Algebraic Topology I: Final Exam (Spring 2005)

Justify your answers fully.

1.

   (a) (15pts.) Find a CW-complex structure of a 3-dimensional lens space $L_p(1,q)$ defined by the action of $T : S^3 \to S^3$ given by $T(z_1, z_2) = (e^{2\pi/p} z_1, e^{2q\pi/p} z_2)$ where S^3 is a unit sphere in C^2 and p, q are relatively prime.

   (b) (15 pts.) Compute the homology groups of $L_p(1,q)$.

2. Compute the homology groups of the following 2-complexes:

   (a) (10 pts.) The quotient space of S^2 obtained by identifying the north and south poles to a point.

   (b) (10 pts.) $S^1 \times (S^1 \vee S^1)$.

   (c) (10 pts.) The quotient space of $S^1 \times S^1$ obtained by identifying the points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotations. (m is a natural number > 1.)

3. (30 pts.) Let $(D, S) \subset (D^n, S^{n-1})$ be a pair of subspaces homeomorphic to (D^k, S^{k-1}) with $D \cap S^{n-1} = S$. Show that the inclusion $S^{n-1} - S \to D^n - D$ induces an isomorphism of the homology group of each dimension.

4. (30 pts.) Use the Lefschetz fixed point theorem to show that a map $S^n \to S^n$ has a fixed point unless its degree equals the degree of $A : S^n \to S^n$ where A is the antipodal map.

5. (30 pts.) Prove the Brouwer fixed point theorem for functions $f : D^n \to D^n$ with $n \geq 3$ using homology theories.