neighborhood A of $I_{\mathcal{R}}$. We draw perpendicular geodesics to $I_{\mathcal{R}}$ foliating a subset of $\tilde{M}$. Each geodesic must meet $\alpha_{\mathcal{R}}$ eventually in A since otherwise $\alpha_{\mathcal{R}} \cup I_{\mathcal{R}}$ do not form a boundary of a domain. Therefore, the first statement is proved.

The second statement follows from the fact that the perpendicular geodesic meets $\tilde{\Sigma} \cap A$ since $\tilde{\Sigma}$ is properly imbedded. □

Corollary 3.17. *Suppose that $\mathcal{R}$ is an outer crescent. Then there exists a convex vertex in $\alpha_{\mathcal{R}}$. That is, the set of vertices of $\mathcal{R}$ cannot consist only of concave vertices and saddle vertices.*

Proof. We choose a function f so that $I_{\mathcal{R}}$ is contained in the zero set and other level sets are totally geodesic. f is bounded on $\alpha_{\mathcal{R}}$ by Proposition 3.16. Hence, there is a maximum point. By tilting the totally geodesic plane by a little, we obtain a strictly convex vertex. □

3.3. **Highest-level crescents.** Given Σ, there is an upper bound to the folding-number of all crescents associated with Σ by Proposition 3.16. We call the maximum the *highest folding number* of Σ. We perturb Σ to minimize the highest folding number which can change only by ± 1 under perturbations. After this, the folding number is constant under small perturbations of Σ. If there are no crescents, then the *folding number* of Σ is defined to be -1.

Also, the union of all crescents for $\tilde{\Sigma}$ is in a uniformly bounded neighborhood of $\tilde{\Sigma}$ with the bound depending only on Σ.

We say that a 0-folded crescent $\mathcal{R}$ is a *highest-level* crescent if it is an innermost crescent of an n-folded crescent $\mathcal{R}'$ where n is the highest-folding number of $\tilde{\Sigma}$ and $\mathcal{R}$ is the innermost one that achieves the highest-level.

Suppose that $\mathcal{R}$ is a compact highest-level crescent. Let $A_1, \ldots, A_n$ be components of $I_{\mathcal{R}}$ with pinched points removed. Recall that $I_{\mathcal{R}}$ lies in a totally geodesic hypersurface. The outermost pinched simple closed curve α_i in the boundary of A_i has a trivial holonomy. Since $\mathcal{R}$ is of highest-level, α_i is an innermost curve itself or bounds some closed curves in $\tilde{\Sigma} \cap \partial I_{\mathcal{R}}$. If each α_i is as in the former case, then $\mathcal{R}$ is said to be an *innermost ball-type crescent*, which is homeomorphic to a 3-ball by the incompressibility condition for Σ.

We have the following important definition:

Definition 3.18. The *outer-folding number* of $\tilde{\Sigma}$ is the maximum of the ambient folding number of an outer highest-level crescents. The *inner-folding number* of $\tilde{\Sigma}$ is the maximum of the ambient folding number of an inner highest-level crescents. The outer-folding number is -1 if there are no highest-level outer crescents. The inner-folding number is -1 if there are no highest-level inner crescents.

By Proposition 3.16, the numbers are finite. The maximum of the both of the numbers are the folding number of $\tilde{\Sigma}$.

3.4. **Outer- and inner-contact points.** We can classify the points of $\tilde{\Sigma} \cap I_{\mathcal{R}}$ when $\tilde{\Sigma}$ is well-triangulated: A point of it is an *outer-contact point* if the point is not in $\partial I_{\mathcal{R}}$ and has a neighborhood in $\tilde{\Sigma}$ outside $\mathcal{R}^o \cup \alpha_{\mathcal{R}}$; a point is an *inner-contact point* if the point is not in $\partial I_{\mathcal{R}}$ and has a neighborhood in $\tilde{\Sigma}$ contained in $\mathcal{R}$. A point is either an outer-point or an inner point or can be both.

The following classifies the set of outer-contact points. (A similar result holds for the set of inner points except for (d).)

Proposition 3.19. *Suppose that $\tilde{\Sigma}$ is well-triangulated. For a highest-level crescent $\mathcal{R}$, the intersection points of $I^o_{\mathcal{R}}$ and $\tilde{\Sigma}$ are either outer-contact points or inner-contact points. The set of outer-contact points of $I_{\mathcal{R}}$ for a highest-level crescent $\mathcal{R}$ is one of the following:*

- (a) *a union of at most three isolated points.*
- (b) *a union of at most one point and a segment or a segment with some endpoints removed.*
- (c) *a union of two segments with a common endpoint with some of the other endpoints removed.*
- (d) *a triangle with some of the vertices or a boundary segments removed.*

The same statement are true for inner-contact points.

Proof. The set of outer-contact points is obviously a union of open cells of dimension 0, 1, or 2. The vertices of the closure of each objects are the vertices of $\tilde{\Sigma}$.

This follows from the general position of vertices of $\tilde{\Sigma}$. If the set of outer-contact points are union of 0- and 1-dimensional objects, then (a), (b), or (c) follows.

If there is a 2-dimensional object, then it contains an open triangle and there cannot be any other objects not in the closure of it.

Either the interior of an edge is in $\partial I_{\mathcal{R}}$ or it is disjoint from $\partial I_{\mathcal{R}}$ since otherwise we might have four coplanar vertices. □

Definition 3.20. Given a crescent $\mathcal{R}$, we define $I^O_{\mathcal{R}}$ to be the $I_{\mathcal{R}}$ with the pinched points, boundary points in the I-part hypersurface, and the segments and triangles in the outer-contact set as above removed. (We don't remove the isolated points.)

Note that $I^O_{\mathcal{R}}$ may not equal the topological interior $I^o_{\mathcal{R}}$ of $I_{\mathcal{R}}$ in the totally geodesic hypersurface.

Remark 3.21. We remark that in cases (b), (c), (d), the set of outer-contact points (inner-contact points) can separate $I_{\mathcal{R}}$. The set is a *disconnecting set of outer-contact points.*

3.5. **Closed geodesics and crescents.**

Proposition 3.22. *Suppose that c is a straight closed geodesic in M not meeting Σ. Let $\mathcal{R}$ be a highest-level crescent. Then*

- *Each component $\tilde c$ of the inverse image of c in $\tilde M$ does not meet $\mathcal{R}$ in its interior and the α-parts.*
- *$\tilde c$ could meet $\mathcal{R}$ in its I-part tangentially and hence be contained in the I-part. In this case, $\mathcal{R}$ is not compact.*
- *If l is a geodesic in $\tilde M$ eventually leaving all compact subsets, then the above two statements hold as well. In particular, this is true if $\tilde M$ is a special hyperbolic manifold and l ends in the limit set of the holonomy group associated with $\tilde M$.*

Proof. If $\tilde c$ meets the α-part of $\mathcal{R}$, then $\tilde c$ meets the interior of $\mathcal{R}$.

If a portion of $\tilde c$ meets the interior of $\mathcal{R}$, then $\tilde c \cap \mathcal{R}$ is a connected arc, say l since $\mathcal{R}$ is a closure of a component cut out by a totally geodesic hyperplane in $\tilde M - \tilde\Sigma$ –(*).

Since $\tilde c$ is disjoint from $\tilde \Sigma$, both endpoints of l must be in $I^O_{\mathcal{R}}$ or in $\mathbf{S}^2_\infty \cap \overline{I_{\mathcal{R}}}$ for the closure $\overline{I_{\mathcal{R}}}$ of $I_{\mathcal{R}}$ in the compactified $\mathbb{H}^3 \cup \mathbf{S}^2_\infty$. If at most one point of l is in $I_{\mathcal{R}}$, then l is transversal to $I_{\mathcal{R}}$ and the other endpoints l must lie in $\alpha_{\mathcal{R}}$ by (*). This is absurd.

If at least two points of l are in $I_{\mathcal{R}}$, then l is a subset of $I_{\mathcal{R}}$. Since $\tilde c$ is disjoint from $\tilde\Sigma$, it follows that $\tilde c$ is a subset of $I_{\mathcal{R}}$, and $\mathcal{R}$ is not compact.

The only remaining possibility for l is the third one that l ends in $\mathbf{S}^2_\infty$, l is a subset of $\mathcal{R}^o$, $\tilde c = l$, and $\mathcal{R}$ is noncompact. Suppose that l is a subset of $\mathcal{R}^o$. A point of $\mathcal{R}^o$ is a point of some compact crescent R_i in $\mathcal{R}$ since a noncompact crescent is a generalized limit of compact ones. Therefore, l meets a compact crescent as above, which was shown to be not possible above. This proves the first two items.

The third item follows similarly.

□

3.6. **A highest-level crescent is included in a secondary highest-level crescent.**

Proposition 3.23. *Let $\tilde\Sigma$ be well-triangulated and $\mathcal{R}$ be a highest-level crescent. Then there exists an innermost crescent $\mathcal{R}'$ containing $\mathcal{R}$ so that*

- *$I_{\mathcal{R}'}$ is connected and has no pinched points or a disconnecting set of outer-contact points, i.e., $I^O_{\mathcal{R}'}$ is connected.*

- *The closure of $\alpha_{\mathcal{R}'}$ is homeomorphic to $I_{\mathcal{R}'}$ and is isotopic to $I_{\mathcal{R}'}$ in $\mathcal{R}'$.*
- *$\mathcal{R}'$ is homeomorphic to $I_{\mathcal{R}'} \times [0,1]$.*
- *$\mathcal{R}'$ is inner-most.*

Proof. We assume the hypothesis 3.1.

The basic idea is to add some domains by disks in $\tilde{\Sigma}$ obtained by incompressibility of $\tilde{\Sigma}$.

By Proposition 3.22, the interior of $\mathcal{R}$ is disjoint from any lifts of $\mathbf{c_1}, \dots, \mathbf{c_n}$.

The first step is to cut off by the I-part hypersurface to simplify the starting crescent: Suppose that $\mathcal{R}$ is compact to begin with. Then $\mathcal{R}$ is disjoint from the lifts of $\mathbf{c_1}, \dots, \mathbf{c_n}$ since $\tilde{\Sigma}$ is disjoint from these. We let $\mathcal{S}$ be a crescent obtained from $\mathcal{R}$ by cutting through the I-part hypersurface P and taking the closure of a component of $\mathcal{R} - \tilde{\Sigma} - P$. We say that $\mathcal{S}$ is a *cut-off* crescent from $\mathcal{R}$. (The ambient folding number and the folding number may change by cutting off.)

Again $\mathcal{S}$ is innermost since otherwise we will have points in $\mathcal{S}$ in the other side of $\tilde{\Sigma}$ than those of $\mathcal{R}$ but $\mathcal{S} \subset \mathcal{R}$.

The ambient level of $\mathcal{S}$ is equal to that of $\mathcal{R}$: To reach a point in $\mathcal{R}$ from any point of $\alpha_{\mathcal{R}'}$ for a super-crescent $\mathcal{R}'$, one needs to traverse on a generic path at least the highest-level times of $\hat{\Sigma}$ in $\mathcal{R}'$. $\mathcal{S}$ is a sub-crescent of the cut-off crescent $\mathcal{S}'$ of $\mathcal{R}'$. Since a path in $\mathcal{S}'$ is a path in $\mathcal{R}'$ and the ambient level is a maximum value, the ambient level may increase. However, since we are already at the highest-level, the equality holds.

We introduce a height function h on $\mathcal{S}$ defined by introducing a parameter of hyperbolic hypersurfaces perpendicular to a common geodesic passing through $I_{\mathcal{S}}$ in the perpendicular manner. (It will not matter which parameter we choose). We may assume that h is Morse in the combinatorial sense. (See Freedman-McMullen [13].)

Now, we fatten up $\mathcal{S}$ a bit so that we can work with surfaces instead of just topological objects: If $I_{\mathcal{S}}$ does not meet any lifts of $\mathbf{c_1}, \dots, \mathbf{c_n}$, let $N_\epsilon(\mathcal{S})$ be the neighborhood of $\mathcal{S}$ in the closure of the component of $\tilde{M} - \tilde{\Sigma}$ containing the interior of $\mathcal{S}$.

We let $N_\epsilon(\alpha_{\mathcal{S}})$ to be the intersection of $\tilde{\Sigma}$ with $N_\epsilon(\mathcal{S})$. $N_\epsilon(\mathcal{S})$ can be chosen so that $N_\epsilon(\alpha_{\mathcal{S}})$ in $\tilde{\Sigma}$ becomes an open surface compactifying to a surface. There exists a part $\mathcal{I}$ in of the boundary which is a complement in the boundary of $N_\epsilon(\mathcal{S})$ of $N_\epsilon(\alpha_{\mathcal{S}})$ and lies on a properly imbedded surface P' perturbed away from the I-part hypersurface P of $\mathcal{S}$.

Topologically, $N_\epsilon(\alpha_{\mathcal{S}})$ is homeomorphic to a surface possibly with 1-handles attached from $\alpha_{\mathcal{S}}$ and $\mathcal{I}$ is obtained from $I_{\mathcal{S}}$ by removing 1-handles corresponding to the pinched points or the disconnecting set of outer-contact points.

Now we aim to show that $N_\epsilon(\mathcal{S})$ is a compression body with $N_\epsilon(\alpha_{\mathcal{S}})$ as the compressible surface in the boundary:

We may extend h to an ϵ-neighborhood of $\mathcal{S}$, which may introduce only saddle type singularity in $N_\epsilon(\alpha_{\mathcal{S}}) - \alpha_{\mathcal{R}}$ where there are only one handles. We modify h so that $\mathcal{I}$ to be in the zero level of h and $h < 0$ in the interior of $\mathcal{S}$.

We show that there are no critical points of positive type for h: If there is a critical point of h with locally positive type where $h < 0$, then we see that in fact there exists a crescent of higher level near the critical point. The critical point is actually below the original I-part hypersurface since only critical points above the I-part hypersurface are of saddle type. The crescent is obtained by a totally geodesic hyperplane containing the level set slightly above the critical point. The level of the new crescent is one more than that of $\mathcal{S}$, which is greater than the highest level, which is absurd.

Since there are no critical point of positive type, $\pi_1(N_\epsilon(\alpha_\mathcal{S})) \to \pi_1(N_\epsilon(\mathcal{S}))$ is surjective as shown by Freedman-McMullen [13]. There exists a compression body in $N_\epsilon(\mathcal{S})$ with a boundary $N_\epsilon(\alpha_\mathcal{S}) \cup S'$ for an incompressible surface S' in the interior of $\mathcal{S}$. Since every closed path in $N_\epsilon(\mathcal{S})$ is homotopic to one in $N_\epsilon(\alpha_\mathcal{S})$, it follows that S' is parallel to I. Hence, $N_\epsilon(\mathcal{S})$ is a compression body homeomorphic to $\mathcal{I}$ times an interval and 1-handles attached at disks disjoint from $\mathcal{I}$. ($\mathcal{S}$ is essentially obtained by pinching some points of $\mathcal{I}$ together and pushing down a bit.)

Next, we reduce the number of components of $\mathcal{I}$:

Suppose now that $\mathcal{I}$ is not connected. This means that there are 1-handles attached to $\mathcal{I}$ times an interval joining the components below $\mathcal{I}$. Then $N_\epsilon(\mathcal{S})$ has a compressing disk D for $N_\epsilon(\alpha_\mathcal{S})$ dual to the 1-handles. Since ∂D bounds a disk D' in $\tilde{\Sigma}$ by the incompressibility of $\tilde{\Sigma}$, the closed curve ∂D is separating in $\tilde{\Sigma}$. Consequently also, $N_\epsilon(\alpha_\mathcal{S})$ is a planar surface.

The irreducibility of $\tilde{M}$ tells us that D and D' bound a 3-ball B in the closure of a component of $\tilde{M} - \tilde{\Sigma}$. Then B contains at least one component of I.

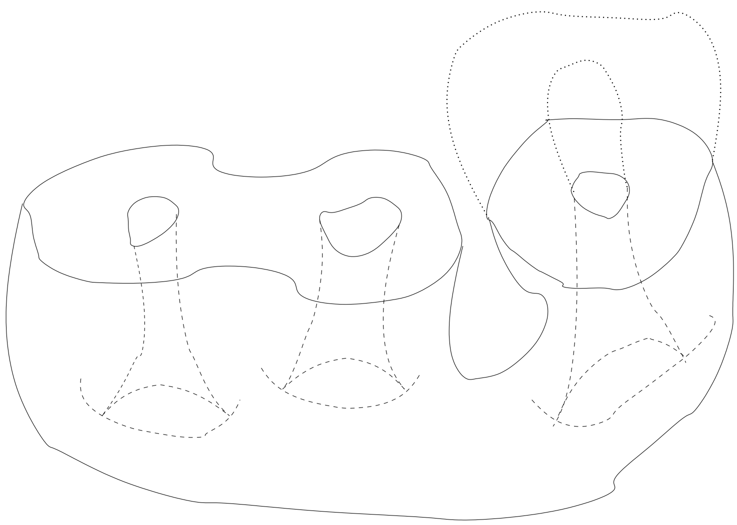

FIGURE 2. The dashed arc indicates the tube from the bottom and the dotted arcs indicated the disks to be attached to the α-parts

By taking a maximal family of compressing disks dual to the 1-handles and regarding the components of the complements as vertices, we see that the 1-handles do not form a cycle. Therefore, we choose the compressing disk D of $N_\epsilon(\alpha_\mathcal{S})$ to be the one such that D and corresponding disk D' in $\tilde{\Sigma}$ bounds a 3-ball B containing a unique component of I.

We take a union of B with $N_\epsilon(\mathcal{S})$. Then it is an N_ϵ-neighborhood of a crescent $\mathcal{S}'$, which contains $\mathcal{S}$ and $\alpha_{\mathcal{S}'}$ containing $\alpha_\mathcal{S}$ and $I_{\mathcal{S}'}$, a subset of $I_\mathcal{S}$.

In fact, $\mathcal{S}'$ is a union of $\mathcal{S}$ and B' with some parts in $\mathcal{I} - \mathcal{I}'$ removed. Also, clearly, $N_\epsilon(\mathcal{S}')$ is a compression body since it is obtained by taking a union of a cell with a compression body obtained from $N_\epsilon(\mathcal{S})$ by splitting along D.

By induction, we obtain a crescent $\mathcal{R}''$ with $I_{\mathcal{R}''}$ in $I_{\mathcal{S}}$ and the surface I' corresponding to I connected. We say that $\mathcal{R}''$ is *derived from* $\mathcal{S}$. Since $\mathcal{R}''$ is homeomorphic to a compression body, $\mathcal{R}''$ is homeomorphic to I times an interval since there are no compressing disks.

If there are any pinched points in $I_{\mathcal{R}''}$ or disconnecting outer-contact points, then I' would be disconnected. $\alpha_{\mathcal{R}''}$ has a closure that is a surface since there are no pinching points.

Since $\mathcal{R}''$ is an I-bundle, it follows that the closure of $\alpha_{\mathcal{R}''}$ and $I_{\mathcal{R}''}$ are homeomorphic surfaces.

Also, $\mathcal{R}''$ is innermost: suppose not. Then there exists a component C of $\mathcal{R}'' - \tilde{\Sigma}$ so that a generic path in $\mathcal{R}''$ from $\alpha_{\mathcal{R}''}$ may meet $\tilde{\Sigma}$ more than once. Then $C \cap \mathcal{R}$ is again a component of $\mathcal{R} - \tilde{\Sigma}$, which is a contradiction.

Now, we go to the final step. Recall that $\mathcal{S}$ was a cut-off crescent from the original $\mathcal{R}$. If there were more than one cut-off crescents $\mathcal{S}$, then we obtain $\mathcal{R}''$ for each $\mathcal{S}$. Suppose that two cut-off crescents $\mathcal{S}$ and $\mathcal{S}'$ adjacent from opposite sides of some of the components of $I_{\mathcal{S}}$. Since the corresponding $\mathcal{R}''$ and $\mathcal{R}'''$ containing $\mathcal{S}$ and $\mathcal{S}'$ respectively does not have any pinched points or separating outer-contact edges, the unique components of $I_{\mathcal{R}''}$ and $I_{\mathcal{R}'''}$ either agree or are disjoint from each other. $\mathcal{R}''$ and $\mathcal{R}'''$ cannot be adjacent from opposite side since we can then form a compact component of $\tilde{\Sigma}$ otherwise. It follows that one of $\mathcal{R}''$ and $\mathcal{R}'''$ is a subset of the other.

Hence, choosing maximal ones among such derived crescents, we see that the conclusions of the proposition hold if $\mathcal{R}$ is compact. The final result is a product of its I-part since the final compression body has no 1-handles. This completes the proof in case $\mathcal{R}$ is compact.

If $\mathcal{R}$ is noncompact, we follow as before but we choose $N_\epsilon(\mathcal{R})$ to be tapered down near infinity.

Recall that $\tilde{M}$ is 2-convex, i.e., the boundary $\partial\tilde{M}$ has only convex or saddle vertices. Since $\tilde{M}$ can be considered a 2-convex affine manifold, recall the main result of [10] that any disk with a boundary in a totally geodesic hypersurface P bounds a disk in P.

Only one component of $\mathcal{I}$ maybe noncompact since the boundary of a compressing disk must bound a compact disk in $\tilde{\Sigma}$: Otherwise, we have a simple closed curve c in $\alpha_{\mathcal{R}}$ which separates the two noncompact components of $\mathcal{I}$ and bounds a compact disk D in $\tilde{M}$ in one side of $\tilde{\Sigma}$. D can be pushed inside $\mathcal{R}$ by the above paragraph. Thus D cannot meet any of the lift of $\mathbf{c_1}, \ldots, \mathbf{c_n}$. Since $\tilde{\Sigma}$ is incompressible in the complement of these lifts, it follows that c also bounds a compact disk in $\tilde{\Sigma}$. This is a contradiction since c separates the two noncompact components of $\mathcal{I}$.

Going a back to finding the ambient crescent with desired properties, components of $\mathcal{I}$ involved in cell-attaching operation as above are compact. So the arguments for noncompact $\mathcal{R}$ are the same as above. □

3.7. **Properties of secondary highest-level crescents.** If a 0-folded crescent $\mathcal{S}$ contains a highest-level crescent $\mathcal{R}$ so that $I^O_{\mathcal{S}}$ is connected and is included in $I^O_{\mathcal{R}}$, then we say that $\mathcal{S}$ is a *highest-level* crescent as well. (Actually, it may not be highest-level since $\mathcal{S}$ may not necessarily be contained in an n-folded crescent but only a part of it.) More precisely, it is a *secondary highest-level crescent*.

Corollary 3.24. *Let $\mathcal{R}$ be a secondary highest-level crescent. Then the statements of Proposition 3.22 hold for $\mathcal{R}$ as well.*

Proof. The proof is exactly the same as that of Proposition 3.22. □

By taking a nearby crescent inside by changing the I-part hypersurface inwards, we see that a highest-level crescent could be *generically chosen* so that the crescent is compact, the I-part and the α-part are surfaces, and I^O-part is truly the interior of the I-part.

Corollary 3.25. *Let $\mathcal{R}$ be the compact secondary highest-level outer (resp. inner) crescent that is generically chosen. Then $\mathcal{R}$ is homeomorphic to the closure of $\alpha_{\mathcal{R}}$ times I, and $I^O_{\mathcal{R}}$ is isotopic to $\alpha_{\mathcal{R}}$ by an isotopy inside $\mathcal{R}$ fixing the boundary of $I_{\mathcal{R}}$.*

3.8. **Intersection properties of highest-level crescents.** We say two crescents $\mathcal{R}$ and $\mathcal{S}$ *face each other* if $I_{\mathcal{R}}$ and $I_{\mathcal{S}}$ agree with each other in some 2-dimensional part and have disjoint one-sided neighborhoods.

Proposition 3.26. *Assume $\tilde{\Sigma}$ is well-triangulated as above. If two highest-level outer (resp. inner) crescents $\mathcal{R}$ and $\mathcal{S}$ face each other, then there exists a (secondary) highest-level outer (resp. inner) crescent $\mathcal{T}$ with connected $I^O_{\mathcal{T}}$ containing both.*

Proof. We replace $\mathcal{R}$ and $\mathcal{S}$ by secondary highest-level crescents with connected I^O-parts. The replacements still face each other or one becomes a subset of another since I-parts are unique boundary sets. In the second case, we are done.

Since $I^O_{\mathcal{R}}$ and $I^O_{\mathcal{S}}$ meet in open subsets, either they are identical or we may assume without loss of generality that the boundary ∂L of their intersection L in $I^O_{\mathcal{R}}$ is not empty. ∂L is a subset of $\tilde{\Sigma}$ and is a 1-complex consisting of pinched arcs. ∂L is a set of outer-contact points of $\mathcal{S}$ since a neighborhood of ∂L in $\tilde{\Sigma}$ must be above $\mathcal{S}$. However, then ∂L must be disjoint from $I^O_{\mathcal{S}}$ since $I^O_{\mathcal{S}}$ is disjoint from outer-contact set by definition. Therefore, $I^O_{\mathcal{S}} = I^O_{\mathcal{R}}$, and $\mathcal{S} \cup \mathcal{R}$ is bounded by a component subsurface of $\tilde{\Sigma}$, which is absurd.

□

Definition 3.27. Two secondary highest-level outer (resp. inner) crescents $\mathcal{R}$ and $\mathcal{S}$ are said to meet *transversally* if $I_{\mathcal{R}}$ and $I_{\mathcal{S}}$ meet in a union of disjoint geodesic segment J, $J \neq \emptyset$, mapping into a common geodesic in $\mathbb{H}^3$, in a transversal manner such that

- The the closure $\nu_{\mathcal{R}}$ of the union of the components of $I_{\mathcal{R}} - J$ in one-side is a subset of $\mathcal{S}$ and the closure $\nu_{\mathcal{S}}$ of the union of those of $I_{\mathcal{S}} - J$ is a subset of $\mathcal{R}$.
- The intersection $\mathcal{R} \cap \mathcal{S}$ is the closure of $\mathcal{S} - \nu_{\mathcal{R}}$ and conversely the closure of $\mathcal{R} - \nu_{\mathcal{S}}$.
- The intersection $\alpha_{\mathcal{R}} \cap \alpha_{\mathcal{S}}$ is a union of components of $\alpha_{\mathcal{R}} - \nu_{\mathcal{S}}$ in one-side of $\nu_{\mathcal{S}}$ and, conversely, is a union of components of $\alpha_{\mathcal{S}} - \nu_{\mathcal{R}}$ in one side of $\nu_{\mathcal{R}}$.

- $\alpha_{\mathcal{R}} \cup \alpha_{\mathcal{S}}$ is an open surface in $\tilde{\Sigma}$.

Proposition 3.28. *Given two secondary highest-level outer (resp. inner) crescents $\mathcal{R}$ and $\mathcal{S}$, there are the following mutually exclusive possibilities:*

- *$\mathcal{R}$ and $\mathcal{S}$ do not meet in $\tilde{M} - \tilde{\Sigma}$.*
- *$\mathcal{R} \subset \mathcal{S}$ or $\mathcal{S} \subset \mathcal{R}$.*
- *$\mathcal{R}$ and $\mathcal{S}$ meet transversally.*

Proof. The reasoning is exactly the same as [8] and [9] in dimension two or three. □

4. The crescent-isotopy

The purpose of this section is to prove Theorem C and Corollary D: Assume that Σ is a closed well-triangulated surface in M which is incompressible in M with a number of closed geodesics removed. In this section, we will describe our crescent-isotopy steps of $\tilde{\Sigma}$. Let $\tilde{\Sigma}$ have a folding number n achieved by outer and/or inner crescents. We may assume without loss of generality that there is an outer highest level crescent coming from an outer or inner crescent. Using such outer crescents, we move first to reduce the outer level by 1.

Subsection 4.1: The first step is to truncate our surface $\tilde{\Sigma}$ along vertices, edges or triangles in order to make highest-level crescents not have outer-contact points. This may make $\tilde{\Sigma}$ only triangulated; however, we make small perturbations of vertices to make it well-triangulated.

Subsection 4.2: We use the secondary highest-level crescents to move isotopy $\tilde{\Sigma}$ by isotopying the closure of the α-parts to the I-parts. One of the outer or inner levels strictly decreases.

Subsection 4.3: The result may have some parts which are pleated with infinitely many and/or infinitely long pleating geodesics. We perturb these parts so that we end up with a triangulated surface but with levels not increasing

Subsection 4.4: We do the above for the level n for inner highest-level crescents. This will decrease the inner level. (The steps are just the same if we reverse the orientation of $\tilde{\Sigma}$.) We keep doing this until our outer and inner level become -1 and we have obtained an saddle-imbedded surface isotopic to $\tilde{\Sigma}$, which proves Theorem C. Finally, we will prove Corollary D.

4.1. Small truncation moves.

4.1.1. *Isotopies.* First, we need:

Lemma 4.1. *Suppose that $\tilde{\Sigma}$ has been isotopied in the outward direction by a sufficiently small amount and $\mathcal{R}$ is an outer crescent. Then there exists a crescent $\mathcal{R}'$ sharing the I-part hypersurface with $\mathcal{R}$ and differs from $\mathcal{R}$ by isotopying the α-part only. Conversely, if $\tilde{\Sigma}$ has been isotopied in the inward direction and $\mathcal{R}$ is an inner crescent, the same can be said.*

Proof. Straightforward. □

We say that $\mathcal{R}'$ is *isotopied from $\mathcal{R}$ with the I-part preserved.* (Of course, this is not literally so.)

4.1.2. *Small truncations.* We may "truncate" Σ at convex vertices and $\tilde{\Sigma}$ correspondingly and perturb: Let v be a convex or concave vertex and H a local half-open ball at v containing $\tilde{\Sigma}$ locally with the side F passing through v.

(a) We may move F inside by a very small amount and then truncate $\tilde{\Sigma}$ using the displaced F and add the trace disk T of the truncation to the surface $\tilde{\Sigma}$.
(b) Then we introduce some equivariant triangulation of T of the truncation and the truncated $\tilde{\Sigma}$ without introducing vertices in the interior of T.
(c) We will have to do this for each vertex which is in the orbit of v so that resulting $\tilde{\Sigma}$ is still equivariant.
(d) Finally, we perturb all the vertices of $\tilde{\Sigma}$ by a sufficiently small amount. Here, the perturbations must be so that the normal vectors to the totally geodesic triangles also move by small amounts, i.e., the normal vectors move continuously as well as the vertices themselves. Moreover, no triangle or edge degenerates to a lower-dimensional object.

The three steps (a)-(c) together are called the *small truncation move.* Together with the final step (d), the move is called the *perturbed small-truncation move.*

For an edge or a triangle e, let F be a neighborhood of e in totally geodesic plane containing e where $F - e$ lies outside $\tilde{\Sigma}$. We may move F inside by a sufficiently small amount and truncate $\tilde{\Sigma}$. The rest is similar to the vertex case. They are also called *small truncation moves* along edges or triangles. After the perturbation, we call the move *perturbed small-truncation move.*

We denote by Σ^ϵ the perturbed Σ where the trace disks are less than an ϵ-distance away from the respective convex vertices and the normal vectors to the triangles are also less than ϵ-distances from the original vertices. Here, we assume that during the perturbations Σ^ϵ is isotopied from Σ and the convexity of the dihedral angles do not change under the isotopy. Thus, if an edge or a vertex is convex after being born, it will continue to be so as $t \to 0$ and as t grows from 0.

We may also assume that the convex vertex move is equivariant on $\tilde{\Sigma}$, i.e., the isotopy is equivariant.

4.1.3. *Small truncation moves and crescents.* An *isotopy* of a crescent as we deform Σ is a one-parameter family of crescents $\mathcal{R}_t$ with α-parts in Σ. The above small truncation moves are isotopies.

We say that a crescent *bursts* if fixing the totally geodesic hypersurface containing the I-part of it and isotopying the α-parts in the isotopied Σ cannot produce a crescent isotopied from the original one.

Such an event happens when a parameter of vertices, edges, or triangles of $\tilde{\Sigma}$ go below the fixed totally geodesic hypersurface from the point of view of the crescent. Of course, a vertex could be a multivertex and all of the new vertices go down. The edge should be on the face that meets the I-parts of the crescents and the vertex on the edge that meets the I-part of the crescent. The event could happen simultaneously but the generic nature of the move shows that at most four vertex submersions, at most three edge submersions, at most two vertices and one edge submersions, or triangle-, edge- or vertex-submersions can happen simultaneously. (Basically, at most four vertices can lie on a totally geodesic plane while deforming.) Moreover, at the event, the vertex and the edge must be in the I-part of the crescent and the triangles of Σ must be placed in certain way in order that the bursting to take place.

Proposition 4.2. *Suppose that $\tilde{\Sigma}$ is well-triangulated. Under a small truncation move in the outer direction, we can isotopy $\tilde{\Sigma}$ to triangulated $\tilde{\Sigma}^\epsilon$ (equivariantly) so that*

(i) *each outer crescent moves into itself by moving the α-part in the outer direction and preserving the I-part hypersurface.*
(ii) *each inner crescent moves into itself union the ϵ-neighborhood of $\tilde{\Sigma}$ by moving the I-part hypersurface in the outer direction or preserving the I-part hypersurface.*

Under a small truncation move in the inner direction, we can deform

(iii) *each inner crescent into itself by moving the α-part in the inner direction and preserving the I-part hypersurface.*
(iv) *each outer crescent into itself union the ϵ-neighborhood of $\tilde{\Sigma}$ by moving the I-part hypersurface in the inner direction or preserving the I-part hypersurface.*

All crescents of $\tilde{\Sigma}^\epsilon$ can be obtained in this way. The highest folding number may decrease only under a convex vertex move, and the union of crescents of all levels strictly decreases under the moves.

Proof. Essentially, the idea is that the move can only "decrease" the associated crescents.

Let $\mathcal{R}$ be an outer crescent and $\tilde{\Sigma}$ moved in the outer direction. Lemma 4.1 implies (i).

Let $\mathcal{R}$ be an inner crescent and $\tilde{\Sigma}$ be moved in the outer direction. Then again an isolated submerging vertex is a convex vertex. In this case, we move the I-part inward so that the submerging vertex stay on the boundary of the I-part. Other cases are treated similarly. This proves (ii).

(iii) and (iv) correspond to (i) and (ii) respectively if we change the orientation of Σ.

To show that all crescents of $\tilde{\Sigma}^\epsilon$ can be obtained in this way: Given an outer crescent for $\tilde{\Sigma}^\epsilon$, we reverse the truncation move. If the I-part of a crescent avoids the trace disks of the truncation moves, then we simply isotopy the α-parts only.

The trace surface has only concave vertices and saddle-vertices.

Let us start by reversing a vertex truncation move: Let P' be a local totally geodesic hypersurface truncating the stellar neighborhood of a convex vertex v of $\tilde{\Sigma}$ at some small distance from v but large compare to our isotopy move distance. Suppose that v were involved in the convex truncation move. We may assume that P' is parallel to the truncating totally geodesic hyperplane near v used to obtain $\tilde{\Sigma}^\epsilon$.

Clearly, P' and a small stellar neighborhood of v in $\tilde{\Sigma}^\epsilon$ bounds a small polyhedron R^ϵ. Let R be the small polyhedron bounded by P' and $\tilde{\Sigma}$.

Suppose that the I-part of a crescent $\mathcal{R}$ for $\tilde{\Sigma}^\epsilon$ are contained in R^ϵ. Then it is contained in a crescent whose I-part meets what are outside the part truncated by P'. It is sufficient to show that the ambient crescent is obtained by the above methods.

We thus assume without loss of generality that the I-part of a crescent $\mathcal{R}$ for $\tilde{\Sigma}^\epsilon$ meets what are outside the part truncated by P'. If the I-part does not meet the trace surface, then we only change the α-parts to obtain a crescent for $\tilde{\Sigma}$ as above. We may assume that the I-part of $\mathcal{R}$ meets the trace surface without loss of generality.

Assuming that our isotopy was very small, since the I-part meets one of the trace surface and P' is separating, the I-part meets P'. P' intersected with the closure of the exterior of $\tilde{\Sigma}^\epsilon$ is a polygonal disk D^ϵ. Then D^ϵ intersected with the I-part is a disjoint union of segments.

Extending the I-part of $\mathcal{R}$ in R until they meet unperturbed $\tilde{\Sigma}$, the set of points in $\mathcal{R}$ extends in R into the polyhedrons bounded by P' and the stellar neighborhood of $\tilde{\Sigma}$.

Since all vertex submersions of $\tilde{\Sigma}^\epsilon$ can happen by vertices near the convex vertices of $\tilde{\Sigma}$ masked off by totally geodesic hypersurfaces such as P', we obtain a crescent $\mathcal{R}'$ for $\tilde{\Sigma}$ preserving the I-part hypersurface of $\mathcal{R}$.

Therefore, $\mathcal{R}$ were obtained from $\mathcal{R}'$ by the convex truncation isotopy preserving the I-part.

For $\tilde{\Sigma}^\epsilon$ obtained from $\tilde{\Sigma}$ by small truncations along edges and triangles, very similar arguments using totally geodesic planes as P' parallel to those used in the truncation process will show the desired results.

Therefore a crescent for $\tilde{\Sigma}^\epsilon$ is one we obtained by the process in (i).

Let $\mathcal{R}$ be an inner crescent for $\tilde{\Sigma}^\epsilon$. Then since the vertices moved outward with respect to $\tilde{\Sigma}$, they move inward when we reverse the process and we see that $\mathcal{R}$ is isotopied to a crescent for $\tilde{\Sigma}$ by Lemma 4.1 by preserving the I-part hypersurface.

To show that the highest folding number can only decrease: For a crescent $\mathcal{R}$ to increase the folding number, a vertex must move into $I_{\mathcal{R}}$ during the isotopy. We see that such a vertex must be a convex one. However, the convex vertex can only move in the direction away from the interior of $\mathcal{R}$. (Even ones after the births obey this rule.)

Also, we can do this construction for $\tilde{\Sigma}$ simplex by simplex so that the final result is an equivariant isotopy. ☐

4.1.4. *Perturbations.*

Definition 4.3. By a perturbation of a triangulated surface, we mean the perturbations of vertices and corresponding edges and faces accordingly.

By definition, there cannot be generations of edges and triangles to lower-dimensional objects under perturbations. This applies the the proof of Proposition 4.4:

Proposition 4.4. *Suppose that $\tilde{\Sigma}$ does not have outer-contact points for its outer highest-level crescents. Then for sufficiently small perturbation, the level of the outer highest-level crescents of $\tilde{\Sigma}$ does not increase. Moreover, the outer highest-level crescents of the perturbed $\tilde{\Sigma}$ do not have outer-contact points.*

Proof. The I-part of the crescent set has an image with a compact closure in the quotient manifold M. Let $\tilde{\Sigma}^\epsilon$ be an equivariantly perturbed surface parameterized by $\epsilon > 0$ where $\tilde{\Sigma} = \tilde{\Sigma}^\epsilon$.

Given $\delta > 0$, we can find $\epsilon > 0$ so that the union of all crescents of $\tilde{\Sigma}^\epsilon$ is in a δ-neighborhood of that of $\tilde{\Sigma}$: If not, there exists a sequence of R_i for $\tilde{\Sigma}^{\epsilon_i}$ converging geometrically to a crescent R for $\tilde{\Sigma}$ where R is not in the δ-neighborhood of the union of all crescents. This is a contradiction.

Suppose that a level increased, say to higher than or equal to $n+1$, for a highest level outer crescent at $\tilde{\Sigma}^\epsilon$ for some $\epsilon > 0$ if the level of $\tilde{\Sigma}$ were n for an integer $n \geq -1$. Then by acting by deck transformation to put the level $(n+1)$-crescents to intersect the nearby fundamental domains of $\tilde{\Sigma}^\epsilon$s, we can find a sequence of level $n+1$ inner-most crescent R_i for $\tilde{\Sigma}^{\epsilon_i}$ converging to a some subset in $\tilde{M}$ as $i \to 0$ where $\epsilon_i \to 0$.

We may assume without loss of generality that

- There exists a sequence of points $p_i \in R_i$ converging to a point p in $\tilde{M}$ where R_is have level $n+1$.
- Let P_i be the totally geodesic hyperplane containing I_{R_i}. P_i has a geometric limit in a totally geodesic hyperplane P and the normal vectors to P_i converges to that of P.
- R_i converges geometrically to a closed subset in $\tilde{M}$ with no interior since otherwise there is a generalized limit crescent of level $n+1$.

A path γ_i from a point of α_{S_i} for the largest ambient crescent S_i to p_i must pass at least $n+1$ components of $\tilde{\Sigma}^{\epsilon_i}$. A subsequence of the closure of the component of $S_i - \tilde{\Sigma}^{\epsilon_i}$ must converge to a closed subset with no interior and and hence a subset of $\tilde{\Sigma}$. There is a point $x_i \in \tilde{\Sigma}^{\epsilon_i}$ on the path γ_i very close to p_i and the path in γ_i between x_i and p_i does not meet $\tilde{\Sigma}^{\epsilon_i}$. A neighborhood of x_i in $\tilde{\Sigma}^{\epsilon^i}$ is a union of triangles and they must all converge to triangles outside the interior of R at the end. Since our perturbation is assumed to be very small, the distance on $\tilde{\Sigma}^\epsilon$ from x and points of $\mathrm{Cl}(\alpha_{R_i})$ does not change much as $\epsilon \to 0$, the distance on $\tilde{M}$ from x to $\mathrm{Cl}(\alpha_{R_i})$ is bounded above as well. Since α_R is a subset of the limit of a subsequence of $\mathrm{Cl}(\alpha_{R_i})$ by Proposition 3.14, x_i converges to a point x in I_R^o, away from the boundary of ∂I_{R_i}. The triangles of $\tilde{\Sigma}$ containing x are outside the interior of R since otherwise we must have level increased. Therefore, there is an outer-contact point x, a contradiction. (See figure 3.)

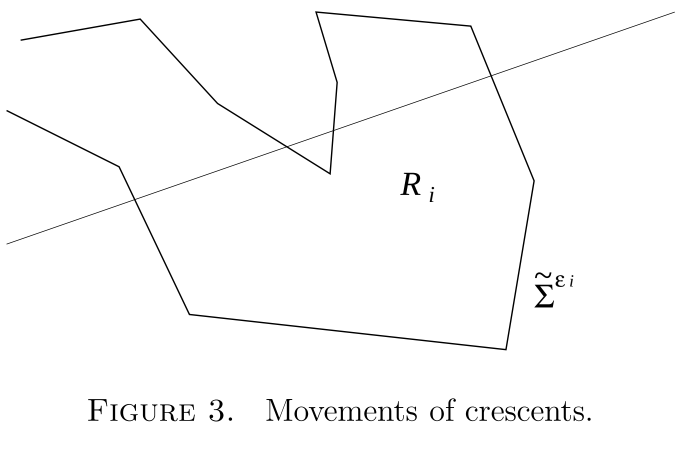

FIGURE 3. Movements of crescents.

Finally, if $\tilde{\Sigma}^\epsilon$ has outer-contact points for all small $\epsilon > 0$, we have a sequence of outer-contact points p_i for highest-level, i.e., level n, crescents R_i for $\tilde{\Sigma}^{\epsilon_i}$ where $\epsilon_i \to 0$.

If R_i does not degenerate, then, similarly to above, we obtain an outer-contact point for a crescent R of $\tilde{\Sigma}$.

If the sequence of R_i degenerates, then a sequence of triangles in the closure of their α-part converges to a triangle which contains a limit point of p_i as well. Since the distance of p_i to the triangle on $\tilde{\Sigma}^{\epsilon_i}$ are bounded below as shown above, this contradicts the imbedded property of $\tilde{\Sigma}$. □

The final result of Subsection 4.1 is:

State 4.5. As a consequence, we now perturb $\tilde{\Sigma}$ after the small truncation moves for outer-contact points for highest-level outer crescents, and the result do not have outer-contact points and have the outer-level kept same or decreased.

4.2. **The crescent isotopy.**

4.2.1. *The crescent-sets.* First, we suppose that there are highest-level crescents whose innermost crescents are outer-direction ones. We will "move" $\tilde{\Sigma}$ in the outer direction first to eliminate the outer highest-level crescents.

If the innermost ones are inner-direction, then we can simply change the orientation of $\tilde{\Sigma}$ and the discussions become the same. Thus we assume the above without loss of generality.

As we did in [8] and [9], we say that two highest-level crescents $\mathcal{R}$ and $\mathcal{S}$ are equivalent if there exists a sequence of transversally intersecting crescents from $\mathcal{R}$ to $\mathcal{S}$; that is,

$$\mathcal{R} = \mathcal{R}_0, \mathcal{R}_i \cap \mathcal{R}_{i+1}^o \neq \emptyset, \mathcal{S} = \mathcal{R}_n \text{ for } i = 1, 2, \ldots, n.$$

- We define $\Lambda(\mathcal{R})$ to be the union of all highest-level crescents equivalent to the highest-level crescent $\mathcal{R}$. As before, $\Lambda(\mathcal{R})$ and $\Lambda(\mathcal{S})$ do not meet in the interior or they are the same.
- We define $\partial_I\Lambda(\mathcal{R})$ to be the boundary of $\Lambda(\mathcal{R})$ removed with the closure of the union of the α-parts of the crescents in it. Then $\partial_I\Lambda(\mathcal{R})$ is a convex surface.
- We define $\partial_\alpha\Lambda(\mathcal{R})$ as the union of the α-parts of the crescents equivalent to $\mathcal{R}$.

Recall that a pleated surface is a surface where through each point passes a geodesic.

Lemma 4.6. *The set*

$$\partial_I\Lambda(\mathcal{R}) \cap \tilde{M} - \tilde{\Sigma}$$

is a properly imbedded pleated surface.

Proof. For each point of x belonging to the above set, x is an element of the interior of $\tilde{M}$ by Proposition 3.11. Let $B(x)$ be a small convex open ball with center at x. Then the crescents equivalent to R meet $B(x)$ in half-spaces. Therefore the complement of their union is a convex subset of $B(x)$ and x is a boundary point. There is a supporting half-space H in x and H belongs to $\Lambda(\mathcal{R})$.

If there were no straight geodesic passing through x in the boundary set $\partial_I\Lambda(\mathcal{R})$, then there exists a totally geodesic disk D in $B(x)$ with ∂D in $\Lambda(\mathcal{R})$ but interior points are not in it.

Since ∂D is in $\Lambda(\mathcal{R})$, each point of ∂D is in some crescent. We can extend D to a maximal totally geodesic hypersurface and we see that a portion of the hypersurface bounds a crescent $\mathcal{T}$ containing D in its I-part and overlapping with the other crescents. Thus $\mathcal{T}$ is a subset of $\Lambda(\mathcal{R})$ and so is D.

Therefore, $\partial_I \Lambda(\mathcal{R}) - \tilde{\Sigma}$ is a pleated surface. □

4.2.2. *Outer-contact points of the crescent-sets.* We may have some so-called outer-contact points of $\tilde{\Sigma}$ at $\partial_I \Lambda(\mathcal{R})$, i.e, those points with neighborhoods in $\tilde{\Sigma}$ outside $\Lambda(\mathcal{R})$. We can classify outer-contact points.

Proposition 4.7. *Assume $\tilde{\Sigma}$ is well-triangulated. The set of outer-contact points on $\partial_I \Lambda(\mathcal{R})$ is a union of the following components:*

- *isolated points,*
- *an arc passing through the pleating locus with at least one vertex.*
- *isolated triangles,*
- *union of triangles meeting each other at vertices or edges.*

Proof. This essentially follows by Proposition 3.19. □

4.2.3. *Crescent isotopy itself.* Recall that the final resulting $\tilde{\Sigma}$ in State 4.5 is a well-triangulated surface whose highest level outer crescents do not have any outer contact points.

First,we show that

$$\partial_I \Lambda(\mathcal{R}) \cup (\tilde{\Sigma} - \partial_\alpha \Lambda(\mathcal{R})) \tag{1}$$

is a properly imbedded pleated surface.

We do this for $\Lambda(\mathcal{R})$ for each highest-level crescent $\mathcal{R}$ obtaining as the end result a properly imbedded surface $\tilde{\Sigma}'$. The deck transformation group acts on $\tilde{\Sigma}'$ since it acts on the union of $\Lambda(\mathcal{R})$. Thus, we obtain a new closed surface Σ'.

Since the union of $\Lambda(\mathcal{R})$ for every highest-level crescent $\mathcal{R}$ is of bounded distance from $\tilde{\Sigma}$ by the boundedness and the fact that M is locally-compact, Σ' is a compact surface.

4.2.4. *The isotopy.* We show that Σ and Σ' are isotopic.

Let N be the ϵ-neighborhood of $\tilde{\Sigma}'$ in the closure of the outer component of $\tilde{M} - \tilde{\Sigma}$. There exists a boundary component $\partial_1 N$ nearer to $\tilde{\Sigma}$ than the other boundary component. The closure of a component K of $\tilde{M} - \tilde{\Sigma} - \tilde{\Sigma}'$ contains $\partial_1 N$. Then K projects to a compact subset of M. We can find a finite collection of generic secondary highest-level compact crescents $\mathcal{R}_1, \dots, \mathcal{R}_n$ and whose images under Γ form a locally finite cover of K.

We label the crescents by $\mathcal{S}_1, \mathcal{S}_2, \dots$. We know that replacing the closure of the α-part of $\mathcal{S}_1$ by the I-part is an isotopy. After this move, $\mathcal{S}_2, \mathcal{S}_3, \dots$ become new generic highest-level crescents by Proposition 3.28 and appropriate truncations.

We define $\partial_I(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \dots)$ as the boundary of $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \dots$ removed with the union of the α-parts of $\mathcal{S}_1, \mathcal{S}_2, \dots$. Again, this is a convex imbedded surface. Therefore, replacing the union of the α-parts of $\mathcal{S}_1, \mathcal{S}_2, \dots$ by $\partial_I(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \dots)$ is an isotopy as above.

We obtain $\Sigma_{\mathcal{R}_1, \dots, \mathcal{R}_n}$ as the image in M, which is isotopic to Σ. If ϵ is sufficiently small, then we see easily that $\partial_I(\mathcal{S}_1 \cup \mathcal{S}_2 \cup \dots)$ in N can be isotopied to $\tilde{\Sigma}'$ using rays perpendicular to $\tilde{\Sigma}'$. Thus, $\Sigma_{\mathcal{R}_1,\ldots,\mathcal{R}_n}$ is isotopic to Σ'. We showed that Σ' is isotopic to Σ. □

4.2.5. *The strict decreasing of the levels.* We show that our isotopy move decreases the level strictly.

Proposition 4.8. *If Σ' is obtained from Σ by the highest-level outer crescent move for level n for an integer $n \geq 0$, then the union of the collection of crescents of $\tilde{\Sigma}$ contains the union of those of $\tilde{\Sigma}'$, and $\tilde{\Sigma}'$ has no outer highest-level crescent of level n or higher for an integer $n \geq 0$. The level of the highest level outer crescents of $\tilde{\Sigma}'$ is now $\leq n-1$. The level for inner highest-level crescents stays the same or may drop. The same statements hold if we replace the word "outer" by "inner".*

Proof. The outer highest-level crescents for $\tilde{\Sigma}'$ can be extended to ones for $\tilde{\Sigma}$ since their I-part can be extended across.

The inner highest level crescents for $\tilde{\Sigma}'$ can be truncated to ones for $\tilde{\Sigma}$ by Lemma 4.1 since the move from $\tilde{\Sigma}'$ to $\tilde{\Sigma}$ is inward and supported by the outer crescents of $\tilde{\Sigma}$. Thus the first statement holds.

If there were outer highest level crescent $\mathcal{R}$ of level n or higher, then we can extend $I_{\mathcal{R}}$ across the I_∂-parts so that we can obtain a level-n or higher-level crescent. Such a crescent would have been included in the crescent set and should have been isotopied away.

If $\mathcal{R}$ were inner highest level one, Lemma 4.1 implies the result. □

We apply our methods of Subsection 4.2.3 to obtain Σ' which we now let it be Σ.

State 4.9. $\tilde{\Sigma}$ now has no outer crescents of level n. However, it may not be triangulated.

4.3. **Convex perturbations.** In this subsection, we modify $\tilde{\Sigma}$, which now has partial pleating, obtained above further. We discuss how a surface with a portion of itself concavely pleated by infinitely long geodesics and the remainder triangulated can be perturbed to a triangulated surface without introducing higher-level crescents. This is done by approximating the union of pleating geodesics by train tracks and choosing normals in the concave direction and finitely many vertices at the normal and pushing the pleating geodesics to become geodesics broken at the vertices.

4.3.1. *Pleated-triangulated surfaces.* We will describe $\tilde{\Sigma}$ as a "pleated-triangulated" surface. We will then perturb the crescent-isotopied Σ into a triangulated surface not introducing any higher-level crescents.

Suppose that Σ' is a closed imbedded surface in M. Σ' is a *pleated-triangulated surface* if

- $\tilde{\Sigma}'$ contains a closed 2-dimensional domain divided into locally finite collection of closed totally geodesic convex domains meeting each other in geodesic segments,
- through each point of the complement in $\tilde{\Sigma}'$ passes a *straight* geodesic in the complement.

We may also add finitely many straight geodesic segments in the surface ending at the domain.

- The domain union with the segments is said to be the *triangulated part* of Σ'. The boundary of the domain is a union of finitely pinched simple closed curves.
- The complement of the domain is an open surface, which is said to be the *pleated part* where through each point passes a straight geodesic.
- The pleated part has a locus where a unique straight geodesic passes through. This part is said to be the *pleating locus*. It is a closed subset of the complement and forms a lamination.
- If we remove the closure of the pleated part from $\tilde{\Sigma}'$, we obtain a locally finite collection of totally geodesic convex domains meeting each other in edges and vertices. The convex domains, edges, and vertices are in general position.

For later purposes, we say that Σ' is *truly pleated-triangulated* if the triangulated part is a union of totally geodesic domains that are convex polygons (i.e., finite-sided) and geodesic segments ending in the domains.

While the triangulated parts and pleating parts are not uniquely determined, we simply choose what seems natural. We also assume that the pleated part is locally convex or locally concave. We usually choose a normal direction so that the surface is locally concave at the pleated part.

If Σ' satisfies all of the above conditions, we say that Σ' is a *concave pleated-triangulated* surface. If we choose the opposite normal-direction, then Σ' is a *convex pleated, triangulated* surface.

Proposition 4.10. *Suppose that Σ is as in State 4.9 where n, $n \geq 0$, is the highest-level for the surface before the applications of methods in Subsection 4.2.3 which is realized by outer-crescents. Then Σ is a concave pleated-triangulated surface in the outer direction. The level of highest-level outer crescents of $\tilde{\Sigma}$ is $\leq n-1$ and the level of highest-level inner crescent of $\tilde{\Sigma}$ is less than or equal to n. Moreover, the statements are true if all "outer" were replaced by "inner" and vise-versa and the word "concave" by "convex".*

Proof. The part $\partial_I \Lambda(\mathcal{R}) - \tilde{\Sigma}$ for crescents $\mathcal{R}$s are pleated by Lemma 4.6. These sets for crescents $\mathcal{R}$ are either identical or disjoint from each other as the sets of form $\Lambda(\mathcal{R})$ satisfy this property. The union of sets of form $\partial_I \Lambda(\mathcal{R})$ comprise the pleated part and the complement in Σ' were in Σ originally and they are the union of totally geodesic 2-dimensional convex domains.

The rest is proved already in Proposition 4.8. □

4.3.2. *The geometry and topology of pleating loci.* Two leaves in a pleating locus are *converging* if one is asymptotic to the other one (see Section 6.1 for definitions); i.e., the distance function from one leaf to the other converges to zero and conversely. By an end of a leaf of a lamination *wrapping around* a closed set, we mean that the leaf converges to a subset of the closed set in the direction of the end.

The main idea of classifying the pleating locus are from those of Thurston as written up in Casson-Bleiler [7].

Lemma 4.11. *Suppose that Σ' is a closed concave pleated-triangulated surface with a triangulated part and pleated part assigned. Suppose that l is a leaf. Then*

- *l is either a leaf of a minimal geodesic lamination or a closed geodesic, or each end of l wraps around a minimal geodesic lamination or a closed geodesic or ends in the boundary of the triangulated part.*
- *if l is isolated from both sides, then l must be a closed geodesic, and*
- *pleating leaves in a neighborhood of l must diverge from l eventually.*

Proof. Since the pleated open surface carries an intrinsic metric which identifies it to a quotient of an open subset of the hyperbolic space, each geodesic in the pleating lamination will satisfy the above properties like the geodesic laminations on the closed hyperbolic surfaces. The first item is done.

For second item, if l is isolated from both sides, then there is a definite positive angle between two totally geodesic hypersurfaces ending at l. Suppose that l is not a simple closed geodesic. Then this angled pair of the hypersurfaces continues to wrap around infinitely in M accumulating at a point of M and the sum of the angles violates the imbeddedness of Σ'. (This is an argument in Thurston [17] to show the similar argument for the boundary of the convex hulls.)

For the third item, if l is not isolated but has converging nearby pleating leaves, the same reasoning will hold as in the second item. □

Proposition 4.12. *Suppose that Σ' is a closed concave pleated-triangulated surface. Let Λ be the set of pleating locus of the pleated part in Σ'. Then Λ decomposes into finitely many components $\Lambda_1, \dots, \Lambda_n$ so that each Λ_i is one of the following:*

- *a finite union of finite-length pleating leaves homeomorphic to a compact set times a line with endpoints in the triangulated part. (A discrete set times a line if Σ' is truly pleated-triangulated.)*
- *a simple closed geodesic.*
- *a minimal geodesic lamination, which is a closed subset of the pleated part isolated away from the triangulated part.*

Here each leaf is either bi-infinite or finite. The union of bi-infinite leaves is a finite union of minimal geodesic laminations and is isolated away from the triangulated part and the union of finite-length leaves.

Proof. Let l be an infinite leaf in the pleating locus. By Lemma 4.11, l is not isolated from both sides and the leaves in its neighborhood is diverging from l. If l is not itself a leaf of a minimal lamination, then an end of l must converge to a minimal lamination or a simple closed geodesic. This means that leaves in a neighborhood also converges to the same lamination in one of the directions. However, this means that they also converges to l, a contradiction. Therefore, each leaf is a leaf of a minimal lamination or a closed geodesic or a finite length line.

The union of all finite length lines in the pleating locus is a closed subset: Its complement in Λ is a compact geodesic lamination in Σ'. If a sequence of a finite length leaves l_i converges to an infinite length geodesic l, then l_i gets arbitrarily close to a minimal lamination or a closed geodesic. If l_i gets into a sufficiently thin neighborhood of one of these, then a neighborhood of an end of l_i must be in a sufficiently thin neighborhood of one of these by the imbeddedness property of l_i, i.e., cannot turn sharply away and go out of the neighborhood. As l_i ends in the triangulated part, the distance from the triangulated part to one of these goes to zero. Since the domains in the triangulated part are in general position, the boundary of the triangulated part cannot contain a closed geodesic or the straight geodesic lamination. This is a contradiction.

Looking at an ϵ-thin neighborhood of Λ, we see that Λ decomposes as described. (See Casson-Bleiler [7] for background informations). □

Remark 4.13. If Σ' is truely pleated-triangulated, then there are only finitely many finite length pleating leaves since their endpoints are on the vertices of the polygons in the triangulated part.

Lemma 4.14. *A line l in the pleating locus of a pleated-triangulated surface cannot end in an interior of a segment s in the boundary of the pleated part.*

Proof. First suppose l is isolated. Then the boundaries of totally geodesic planes in its side must contain open segments in s, and the planes have to be identical, contradicting that l is in the pleating locus.

Suppose that l is not isolated. Then the nearby leaves of the foliation must end at the same place as l; otherwise, we get that the nearby totally geodesic planes are identical. If they all end at the same place, again a similar reasoning shows that the nearby geodesic planes are identical. These contradicts the fact that l is in the pleating locus. □

4.3.3. *The perturbation moves.* Recall that our surface $\tilde{\Sigma}$ is pleated triangulated and has level n, achieved by innermost inner-crescents, for an integer $n \geq 0$. The outer level is less than or equal to $n-1$.

A train track is obtained by taking a thin neighborhood of the lamination. We can think of the train track as a union of segment times an interval, so-called branches, joined up at the end of each segment times the intervals so that the intervals stacks up and matches. A point times the interval is said to be a *tie* and a tie where more than one branches meet a *switch*. One can collapse the interval direction to obtain a union of graphs and circles. For more details, consult Casson-Bleiler [7].

We will perturb the pleating locus to obtain our well-triangulated surface. We describe our results divided in Theorem 4.16.

The first step (I) of the perturbation move: Let $l_1, \ldots, l_k$ be the thin strips containing all the finite length open leaves ending at the triangulated parts. We find a thin totally geodesic hypersurface P_i near l_is nearly parallel to $l_i s$. Then we cut off the neighborhood of l_i in Σ by P_i and replace the lost part with the portion in P_i. This introduces squares which are triangulated into pairs of triangles.

We now remove the union of the squares from the pleated part and add the union to the triangulated part. Now we retriangulate the triangulated part. They will be immobile during the pertubations now.

This forms a generalization of a small truncation move. We still call it a *small truncation move.* –(*)

As in Proposition 4.2, we don't increase the level and we remove some of the outer-contact points.

Finally, we do remove all of the outer-contact points in the triangulated part, i.e., the closed set consisting of the compact totally geodesic domains.

The step (II) of the perturbation move: Now, every component of the pleating locus is contained in a minimal lamination. The set of pleated locus is a union of finitely many minimal laminations.

Definition 4.15. In the pleated part, we define the *minimal pleated part* as the intrinsic convex hull in the closure of the pleated part of the union of bi-infinite pleating geodesics with respect to the intrinsic metric obtained by piecing the pleated parts together. The minimal pleated part is a subsurface of the pleated part which is the closure of the union of totally geodesic subsurfaces bounded by the bi-infinite pleating geodesics.

The boundary of the minimal pleated parts is a union of simple closed curve which might be a broken geodesic or just a geodesic. We triangulate the closure of the complement of the minimal pleated part, which will be added shortly to the triangulated part. This may introduce vertices at the boundary of the minimal pleated part.

We now add finite leaves of infinite length in the minimally pleated part so that the components of the complement of the union of the pleating locus and these leaves are all open triangles. These can start from the vertices at the boundary of the minimal pleated part. This can be done even though the boundary of the pleated part is not geodesic.

By choosing sufficiently small ϵ-thin-neighborhoods of the union for $\epsilon > 0$, we obtain a train track. We obtain a maximal train track.

We first choose switches for the endpoint of the finite length geodesics in the squares and added infinite length finite leaves. (The switches are transverse arcs.) We choose switches for the rest. We label them $I_1, \dots, I_m$. We may have a uniform bound on m depending only on the Euler characteristic of the open pleated part surface and a uniform lower bound to the distances between any two of I_i. (Note here m is bounded above by a constant depending only on Σ as Σ has CAT(-1) metric and Thurston's theory of geodesic laminations hold for such surfaces)

By choosing $\epsilon > 0$ sufficiently small, we can assume that the outer-normal vectors to totally geodesic hypersurfaces meeting I_i are δ-close for a small $\delta > 0$ except the outer-normal vectors to the totally geodesic hypersurfaces corresponding to the complementary regions of the train track. (Here the outer-normal is in the concave directions.)

The union of I_is with the leaves of the train tracks divides the pleated part into infinitely squares with two sides within I_is and polygons with some edges in I_is. We regard the endpoints of I_i in edges of some polygons to be a vertex and we retriangulated these accordingly. We can assume that there is a lower bound to the length of each edges which are not in I_is. In fact, we can assume that the ratios of the edges not in I_i to those in I_i are greater than 1000.

Topologically, the train track collapses to a union of graphs with vertices corresponding to $I_1, \dots, I_n$ and closed geodesics, and the complement becomes a union of totally geodesic triangles.

We now do the collapsing geometrically: We choose one of the normal vectors and a generically chosen point x_i on the normal vector γ-close to I_i, where γ is a small positive number. We push all the points of I_i to x_i to obtain a train track $\tau_{\epsilon,\delta,\gamma}$ and the complementary regions move accordingly to disks divided into compact totally geodesic geodesic triangles with vertices in the train track $\tau_{\epsilon,\delta,\gamma}$ and in triangulated part. (x_is are said to be *pleated part vertices.*)

We claim that then the triangles are very close to the original triangles in their normal directions as well as in the Hausdorff distance since the edge lengths of the triangles are bounded below. Since there are no rapid turning of the complementary geodesic triangles, we can be assured that the new surface is imbedded by integrations. We can see this by looking at a cross-section, which is a function of bounded variations.

The leaves of the laminations are moved to become a train track in the normal direction which is a concave direction. Thus the leaf is bent against the concave direction. The triangles meeting the endpoints of I_i of almost the same direction as before and hence have angles $< \pi$ by concavity. Lemma 2.3 shows that the pleated part vertices are strict saddle vertices.

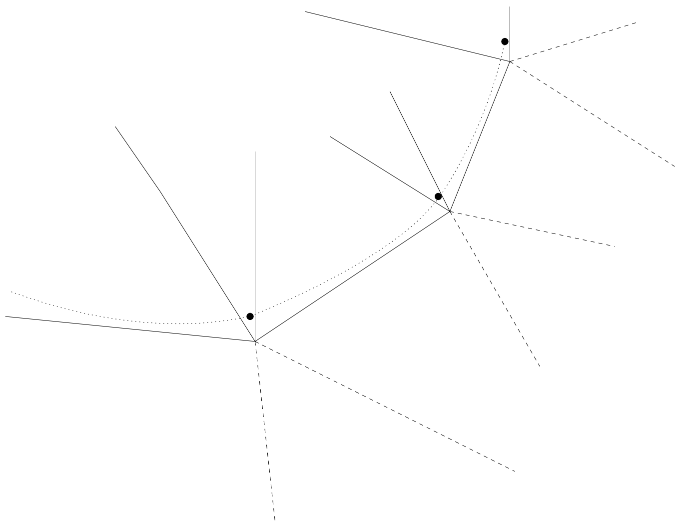

FIGURE 4. Making the boundary vertices of the pleated part into saddle-vertices.

Of course, here there are many choices and the combinatorial structure of the triangulations will change according to the choices.

There is a slight modification when $n = 0$: Since we started from well-triagulated $\tilde{\Sigma}$. All vertices in the interior of the triangulated parts are strict saddle vertices or convex vertices and the vertices in the boundary of the triangulated part are saddle vertices as there are no outer crescents by Corollary 3.17. By Lemma 2.2, we perturb the vertices in the boundary of the triangulated part to be strict saddle vertices. The adjacent component of the pleated part with the pleating locus removed is totally geodesic. By introducing finite geodesic leaves, we see that the pleating locus will become larger including these leaves. By taking a sufficiently small perturbation at the boundary vertices, this will not change the vertex type of all vertices in the triangulated part. Also, we choose a train track for the pleating minimal lamination. The angles between the adjacent totally geodesic planes across the switches square are less than π. We further require that we perturb the boundary vertices by sufficiently small amount so that the angles are still less than π. Finally, we perturb the pleated parts as above introducing pleated part vertices.

(Clearly, we can do the above collapsing and the perturbations equivariantly).

Theorem 4.16. *Let Σ' be a closed concave pleated, triangulated surface where the outer highest level crescents have level $n-1$ for an integer n, $n \geq 0$. Then one can find an imbedded isotopic well-triangulated surface Σ'' in any ϵ-neighborhood of Σ' by the above methods of perturbations so that the following hold.*

(i) *The union of the set of crescents for $\tilde{\Sigma}''$ is in the ϵ-neighborhood of that of $\tilde{\Sigma}'$ and vice-versa for a small ϵ if we choose $\tilde{\Sigma}''$ sufficiently close to $\tilde{\Sigma}'$.*

(ii) *The pleated part vertices are strict saddle-vertices.*

(iii) *The level of the outer highest-level crescents for the resulting surface $\tilde{\Sigma}''$ is less than or equal to $n-1$ and the level of inner highest level crescents for $\tilde{\Sigma}''$ is less than or equal to that of $\tilde{\Sigma}'$.*

(vi) *In particular, if $\tilde{\Sigma}'$ has no outer (resp. inner) crescent, then $\tilde{\Sigma}''$ contains no outer (resp. inner) crescent. If there were no outer and inner crescents, $\tilde{\Sigma}''$ is saddle-shaped.*

Proof. (i) This matters for crescents that are inner if the perturbations are inner and ones that are outer if the perturbations are outer: In other cases, Lemma 4.1 shows that reversing the perturbation process gives us back all of our old crescents preserving the I-part hypersurface.

Suppose that we have the perturbed sequence Σ'_i closer and closer to Σ', and there exists a sequence of crescent $\mathcal{R}_i$ for Σ'_i not contained in a certain neighborhood of the union of crescents for Σ'. Then the limit $\mathcal{R}$ of $\mathcal{R}_i$ is still a crescent for Σ' and is not in the neighborhood. This is absurd.

(ii) This is proved above.

(iii) Let $n-1$ be the level of outer highest-level crescents of Σ'.

As stated earlier, the perturbation step (I) does not increase the level.

Let us discuss what are the outer-contact points of outer highest-level crescents, where we are no longer assuming the general position property of $\tilde{\Sigma}'$:

Let $\mathcal{R}$ be a highest-level crescent with outer-contact points. There could be points of the minimal sets pleated parts meeting tangentially the I-part of $\mathcal{R}$. If not, then they are on the triangulated part. In these cases, we do the small truncation move. This will not increase the folding number.

The minimal sets in pleated part may meet the crescents only tangentially at the I-part or meet the closures of the α-parts of the crescents: If not, then the minimal pleated part must pass through the interior of a crescent and this implies that a bi-infinite pleating geodesic pass through the interior. By Proposition 3.22, this is a contradiction.

If a minimal set in the pleated part meets the I-part of a crescent, then the closure of a certain number of pleated leaves are in the I-part. Thus, it is clear that the complete geodesic leaf or the closure of a complement of the pleating locus in the minimally pleated part equals the set of outer-contact points of the I-part.

We now move vertices of the train tracks of the pleating laminations by a very small amount according to the perturbation step (II). (The crescents do move in its ϵ-neighborhood.)

We choose the ϵ-neighborhood sufficiently small so that any new component of $\tilde{\Sigma}'$ intersected with crescents may not arise as $\tilde{\Sigma}'$ is deformed: The small truncated places may be avoided by taking ϵ-sufficiently small.

Since any crescent during the perturbation cannot meet the minimal pleated part in its interior and its α-part by Proposition 3.22, the I-part of crescents and crescents themselves close to the minimal pleated part lie below the minimal pleated part or may meet the minimal pleated part but cannot pass through it.

Let us choose a sequence $\epsilon_i \to 0$ of positive real numbers ϵ_i and a sequence of approximating isotopic surfaces $\Sigma^{\epsilon_i} \to \Sigma'$. We assume that they all have the same number of vertices by introducing finer triangulations if necessary. We can complete the sequence Σ^{ϵ_i} to a one-parameter family since any two triangulation with same number of vertices on a closed surfaces are related by elementary moves. (Of course, there is a skipping around a tetrahedron during the elementary move. See Figure 5)

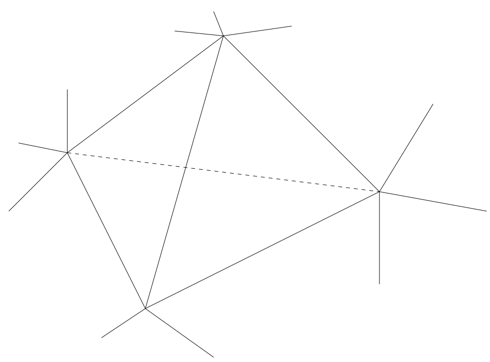

FIGURE 5. An elementary move and a skipping a tetrahedron.

Suppose that the outer-level became n or higher for Σ^{ϵ_i} for every i. Choose an index i. Then there is a vertex v of $\tilde{\Sigma}^{\epsilon_i}$ in the interior of a crescent R of $\tilde{\Sigma}^{\epsilon_i}$ of level n so that a component C of $\tilde{\Sigma}^{\epsilon_i} \cap R^o$ containing v is in the α-part of an innermost crescent realizing the level n.

If v is from the triangulated part and $n \geq 1$, then by letting $i \to \infty$, we obtain an outer-contact point as in Proposition 4.4. This was ruled out by the perturbation step (I).

Suppose that $n = 0$. Then all vertices in the interior of the triangulated parts are strict saddle vertices or convex vertices and the vertices in the boundary of the triangulated part are saddle vertices after the perturbation step (II) by construction.

The vertex v cannot come from the triangulated part.

Since the vertex v is from the pleated part of $\tilde{\Sigma}'$, it follows that v is a saddle-type vertex. Moreover each vertex of C are saddle-type by the same reason. Corollary 3.17 gives us a contradiction.

Since our move is toward outside only, we see that the set of inner crescents decrease only and hence the inner level do not increase.

(iv) The first part follows from (iii).

If $\tilde{\Sigma}'$ has no inner or outer crescents, the first part of (iv) shows that we can construct $\tilde{\Sigma}''$ with no crescents. This implies that $\tilde{\Sigma}''$ is saddle-imbedded.

□

We now do small truncation moves to $\tilde{\Sigma}$ so that outer highest-level crescent of level now $n-1$ has no outer-contact points. We now perturb the triangles so that resulting $\tilde{\Sigma}''$ is well-triangulated. As in Application 4.5, we see that the outer level of $\tilde{\Sigma}''$ is still $n-1$.

State 4.17. Applying the method of this section, we obtain a well-triangulated $\tilde{\Sigma}$ with outer level $n-1$ and the inner level $\leq n$.

4.4. **The proof of Theorem C and Corollary D.**

4.4.1. *Isotopy sequences.* We review the outer level n crescent moves. We temporarily denote the result of the move after each move —(*):

(i) the small truncation moves for outer highest-level crescents: $\tilde{\Sigma}^{(i)}$.
(ii) the crescent-isotopy for outer highest-level crescents: $\tilde{\Sigma}^{(ii)}$.
(iii) the convex perturbations which involves small truncation moves: $\tilde{\Sigma}^{(iii)}$.

Let n be the highest level for $\tilde{\Sigma}$. First, we do the above outer highest-level crescent moves of level n.

We should worry about one issue here with regards to $\mathbf{c_1}, \dots \mathbf{c_n}$: After the step (ii), we may find $\tilde{\Sigma}^{(ii)}$ meeting a lift of one of these. In which case, a lift l_i for some i is contained in the I-part of a secondary highest level crescent. In this case, $\mathbf{c_{i_1}}, \dots, \mathbf{c_{i_k}}$ is a subset of the image surface $\Sigma^{(ii)}$ for some integer $k \geq 1$.

Choose an arbitrary j. The deck transformation acting on l_{i_j} preserves the I-part of the secondary highest level crescent since otherwise we get more secondary highest level crescent around l_{i_j} completing the 2π-angle and hence l_{i_j} can't be in $\tilde{\Sigma}^{(ii)}$. Thus the holonomy of $\mathbf{c_{i_j}}$ is hyperbolic, i.e., loxodromic without rotational part.

Let $P_1, \dots, P_m$ be the 1-complexes which are components of the union of $\mathbf{c_{i_1}}, \dots, \mathbf{c_{i_k}}$, which could just be the disjoint union of the closed curves themselves.

After the convex perturbation (iii), we find the 1-complex P_i' in $\Sigma^{(iii)}$ approximating P_i: What happens actually is that in case P_i is a simple closed geodesic, then P_i' is perturbed to have saddle vertices only, and in case P_i is a true 1-complex with singularities, then there is a totally geodesic subsurface S_i with boundary, and the boundary vertices are perturbed to become saddle vertices and S_i is isotoped accordingly to S_i'

without introducing any interior vertices and P_i is isotopied to P'_i on S'_i accordingly. Since the holonomies of closed curves on S_i are without rotational parts, we assume without loss of generality that we moved P_i to P'_i in a fixed angle to the $\Sigma^{(ii)}$ (perpendicularly in the second case). Let $\tilde{P}_i$ be the inverse image of P_i in $\tilde{\Sigma}^{(ii)}$, and $\tilde{P}'_i$ be that of P'_i. Thus $\tilde{P}_i$ and $\tilde{P}'_i$ lie on a union of totally geodesic planes L_i in a fixed angle to the I-part of a crescent for $\tilde{\Sigma}^{(ii)}$, where a 2-complex D_i is bounded by $\tilde{P}_i$ and $\tilde{P}'_i$. (D_i maps to a compact 2-complex in M.)

Recall that our isotopies are in the outer-direction and the convex perturbation was also in the outer-direction.

Lemma 4.18.
- *If P'_i is a closed geodesic, then a highest level crescent $\mathcal{R}$ of $\tilde{\Sigma}^{(iii)}$ may meet a lift of P'_i in $\tilde{M}$ only in an α-part by a connected subarc.*
- *If P'_i is a true 1-complex with singularities, then a highest level crescent $\mathcal{R}$ of $\tilde{\Sigma}^{(iii)}$ may meet a component of the inverse image of S'_i in $\tilde{M}$ in a connected surface.*
- *In both case, when meeting, the I-part of a crescent for the isotopied $\tilde{\Sigma}$ passes D_i and hence the I-part is in the inner direction of the isotopied $\tilde{\Sigma}$.*

Proof. If P'_i is a simple closed curve, then P'_i is an arc bent in the inner-direction. Also, S'_i is bent in the inner-direction since the deck transformations corresponding to S'_i acts on a totally geodesic hypersurface H_i containing l_{i_j}s and hence is a Fuchsian subgroup. The result follows from this. □

Finally, we let $\Sigma = \Sigma^{(iii)}$. During the isotopy, Σ may pass through these $\mathbf{c_{i_j}}$s. Thus, Σ may no longer be incompressible.

The next step is that we do the inner highest-level crescent moves of level n.

Here, the incompressibility may not hold: however, the only place we need incompressibility is when we show that there is a secondary highest-level crescent containing a given highest-level crescent: We need to show that each boundary curve c of the perturbed I-part of a highest-level crescent which is an innermost component in the perturbed I-part bounds a disk in $\tilde{\Sigma}$. If c does not meet $\tilde{P}'_i$, then the argument is the same since $\tilde{\Sigma} - \tilde{P}'_i$ is incompressible. If c does meet $\tilde{P}'_i$, then we reverse the convex perturbation move and the isotopied c' from c is a closed curve in $\tilde{M}$ and hence bounds a disk D in $\tilde{\Sigma}^{(ii)}$ before the convex perturbations (regarding the $\mathbf{c_{i_j}}$ to be pushed outward from $\tilde{\Sigma}^{(ii)}$ by a small amount). D meets the the corresponding inverse images of P_i and H_i in regions. Since P_i and H_i are isotopied by a very small amount, we can isotopy D to D'. Thus, c bounds a disk in the convex perturbed $\tilde{\Sigma}^{(iii)}$.

If $\tilde{\Sigma}^{(iii)}$ passed over P_i, then Lemma 4.18 shows that the inner level n crescent move can only move P'_i closer to P_i since the direction of the crescent is now reversed but cannot pass P_i. Since the perturbation can be controlled, we see that the result $\tilde{\Sigma}'$ of the inner level n crescent move, there is P''_i very close to P_i also.

We go to the level $n-1$ and so on. We see that the inner and outer level strictly decreases until there are no more crescents. Therefore, the final result is saddle-imbedded.

Moreover the union of the set of crescents is contained in the ϵ-neighborhood of the union of the set of crescents in the previous step.

This completes the proof of Theorem C. □

We now prove Corollary D: Assume as in the premise of the corollary that M is a codimension 0 compact submanifold of a general hyperbolic manifold N.

We modify the boundary surface ∂M according to Theorem C. The final result is an imbedded closed surface Σ bounding a compact region by our construction. We let this region to be our isotopied manifold M'. □

Remark 4.19. The vertices of the boundary of M' are strict saddle-vertices: We start from general position ∂M so that each saddle-vertex is a strict saddle-vertex (see Theorem 4.16). Newly created vertices in the interior of the pleated part after the crescent isotopies are all strict saddle-vertices. The boundary vertices of the pleated part at level 0 are perturbed to be strict saddle-vertices. Finally, we no longer have any convex or concave vertex left at M'.

Part 2. General hyperbolic 3-manifolds and convex hulls of their cores

5. Introduction to Part 2

A *hyperbolic triangle* is a subset of a metric space isometric with a geodesic triangle in the hyperbolic plane $\mathbb{H}^2$. If the ambient space is a 3-dimensional metric space, then we require it to be totally geodesic as well and develop into a totally geodesic plane in the hyperbolic space $\mathbb{H}^3$.

A *general hyperbolic manifold* M is a metric space with a locally isometric immersion $\mathbf{dev} : \tilde{M} \to \mathbb{H}^3$ from its universal cover $\tilde{M}$. $\mathbf{dev}$ has an associated homomorphism $h : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ given by $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for each deck transformation $\gamma \in \pi_1(M)$. We require that the boundary of M triangulated by totally geodesic hyperbolic triangles.

A 2-*convex* general hyperbolic manifold is a general hyperbolic manifold so that a isometric imbedding from $T^o \cup F_1 \cup F_2 \cup F_3$ where T^o is a hyperbolic tetrahedron in the standard hyperbolic space $\mathbb{H}^3$ and F_1, F_2, F_3 its faces extend to an imbedding from T.

The main result of Part 2 is:

Theorem 5.1. *The universal cover of a* 2*-convex general hyperbolic manifold is* $\mathrm{CAT}(-1)$.

A *hyperbolic-map* from a triangulated hyperbolic surface is a map sending hyperbolic triangles to hyperbolic triangles and the sum of angles of image triangles around a vertex is greater than or equal to 2π.

Theorem 5.2. *Let Σ be a compact hyperbolically-mapped surface relative to $v_1, \dots, v_n$ into a general hyperbolic manifold and suppose that each arc in $\partial\Sigma - \{v_1, \dots, v_n\}$ is geodesic. Let θ_i be the exterior angle of v_i with respect to geodesics in the boundary of Σ. Then*

$$\mathrm{Area}(\Sigma) \leq \sum_{\mathrm{i}} \theta_{\mathrm{i}} - 2\pi\chi(\Sigma). \tag{2}$$

Note that in this part, by a geodesic we mean the geodesic with respect to the ambient manifold M unless we state otherwise that it is a geodesic in an submanifold, say of codimension-one.

The convex hull of a homotopy-equivalent closed subset of a general hyperbolic manifold M is the image in M of the smallest closed convex subset of containing the inverse image of the closed subset in the universal cover $\tilde{M}$ of M. We can always choose the core $\mathcal{C}$ to be a subset of M^o.

Theorem 5.3. *Let* $\mathrm{convh}(\mathcal{C})$ *be the convex hull of the core* $\mathcal{C}$ *of a* 2*-convex general hyperbolic manifold. We assume that* $\mathcal{C}$ *is chosen to be a subset of* M^o *and* $\partial\mathcal{C}$ *is saddle-imbedded. Suppose that* $\mathrm{convh}(\mathcal{C})$ *is compact. Then* $\mathrm{convh}(\mathcal{C})$ *is homotopy equivalent to the core and the boundary is a truly pleated-triangulated hyperbolic-surface.*

In the preliminary, Section 6, we recall the definition of CAT(κ)-spaces for $\kappa \in \mathbb{R}$ using geodesics and triangles. We also define M_κ-spaces, the simplicial metric spaces needed in this paper. We discuss the link conditions to check when M_κ-space is CAT(κ)-space, the Cartan-Hadamard theorem, and Gromov boundaries of these spaces. Next, we discuss the 2-dimensional versions of these spaces. Define the interior angles, and prove the Gauss-Bonnet theorem. Finally, we discuss general hyperbolic manifolds.

In Section 7, we show that the universal cover of a 2-convex general hyperbolic manifold, which we used a lot in Part 1, is a M_{-1}-simplicial metric space and a CAT(-1)-space and a visibility manifold. Next, we define hyperbolic-maps of surfaces. These are similar to hyperbolic surfaces as defined by Bonahon, Canary, and Minsky. We define Alexandrov nets and A-nets as generalized triangles. We show that maps from surfaces can be homotopied to hyperbolic-maps relative to a collection of boundary points in 2-convex general hyperbolic manifolds. We prove the Gauss-Bonnet theorem for such surfaces and find area bounds for polygons.

In Section 8, we discuss the convex hull of the core $\mathcal{C}$ in a general hyperbolic manifold. First, we show that the convex hull and $\mathcal{C}$ is homotopy equivalent. We finally show that the boundary of the convex hull can be deformed to a nearby hyperbolic-imbedded surface, which is truly pleated-triangulated. We show this by finding a geodesic in the boundary of the convex hull through each point of the boundary.

6. Hyperbolic metric spaces

6.1. **Metric spaces, geodesic spaces, and cat(-1)-spaces.** We will follow Bridson-Haefliger [5].

Let (X, d) be a metric space. A *geodesic path* from a point x to y, $x, y \in X$ is a map $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

A *local geodesic* is a map $c : I \to X$ from an interval I with the property that for every $t \in I$ there exists $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for t', t'' in the ϵ-neighborhood of t in I.

(X, d) is a *geodesic metric space* if every pair of points of X is joined by a geodesic.

We denote by $\mathbb{E}^2$ the plan $\mathbb{R}^2$ with the standard Euclidean metric. A *comparison triangle* in $\mathbb{E}^2$ of a triple of points (p, q, r) in X is a triangle in $\mathbb{E}^2$ with vertices $\bar{p}, \bar{q}, \bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$, and $d(r, p) = d(\bar{r}, \bar{p})$, which is unique up to isometries of $\mathbb{E}^2$.

The interior angle of the comparison triangle at $\bar{p}$ is called the comparison angle between q and r at p and is denoted $\bar{\angle}_p(q, r)$.

Let $c : [0, a] \to X$ and $c' : [0, b] \to X$ be two geodesics with $c(0) = c'(0)$. We define the *upper angle* $\angle_{c,c'} \in [0, \pi]$ between c and c' to be

$$\angle(c, c') := \limsup_{t,t' \to 0} \bar{\angle}_{c(0)}(c(t), c'(t')) \tag{3}$$

The angle exists in strict sense if the limsup equals the limit.

The angles are always less than or equal to π by our construction. We define angles greater than π in two-dimensional spaces by specifying sides and dividing the side into many parts (see Subsection 6.2).

We say that a sequence of closed subsets $\{K_i\}$ converge to a closed subset K if for any compact subset A of X, $\{A \cap K_i\}$ converges to $A \cap K$ in Hausdorff sense.

Let (X, d) be a metric. We can define a *length-metric* $\bar{d}$ so that $\bar{d}(x, y)$ for $x, y \in X$ is defined as the infimum of the lengths of all rectifiable curves joining x and y. We note that $d \leq \bar{d}$ and (X, d) is said to be a length space if $\bar{d} = d$.

Proposition 6.1 (Hopf-Rinow Theorem)**.** *Let (X, d) be a length space. If X is complete and locally compact, then every closed bounded subset of X is compact and X is a geodesic space.*

As an example, a Riemannian space with path-metric is a geodesic metric space. A covering space of a length space has an obvious induced length metric

We define M_κ to be the 3-sphere of constant curvature k, Euclidean space, or the hyperbolic 3-space of constant curvature k depending on whether $\kappa > 0, = 0, < 0$ respectively.

Let D_κ denote the diameter of M_κ if $\kappa > 0$ and let $D_\kappa = \infty$ otherwise.

Let (X, d) be a metric space. Let $\triangle$ be a geodesic triangle in X with parameter less than $2D_\kappa$ and $\bar{\triangle}$ the comparison triangle in M_κ. Then $\triangle$ is said to satisfy CAT(κ)-*inequality* if $d(x, y) \leq d(\bar{x}, \bar{y})$ for all x, y in the edges of $\triangle$ and their comparison points $\bar{x}, \bar{y}$, i.e., of same distance from the vertices, in $\bar{\triangle}$. If $\kappa < 0$, a *CAT(κ)-space* is a geodesic space all of whose triangles bounded by geodesics satisfy CAT(κ)-inequality. If $\kappa > 0$, then X is called a CAT*(κ)-space* if X is D_κ geodesic and all geodesic triangles in X of perimeter less than $2D_\kappa$ satisfy the CAT(κ)-inequality.

Angles exists in the strict sense for CAT(κ)-spaces if $\kappa \leq 0$.

A CAT(κ)-space is a CAT(κ')-space if $\kappa \leq \kappa'$.

A CAT(0)-space X has a metric $d : X \times X \to \mathbb{R}$ that is convex; i.e., given any two geodesics $c : [0, 1] \to X$ and $c' : [0, 1] \to X$ parameterized proportional to length, we have

$$d(c(t), c'(t)) \leq (1 - t)d(c(0), c'(0)) + td(c(1), c'(1)). \tag{4}$$

A geodesic n-simplex in M_κ is the convex hull of $n + 1$ points in general position.

An *M_κ-simplicial complex* K is defined to be the quotient space of the disjoint union X of a family of geodesic n-simplicies so that the projection $q : X \to K$ induces the injective projection p_λ for each simplex λ and if $p_\lambda(\lambda) \cap p_{\lambda'}(\lambda') \neq \emptyset$, there exists an isometry $h_{\lambda,\lambda'}$ from a face of λ to λ' such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.

In this paper, we will restrict to the case when locally there are only finitely many simplicies, i.e., X is locally convex. We do not assume that we have a finite isometry types of simplicies as Bridson does in [4].

A *geodesic link* of x in K, denoted by $L(x,K)$ is the set of directions into the union of simplicies containing x. The metric on it is defined in terms of angles. (For details, see Chapter I.7 of [5].)

Definition 6.2. An M_κ-simplicial complex satisfies the link condition if for every vertex v in K, the link complex $L(v,K)$ is a CAT(1)-space.

The following theorem can be found in Bridson-Haefliger [5]:

Theorem 6.3 (Ballman)**.** *Let K be a locally compact M_κ-simplicial complex. If $\kappa \leq 0$, then the following conditions are equivalent*:

(i) *K is a* CAT*(κ)-space.*
(ii) *K is uniquely geodesic.*
(iii) *K satisfies the link condition and contains no isometrically imbedded circle.*
(iv) *K is simply connected and satisfies the link condition.*

If $\kappa > 0$, then the following conditions are equivalent :

(v) *K is a* CAT*(κ)-space.*
(vi) *K is $\pi/\sqrt{\kappa}$-uniquely geodesic.*
(vii) *K satisfies the link condition and contains no isometrically embedded circles of length less than $2\pi/\sqrt{\kappa}$.*

Proof. See Ballmann [2] or Bridson-Haefliger [5]. □

Lemma 6.4. *A 2-dimensional M_κ-complex K satisfies the link condition if and only if for each vertex $v \in K$, every injective loop in $Lk(v,K)$ has length at least 2π.*

Definition 6.5. A metric space X is said to be of *curvature* $\leq \kappa$ if it is locally isometric to a CAT(κ)-space, i.e., for each point x of X, there exists a ball which is a CAT(κ)-space.

Theorem 6.6 (Cartan-Hadamard)**.** *Let X be a complete metric space.*

(i) *If the metric on X is locally convex, then the induced length metric on the universal cover $\tilde{X}$ is globally convex.* (*In particular, there is a unique geodesic connecting two points of $\tilde{X}$, and geodesic segments in $\tilde{X}$ vary continuously with respect to their endpoints.*)
(ii) *If X is of curvature $\leq \kappa$ where $\kappa \leq 0$, then $\tilde{X}$ is a* CAT(κ)*-space.*

Let δ be a positive real number. A geodesic triangle in a metric space X is said to be *δ-slim* if each of its sides is contained in the δ-neighborhood of the union of the other two sides.

For $\kappa < 0$, CAT(κ)-space is δ-hyperbolic.

For positive real numbers λ, ϵ, (λ,ϵ)-*quasi-geodesic* in X is a map $c : I \to X$ such that

$$1/\lambda|t-t'| - \epsilon \leq d(c(t),c(t')) \leq \lambda|t-t'| + \epsilon \text{ for any pair} t,t' \in \mathbb{R} \tag{5}$$

Let X be a δ-hyperbolic space. Two quasi-geodesic rays c, c' are *equivalent* or *asymptotic* if their Hausdorff distance is finite, or, equivalently $\sup_{t>0} d(c(t),c'(t))$ is finite. We define the *Gromov boundary* ∂X as the space of equivalence classes of quasi-geodesic rays in X. One can show that ∂X is the space of equivalence classes of geodesics rays as well.

If X is a proper metric space, then X is a visibility space: For each pair of points x and y in ∂X, there exists a geodesic limiting to x and y. Topology and metrics are given on ∂X to compactify $X \cup \partial X$. The group of isometry acts as homeomorphisms on ∂X.

6.2. **Sigular hyperbolic surfaces.** A *hyperbolic triangle* in a metric space is a subset isometric to a triangle in $\mathbb{H}^2$ bounded by geodesics. Sometimes, we need a *degenerate hyperbolic triangle*. It is defined to be a straight geodesic segment or a point where the vertices are defined to be the two endpoints and a point, which may coincide.

A *hyperbolic tetrahedron* in a metric space is a subset isometric to a tetrahedron in $\mathbb{H}^3$ bounded by four totally geodesic planes with six edges geodesic segments and four vertices. Again degenerate ones can obviously be defined on a hyperbolic triangle, a segment, and a point with various vertex and edge structures.

A *hyperbolic cone-neighborhood* of a point x in a surface Σ with a metric is a neighborhood of x which divides into hyperbolic triangles with vertices at x. The *cone-angle* is the sum of angles of the triangles at x. The set of singular points is denoted by $\mathrm{sing}(\Sigma)$ and the cone-angle at x by $\theta(x)$.

By a *singular hyperbolic surface*, we mean a complete metric space X locally isomorphic to a hyperbolic plane or a hyperbolic cone-neighborhood with cone-angle $\geq 2\pi$ so that the set of singular points are discrete. We will also require that X is triangulated by hyperbolic triangles in this paper (i.e., is a metric simplicial complex in the terminology of [5]).

By Lemma 6.4, the universal cover $\tilde{X}$ of X is a CAT(-1)-space.

Definition 6.7. Let X be a singular hyperbolic surface. Clearly, X has an induced length metric and is a geodesic space.

- We say that a geodesic in X is *straight* if it is a continuation of geodesics in hyperbolic triangles meeting each other at π-angles in the intrinsic sense.
- We can measure angles greater than π in singular hyperbolic surface by dividing the angle into smaller ones. In this case, we need to specify which side you are working on. In general a path is *geodesic* if it is a continuation of straight geodesic meeting each other at greater than or equal to π-angles from both sides.
- We also say that a boundary point x is *bent* if the two straight geodesics end at the point not at π-angle in the interior.
- For $x \in \partial X$, we define the *interior angle* to be the sum of angles of triangles with vertices at x and the *exterior angle* $\theta(x)$ to be π minus the interior angle. It could be negative.
- We will denote by sing(X) the set of singular points in the interior of X and sing(∂X) the set of bending points of ∂X.

Proposition 6.8 (Gauss-Bonnet Theorem)**.** *Let Σ be a compact singular hyperbolic surface with piecewise straight geodesic boundary. Then*

$$-\text{Area}(\Sigma) + \sum_{\text{v}\in\text{sing}(\Sigma)} (2\pi - \theta(\text{v})) + \sum_{\text{v}\in\text{sing}(\partial\Sigma)} \theta(\text{v}) = 2\pi\chi(\Sigma). \tag{6}$$

From the Gauss-Bonnet theorem, we can show that there exists no disk bounded by two geodesics. This follows since if such a disk exists, then $\theta(v) \geq 2\pi$ for all singular, the exterior angles at virtual vertices ≤ 0, the exterior angles at common end points $< \pi$, and the area is less than 0.

This implies: Given a compact singular hyperbolic space and a closed curve, we can homotopy the curve into a closed geodesic, and the closed geodesic is unique in its homotopy class.

Moreover, two closed geodesics meet in a minimal number of times up to arbitrarily small perturbations: that is, the minimum of geometric intersection number under small perturbation is the true minimum under all perturbations. (We may have two geodesics agreeing on an interval and diverging afterward unlike the hyperbolic plane.)

6.3. **General hyperbolic 3-manifolds.** By a *general hyperbolic manifold*, we mean a manifold M with an atlas of charts to $\mathbb{H}^3$ with transition maps in $\text{Isom}(\mathbb{H}^3)$. The metric on it will be the length metric given by the induced Riemannian metric. We require the metric to be complete. As a consequence, this is a geodesic space by local compactness [5]. In general we assume that ∂M is not empty. If it is not geodesically complete, M need not be a quotient of $\mathbb{H}^3$ which are the usual subject of the study in 3-manifold theory.

Also, we will require general hyperbolic manifolds to have hyperbolic triangulations, i.e., it has a triangulation so that each tetrahedron is isometric with a hyperbolic tetrahedron in $\mathbb{H}^3$. Moreover, we assume that the vertices of the triangulations are discrete and the induced triangulation on the universal cover map under **dev** to a collection of tetrahedra in general position in $\mathbb{H}^3$. We also require the following mild condition: Every boundary point of a general hyperbolic manifold has a neighborhood isometric with a subspace of a metric-ball in $\mathbb{H}^3$. By subdivisions and small modifications, we can always achieve this condition.

We will say that M is *locally convex* if there is an atlas of charts where chart images are convex subsets of $\mathbb{H}^3$. Thus, M is locally convex if ∂M is empty. (In this paper, we will be interested in the non-locally-convex manifolds.)

Given a general hyperbolic manifold M, its universal cover $\tilde{M}$ has an immersion $\mathbf{dev} : \tilde{M} \to \mathbb{H}^3$, which is not in general an imbedding or a covering map, and a homomorphism h from the deck transformation group $\pi_1(M)$ to $\text{Isom}(\mathbb{H}^3)$ satisfying

$$\mathbf{dev} \circ \vartheta = h(\vartheta) \circ \mathbf{dev}, \vartheta \in \pi_1(M). \tag{7}$$

dev is said to be a *developing map* and h a holonomy homomorphism.

Theorem 6.9 (Thurston)**.** *Let M be a metrically complete general hyperbolic 3-manifold and is locally convex. Then its developing map* **dev** *is an imbedding onto a convex domain, and M is isometric with a quotient of a convex domain in $\mathbb{H}^3$ by an action of a Kleinian group.*

Proof. See [17]. □

In this paper, we will often meet *drilled hyperbolic manifolds* obtained by removing the interior of a codimenion-zero submanifold of a general hyperbolic manifolds. They are of course general hyperbolic manifolds.

Of course, special hyperbolic manifolds are general hyperbolic manifolds and drilled hyperbolic manifolds.

Since a general hyperbolic manifold has a geodesic metric, we can define geodesics. A *straight* geodesic is a geodesic which maps to geodesic in $\mathbb{H}^3$ under the charts. Geodesics are in general a union of straight geodesics. Thus, it has many bent points in general. The bent points in the interior of the geodesic segments are said to be *virtual vertices.*

We define angles as above for metric spaces. Then at a virtual vertex, the angle is equal to π since if not, then we can shorten the geodesics.

Proposition 6.10. *Let l be a geodesic with a bent point x in its interior. Let S be a simplicially immersed surface containing l in its boundary. Then for S with an induced length metric, the interior angle at x in S is always greater than or equal to π.*

Proof. If the angle is less than π, we can shorten the geodesic. □

Given an oriented geodesic l_1 ending at a point x and an oriented geodesic l_2 starting from x, we define an *exterior angle* between l_1 and l_2 to be π minus the angle between the geodesic l_1' with reversed orientation and the other one l_2.

7. 2-convex general hyperbolic manifolds and h-maps of surfaces

7.1. 2-convexity and general hyperbolic manifolds.

In Part 1, we showed that a general hyperbolic manifold was 2-convex if the vertices of the boundary were either saddle-vertices or convex vertices.

We recall the definition of 2-convexity:

Definition 7.1. A general hyperbolic manifold is 2-*convex* if given a compact subset K mapping to a union of three sides and the interior T^o of a tetrahedron T in $\mathbb{H}^3$ under a chart ϕ of the atlas, there exists a subset T' mapping to T by a chart extending ϕ.

Proposition 7.2. *If M is a 2-convex general hyperbolic manifold, then M is a $K(\pi_1(M))$, i.e., its universal cover is contractible.*

Proof. Since the universal cover $\tilde{M}$ has an affine structure with trivial holonomy induced from the affine space containing $\mathbb{H}^3$ from the Klein model, this follows from [10]. Also, this follows from Theorem 7.3. □

Theorem 7.3. *Let M be a 2-convex general hyperbolic manifold. Then its universal cover $\tilde{M}$ is a M_{-1}-simplicial complex and a* CAT(−1)*-space.* (M *has a curvature* ≤ -1.)

Proof. Using Theorem 6.3 (iv), we need to show that for each vertex x of $\tilde{M}$, the link $P = L(x, \tilde{M})$ is a CAT(1)-space.

To show P is a CAT(1)-space, we use (vii) of Theorem 6.3; i.e., we show that P satisfies the link condition and contains no isometric circle of length $< 2\pi$. By the boundary condition on M, P is isometric to the unit sphere if x is the interior point or is isometric to a subspace of the unit sphere if x is the boundary point. Clearly the former satisfy the 2-dimensional link condition.

Let P be a proper subspace of a unit sphere $\mathbf{S}^2$ and c an isometrically imbedded circle of length $< 2\pi$. By Lemma 7.4, c is disjoint from a closed hemisphere H in $\mathbf{S}^2$.

The closed curve c meets ∂P since otherwise c has to be a great circle of length 2π being a geodesic. As c may never cross-over the circle ∂P, let D_c^1 and D_c^2 denote the disks in $\mathbf{S}^2$ bounded by c. Then ∂P is a subset of D_c^1 or D_c^2. Assume without loss of generality that the former is true.

Since c is disjoint from the hemisphere H, it follows that H is a subset of D_c^1 or D_c^2. In the second case, $H \subset P$. Looking at this situation, from the vertex $x \in M$ again, we see that 2-convexity is violated since we can find a triangle in M containing x in its interior whose one-sided neighborhood with x removed is in the interior of M.

Suppose that H is a subset of D_c^1. Let H' be the complementary open hemisphere. Then $c \subset H'$ and ∂P is outside the disk D_c^2 in H' bounded by c. Since H' has a natural affine structure, and c is compact, it follows that the convex hull K' of c in H' is compact. Let y be an extreme point of K, where $y \in c$ as well. In the one-sided neighborhood of y inside c, there are no points of ∂P implying that we can shorten c in P contrary to the fact that c is isometrically imbedded. □

Lemma 7.4. *Let γ be a broken geodesic loop in the sphere $\mathbf{S}^2$ of radius 1. If the length of γ is less than 2π, then there exists an open hemisphere containing it (and hence a disjoint closed hemisphere).*

Proof. We can shorten the loop without increasing the number of broken points to a loop as short as we want. A sufficiently short loop is contained in an open hemisphere.

Let $l_t, t \in [0,1]$ be a homotopy so that l_1 is the original loop and l_0 is a constant loop. Then let A be the maximal connected set containing 0 so that l_t for $t \in A$ is contained in an open hemisphere, say H_t.

The set A is an open set since the small change in l_t does not violate the condition. Suppose that the complement of A is not empty, and let t_0 be the greatest lower bound of the complement of A. Then l_{t_0} is contained in a closed hemisphere, say H', since we can find a geometric limit of the closure of H_ts.

Suppose that $\partial H' \cap l_{t_0}$ is contained in a subset of length strictly less than π. Then we can rotate H' along a pivoting antipodal pair of points on $\partial H'$ outside the subset. Then the new hemisphere contains l_{t_0} in its interior, a contradiction.

Suppose that $\partial H' \cap l_{t_0}$ contains a pair of antipodal points. Let s_1 and s_2 be the corresponding points of $[0,1]$ and suppose $0 < s_1 < s_2 < 1$ without loss of generality. Then two arcs $l_{t_0}|[s_1, s_2]$ and $l_{t_0}|[s_2, 1] \cup [0, s_1]$, must have length greater than or equal to π, a contradiction. Therefore, no subsegment in $\delta H'$ of length $\leq \pi$ contains $\partial H' \cap l_{t_0}$.

Suppose now that there are three points p_1, p_2, p_3 in $\delta H' \cap l_{t_0}$ are not contained in a subsegment in $\delta H'$ of length $\leq \pi$ and no pair of them are antipodal.

The sum of lengths of segments $\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_1}$ equals 2π. This is clearly less than or equal to that of l_{t_0} since the shortest arcs connecting the pairs (p_1, p_2), (p_2, p_3), (p_3, p_1) are these segments respectively. This is again a contradiction.

Thus A must be all of $[0,1]$. □

The following proves Theorem 5.1 in detail.

Proposition 7.5. *Let $\tilde{M}$ be a universal cover of a compact 2-convex general hyperbolic manifold M. Then the following hold:*

- *$\tilde{M}$ is uniquely geodesic.*
- *Geodesic segments of $\tilde{M}$ depend continuously on their endpoints.*
- *The metric is locally convex.*
- *$\tilde{M}$ is δ-hyperbolic and hence it is a visibility manifold.*
- *M has curvature ≤ -1.*
- *Given any path class on M, there exists a unique geodesic segment, which depends continuously on endpoints.*

Proof. These are direct consequences of the fact that $\tilde{M}$ is a CAT(-1)-space. □

7.2. **Hyperbolic-maps of surfaces into 2-convex general hyperbolic manifolds.** A *triangulated hyperbolic surface* is a metric surface with or without boundary triangulated and each triangle is isometric with a hyperbolic triangle or a degenerate hyperbolic triangle in $\mathbb{H}^2$. A *half-space* of $\mathbb{H}^3$ is a subset bounded by a totally geodesic plane.

Definition 7.6. Let Σ be a compact triangulated hyperbolic surface, M a general hyperbolic 3-manifold, and $f : \Sigma \to M$ a map which sends each triangle to a hyperbolic triangle in M. Let $\partial\Sigma$ have distinguished vertices $v_1, \ldots, v_n$. Then f is a *hyperbolic-map relative to* $\{v_1, \ldots, v_n\}$ if the sum of the angles of the image triangles of the stellar neighborhood of each interior vertex v is greater than or equal to 2π and the sum of angles of the image triangles of the stellar neighborhood of the boundary vertex v, $v \neq v_i$, is greater than or equal to π.

A hyperbolic-map is a completely analogous concept to a hyperbolic-map by Bonahon [3], Canary and Minsky and so on. Note that if the boundary portion between v_i and v_{i+1} is geodesic for each i, then the boundary angle conditions are satisfied also.

Definition 7.7. Given an arc or a point α and an arc β in M, an *Alexandrov net with ends α and β* is a map $f : I \times I \to M$ so that $s \in I \mapsto f(t, s)$ is geodesic for each s and $t \mapsto f(t, 0)$ is α and $t \mapsto f(t, 1) \in \beta$.

Lemma 7.8. *Let M be a 2-convex general hyperbolic manifold. Let γ be a geodesic in M. Then for any geodesic γ' sufficiently close to γ, there exists a homotopy $H : I \times I \to M$ so that the following hold :*

- *$s \mapsto H(0, s)$ is γ and $s \mapsto H(1, s)$ is γ'.*
- *H is a simplicial map with a triangulation of $I \times I$ with all vertices at $\{0, 1\} \times I$.*

Proof. For each virtual vertex of γ, we choose a real number $\epsilon > 0$ such that the ϵ-neighborhood of the vertex is a stellar neighborhood. If γ' is in an ϵ-neighborhood of γ for $\epsilon > 0$ for any such ϵs, then we can find the desired H. □

Given a point or an arc α and another arc β, an *A-net* $f : I \times I \to M$ with ends α and β is a map such that

- $s \mapsto f(t_i, s)$ for a finite subset $\{t_1 = 0, t_2, \ldots, t_n = 1\}$ of I is a geodesic for each i,

- $t \mapsto f(t, 0)$ is α and $t \mapsto f(t, 1)$ is β.
- f is a hyperbolic-map relative to vertices of the arcs α and β with a triangulation of $I \times I$ with all the vertices in $\{t_1, \ldots, t_n\} \times I$.

Proposition 7.9. *Given a point or an arc α and another arc β, there exists an A-net with ends α and β.*

Proof. We find an Alexandrov net $f : I \times I \to M$ with ends α and β. We take sufficiently many t_i's so that geodesics $s \mapsto f(t_i, s)$ are very close. By Lemma 7.8, we can find a simplicial map $F : I \times I \to M$. Since $s \mapsto F(t_i, s) = f(t_i, s)$ are geodesics, the sum of angles at each of the sides of a vertex on this geodesic is greater than π. Hence, the sum of angles at an interior vertex is greater than or equal to 2π. At the vertices of $s \mapsto F(0, s)$ or $s \mapsto F(1, s)$, the sum of angles are greater than π. Therefore, F is a hyperbolic-map. □

Proposition 7.10. *Let Σ be a compact triangulated surface, M a general hyperbolic 3-manifold, and let $f : \Sigma \to M$ be a map with an injective induced homomorphism $f_* : \pi_1(\Sigma) \to \pi_1(M)$.*

- *Let $v_1, \ldots, v_n$ be the distinguished vertices in $\partial\Sigma$ and l be a union of disjoint simple closed curve in Σ which is disjoint from $\{v_1, \ldots, v_n\}$ and is a component of $\partial\Sigma$ or is disjoint from $\partial\Sigma$.*
- *We suppose that $\{v_1, \ldots, v_n\} \cup l$ is not empty. Suppose that f maps each arc in $\partial\Sigma$ connecting two distinguished vertices to a geodesic and each component of l or $\partial\Sigma$ without any of $v_1, \ldots, v_n$ to a closed geodesic.*

Then in the relative homotopy class of f with $f|\partial\Sigma$ fixed, there exists a hyperbolic-map $f' : \Sigma' \to M$ relative to $v_1, \ldots, v_n$ where Σ' is Σ with a different triangulation in general and f' agrees with f on $\partial\Sigma \cup l$.

From now on, we will just use v_i for $f(v_i)$ and so on since the reader can easily recognize the difference. By the angle of a triangle, we mean the corresponding angle measured in the image triangle of f.

Proof. First, we find a topological triangulation Σ so that all the vertices are in the union of $\{v_1, \ldots, v_n\} \cup l \cup \partial\Sigma$. We find a geodesic in the right path-class for each of the edges of the triangulations. For each triangle, we extend by choosing a vertex and the opposite geodesic edges and finding A-nets with these ends.

At each interior point of an edge, we see that the sum of angles of any of its side is greater than or equal to π since the edge is geodesic. Since A-nets are hyperbolic-maps, we see that the whole map is a hyperbolic-map. □

7.3. **Gauss-Bonnet theorem for hyperbolic-maps.**

Proposition 7.11. *Let Σ be a compact hyperbolically-mapped surface relative to $v_1, \ldots, v_n$. Let θ_i be the exterior angle of v_i with respect to geodesics in the boundary of Σ. Then*

$$\mathrm{Area}(\Sigma) \leq \sum_{\mathrm{i}} \theta_{\mathrm{i}} - 2\pi\chi(\Sigma). \tag{8}$$

Proof. The interior angle with respect to Σ is larger than the angle in M itself. Thus the exterior angle with respect to Σ is smaller than the exterior angle in M.

Since the interior vertices have the angle sums greater than or equal to 2π and the boundary virtual vertices have the angle sum greater than or equal to π, the proposition follows from the Gauss-Bonnet theorem. □

An n-gon is a disk with boundary a union of geodesic segments between n vertices.

Corollary 7.12. *Let S be a hyperbolically-mapped n-gon. Then* $\mathrm{Area(S)} \leq (\mathrm{n}-2)\pi$.

Proof. The exterior angle of a bent virtual vertex on a geodesic is always less than π. □

8. Convex hulls in 2-convex general hyperbolic manifolds

Let M be a 2-convex general hyperbolic manifold with finitely generated fundamental group, and $\mathcal{C}$ denote a core of M.

Let $\tilde{M}$ be the universal cover of M. Since $\mathcal{C} \to M$ is a homotopy equivalence the subset $\tilde{\mathcal{C}}$ in $\tilde{M}$ which is the inverse image of $\mathcal{C}$ is connected and is a universal cover of $\mathcal{C}$. A subset of $\tilde{M}$ is *convex* if any two points can be connected by a geodesic in the subset.

The *convex hull* $\mathrm{convh(K)}$ of a subset K of $\tilde{M}$ is the smallest closed convex subset containing K. Since $\tilde{\mathcal{C}}$ is deck-transformation group invariant, and the convex hull is the smallest convex subset, $\mathrm{convh}(\tilde{\mathcal{C}})$ is deck-transformation group invariant. Therefore, $\mathrm{convh}(\tilde{\mathcal{C}})$ covers its image. We define the image as $\mathrm{convh}(\mathcal{C})$, i.e., $\mathrm{convh}(\tilde{\mathcal{C}})$ quotient by the deck-transformation group action.

Since $\mathcal{C}$ is a 3-dimensional domain, $\mathrm{convh}(\mathcal{C})$ is a 3-dimensional closed set.

Proposition 8.1. *The convex hull* $\mathrm{convh}(\mathcal{C})$ *of the compact core $\mathcal{C}$ of M is homotopy equivalent to $\mathcal{C}$.*

Proof. Let $\tilde{\mathcal{C}}$ be the inverse image of $\mathcal{C}$ in the universal cover $\tilde{M}$ of M. Then $\tilde{\mathcal{C}}$ and $\tilde{M}$ are both contractible as M and $\mathcal{C}$ are irreducible 3-manifolds.

A closed curve in the convex hull $\mathrm{convh}(\tilde{\mathcal{C}})$ of $\tilde{\mathcal{C}}$ bounds a disk in $\mathrm{convh}(\tilde{\mathcal{C}})$ since a distinguished point on the curve can be connected by a geodesic in any other point of the curve. Similarly, a sphere always bounds a 3-ball. Therefore, $\mathrm{convh}(\tilde{\mathcal{C}})$ is contractible. □

A surface is *pleated* if through each point of it passes a straight geodesic.

Recall that the pleated-triangulated surface is an imbedded surface where a closed subdomain is a union of a locally finite collection of totally geodesic convex domains meeting each other in geodesic segments and the complementary open surface is pleated.

A pleated-triangulated surface is *truly pleated-triangulated* if the triangulated part are union of totally geodesic polygons in general position.

A *truly pleated-triangulated hyperbolic-surface* is a truly pleated-triangulated surface where each vertex of the triangles is a hyperbolic-vertex.

Lemma 8.2. *If a geodesic in M contained in S passes through a vertex in the triangulated part of S, then the vertex is a hyperbolic-vertex.*

Proof. A neighborhood of a point of the triangulated part is stellar. If a geodesic passes through, the angles in both sides are greater than or equal to π: otherwise, we can shorten the geodesic. Hence, the sum of the angles is greater than or equal to 2π. □

The following proves Theorem 5.3:

Proposition 8.3. *Let K be a deck-transformation-group invariant codimension 0 submanifold of $\tilde{M}$ with ∂K saddle-imbedded. Also, suppose K is a subset of $\tilde{M}^o$. The boundary of* convh(K) *can be given the structure of a convex truly pleated-triangulated hyperbolic-surface.*

Proof. We will show that

- through each point of ∂convh(K) a geodesic in ∂convh(K) passes or
- the point is in the triangulated part and is a saddle-vertex or a hyperbolic-vertex.

Let x be a boundary point of convh(K): Suppose that x is a point of the manifold-interior of $\tilde{M}$. Take a ball $B_\epsilon(x)$ in the interior for a sufficiently small ϵ. Then $\mathrm{convh(K)} \cap \mathrm{B}_\epsilon$ is the convex hull of itself. Since B_ϵ is isometric with a small open subset of $\mathbb{H}^3$, the ordinary convex hull theory shows that there exists a geodesic in the boundary of the convex hull through x: If not, we can find a small half-open ball to decrease the convex hull as the side of the half-open ball cannot meet ∂K by the saddle-imbeddedness of ∂K.

Suppose that x is in the topological interior of convh(K) but in $\partial \tilde{M}$. There exists a neighborhood of x in convh(K) with manifold-boundary in $\partial \tilde{M}$. If x is in the interior of an edge or a face of $\partial \tilde{M}$, then there is a geodesic through x obviously. Suppose that x is a vertex of $\partial \tilde{M}$. x can be a saddle-vertex or a convex vertex (see Proposition 2.5).

- If x is a convex vertex of $\partial \tilde{M}$, we can find a truncating totally geodesic hyperplane and a sufficiently small disk in it bounding a neighborhood of x in $\tilde{M}$. Since K is disjoint from the disk, we see that x is not in the convex hull. This is absurd.
- If x is a saddle-vertex of $\partial \tilde{M}$, then x is a saddle-vertex of ∂convh(K).

Assume from now on that x is a point in the topological boundary of convh(K) and on $\partial \tilde{M}$. This means that x is in the frontier of the open surface $C = \partial\mathrm{convh(K)} - \partial\tilde{\mathrm{M}}$.

(a) Suppose x is a point of the interior of a triangle T in $\partial \tilde{M}$. The set $T \cap \mathrm{convh(K)}$ is a convex subset and x lies in the boundary. The boundary must be a geodesic since we can use a small half-open ball to decrease the convex hull otherwise. Hence there is a geodesic through x.

(b) Suppose that x is a point of the interior of an edge in $\partial \tilde{M}$. We take a small ball $B_\epsilon(x)$ around x, which is isometric with a ball in $\mathbb{H}^3$ of the same radius and a wedge removed. The line l of the wedge passes through x.

Let P_1 and P_2 be the totally geodesic plane extended in $B_\epsilon(x)$ from the sides of the wedge. We denote by P_1' the set $\partial B_\epsilon(x) \cap P_1$ and P_2' the set $\partial B_\epsilon(x) \cap P_2$. We can form two convex subsets L_1 and L_2 in $B_\epsilon(x)$ that are the closures of the components of $B_\epsilon(x) - P_1 - P_2$ and adjacent to P_1' and P_2' respectively.

The set $L_1 \cap \text{convh(K)}$ is a convex subset of L_1 and $L_2 \cap \text{convh(K)}$ one of L_2. The open surface C may intersect L_1 or L_2 or both.

If C is disjoint from L_1, then it maybe that a one-sided neighborhood of x in a triangle in $\partial\tilde{M}$ is a subset of convh(K) and an edge of the triangle is a geodesic through x. (The side $P_1 - P_1^{\prime,o}$ of $B_\epsilon(x)$ is in convh(K).) Otherwise, convh(K) is contained in a convex subset of $B_\epsilon(x)$ bounded by P_1. In this case, the ordinary convexity in $\mathbb{H}^3$ holds and there is a geodesic in ∂convh(K) through x.

Since the same argument holds with L_2 as well, we assume that $C \cap L_1$ and $C \cap L_2$ are both not empty.

If $C \cap L_1$ or $C \cap L_2$ are totally geodesic surfaces, then x is on a pleating locus that is the edge of the wedge.

We may assume without loss of generality that $C \cap L_1$ is not totally geodesic in a ball of radius ϵ about x for every sufficiently small $\epsilon > 0$. Then there exists a sequence of points $\{x_i \in L_1 \cap \tilde{M}^o\}$ converging to x and a sequence of pleating lines

$$\{l_i \subset \partial\text{convh(K)} \cap \tilde{\text{M}}^{\text{o}}\}$$

so that $x_i \in l_i$. By Lemma 8.4, we only have to worry about the case when all l_is end at x. In this case there exists a small neighborhood $B(x)$ of x such that $C \cap L_1 \cap B(x)$ is a stellar set with vertex at x.

By a same argument, $C \cap L_2 \cap B(x)$ is a stellar set also with a vertex at x. Considering $C \cap L_1 \cap B(x)$ and $C \cap L_2 \cap B(x)$ at the same time, in order that at x, the convexity to hold true and x to be in ∂convh(K), we see that $C \cap L_1 \cap B(x)$ has to have a unique pleating geodesic with a convex dihedral angle as seen from convh(K) and so does $C \cap L_2 \cap B(x)$. Furthermore, their unique pleating geodesics must extend each other as geodesics passing through x.

(c) Now assume that x in ∂convh(K) is a vertex of $\partial\tilde{M}$.

Let $B_\epsilon(x)$ be a small neighborhood of x so that $B_\epsilon(x) \cap \tilde{M}$ is a stellar set from x. As before x is in the boundary of C.

Suppose first that there are no pleating lines with a sequence of points on them converging to x. We can choose a small ϵ so that $B_\epsilon(x) \cap \text{convh(K)}$ is a stellar set.

Let M' be an ambient general hyperbolic manifold containing M in its interior which is homeomorphic to the interior of M as there are always such a manifold. We claim that x is a saddle-vertex of $B_\epsilon(x) \cap \partial\text{convh(K)}$: If not, we can find a small half-open ball B in $\tilde{M}'$ with a totally-geodesic side passing through x with B^o disjoint from ∂convh(K). By stellarity, B^o is disjoint from convh(K) and we can decrease convh(K) if $x \notin K$, which is a contradiction. If $x \in K$, then there is no such B as x is a saddle vertex of K itself. Therefore, x is a saddle-vertex.

We assume that $B_\epsilon(x) \cap \text{convh(K)}$ is not a stellar set. Suppose now that there exists a sequence of points $\{x_i \in l_i\}$ converging to x where l_i is a distinct pleating line for each i and does not end at x. Here, l_i are infinitely many. Lemma 8.4 shows that the endpoints of l_i are bounded away from x. Therefore, a subsequence of l_i converges to a geodesic l passing through x.

We proved the two items above, and ∂convh(K) is a pleated-triangulated surface.

As a final step, we show that ∂convh(K) is a truly pleated-triangulated surface: Let A be the closure of the set of all points in ∂convh(K) intersected with the interiors of triangles in $\partial\tilde{M}$. Then A is a locally finite union of totally geodesic polygons and segments. The complement of A in ∂K is pleated since it lies in the interior of $\tilde{M}$. Any pleated lines in $\tilde{M}^o$ must end at A or is infinite. By Remark 4.13, the set of pleating lines ending at A is isolated.

By Lemma 8.2, it is a hyperbolic-surface as well. The convexity is obvious. □

Lemma 8.4. *Let l_i, $i \in I$, be a collection of mutually distinct straight pleating lines $\partial\mathrm{convh(K)} - \partial\tilde{\mathrm{M}}$ for a convex hull $\mathrm{convh(K)}$ of a closed subset K of $\tilde{M}$ and a countably-infinite index set I. Suppose $x_i \in l_i$ form a sequence converging to x but x is not on l_is. Then the endpoints of l_is are bounded away from x and a subsequence of l_i converges to a line segment in the pleating locus containing x in its interior.*

Proof. Suppose that the endpoints of l_i are bounded away from x. Then the second statement holds obviously.

Suppose that the endpoints q_i of l_i form a sequence converging to x. Then we may assume without loss of generality that q_i lies in an arc or a point α in a triangle in $\partial\tilde{M}$. If the arc α is a convex curve, we can decrease convh(K) further, a contradiction. Thus α is a geodesic or a point.

By Lemma 4.14, α cannot be a line. If α is a point, then $\alpha \neq x$, and the conclusion holds. □

Part 3. The proof of the tameness of hyperbolic 3-manifolds

9. Outline of the proofs

We will prove Theorems A and B in this part: The strategy is as follows. Suppose that the unique end E of M not associated with incompressible surface is not geometrically finite and is not tame. We find an exhausting sequence M'_i in M so that M'_i contains neighborhoods of all tame and geometrically finite ends and meets the neighborhood of the infinite end in a compact subset.

Step 1: Using the work of Freedman-Freedman [12], we can modify M'_is to be compression bodies: Since E is not geometrically finite, we can choose a sequence of closed geodesics $\mathbf{c_i} \to \infty$. We fix a sufficiently small Margulis constant ϵ. We assume without loss of generality that $\mathbf{c_i} \subset \mathbf{M'_i}$. Let μ_i be the union of closed geodesics that are in the Margulis tubes in M_i. We further modify M'_i so that $\partial M'_i$ is incompressible in $M - \mathbf{c_1} - \cdots \mathbf{c_i} - \mu_\mathbf{i}$ with the compact core $\mathcal{C}$ removed.

The manifold N_i is obtained from compressing disks for M'_i in $M - \mathbf{c_1} - \cdots - \mathbf{c_i} - \mu_\mathbf{i} - \mathcal{C}$. Let A_i be a homotopy in M between $\mathbf{c_i}$ and the closed curve in $\mathcal{C}$, which can be homotopied to be in N_i.

Now we modify N_i to M''_i so that $\mathbf{c_i} \subset \mathbf{M''_i}$ and M''_i are compression bodies and the boundary component of M''_i corresponding to E is incompressible in M removed with the closed geodesics $\mathbf{c_1} \ldots, \mathbf{c_i}$ and $\mathcal{C}$ and contains any Margulis tubes that M''_i meets. (See Subsection 10.1.)

Step 2: As the boundary is incompressible, we take a 2-convex hull M_i of M''_i using crescents (see Part 1). This implies that M_i is a polyhedral hyperbolic space and hence is CAT(-1). Since M_i is isotopic to M_i'', the homotopy A_i between $\mathbf{c_i}$ and a closed curve in $\mathcal{C}$ still exists.

We show that we can choose a Margulis constant independent of i and the thin part of M_i are contained in the original Margulis tubes of M and are homeomorphic to solid tori with nontrivial homotopy class in the original tubes. (See Subsection 10.2.)

Step 3: We take the cover L_i of M_i corresponding to the fundamental group of the fixed compact core $\mathcal{C}$ of M. Since M_i is tame, the cover L_i is shown to be tame (this is from ideas of Agol).

The core $\mathcal{C}$ lifts to the cover L_i and can be considered a subset. We take a convex hull K_i of $\mathcal{C}$ in L_i, which is shown to be compact. Since K_i is homotopy equivalent to $\mathcal{C}$ by Proposition 8.1 in Part 2, the boundary component Σ_i of K_i corresponding to E has the same genus as that of a boundary component of $\mathcal{C}$. Σ_i is a "hyperbolic surface" (see Part 2). Since $\mathbf{c_i}$ is an exiting sequence, and an ϵ-neighborhood of M_i contains $\mathbf{c_i}$, it follows that $p_i|\Sigma_i$ is an exiting sequence of surfaces. This proves Theorem A. (See Subsection 10.3.)

Step 4: We push $\mathcal{C}$ inside itself so that $\mathcal{C}$ does not meet ∂K_i. We now remove the core from K_i to obtain $K_i - \mathcal{C}^o$. We can find a simple closed curve α in Σ_i compressible in K_i. We realize α by a closed geodesic α^* in $K_i - \mathcal{C}^o$. Using it, as Bonahon does [3], we obtain a simplicial hyperbolic surface T_i meeting $\partial\mathcal{C}$.

Then by compactness of bounded simplicial hyperbolic surfaces of Souto [16], infinitely many immersed T_is are isotopic in $M - \mathcal{C}^o$ and hence infinitely many of $p_i|\Sigma_i$ are isotopic. Since $p_i|\Sigma_i$ are exiting and are isotopic in $M - \mathcal{C}^o$, this implies that the end E is topologically tame, proving Theorem B. (See Subsection 10.4.)

10. The Proof of Theorem A

10.1. Choosing the right exhausting sequence.

10.1.1. *Choosing the core.* As a preliminary step, we choose the compact core more carefully so that $\partial\mathcal{C}$ is saddle-imbedded: We choose incompressible closed surfaces F_i associated with incompressible ends E_i to be strictly saddle-imbedded by Theorem C and disjoint from one another (see Remark 4.19). We choose a number of closed geodesics in E_i and choose a mutually disjoint submanifold homeomorphic to $F_i \times I$ disjoint and between these curves for each i. Then by Theorem C, we find a mutually disjoint collection of manifolds in the respective neighborhoods of E_i between these curves whose boundary components are strictly saddle-imbedded.

Essentially $\partial\mathcal{C}$ is considered as a regular neighborhood of the union of the essential surfaces $F_1, \ldots, F_n$ and a number of arcs connecting them in some manner.

We choose each of the arcs to be the shortest path in M among the arcs connecting the surfaces F_is with the respectively given homotopy classes. Their endpoints must be in the interior of an edge of a triangle. By perturbing F_i if necessary, we may assume that they are all disjoint geodesics. We first take thin regular neighborhoods of F_is which are triangulated. We take thin regular neighborhoods of the geodesics which are triangulated and all of whose vertices lie in F_is.

We take the union of the regular neighborhood of these geodesics with those of F_is to be our core $\mathcal{C}$. We may assume that $\partial\mathcal{C}$ is strictly saddle-imbedded as well. (We may need to modify a bit where the neighborhoods meet.) As stated above, we choose $\mathcal{C}$ to be in the interior M^o. Obviously, if necessary, we push $\mathcal{C}$ inward itself without violating strict saddle-imbeddedness of $\partial\mathcal{C}$.

10.1.2. *Choosing a compression body exhaustion.* Let M be as in the introduction, and let $U_1, \ldots, U_n$ be mutually disjoint neighborhoods of incompressible ends $E_1, \ldots, E_n$. Suppose that the end E is a geometrically infinite but not geometrically tame.

Let $\hat{M}$ be the 2-convex hull of M with $U_1, \ldots, U_n$ removed. The boundary components $F_1, \ldots, F_n$ corresponding to $U_1, \ldots, U_n$ of $\hat{M}$ are saddle-imbedded respectively.

Let M_i' be an exhaustion of M by compact submanifolds containing $\hat{M}$. We extend M_i' by taking a union with $U_1, \ldots, U_n$ so that M_i' meets neighborhoods of E in compact subsets or in the empty set. We assume that M_i' contains the boundary components $F_1, \ldots, F_n$ and contains the core $\mathcal{C}$ of M always.

Lemma 10.1. *A disk with boundary outside the union of Margulis tubes may be isotopied with the boundary of the disk fixed so that the intersection is the union of meridian disks.*

Proof. First, we perturb the disk to obtain transversal intersection. A disk may meet the Margulis tubes in a union of planar surfaces. The boundary of the Margulis tube meets the disk in a union C of circles. If an innermost component is outside the tube, then since the boundary tube is incompressible in M with the interior of the Margulis tubes removed, it follows that we can isotopy it inside. This means that the innermost components are disks.

If an innermost component of C is a circle bounding a disk in the boundary of the Margulis tube, then we can isotopy the bounded disk away from the tube. Now, each component of C is a meridian circle. □

We fix a small Margulis constant $\epsilon_M > 0$.

Proposition 10.2 (Freedman-Freedman, Ohshika)**.** *We obtain a new exhaustion M_i' so that each M_i' is homeomorphic to a compression-body with incompressible boundary components removed. M_i' contains the Margulis tubes that M_i' meets by taking union with these.*

Proof. We essentially follow Theorem 2 of Freedman-Freedman [12]. We assume that each M_i' includes any Margulis tubes it meets. ∂M has no incompressible closed surface other than ones parallel to $F_1, \ldots, F_n$. Hence, we compress the boundary component $\partial M_i'$ until we obtain a union of balls and manifolds homeomorphic to F_i times an interval. An exterior disk compression adds a disk times I to the compressed manifold from M_i' but an interior disk compression removes a disk times I from the manifold. The exterior disk may meet Margulis tubes outside M_i'. By Lemma 10.1, if the disk meets a Margulis tube at an essential disk, then we include the Margulis tube. If not, we push the disk off the Margulis tube. This operation amounts to adding 1-handles to the manifolds.

We recover our loss to M_i' by interior disk compressions by attaching 1-handles each time we do interior compressions. The core arcs of 1-handles may meet the exterior compression disk many times. We add a small neighborhood of the cores first. Then we isotopy to make it larger and larger to recover the loss due to interior disk compression while fixing the Margulis tubes outside M_i. (This may push the exterior compression disks.) We also recover all the Margulis tubes originally in M_i'.

From the surface times the interval components, we add all the 1-handles to obtain the desired compression body.

□

Remark 10.3. We do interior compressions first and then exterior compressions. This is sufficient to obtain the union of 3-balls and F_i times the intervals. The reason is that we can isotopy any interior compressing curve away from the traces of exterior compression disks.

Since E is geometrically infinite, there exists a sequence of closed geodesics $\mathbf{c_i}$ tending to E by Bonahon [3]. We assume that $\mathbf{c_i} \subset \mathbf{M_i'}$ for each i since M_i' is exhausting. Let $\mathcal{C}_i$ denote the union of $\mathbf{c_1}, \dots, \mathbf{c_i}$. We assume without loss of generality that M_i' has a free-homotopy A_i between $\mathbf{c_i}$ and a closed curve in $\mathcal{C}$ since M_i's form an exhausting sequence. Let μ_i be the union of the simple closed geodesics in the Margulis tubes that M_i' contains.

Remark 10.4. We know that given a surface there exists a finite maximal collection of exterior essential compressing disk so that any other exterior essential compressing disks can be pushed inside the regular neighborhood of their union and some arcs on the surface connecting them. This is from the uniqueness of the compression body. (See Theorem 1 in Chapter 5 of McCullough [14].)

10.1.3. *Compression bodies M_i'' with "incompressible" boundaries.*

N_i with "incompressible" boundaries: If we remove the interior $\mathcal{C}^o$ of the core from M_i', and compress the boundary $\partial M_i'$ in $M - \mathcal{C}^o - \mathcal{C}_i - \mu_i$ and remove resulting cells to obtain a manifold with incompressible boundary containing $\partial\mathcal{C}$. Then we join the result with $\mathcal{C}$, $\mathcal{C}_i$ and μ_i. Let us call the resulting 3-manifold N_i. Note that N_i need not be a compression body and may not form an exhausting sequence. However, N_i contains $\mathcal{C}, \mathcal{C}_i, \mu_i$ for each i.

The exterior compressing disk of M_i' may meet the Margulis tubes outside M_i'. We may assume that N_i meets these Margulis tubes in meridian disks times intervals by Lemma 10.1.

The manifold N_i is obtained from compressing disks for M_i'. Let A_i be a homotopy in M_i' between $\mathbf{c_i}$ and a closed curve in $\mathcal{C}$. Then a compressing disk for the sequence of manifolds obtained from M_i' by disk-compressions in $M - \mathcal{C}^o - \mathcal{C}_i - \mu_i$ may meet A_i in immersed circles. Since the circles bound immersed disks in the compressing disk, and $\mathbf{c_i}$ is not null-homotopic in M, we can modify A_i so that A_i has one less number of components where A_i meets the compressing disk. In this manner, we can find A_i in N_i.

N_i^{III}: We do the following steps:

- We find a maximal collection of essential interior compressing disks for N_i and do disk compressions. By incompressibility, the disk must meet one of $\mathcal{C}, \mathcal{C}_i, \mu_i$ essentially. We call N_i^I the component of the result containing $\partial\mathcal{C}$.
- We find a maximal collection of essential exterior compressing disk for the result of the first step and do disk compressions. We call N_i^{II} the component containing $\partial\mathcal{C}$.
- We add 1-handles lost in the first step. We call the result N_i^{III}.

Clearly N_i^{III} includes N_i.

Proposition 10.5. *The submanifold N_i^{III} is homeomorphic to a compression body and is contained in a compression body M_i'' whose boundary component is incompressible in $M - \mathcal{C}^o - \mathcal{C}_i - \mu_i$. A Margulis tube is either contained in M_i'' or the tube meets M_i'' in meridian disk times intervals.*

Proof. The fact that N_i^{III} is a compression body follows as before. Using the fact that N_i^{III} is contained in some compression body M_j' for a large j, we will now show that N_i^{III} is contained in a compression body M_i'' with the above property.

By construction of N_i^{III}, it follows that an interior compression disk can be isotopied inside the regular neighborhood of the union the disks of the 1-handles and arcs in the boundary. Therefore, each interior compressing disk must intersect at least one of $\mathcal{C}, \mathcal{C}_i, \mu_i$ essentially. Hence ∂N_i^{III} has no interior compression disk in $M - \mathcal{C}^o - \mathcal{C}_i - \mu_i$.

There could be an exterior compression disk for N_i^{III}. We take a maximal mutually disjoint family $D_1, \dots, D_n$ of them where no two of ∂D_i are parallel. We choose j sufficiently large so that a compression body M_j' includes all of them as M_i's form an exhausting sequence.

We find a 3-manifold X isotopic to N_i^{III} in M_j': M_j' decomposes into a union of cells or submanifolds homeomorphic to F_l times intervals by a maximal family of interior compression disks. We suppose that no two of the disks are parallel and call D the union of these disks.

We consider X to be a thin regular neighborhood of a 1-complex with the unique vertex in a fixed base cell $\mathcal{B}$ of M_j', fix a handle decomposition of X corresponding to the 1-complex structure, define the complexity of the imbedding of X in M_j' by the number of components of $X \cap D$, choose X with minimal complexity, and put all things in general positions. For each disk D_k, we first get rid of any closed circles by the innermost circle arguments. We may find an edgemost arc if ∂D_k meets D bounding a component of $D_k - D$. Then there must be a handle of X following the arc in ∂D_k. This handle can be isotopied away, and then using the innermost circle argument again if necessary, we can reduce the complexity. Therefore, it follows that each D_k is in the fixed base cell of M_j'. Also, a handle where D_k passes essentially must lie in the base cell also.

We look at the handles in the base cell $\mathcal{B}$ and the disks. The union of the handles and the ball around the basepoint is a handle body. Our disks $D_1, \dots, D_n$ are in the cell.

From Corollary 3.5 or 3.6 of Scharlemann-Thomson [15], we see that there exists an unknotted cycle in the 1-complex or the 1-complex has a separating sphere. In the first case, we cancel the corresponding cycle by an exterior compression. In the later case, a sphere bounds a ball which we add to X, i.e., we engulf it. We do the corresponding topological operations to N_i^{III} while X and N_i^{III} are still compression bodies. In both cases, we can reduce the genus of the boundary of the handle body X or N_i^{III}. We do this operations until there are no more exterior compressing disks. (The genus complexity shows that the process terminates.) We let the final result to be M_i''.

The imbedding $\partial M_i'' \rightarrow M - M_i''^{,o}$ is incompressible since the boundary of any exterior compressing disk for $\partial M_i''$ can be made to avoid the traces of handle-canceling exterior compressions which are pairs of disks or the disks from the 3-ball engulfing. Therefore these must be exterior compressing disks for N_i^{III}.

The imbedding $\partial M_i'' \rightarrow M_i''^{,o} - \mathcal{C}^o - \mathcal{C}_i - \mu_i$ is incompressible since the boundary of any interior compressing disk can be made to avoid the traces of handle-canceling exterior compressions.

The statement about Margulis tubes follows by Lemma 10.1. □

10.2. **Crescent-isotopy.** We will now modify M_i'' by crescent-isotopy.

Lemma 10.6. *A secondary highest-level crescent of $\tilde{\Sigma}$ does not meet the interior of $\tilde{\mathcal{C}}$.*

Proof. If not, then $\tilde{\mathcal{C}}^o$ meets $I_{\mathcal{R}}$ for a secondary highest-level crescent $\mathcal{R}$. We may assume that $\mathcal{R}$ is compact by an approximation inside. Again find a Morse function by totally geodesic planes parallel to $I_{\mathcal{R}}$. $\mathcal{C} \cap \mathcal{R}$ has a maximum inside as $\mathcal{C} \cap \mathcal{R}$ is compact. But at the maximum point, a totally geodesic plane bounds a local half open ball disjoint from $\mathcal{C}^o$. This contradicts the saddle-imbeddedness of $\partial\tilde{\mathcal{C}}$. □

We define M_i to be the 2-convex hull of M_i''. The boundary components of M_i are saddle-imbedded. Let $\mathcal{C}'$ be the core obtained from $\mathcal{C}$ by pushing $\partial\mathcal{C}$ inside $\mathcal{C}$ by an ϵ-amount. Note that during the crescent-isotopy $\mathcal{C}'$ is not touched by the interior of secondary highest level crescents since $\partial\mathcal{C}$ is saddle-imbedded by Lemma 10.6.

Define M_i^ϵ be the regular ϵ-neighborhood of M_i. $\mathcal{C}_i, \mu_i \subset M_i^\epsilon$ by Proposition 3.22 in Part 1. We may assume A_i is in M_i^ϵ since we isotopied M_i'' to obtained M_i.

The universal cover $\tilde{M}_i$ of M_i with the universal covering map p_i is an M_{-1}-space and is δ-hyperbolic.

We define the *thin part* of M_i as the subset of M_i where the injectivity radius is $\leq \epsilon$. Since $\tilde{M}_i$ is a uniquely geodesic, through each point of the thin part of M_i there exists a closed curve of length $\leq \epsilon$ which is not null-homotopic in M_i.

Proposition 10.7. *The ϵ-thin part of M_i is homeomorphic to a disjoint union of solid tori in Margulis tubes in M parallel to a multiples of the shortest geodesics in the respective tubes. Furthermore, we can choose $\epsilon > 0$ independent of i.*

Proof. Let γ be a closed curve of length $\leq \epsilon$ which is not null-homotopic in M_i. Then if γ has nontrivial holonomy, then γ lies in a Margulis tube of M which either is in M_i or disjoint from M_i.

Suppose that γ is null-homotopic in M. Then γ bounds an immersed disk D in M. The diameter of D is $\leq \epsilon$.

Suppose that D cannot be isotopied into M_i. By incompressibility of ∂M_i, D must meet $\mathbf{c_i}$ or μ_i or $\mathcal{C}$. Also, D/M_i must be nonempty. There must be an innermost disk

D' such that $\partial D'$ maps to ∂M_i and D' maps into M_i and meets μ_i, $\mathcal{C}$ or $\mathcal{C}_i$ and a component of $D/\partial M_i$ adjacent to D' lies outside M_i.

If D' meets $\mathcal{C}$, then the diameter of $\partial D'$ is not so small, and hence that of D. If D' meets $\mathcal{C}_i$, then since A_i is in M_i D' cannot be bounded outside by a component in M/M_i^o.

Suppose that an innermost disk D' meet μ_i. Since $\partial D'$ is very close to μ_i due to its size, and the distance from μ_i to $\partial M_i''$ is bounded below by a certain constant, it follows that the boundary of D' lies in the union of I-parts of some crescents or its perturbed images obtained during the crescent-isotopies. Since the length of components of μ_i is short, we see that the the I-parts meeting $\partial D'$ would extend for long lengths along the geodesics near a component of $\tilde{\mu}_i$. Therefore, we see that at the last stage of the isotopies, we have the inverse image of torus bounding a component of μ_i. Since our crescent moves are isotopies and $\partial M_i'$ is not homeomorphic to a torus, this is a contradiction.

We conclude that γ bounds a disk in M_i. Since γ is not null-homotopic in M_i, this is a contradiction. Therefore, the thin part of M_i is in the intersection of the Margulis tubes of M with M_i.

Note here that the Margulis constant $\epsilon > 0$ could be chosen independent of i as the above argument passes through once ϵ is sufficiently small regardless of i. □

By above discussions, it follows that any ϵ-short closed curve in M_i is a multiple of the simple closed geodesic in a Margulis tube. We may assume that given a component of the thin part of M_i, an ϵ-short closed curve of a fixed homotopy class passes through each point of the component. Therefore, components of thin parts are solid tori in Margulis tubes.

During the crescent move, the shortest closed geodesic in the Margulis tube may go outside particularly during the convex perturbations. But there are short closed curves in the result homotopic to the closed geodesic. Therefore the thin part are union of solid tori parallel to some multiples of the shortest geodesics.

10.3. **Covers L_is.** Assume without loss of generality that we have an inclusion map $i : \mathcal{C} \to M_i$ for each i. Let L_i be $\tilde{M}_i/i_*\pi_1(\mathcal{C})$ with the covering map $p_i : L_i \to M_i$. L_i has ends corresponding to $F_1, \dots F_n$, and another end E corresponding to E. (We abused notation a little here.)

Proposition 10.8. *The convex hull K_i of $\mathcal{C}$ in L_i is compact.*

Proof. Since M_i is tame, its cover L_i is tame. For any compact set, we can find a compact core of L_i containing it. By choosing a large compact subset of M_i, we obtain a compact core $\mathcal{C}'$ of L_i containing it which is obtained as the closure of the appropriate component of L_i with a finite number of disks removed.

Certainly $\mathcal{C}$ is a subset of it. The disks lifts to disks in the universal cover $\tilde{L}_i$ of L_i. They bound the universal cover $\tilde{\mathcal{C}}'$ of $\mathcal{C}'$.

The convex hull of a disk is a compact subset of $\tilde{L}_i$ since the convex hull of a compact subset is compact in the universal cover. Since the convex hull of $\tilde{\mathcal{C}}'$ is in the union of $\tilde{\mathcal{C}}'$ and convex hulls of the boundary disks, the convex hull of $\mathcal{C}$ itself is compact being a subset of the union of a compact set $\mathcal{C}'$ and finitely many compact sets. □

Since $\mathcal{C}$ is homotopy equivalent to L_i, there exists a convex hull K_i of $\mathcal{C}$ in L_i homotopy equivalent to $\mathcal{C}$ by Proposition 8.1 in Part 2. Obviously, K_i contains $F_1, \ldots, F_n$. Let Σ_i be the unique boundary component of ∂K_i associated with E.

Any closed geodesic homotopic to a closed curve in $\mathcal{C}$ in L_i is contained in K_i: If not, we can find a hyperbolic-imbedded annulus B_i with boundary consisting of the closed geodesic and a closed curve on ∂K, intrinsically geodesic, where the interior angles in B_i are always greater than or equal to π. Such an annulus clearly cannot exist.

We can find a quasi-geodesic c_i' ϵ-close to $\mathbf{c_i}$ in M_i. Since c_i' is homotopic to a closed curve in $\mathcal{C}$ by a homotopy A_i' in M_i modified from A_i, we have that $c_i' \subset L_i$. Choose a geodesic representative c_i'' in L_i which is again arbitrarily close to c_i' and hence to $\mathbf{c_i}$. Therefore $c_i'' \subset K_i$ for each i.

Since Σ_i is a truly pleated-triangulated convex hyperbolic-surface, the intrinsic metric in the pleated part is a Riemannian hyperbolic ones. Thus, Σ_i carries a triangulated hyperbolic-surface structure intrinsically. Since Σ_i is intrinsically a hyperbolic-imbedded surface and c_i'' forms an exiting sequence, $p|\Sigma_i$ is one also and hence form an exiting sequence in E.

More precisely, the parts of boundary of the image of K_i form an exiting sequence in E. Any part of the boundary of the image of K_i is in the image of Σ_i. Hence, there exists an exiting sequence of parts of Σ_i. By the uniform boundedness of Σ_i, it follows that Σ_i form an exiting sequence in E.

Remark 10.9. The uniform nature of the Margulis constant plays a role here. Any ϵ-thin part of a hyperbolic-immersed surface must be inside a Margulis tube in L_i and by incompressibility the thin parts are homeomorphic to essential annuli. Since Σ_i is incompressible in $L_i - \mathcal{C}^o$, we see that the thin part of Σ_i is a union of essential annuli which are not homotopic to each other. Thus, outside the Margulis tubes, Σ_is have bounded diameter independent of i.

10.4. **The Proof of Theorem B.** We recall that $\mathcal{C}$ was pushed inside itself somewhat so that Σ_i and ∂K_i does not meet $\mathcal{C}$. In K_i, we may remove $\mathcal{C}^o$ and we obtain a compact submanifold Q_i of codimension 0 bounded by saddle-surfaces including Σ_i since $\partial\mathcal{C}$ is saddle-imbedded. Q_i is a 2-convex CAT(-1)-space. Finally, Σ_i is incompressible in Q_i since any compressing disk of Σ_i not meeting the core would reduce the genus of Σ_i but the genus of Σ_i is the same as that of the component of $\partial\mathcal{C}$ corresponding to the end E since K_i is homotopy equivalent to $\mathcal{C}$.

As K_i is homeomorphic to a compression body, we choose a compressing curve α in Σ_i. Then α bounds a disk D in K_i and the core $\mathcal{C}$ must meet D in its interior. Let $\hat{\alpha}$ be the geodesic realization of α in $K_i - \mathcal{C}^o$, which must be in K_i.

If $\hat{\alpha}$ does not meet $\partial\mathcal{C}$, then it maps to a geodesic in M, which is absurd since the holonomy of α is the identity. Therefore, $\hat{\alpha}$ meets $\partial\mathcal{C}$.

We form a triangulation of Σ_i with the only vertex p at a point of α and including α as an edge. Then choosing a vertex $\hat{p}$ in $\hat{\alpha}$ and a path from p to $\hat{p}$, we isotopy each edge of the triangulation to a geodesic loop in $K_i - \mathcal{C}^o$ based at $\hat{p}$. Each triangle is isotopied to an A-net spanned by new geodesic edges. The resulting surface T_i is a hyperbolic-surface since each of the triangles is an A-net and a geodesic passes through each point of the 1-complex.

Each surface $q_i : T_i \to M - \mathcal{C}^o$ has the same genus and is homotopic to $p_i|\Sigma_i$ in $M - \mathcal{C}^o$. Since they are hyperbolic-imbedded, and meet $\partial\mathcal{C}$, they are in a bounded neighborhood of $\mathcal{C}$ by the boundedness of hyperbolic-imbedded surfaces. They form a pre-compact sequence. Thus infinitely many of $q_i|T_i$ are isotopic in $M - \mathcal{C}^o$. Therefore, infinitely many of $p_i|\Sigma_i$ are isotopic in $M - \mathcal{C}^o$. Since Σ_i bounds larger and larger domains in a cover of M and Σ_i projects to a surface far from $\mathcal{C}$, the above fact shows that our end E is tame as shown by Thurston [17]. (This is essentially the argument of Souto [16] simplifiable in our setup.)

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