Real projective orbifolds with ends and their deformation theory:
The deformation theory for nicest ones (Teaching copy for 2024)

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Preface

Warning: This draft is still temporary and not final.

Let G be a Lie group acting transitively on a manifold X. An (X,G)-geometry is given by this pair. Furthermore, an (X,G)-structure on an orbifold or a manifold is an atlas of charts to X with transition maps in G. Here, we are concerned with $G = \mathsf{PGL}(n+1,\mathbb{R})$ and $X = \mathbb{RP}^n$.

Cartan, Ehresmann, and others started the field of (X,G)-structures. Subjects of (X,G)-structures were popularized by Thurston and Goldman among many other people. These structures provide a way to understand representations and their deformations giving us viewpoints other than algebraic ones. Our deformation spaces often parameterize significant parts of the space of representations.

Since the examples are easier to construct, even now, we will be studying orb, a natural generalization of manifolds. Also, computations can be done fairly well for simple examples. We began our study with Coxeter orbifolds where the computations are probably the simplest possible.

Thurston did use the theory of orbifolds in a deep way. The hyperbolization of Haken 3-manifold requires the uses of the deformation theory of orbifolds where we build from hyperbolic structures from handlebodies with "scalloped" orbifold structures. (See Morgan [133].) We do not really know how to escape this step, which was a very subtle point that some experts misunderstood. Also, orbifolds are natural objects obtained when we take quotients of manifolds by fibrations and so on. These are some of the reasons we study orbifolds instead of just manifolds.

Classically, conformally flat structures were studied much by differential geometers. Projectively flat structures were also studied from Cartan's time. However, our techniques are much different from their approaches.

Convex real projective orbifolds are quotient spaces of convex domains on a projective space $\mathbb{RP}^n$ by a discrete group of projective automorphisms. Hyperbolic manifolds and many symmetric manifolds are natural examples. These can be deformed to one not coming from simple constructions. The study was initiated by Kuiper [116], Koszul [114], Benzécri [25], Vey [151], and Vinberg [153], accumulating some class of results. Closed manifolds or orbifolds admit many such structures as shown first by Kac-Vinberg [107], Goldman [88], and Cooper-Long-Thistlethwaites [64], [65]. Some parts of the theory for closed orbifolds were completed by Benoist [20] in the 1990s.

The topics of convex real projective structures on manifolds and orbifolds are currently developing. We present some parts. This book is mainly written for researchers in this field.

There are surprisingly many such structures coming from hyperbolic ones and deforming as shown by Vinberg for Coxeter orbifolds, Goldman for surfaces, and later by Cooper-Long-Thistlethwaite for 3-manifolds.

We compare these theories to the Mostow or Margulis type rigidity for symmetric spaces. The rigidity can be replaced by what is called the *Ehresmann-Thurston-Weil principle* that

- a subspace of the G-character space (variety) of the fundamental group of a manifold or orbifold M classifies the (X,G)-structures on M under the map

$$\mathrm{hol} : \mathrm{Def}_c(M) \to \mathrm{Hom}_c(\pi_1(M), G)/G$$

  where
  - we define the deformation space

$$\mathrm{Def}_c(M) := \{(X,G)\text{-structures on } M \text{ satisfying some conditions denoted by } c\}/\sim$$

    where $\sim$ is the isotopy equivalence relation, and
  - $\mathrm{Hom}_c(\pi_1(M), G)/G$ is the subspace of the character space $\mathrm{Hom}(\pi_1(M), G)/G$ satisfying the corresponding conditions to c.

For closed real projective orbifolds, it is widely thought that Benoist's work is quite an encompassing one. Hence, we won't say much about this topic. (See Choi-Lee-Marquis for a survey [61].)

We focus on convex real projective orbifolds with ends, which we have now accumulated some number of examples. Basically, we will prove an Ehresmann-Thurston-Weil principle: We will show that the deformation space of properly convex real projective structures on an orbifold with some end conditions identifies under a map with the union of components of the subset of character spaces of the orbifold satisfying the corresponding conditions on end holonomy groups. Our conditions on the ends are probably very generic ones, and we have many examples of such deformations.

In fact, we are focusing on generic cases of lens-type or horospherical ends. To complete the picture, we need to consider all types of ends. Even with end vertex conditions, we are still to complete the picture leaving out the NPNC-ends. We hope to allow these types for our deformation spaces in the near future.

The book is divided into three parts:

(Part I): We will give some introduction and survey our main results and give examples where our theory is applicable.

(Part II): We will classify the types of ends we will work with. We use the uniform middle eigenvalue conditions. The condition is used to prove the preservation of the convexity of the deformations.

(Part III): We will try to prove the Ehresmann-Thurston-Weil principle for the deformation spaces for our type of orbifolds. We show the local homeomorphism property and the closedness of the images for the maps from the deformations spaces to the character spaces restricted by the end conditions.

We will try to follow the strictly logical progression of the material. However, for each chapter, we will introduce the main results first.

As an application, we will use the results of the whole of the monograph Chapter 12, which are the nicest cases. One can consider these as the conclusions of the monograph.

The logical dependence of the monograph is as ordered by the order of appearance. Appendix A depends only on Chapter 1, and the results are used in the monograph except for Chapter 1.

We give an outline at the beginning of each part.

As a motivation for our study, we say about some long-term goal: Deforming a real projective structure on an orbifold to an unbounded situation results in the actions of the fundamental group on affine buildings. This hopefully will lead us to some understanding of orbifolds and manifolds in particular of dimension three as indicated by Ballas, Cooper, Danciger, G.S. Lee, Leitner, Long, Thistlethwaite, and Tillmann.

There is a concurrent work by the group consisting of Cooper, Long, and Tillmann with Ballas and Leitner on the same subjects but with different conditions on ends. They impose the condition that the end fundamental groups to be amenable. However, we do not require the same conditions in this paper but instead we will use some type of norms of eigenvalue conditions to guarantee the convexity during the deformations. We note that their deformation spaces are somewhat differently defined. Of course, we benefited much from their work and insights in this book and are very grateful for their generous help and guidance. We also appreciate much help from Crampon and Marquis working also independently of the above group and us.

We need to lift the objects to $\mathbb{S}^n$ using Section 1.1.8. We give proofs in the book by considering objects to be in $\mathbb{S}^n$ and using the projective automorphism group $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$. We will use proof symbols:

- $\mathbb{S}^n$**S:** at the end of the proof indicates that it is sufficient to prove for $\mathbb{S}^n$ since the conclusion does not involve $\mathbb{RP}^n$ nor $\mathbb{S}^n$.
- $\mathbb{S}^n$**T:** indicates that the version of the theorem, proposition, or lemma for $\mathbb{S}^n$ implies one for $\mathbb{RP}^n$ often with the help of Proposition 1.4.2.
- $\mathbb{S}^n$**P:** indicates that the proof of the theorem for $\mathbb{S}^n$ implies one for $\mathbb{RP}^n$ often with the help of Proposition 1.4.2.

If we do not need to go to $\mathbb{S}^n$ to prove the result, we leave no mark except for the end of the proof.

This book generalizes and simplifies the earlier preprints of the author. We were able to drop many conditions in the earlier versions of the theorems overcoming many limitations. Some of the results were announced in some survey articles [52] and [53].

Finally, to better communicate the ideas, the author made some effort to make the material more clear and precise, entailing the trade-off of the writing being long, somewhat technical, and sometimes redundant.

Daejeon,

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Part 1

Introduction to orbifolds and real projective structures.

Part I aims to survey some preliminary definitions and elementary used facts. These are standard materials, and there are no new results.

In Chapter 1, we go over basic preliminary materials. We begin with defining geometric structures and real projective structures, in particular convex ones. We discuss the ends of orbifolds. Affine orbifolds and affine suspensions of real projective orbifolds are defined. We discuss the linear algebra and estimations using it, orthopotent actions of Lie groups, higher-convergence group actions, attracting and repelling sets, convexity, the Benoist theory on convex divisible actions, and so on. We discuss the dual orbifolds of a given convex real projective orbifolds as given by Vinberg. Finally, we extend duality to all convex compact sets and discuss the geometric limits of the dual convex sets. Here we will use a slightly more generalized version of convexity.

In Chapter 2, we give some examples, where our full theory applies. We will fully explain this in Chapter 12. Coxeter orbifolds and the orderability theory for Coxeter orbifolds using the Vinberg theory will be explained. We discuss the work jointly done with Gye-Seon Lee, Hodgson, and Greene. We state the work of Heusner-Porti on projective deformations of hyperbolic link complements. Also, we state some results on finite-volume convex real projective structures by Cooper-Long-Tillmann and Crampon-Marquis that these admit thick and thin decompositions.

CHAPTER 1

Preliminaries

We will go over the underlying theory. In Section 1.1, we discuss the Hausdorff convergences of sequences of compact sets, Hilbert metrics, some orbifold topology, geometric structures on orbifolds, real projective structures on orbifolds, spherical real projective structures and liftings. We also classify compact convex subsets of $\mathbb{S}^n$ in Proposition 1.1.4. In Section 1.2, we discuss that affine structures and affine suspensions of real projective orbifolds. In Section 1.3, we discuss the linear algebra and estimation to find convergences, orthopotent groups, proximal and semi-proximal actions, semi-simplicity, and the higher convergence groups. Higher convergence groups are generalizations of convergence groups. In Section 1.4, we explain the comprehensive Benoist theory on convex orbifolds, where he completed theories of Kuiper, Koszul, Vey, Vinberg, and so on, on divisible actions on convex linear cones as he terms them. In particular, the strict-join decomposition of properly convex orbifolds will be explained. Lemma 1.4.16 shows that the properly convex real projective structures are uniquely determined by holonomy groups, which is a somewhat commonly overlooked fact. In Section 1.5, we explain the duality theory of Vinberg. We introduce the augmented boundary of properly convex domains as the set of boundary points and the sharply supporting hyperplanes associated with these points. The duality map is extended to the augmented boundary. Duality is extended to sweeping actions also. The duality is extended to every compact convex set in $\mathbb{S}^n$, and we discuss the relationship between the duality and the geometric convergences of the sequences of properly convex sets.

1.1. Preliminary definitions

As usual, we denote by $\mathbb{RP}^n$ the projectivization of $\mathbb{R}^{n+1}$. There is a group $\mathsf{PGL}(n+1,\mathbb{R})$ acting effectively and transitively on $\mathbb{S}^n$.

Given a vector space V, we denote by $\mathbb{S}(V)$ the quotient space of

$$(V-\{O\})/\sim \text{ where } v\sim w \text{ iff } v = sw \text{ for } s>0.$$

We denote by $\mathbb{S}^n := \mathbb{S}(\mathbb{R}^{n+1})$. We will represent each element of $\mathsf{PGL}(n+1,\mathbb{R})$ by a matrix of determinant ± 1; i.e., $\mathsf{PGL}(n+1,\mathbb{R}) = \mathrm{SL}_{\pm}(n+1,\mathbb{R})/\langle \pm \mathrm{I}\rangle$. Recall the covering map $p_{\mathbb{S}^n} : \mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1}) \to \mathbb{RP}^n$.

The following notation is used in the monograph. For a subset A of a space X, we denote by $\mathrm{Cl}_X(A)$ the closure of A in X and $\mathrm{bd}_X A$ the boundary of A in X. We will omit the subscript X if X is clear from the context. If A is a domain of a subspace of $\mathbb{RP}^n$ or $\mathbb{S}^n$, we denote by $\mathrm{bd}A$ the topological boundary in the subspace. The closure $\mathrm{Cl}(A)$ of a subset A of $\mathbb{RP}^n$ or $\mathbb{S}^n$ is the topological closure in $\mathbb{RP}^n$ or in $\mathbb{S}^n$. We will also denote by K^o the manifold or orbifold interior for a manifold or orbifold K. Also, we may use K^o as the interior relative to the topology of P when K is a domain K in a totally geodesic subspace P in $\mathbb{S}^n$ or $\mathbb{RP}^n$. Define ∂A for a manifold or orbifold A to be the *manifold or orbifold boundary*. (See Section 1.1.4.)

Let $p, q \in \mathbb{S}^n$. We also denote by $\overline{pq}$ a minor arc connecting p and q in a great circle in $\mathbb{S}^n$. If $q \neq p_-$, this is unique. Otherwise, we need to specify a point in $\mathbb{S}^n$ not antipodal to both. We denote by $\overline{pzq}$ the unique minor arc connecting p and q passing z.

If $p, q \in \mathbb{RP}^n$, then $\overline{pq}$ denote one of the closures of a component of $l - \{p, q\}$ for a one dimensional projective line containing p, q.

1.1.1. Convex sets in $\mathbb{RP}^n$ and $\mathbb{S}^n$. Recall that an affine path in $\mathbb{RP}^n$ is a complement of a codimension-one subspace. It has a canonical geodesic structure where each projective geodesics corresponds to affine geodesics up to parameterizations and conversely. A *convex set* in $\mathbb{RP}^n$ is a convex subset of an affine patch of $\mathbb{RP}^n$.

We use a slightly different definition of convexity for $\mathbb{S}^n$.

DEFINITION 1.1.1. A *convex segment* is an arc contained in a great segment. A *convex subset* of $\mathbb{S}^n$ is a subset A where every pair of points of A connected by a convex segment.

It is easy to see that either a convex subset of $\mathbb{S}^n$ is contained in an affine subspace, it is in a closed hemisphere, or it is a great sphere of dimension ≥ 1. In the first case, the set embeds to a convex set in $\mathbb{RP}^n$ under the covering map.

Since an affine patch of $\mathbb{RP}^n$ always lifts to an open hemisphere in $\mathbb{S}^n$. Hence, a convex subset of $\mathbb{RP}^n$ always lifts to a convex subset of $\mathbb{S}^n$ which maps to it homeomorphically under the projection $\mathbb{S}^n \to \mathbb{RP}^n$. Hence, the convex subsets of $\mathbb{RP}^n$ corresponds to convex subset of $\mathbb{S}^n$ contained in an open hemisphere in a one-to-one manner.

DEFINITION 1.1.2. Given a convex set D in $\mathbb{RP}^n$, we obtain a connected cone $C(D)$ in $\mathbb{R}^{n+1} - \{O\}$ mapping to D, determined up to the antipodal map. For a convex domain $D \subset \mathbb{S}^n$, we have a unique domain $C(D) \subset \mathbb{R}^{n+1} - \{O\}$.

A *join* of two properly convex subsets A and B in a convex domain D of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is defined as

$$A * B := \{[t\vec{x} + (1-t)\vec{y}] | \vec{x}, \vec{y} \in C(D), [\vec{x}] \in A, [\vec{y}] \in B, t \in [0,1]\}$$

$$(\text{resp. } A * B := \{(\!(t\vec{x} + (1-t)\vec{y})\!) | \vec{x}, \vec{y} \in C(D), (\!(\vec{x})\!) \in A, (\!(\vec{y})\!) \in B, t \in [0,1]\})$$

where $C(D)$ is a cone corresponding to D in $\mathbb{R}^{n+1}$. The definition is independent of the choice of $C(D)$ in $\mathbb{S}^n$. In $\mathbb{RP}^n$, the join may depend on the choice $C(D)$. Note we use $p * B = \{p\} * B$ interchangeably for a point p.

DEFINITION 1.1.3. Let $C_1, \dots, C_m$ respectively be cones in a set of independent vector subspaces $V_1, \dots, V_m$ of $\mathbb{R}^{n+1}$. In general, a *sum* of convex sets $C_1, \dots, C_m$ in $\mathbb{R}^{n+1}$ in independent subspaces V_i is defined by

$$C_1 + \cdots + C_m := \{v | v = \vec{c}_1 + \cdots + \vec{c}_m, \vec{c}_i \in C_i\}.$$

A *strict join* of convex sets Ω_i in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) is given as

$$\Omega_1 * \cdots * \Omega_m := \Pi'(C_1 + \cdots + C_m) \text{ (resp. } \Pi(C_1 + \cdots + C_m))$$

where each $C_i - \{O\}$ is a convex cone with image Ω_i for each i for the projection Π' (resp. Π).

PROPOSITION 1.1.4. *A closed convex subset K of $\mathbb{S}^n$ is either a great sphere $\mathbb{S}^{i_0}$ of dimension $i_0 \geq 1$, or is contained in a closed hemisphere H^{i_0} in $\mathbb{S}^{i_0}$ and is one of the following:*

- *There exist a great sphere $\mathbb{S}^{j_0}$ of dimension $j_0 \geq 0$ in the boundary* bd$K$ *and a compact properly convex domain K_K in an independent subspace of $\mathbb{S}^{j_0}$ and $K = \mathbb{S}^{j_0} * K_K$, a strict join. Moreover, $\mathbb{S}^{j_0}$ is a unique maximal great sphere in K.*

- *K is a properly convex domain in the interior of* i_0+1*-hemisphere. for some i.*
- *Unless K is a great sphere, $\partial K = \mathrm{bd}_{\mathbb{S}^n} K$ and K is homeomorphic to a cell. For a properly convex compact domain K in $\mathbb{RP}^n$, $\partial K = \mathrm{bd}_{\mathbb{RP}^n} K$ and K is homeomorphic to a cell.*

PROOF. Let $\mathbb{S}^{i_0}$ be the span of K. Then K^o is not an empty domain in $\mathbb{S}^{i_0}$. The map $x \mapsto \mathbf{d}(x,K)$ is continuous on $\mathbb{S}^{i_0}$. Choose a maximum point x_0. If the maximum is $< \pi/2$, then the elliptic geometry tells us that there at least two point y,z of K closest to x_0 of same distance from x_0 since otherwise we can increase the value of $\mathbf{d}(\cdot,K)$ by moving x_0 slightly. Then there is a closer point on $\overline{xy}^o$ in K to x_0. This is a contradiction. Hence, $K = \mathbb{S}^{i_0}$. Otherwise, K is a subset of an i_0-hemisphere in $\mathbb{S}^{i_0}$. (See [43] also.)

The second part follows from Section 1.4 of [36]. (See also [71].) Hence, we obtain a unique maximal great sphere $\mathbb{S}^{j_0}$ in K which is contained in $\mathrm{bd}K$, and K is a union of j_0+1-hemispheres with common boundary $\mathbb{S}^{j_0}$.

By choosing an independent subspace $\mathbb{S}^{n-j_0-1}$ to $\mathbb{S}^{j_0}$, each j_0+1-hemisphere in K is transverse to $\mathbb{S}^{n-j_0-1}$ and hence meets it in a unique point. We let K_K denote the set of intersection points. Therefore, $K = \mathbb{S}^{j_0} * K_K$.

There is a map $K \to K_K$ given by sending a j_0+1-hemisphere to the intersection point. Obviously, this is a restriction of projective diffeomorphism from the space of j_0+1-hemispheres with boundary $\mathbb{S}^{j_0}$ to $\mathbb{S}^{n-j_0-1}$. Since K cannot contain a higher-dimensional great sphere, it follows that K_K is properly convex also.

For the final item, the fact that K is a join of a great sphere with a properly convex domain implies this. □

Given a vector space $\mathbf{V}$, we let $\mathbb{P}(\mathbf{V})$ denote the space obtained by taking the quotient space of $\mathbf{V} - \{O\}$ under the equivalence relation

$$\vec{v} \sim \vec{w} \text{ for } \vec{v}, \vec{w} \in \mathbf{V} - \{O\} \text{ iff } \vec{v} = s\vec{w}, \text{ for } s \in \mathbb{R} - \{0\}.$$

We let $[\vec{v}]$ denote the equivalence class of $\vec{v} \in \mathbf{V} - \{O\}$. For a subspace $\mathbf{W}$ of $\mathbf{V}$, we denote by $\mathbb{P}(\mathbf{W})$ the image of $\mathbf{W} - \{O\}$ under the quotient map, also said to be a *subspace*.

Recall that the projective linear group $\mathsf{PGL}(n+1,\mathbb{R})$ acts on $\mathbb{RP}^n$, i.e., $\mathbb{P}(\mathbb{R}^{n+1})$, in a standard manner.

Recall that $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ is isomorphic to $\mathsf{GL}(n+1,\mathbb{R})/\mathbb{R}_+$. Then this group acts on $\mathbb{S}^n$ to be seen as a quotient space of $\mathbb{R}^{n+1} - \{O\}$ by the equivalence relation

$$\vec{v} \sim \vec{w}, \vec{v}, \vec{w} \in \mathbb{R}^{n+1} - \{O\} \text{ if } \vec{v} = s\vec{w} \text{ for } s \in \mathbb{R}^+.$$

We let $(\!(\vec{v})\!)$ denote the equivalence class of $\vec{v} \in \mathbb{R}^{n+1} - \{O\}$. Given a vector subspace $V \in \mathbb{R}^{n+1}$, we denote by $\mathbb{S}(V)$ the image of $V - \{O\}$ under the quotient map. The image is called a *subspace*. A set of antipodal points is a subspace of dimension 0. There is a double covering map $p_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{RP}^n$ with the deck transformation group generated by $\mathscr{A}$. This gives a projective structure on $\mathbb{S}^n$. The group of projective automorphisms is identified with $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. The notion of geodesics are defined as in the projective geometry: they correspond to arcs in great circles in $\mathbb{S}^n$.

A collection of subspaces $\mathbb{S}(V_1), \dots, \mathbb{S}(V_m)$ (resp. $\mathbb{P}(V_1), \dots, \mathbb{P}(V_m)$) are *independent* if the subspaces $V_1, \dots, V_m$ are independent.

The group $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ of linear transformations of determinant ± 1 maps to the projective group $\mathsf{PGL}(n+1,\mathbb{R})$ by a double covering homomorphism $\hat{q}$, and $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ acts on $\mathbb{S}^n$ lifting the projective transformations. The elements are also *projective transformations*.

REMARK 1.1.5. For each $g \in \mathsf{PGL}(n+1,\mathbb{R})$ acting on a convex open domain Ω, there is a unique lift in $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$ preserving each component of the inverse image of Ω under $\mathbb{S}^n \to \mathbb{RP}^n$. We will use this representative.

1.1.2. The Hausdorff distances used. We will be using the standard elliptic metric $\mathbf{d}$ on $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) where the set of geodesics coincides with the set of projective geodesics up to parameterizations. Sometimes, these are called Fubini-Study metrics.

DEFINITION 1.1.6. Given a set, we define

$$N_\varepsilon(A) := \{x \in \mathbb{S}^n | \mathbf{d}(x,A) < \varepsilon\} \text{ (resp. } N_\varepsilon(A) := \{x \in \mathbb{RP}^n | \mathbf{d}(x,A) < \varepsilon\}.)$$

Given two subsets K_1 and K_2 of $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$), we define the *Hausdorff distance* $\mathbf{d}_H(K_1,K_2)$ between K_1 and K_2 to be

$$\inf\{\varepsilon > 0 | K_2 \subset N_\varepsilon(K_1), K_1 \subset N_\varepsilon(K_2)\}.$$

The simple distance $\mathbf{d}(K_1,K_2)$ is defined as

$$\inf\{\mathbf{d}(x,y) | x \in K_1, K_2\}.$$

We say that a sequence $\{A_i\}$ of compact sets *converges* to a compact subset A if $\{\mathbf{d}_H(A_i,A)\} \to 0$. Here the limit is unique. Recall that every sequence of compact sets $\{A_i\}$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) has a convergent subsequence. The limit A is characterized as follows if it exists:

$$A := \{a \in H | a \text{ is a limit point of some sequence } \{a_i | a_i \in A_i\}\}. \tag{1.1.1}$$

See Proposition E.12 of [15] for a proof since the Chabauty topology for a compact space is the Hausdorff topology (See also Munkres [136].)

We will use the same notation even when A_i and A are closed subsets of a fixed open domain using $\mathbf{d}_H$ and $\mathbf{d}$.

PROPOSITION 1.1.7 (Benedetti-Petronio). *A sequence $\{A_i\}$ of compact sets in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) converges to A in the Hausdorff topology if and only if the both of the following hold*:

- *If $x_{i_j} \in A_{i_j}$ and $\{x_{i_j}\} \to x$, where $i_j \to \infty$, then $x \in A$.*
- *If $x \in A$, then there exists $x_i \in A_i$ for each i such that $\{x_i\} \to x$.*

PROOF. Since $\mathbb{S}^n$ and $\mathbb{RP}^n$ are compact, the Chabauty topology is same as the Hausdorff topology. Hence, this follows from Proposition E.12 of Benedetti-Petronio [15]. □

LEMMA 1.1.8. *Let $\{g_i\}$ be a sequence of elements of $\mathsf{PGL}(n+1,\mathbb{R})$ (resp. $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$) converging to g_∞ in $\mathsf{PGL}(n+1,\mathbb{R})$ (resp. $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$). Let $\{K_i\}$ be a sequence of compact set and let K be another one. Then $\{K_i\} \to K$ if and only if $\{g_i(K_i)\} \to g_\infty(K)$.*

PROOF. We use the above point description of the geometric limit. [$\mathbb{S}^n$S] □

An n-hemisphere H in $\mathbb{S}^n$ *supports* a domain D if H contains D. H is called a *supporting* hemisphere. An oriented hyperspace S in $\mathbb{S}^n$ *supports* a domain D if the closed hemisphere bounded in an inner-direction by S contains D. S is called a *supporting* hyperspace. If a supporting hyperspace contains a (not necessarily unique) boundary point x of D, then it is called a *sharply-supporting* hyperspace at x. If the boundary of a supporting n-hemisphere is sharply supporting at x, then the hemisphere is called a *sharply-supporting hemisphere* at x.

PROPOSITION 1.1.9. *Let K_i be a sequence of compact convex sets (resp. cells) of $\mathbb{S}^n$. Then up to choosing a subsequence $K_i \to K$ to a compact convex set (resp. cell) K of $\mathbb{S}^n$. Also, a geometric limit must be a compact convex cell when K_i are compact convex cells. If K_i is in a fixed n-hemisphere, then so is K.*

PROOF. By Proposition 1.1.7, we can show this when K_i is a hemisphere. For other cases, consider sequences of segments and Proposition 1.1.7. $\square$

The following is probably well-known.

LEMMA 1.1.10. *Suppose that one of the following holds*:

- *K_i for each i, $i = 1, 2, \dots$, is a compact convex domain, and K is one also in $\mathbb{S}^n$.*
- *K_i is a convex open domain, and K is one also in $\mathbb{S}^n$.*
- *K_i is a properly convex domain, and K is one also in $\mathbb{RP}^n$.*

Suppose that a sequence $\{K_i\}$ geometrically converging to K with nonempty interior. Then $\{\mathrm{bd} K_i\} \to \mathrm{bd} K$.

PROOF. We prove for $\mathbb{S}^n$. Suppose that a point p is in $\mathrm{bd} K$. Let $B_\varepsilon(p)$ be an open ε-ball of p. Suppose $B_\varepsilon(p) \cap K_i = \emptyset$ for infinitely many i. Then p cannot be a limit point of K by Proposition 1.1.7. This is a contradiction. Thus, $B_\varepsilon(p) \cap K_i \neq \emptyset$ for $i > N$ for some N. Suppose that $B_\varepsilon(p) \subset K_i$ for infinitely many i. Then each point in $B_\varepsilon(p) - K$ is a limit point of some sequence $p_i, p_i \in K_i$. and hence $B_\varepsilon(p) \subset K, p \in K^o$, a contradiction. Hence, given $\varepsilon > 0$, $B_\varepsilon(p) \cap \mathrm{bd} K_i \neq \emptyset$ for $i > M$ for some M. Then p is a limit of a sequence $p_i, p_i \in \mathrm{bd} K_i$.

Conversely, suppose that a sequence $\{p_{i_j}\}, p_{i_j} \in \mathrm{bd} K_{i_j}$ where $i_j \to \infty$ as $j \to \infty$, converges to p. Then $p \in K$ clearly. Suppose that $p \in K^o$. Then there is $\varepsilon, \varepsilon > 0$, with $B_\varepsilon(p) \subset K$. Now, K_{i_j} has a sharply supporting closed hemisphere H_{i_j} at p_{i_j} with $K_{i_j} \subset H_{i_j}$. Since $\{p_{i_j}\} \to p$, we may choose a subsequence k_j so that $\{H_{k_j}\} \to H_\infty$ and $\mathbf{d}_H(H_{k_j}, H_\infty) < \varepsilon/4$ for a hemisphere H_∞. Let $q \in B_{3\varepsilon/4}(p) - H_\infty$ so that $\mathbf{d}_H(q, H_\infty) > \varepsilon/4$. Hence, $B_{\varepsilon/4}(q) \in B_\varepsilon(p) - H_{k_j}$ for all j. Since $K_{k_j} \subset H_{k_j}$, no sequence $\{q_{k_j}\}, q_{k_j} \in K_{k_j}$ converges to q. However, since $\{K_{k_j}\} \to K$ and $q \in K$, this is a contradiction to Proposition 1.1.7. Hence, $p \in \mathrm{bd} K$. Now, Proposition 1.1.7 proves $\{\mathrm{bd} K_i\} \to \mathrm{bd} K$.

When K_i is an open domain in $\mathbb{S}^n$, we just need to take its closure and use the first part.

For the $\mathbb{RP}^n$-version, we lift K_i to $\mathbb{S}^n$ to properly convex domains K_i'. Now, we may also choose a subsequence so that $\{K_i'\}$ geometrically converges to a choice of a lift K' of K by Proposition 1.1.7. Since K' is properly convex, K' is in a bounded subset of an affine subspace of $\mathbb{S}^n$. Then the result follows from the $\mathbb{S}^n$-version. $\square$

We note that the last statement is false if $\{K_i\}$ geometrically converges to a hemisphere when lifted to $\mathbb{S}^n$.

THEOREM 1.1.11. *Suppose that K_i and K are (resp. properly) convex compact balls of the same dimension in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Suppose that $\{K_i\} \to K$. It follows that*

$$\{\mathrm{bd} K_i\} \to \mathrm{bd} K. \tag{1.1.2}$$

This holds also provided K_i and K are properly compact convex in $\mathbb{RP}^n$ with $\{K_i\} \to K$.

PROOF. Since K_i and K are of the same dimension, we find $g_i \in \mathrm{SL}_\pm(n+1, \mathbb{R})$ so that $g_i(\langle K_i \rangle) = \langle K \rangle$ and $\{g_i\} \to g_\infty$ for $g_\infty \in \mathrm{SL}_\pm(n+1, \mathbb{R})$. Then $\{g_i(K_i)\} \to g_\infty(K)$. Then $\{\mathrm{bd} g_i(K_i)\} \to \mathrm{bd} g_\infty(K)$ by Lemma 1.1.10. Hence, $\{\mathrm{bd} K_i\} \to \mathrm{bd} K$ by Lemma 1.1.8. [$\mathbb{S}^n$P] $\square$

1.1.3. The Hilbert metric. Let Ω be a convex open domain. A line or a subspace of dimension-one in $\mathbb{RP}^n$ has a 2-dimensional homogeneous coordinate system. Let $[o,s,q,p]$ denote the cross ratio of four points on a line as defined by

$$\frac{\bar{o}-\bar{q}}{\bar{s}-\bar{q}}\frac{\bar{s}-\bar{p}}{\bar{o}-\bar{p}}$$

where $\bar{o},\bar{p},\bar{q},\bar{s}$ denote respectively the first coordinates of the homogeneous coordinates of o,p,q,s provided that the second coordinates equal 1. Define a pseudo-metric for $p,q \in \Omega$, $d_\Omega(p,q) = \log|[o,s,q,p]|$ where o and s are endpoints of the maximal segment in Ω containing p,q where o,q separates p,s. If Ω is properly convex, then it is a metric and a Finsler metric (See [112].) If Ω is complete affine, the metric is zero always.

LEMMA 1.1.12. *Assume that $\{K_i\} \to K$ geometrically for a sequence of properly convex compact domains K_i and a properly convex compact domain K. Suppose that two sequences of points $\{x_i|x_i \in K_i^o\}$ and $\{y_i|y_i \in K_i^o\}$ converge to $x,y \in K^o$ respectively. Then*

$$\{d_{K_i^o}(x_i,y_i)\} \to d_{K^o}(x,y). \tag{1.1.3}$$

PROOF. Let z_i and t_i denote the endpoint of the maximal line containing x_i and y_i in K_i. Let z and t denote the endpoint of one containing x and y in K. It is easy to see $z_i \to z$ and $t_i \to t$. Let l_i and m_i denote the supporting great hypersphere at z_i and t_i for K_i. Then l_i and m_i must converge up to subsequences a supporting great hypersphere at z and t respectively since the closures of components of their complements are disjoint from K_i^o. This implies that $\overline{z_it_i}$ converges to a subsegment of $\overline{zt}$ up to subsequences. However, a limit cannot be a proper segment since otherwise a boundary point of K_i converges to an interior point of K contradicting Theorem 1.1.11. This implies the result. $\square$

LEMMA 1.1.13 (Cooper-Long-Tillman [67]). *Let U be a convex subset of a properly convex domain V in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Let*

$$U_\varepsilon := \{x \in V | d_V(x,U) \le \varepsilon\}$$

for $\varepsilon > 0$. Then U_ε is properly convex.

PROOF. Given $u,v \in U_\varepsilon$, we find

$$w,t \in \Omega \text{ so that } d_V(u,w) < \varepsilon, d_V(v,t) < \varepsilon.$$

Then each point of $\overline{uv}$ is within ε of $\overline{wt} \subset U$ in the d_V-metric. By Lemma 1.8 of [67], this follows. [$\mathbb{S}^n$S] $\square$

PROPOSITION 1.1.14. *Let Ω be a properly convex domain in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Then the group* $\mathbf{Aut}(\Omega)$ *of projective automorphisms Ω is closed in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ (resp. $\mathsf{PGL}(n+1,\mathbb{R})$) acts on Ω. Also, the set of elements of g of* $\mathbf{Aut}(\Omega)$ *so that $g(x) \in K$ for a compact subset K of Ω is compact.*

PROOF. We prove for $\mathbb{S}^n$. Clearly, the limit of a sequence of elements in $\mathbf{Aut}(\Omega)$ is an isometry of the Hilbert metric of Ω. Hence, it acts on Ω.

For the second part, we take an n-simplex σ with a point x in the interior as a base point.

The space $\mathscr{S}^n$ of nondegenerate convex n-simplices with base points in their interiors with the Hausdorff topology is homeomorphic to $\mathsf{SL}_\pm(n+1,\mathbb{R})$ since the action of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ is simply transitive on $\mathscr{S}^n$.

The subspace of simplices of form $g(\sigma)$ for g with $g(x) \in K$, $g \in \mathbf{Aut}(\Omega)$ is compact by the existence of the Hilbert metric: We can show this by using the invariants. The edge lengths are invariants. The distance from each vertex to the hyperspace containing the remaining vertices is an invariant of the action.

We can see that the set of such g is bounded: We can find the bounded set

$$\{h_g \in \mathbf{Aut}(\mathbb{S}^n) | h_g \circ g(\sigma, x) = (\sigma, x)\}$$

since the set of $g(\sigma)$ do not degenerate and $g(x)$ is uniformly bounded away from the boundary of $g(\sigma)$. Since the simplex σ and the basepoint x is fixed, we have $h_g \circ g = \mathrm{I}$ for $g(x) \in K$. Hence the set of $\{g | g(x) \in K\}$ is uniformly bounded.

Since $\mathscr{S}^n$ is diffeomorphic to $\mathsf{SL}_\pm(n+1, \mathbb{R})$, the closedness of $\mathbf{Aut}(\Omega)$ proves the result. □

PROPOSITION 1.1.15. *Let Ω be a properly convex domain in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Suppose that a discrete subgroup Γ of $\mathsf{SL}_\pm(n+1, \mathbb{R})$ (resp. $\mathsf{PGL}(n+1, \mathbb{R})$) acts on Ω. Then Ω/Γ is an orbifold.*

PROOF. The second part of Proposition 1.1.14 implies that Γ acts properly discontinuously. We obtain that Ω/Γ is again a closed orbifold. (We need a slight modification of Proposition 3.5.7 of Thurston [149].) □

1.1.4. Topology of orbifolds. We summarize Chapter 4 of [51]. We will only briefly go over it. An n-dimensional *orbifold structure* on a Hausdorff space X is given by maximal collection of charts (U, ϕ, G) satisfying the following conditions:

- U is an open subset of $\mathbb{R}^n$ and $\phi : U \to X$ is a map and G is a finite group acting on U,
- the chart $\phi : U \to X$ induces a homeomorphism U/G to an open subset of X,
- the sets of form $\phi(U)$ covers X.
- for any pair of models (U, ϕ, G) and (V, ψ, H) with an inclusion map $\iota : \phi(U) \to \phi(V)$ lifts to an embedding $U \to V$ equivariant with respect to an injective homomorphism $G \to H$. (compatibility condition)

An *orbifold* $\mathscr{O}$ is a topological space with an orbifold structure. The boundary $\partial\mathscr{O}$ of an orbifold is defined as the set of points with only half-open sets as models. (These are often distinct from topological boundary.) A *suborbifold* N of $\mathscr{O}$ is a subspace of X equipped with maximal collection of charts containing the orbifold charts of form $(U \cap N, \phi | U \cap N, G | U \cap N)$ from $\mathscr{O}$. (See Definition 4.4.2 of [51].) (Note this is more general than other defintiions.) A boundary components of $\mathscr{O}$ is a suborbifold.

Orbifolds are stratified by manifolds. Let $\mathscr{O}$ denote an n-dimensional orbifold with finitely many ends. We will require that $\mathscr{O}$ is *strongly tame*; that is, $\mathscr{O}$ has a compact suborbifold K so that $\mathscr{O} - K$ is a disjoint union of end neighborhoods homeomorphic to closed $(n-1)$-dimensional orbifolds multiplied by open intervals. Hence $\partial\mathscr{O}$ is a compact suborbifold. (See [148], [1], [108] and [51] for details.)

An *orbifold covering map* $p : \mathscr{O}_1 \to \mathscr{O}$ is a map so that for any point on $\mathscr{O}$, there is a connected open set $U \subset X$ with model $(\tilde{U}, \phi, G)$ as above whose inverse image $p^{-1}(U)$ is a union of connected open set U_i of $\mathscr{O}_1$ with models $(\tilde{U}, \phi_i, G_i)$ for a subgroup $G_i \subset G$ and the induced chart $\phi_i : \tilde{U} \to \tilde{U}_i$.

We say that an orbifold is a manifold if it has a subatlas of charts with trivial local groups. We will consider good orbifolds only, i.e., covered by simply connected manifolds. In this case, the universal covering orbifold $\tilde{\mathscr{O}}$ is a manifold with an orbifold covering map $p_{\mathscr{O}} : \tilde{\mathscr{O}} \to \mathscr{O}$. The group of deck transformations will be denote by $\pi_1(\mathscr{O})$ or $\mathbf{\Gamma}$, and is said to be the *fundamental group* of $\mathscr{O}$. They act properly discontinuously on $\tilde{\mathscr{O}}$ but not necessarily freely.

We will follow Section 4.4.2 of [51]. (See Chapter 4 of [100] for manifolds.) A *neat suborbifold* $N \subset \mathscr{O}$ is a suborbifold such that $\partial N \subset \partial \mathscr{O}$ and the tangent spaces to N at ∂N is transversal to the tangent spaces of $\partial \mathscr{O}$. Of course, if $\partial N = \emptyset$, a suborbifold N is considered neat. Let $N(N)$ denote the subspace of the tangent bundle of $\mathscr{O}$ over N consisting of vectors perpedicular to N. Let $\varepsilon : N \to [0,\infty)$ denote a real valued function. We denote by $N_\varepsilon(N)$ the subspace of normal vectors to N of length $\leq \varepsilon(x)$ at each $T_x\mathscr{O}$, $x \in N$. The exponential map is an embedding from $N_\varepsilon(N)$ to $\mathscr{O}$ for sufficiently small ε. We call the image *tubular neighborhood* of N.

PROPOSITION 1.1.16. *We can give a Riemannian metric on $\mathscr{O}$ so that $\partial\mathscr{O}$ is totally geodesic and a neat submanifold N to be totally geodesic perpendicular to $\partial\mathscr{O}$. A tubular neighborhoods are always diffeomorphic to the orbifold product $N \times (-1,1)$ or one $N \times [0,1)$ provided N is a union of boundary components.*

PROOF. See Section 4.4.2 and Lemma 4.4.1 of [51]. □

1.1.5. Geometric structures on orbifolds. An (X,G)-*structure* on an orbifold $\mathscr{O}$ is an atlas of charts from open subsets of X with finite subgroups of G acting on them, and the inclusions always lift to restrictions of elements of G in open subsets of X. This is equivalent to saying that the orbifold $\mathscr{O}$ has a simply connected manifold cover $\tilde{\mathscr{O}}$ with an immersion $D : \tilde{\mathscr{O}} \to X$ and the fundamental group $\pi_1(\mathscr{O})$ acts on $\tilde{\mathscr{O}}$ properly discontinuously so that $h : \pi_1(\mathscr{O}) \to G$ is a homomorphism satisfying $D \circ \gamma = h(\gamma) \circ D$ for each $\gamma \in \pi_1(\mathscr{O})$. Here, $\pi_1(\mathscr{O})$ is allowed to have fixed points with finite stabilizers. (We shall use this second more convenient definition here.) $(D, h(\cdot))$ is called a *development pair* and for a given (X,G)-structure, it is determined only up to an action

$$(D, h(\cdot)) \mapsto (k \circ D, kh(\cdot)k^{-1}) \text{ for } k \in G.$$

Conversely, a development pair completely determines the (X,G)-structure. (See Thurston [149] for the general theory of geometric structures.)

Thurston showed that an orbifold with an (X,G)-structure is always good, i.e., covered by a manifold with an (X,G)-structure. (See Proposition 13.2.1 of Chapter 13 of Thurston [148].) Hence, every geometric orbifold is of form $\tilde{M}/\Gamma$ for a discrete group Γ acting on a simply connected manifold $\tilde{M}$. Here, we have to understand $\tilde{M}/\Gamma$ as having an orbifold structure coming from an atlas where each model set is based on a precompact open cell of $\tilde{M}$ on which a finite subgroup of Γ acts. (See Theorem 4.23 of [51] for details.)

1.1.6. Real projective structures on orbifolds. A cone C in $\mathbb{R}^{n+1} - \{O\}$ is a subspace so that given a vector $x \in C$, $sx \in C$ for every $s \in \mathbb{R}_+$. A *convex cone* is a cone that is a convex subset of $\mathbb{R}^{n+1}$ in the usual sense. A *properly convex cone* is a convex cone not containing a complete affine line.

This seems repeated

Recall the real projective space $\mathbb{RP}^n$ is defined as $\mathbb{R}^{n+1} - \{O\}$ under the quotient relation $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{O\}$.

- Given a vector $\vec{v} \in \mathbb{R}^{n+1} - \{O\}$, we denote by $[\vec{v}] \in \mathbb{RP}^n$ the equivalence class. Let $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n$ denote the projection.
- Given a connected subset A of an affine subspace of $\mathbb{RP}^n$, a cone $C(A) \subset \mathbb{R}^{n+1}$ of A is given as a connected cone in $\mathbb{R}^{n+1}$ mapping onto A under the projection $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n$.
- $C(A)$ is unique up to the antipodal map $\mathscr{A} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $\vec{v} \to -\vec{v}$.

The general linear group $\mathsf{GL}(n+1,\mathbb{R})$ acts on $\mathbb{R}^{n+1}$ and $\mathsf{PGL}(n+1,\mathbb{R})$ acts faithfully on $\mathbb{RP}^n$. Denote by $\mathbb{R}_+ = \{r \in \mathbb{R} | r > 0\}$. The *real projective sphere* $\mathbb{S}^n$ is defined as the quotient of $\mathbb{R}^{n+1} - \{O\}$ under the quotient relation $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R}_+$. We will also use $\mathbb{S}^n$ as the double cover of $\mathbb{RP}^n$. Then $\mathbf{Aut}(\mathbb{S}^n)$, isomorphic to the subgroup $\mathsf{SL}_\pm(n+1,\mathbb{R})$ of $\mathsf{GL}(n+1,\mathbb{R})$ of determinant ± 1, double-covers $\mathsf{PGL}(n+1,\mathbb{R})$. $\mathbf{Aut}(\mathbb{S}^n)$ acts as a group of projective automorphisms of $\mathbb{S}^n$. A *projective map* of a real projective orbifold to another is a map that is projective by charts to $\mathbb{RP}^n$. Let $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{RP}^n$ be a projection and let $\Pi' : \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$ denote one for $\mathbb{S}^n$. An infinite subgroup Γ of $\mathsf{PGL}(n+1,\mathbb{R})$ (resp. $\mathsf{SL}_\pm(n+1,\mathbb{R})$) is *strongly irreducible* if every finite-index subgroup is irreducible. A *subspace* S of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is the image of a subspace with the origin removed under the projection Π (resp. Π').

A line in $\mathbb{RP}^n$ or $\mathbb{S}^n$ is an embedded arc in a 1-dimensional subspace. A *projective geodesic* is an arc in a projective orbifold developing into a line in $\mathbb{RP}^n$ or to a one-dimensional subspace of $\mathbb{S}^n$. A *great segment* is an embedded geodesic connecting a pair of antipodal points in $\mathbb{S}^n$ or the complement of a point in a 1-dimensional subspace in $\mathbb{RP}^n$. Sometimes open great segment is called a *complete affine line*. An affine space $\mathbb{A}^n$ can be identified with the complement of a codimension-one subspace $\mathbb{RP}^{n-1}$ so that the geodesic structures are same up to parameterizations. A *convex subset* of $\mathbb{RP}^n$ is a convex subset of an affine subspace in this paper. A *properly convex subset* of $\mathbb{RP}^n$ is a precompact convex subset of an affine subspace. $\mathbb{R}^n$ identifies with an open half-space in $\mathbb{S}^n$ defined by a linear function on $\mathbb{R}^{n+1}$. (In this paper an affine subspace is either embedded in $\mathbb{RP}^n$ or $\mathbb{S}^n$.)

An *i-dimensional complete affine subspace* is a subspace of a projective orbifold projectively diffeomorphic to an i-dimensional affine subspace in some affine subspace $\mathbb{A}^n$ of $\mathbb{RP}^n$ or $\mathbb{S}^n$.

Again an affine subspace in $\mathbb{S}^n$ is a lift of an affine subspace in $\mathbb{RP}^n$, which is the interior of an n-hemisphere. Convexity and proper convexity in $\mathbb{S}^n$ are defined in the same way as in $\mathbb{RP}^n$.

The complement of a codimension-one subspace W in $\mathbb{RP}^n$ can be considered an affine space $\mathbb{A}^n$ by correspondence

$$[1, x_1, \dots, x_n] \to (x_1, \dots, x_n)$$

for a coordinate system where W is given by $x_0 = 0$. The group $\mathbf{Aff}(\mathbb{A}^n)$ of projective automorphisms acting on $\mathbb{A}^n$ is identical with the group of affine transformations of form

$$\vec{x} \mapsto A\vec{x} + \vec{b}$$

for a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$. The projective geodesics and the affine geodesics agree up to parametrizations.

A subset A of $\mathbb{RP}^n$ or $\mathbb{S}^n$ *spans* a subspace S if S is the smallest subspace containing A. We write $S = \langle A \rangle$. Of couse, we use the same term for affine and vector spaces as well.

We will consider an orbifold $\mathscr{O}$ with a real projective structure: This can be expressed as

- having a pair $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ is an immersion equivariant with respect to
- the homomorphism $h : \pi_1(\mathscr{O}) \to \mathsf{PGL}(n+1,\mathbb{R})$ where $\tilde{\mathscr{O}}$ is the universal cover and $\pi_1(\mathscr{O})$ is the group of deck transformations acting on $\tilde{\mathscr{O}}$.

$(\mathbf{dev}, h)$ is only determined up to an action of $\mathsf{PGL}(n+1,\mathbb{R})$ given by

$$g \circ (\mathbf{dev}, h(\cdot)) = (g \circ \mathbf{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R}).$$

$\mathbf{dev}$ is said to be a *developing map* and h is said to be a *holonomy homomorphism* and $(\mathbf{dev}, h)$ is called a *development pair*. We will usually use only one pair where $\mathbf{dev}$ is an embedding for this paper and hence identify $\tilde{\mathscr{O}}$ with its image. A *holonomy* is an image of an element under h. The *holonomy group* is the image group $h(\pi_1(\mathscr{O}))$.

We denote by $\mathbf{Aut}(K)$ the group of projective automorphisms of a set K in some space with a projective structure. The Klein model of the hyperbolic geometry is given as follows: Let $x_0, x_1, \ldots, x_n$ denote the standard coordinates of $\mathbb{R}^{n+1}$. Let $\mathbb{B}$ be the interior in $\mathbb{RP}^n$ or $\mathbb{S}^n$ of a standard ball that is the image of the positive cone of $x_0^2 > x_1^2 + \cdots + x_n^2$ in $\mathbb{R}^{n+1}$. Then $\mathbb{B}$ can be identified with a hyperbolic n-space. The group of isometries of the hyperbolic space equals the group $\mathbf{Aut}(\mathbb{B})$ of projective automorphisms acting on $\mathbb{B}$. Thus, a complete hyperbolic manifold carries a unique real projective structure and is denoted by $\mathbb{B}/\Gamma$ for $\Gamma \subset \mathbf{Aut}(\mathbb{B})$. Actually, $g(\mathbb{B})$ for any $g \in \mathsf{PGL}(n+1,\mathbb{R})$ will serve as a Klein model of the hyperbolic space, and $\mathbf{Aut}(g\mathbb{B}) = g\mathbf{Aut}(\mathbb{B})g^{-1}$ is the isometry group. (See [51] for details.)

A *totally geodesic hypersurface* A in $\tilde{\mathscr{O}}$ is a suborbifold of codimension-one where each point p in A has a neighborhood U in $\tilde{\mathscr{O}}$ so that $D|\tilde{A}$ has the image in a hyperspace. A suborbifold A is a *totally geodesic hypersurface* if it is covered by a one in $\tilde{\mathscr{O}}$.

1.1.7. Spherical real projective structures. We now discuss the standard lifting: A real projective structure on $\mathscr{O}$ provides us with a development pair $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ is an immersion and $h : \pi_1(\mathscr{O}) \to \mathsf{PGL}(n+1,\mathbb{R})$ is a homomorphism. Since $p_{\mathbb{S}^n}$ is a covering map and $\tilde{\mathscr{O}}$ is a simply connected manifold, $\mathscr{O}$ being a good orbifold, there exists a lift $\mathbf{dev}' : \tilde{\mathscr{O}} \to \mathbb{S}^n$ unique up to the action of $\{\mathrm{I}, \mathscr{A}\}$. This induces a spherical real projective structure on $\tilde{\mathscr{O}}$ and $\mathbf{dev}'$ is a developing map for this real projective structure. Given a deck transformation $\gamma : \tilde{\mathscr{O}} \to \tilde{\mathscr{O}}$, the composition $\mathbf{dev}' \circ \gamma$ is again a developing map for the real projective structure and hence equals $h'(\gamma) \circ \mathbf{dev}'$ for $h'(\gamma) \in \mathsf{SL}_{\pm}(n+1,\mathbb{R})$. We verify that $h' : \pi_1(\mathscr{O}) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})$ is a homomorphism. Hence, $(\mathbf{dev}', h')$ gives us a spherical real projective structure, which induces the original real projective structure.

Given a real projective structure where $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ is an embedding to a properly convex open subset D, the developing map $\mathbf{dev}$ lifts to an embedding $\mathbf{dev}' : \tilde{\mathscr{O}} \to \mathbb{S}^n$ to an open domain D without any pair of antipodal points. D is determined up to $\mathscr{A}$.

We will identify $\tilde{\mathscr{O}}$ with D or $\mathscr{A}(D)$ and $\pi_1(\mathscr{O})$ with $\mathbf{\Gamma}$. Then $\mathbf{\Gamma}$ lifts to a subgroup $\mathbf{\Gamma}'$ of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ acting faithfully and discretely on $\tilde{\mathscr{O}}$. There is a unique way to lift so that $D/\mathbf{\Gamma}$ is projectively diffeomorphic to $\tilde{\mathscr{O}}/\mathbf{\Gamma}'$.

THEOREM 1.1.17. *There is a one-to-one correspondence between the space of real projective structures on an orbifold $\mathscr{O}$ with the space of spherical real projective structures on $\mathscr{O}$. Moreover, a real projective diffeomorphism of real projective orbifolds is an $(\mathbb{S}^n, \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$-diffeomorphism of spherical real projective orbifolds and vice versa.*

PROOF. Straightforward. See p. 143 of Thurston [149] (see also Section 1.1.8). □

Again, we can define the *radial end structures*, *horospherical*, and *totally geodesic ideal boundary* for spherical real projective structures in obviously. Also, each end has $\mathscr{R}$-type or $\mathscr{T}$-type assigned accordingly compatible with these definitions. They correspond directly in the following results also.

PROPOSITION 1.1.18 (Selberg-Malcev). *The holonomy group of a convex real projective orbifold is residually finite.*

PROOF. In this case, $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ always lifts an embedding to a domain in $\mathbb{S}^n$. Γ also lifts to a group of projective automorphisms of the domain in $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$. The lifted group is residually finite by by Malcev [122]. Hence, Γ is thus always residually finite. [$\mathbb{S}^n$S] □

THEOREM 1.1.19 (Selberg). *A real projective orbifold S is covered finitely by a real projective manifold M and S is real projectively diffeomorphic to M/G_1 for a finite group G_1 of real projective automorphisms of M. An affine orbifold S is covered finitely by an affine manifold N, and S is affinely diffeomorphic to N/G_2 for a finite group G_2 of affine automorphisms of N. Finally, given a two convex real projective or affine orbifold S_1 and S_2 with isomorphic fundamental groups, one is a closed orbifold if and only if so is the other.*

PROOF. Since $\mathbf{Aff}(\mathbb{A}^n)$ is a subgroup of a general linear group, Selberg's Lemma [142] shows that there exists a torsion-free subgroup of the deck transformation group. We can choose the group to be a normal subgroup and the second item follows.

A real projective structure induces an $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-structure and vice versa by Theorem 1.1.17. Also a real projective diffeomorphism of orbifolds is an $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-diffeomorphism and vice versa. We regard the real projective structures on S and M as $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-structures. We are done by Selberg's lemma [142] that a finitely generated subgroup of a general linear group has a torsion-free normal subgroup of finite-index.

For the final item, we can take a torsion-free subgroup and the finite covers of S_1 and S_2 are manifold which are $K(\pi,1)$ for identical π. Hence, the conclusion follows. [$\mathbb{S}^n$S] ☐

1.1.8. A comment on lifting real projective structures and conventions. We sharpen Theorem 1.1.17. Let $\mathsf{SL}_-(n+1,\mathbb{R})$ denote the component of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ not containing I. A projective automorphism g of $\mathbb{S}^n$ is orientation preserving if and only if g has a matrix in $\mathsf{SL}(n+1,\mathbb{R})$. For even n, the quotient map $\mathsf{SL}(n+1,\mathbb{R}) \to \mathsf{PGL}(n+1,\mathbb{R})$ is an isomorphism and so is the map $\mathsf{SL}_-(n+1,\mathbb{R}) \to \mathsf{PGL}(n+1,\mathbb{R})$ for the component of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ with determinants equal to -1. For odd n, the quotient map $\mathsf{SL}(n+1,\mathbb{R}) \to \mathsf{PGL}(n+1,\mathbb{R})$ is a 2 to 1 covering map onto its image component with deck transformations given by $A \to \pm A$.

THEOREM 1.1.20. *Let M be a strongly tame n-orbifold. Suppose that $h : \pi_1(M) \to \mathsf{PGL}(n+1,\mathbb{R})$ is a holonomy homomorphism of a real projective structure on M with radial or lens-shaped totally geodesic ends. Then the following hold:*

- *Suppose that M is orientable. We can lift to a homomorphism $h' : \pi_1(M) \to \mathsf{SL}(n+1,\mathbb{R})$, which is a holonomy homomorphism of the $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-structure lifting the real projective structure.*
- *Suppose that M is not orientable. Then we can lift h to a homomorphism $h' : \pi_1(M) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$ that is the holonomy homomorphism of the $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-structure lifting the real projective structure so that the condition $(*)$ is satisfied.*
  - $(*)$ *a deck transformation goes to a negative determinant matrix if and only if it reverses orientations.*

In general a lift h' is unique if we require it to be the holonomy homomorphism of the lifted structure. For even n, the lifting is unique if we require the condition $(*)$.

PROOF. For the first part, recall $\mathsf{SL}(n+1,\mathbb{R})$ is the group of orientation-preserving linear automorphisms of $\mathbb{R}^{n+1}$ and hence is precisely the group of orientation-preserving projective automorphisms of $\mathbb{S}^n$. Since the deck transformations of the universal cover $\tilde M$ of the lifted $(\mathbb{S}^n, \mathsf{SL}_\pm(n+1,\mathbb{R}))$-orbifold are orientation-preserving, the holonomy of the lift are in $\mathsf{SL}(n+1,\mathbb{R})$. We use as h' the holonomy homomorphism of the lifted structure.

For the second part, we can double cover M by an orientable orbifold M' with an orientation-reversing $\mathbb{Z}_2$-action of the projective automorphism group generated by ϕ :

$M' \to M'$. ϕ lifts to $\tilde{\phi} : \tilde{M}' \to \tilde{M}'$ for the universal covering manifold $\tilde{M}' = \tilde{M}$ and hence $h(\tilde{\phi}) \circ \mathbf{dev} = \mathbf{dev} \circ \tilde{\phi}$ for the developing map **dev** and the holonomy

$$h(\tilde{\phi}) \in \mathsf{SL}_-(n+1,\mathbb{R}).$$

Then it follows from the first item since **dev** preserves orientations for a given orientation of $\tilde{M}$. (See p. 143 of Thurston [149].)

The proof of the uniqueness is straightforward. □

REMARK 1.1.21 (Convention on using spherical real projective structures). Suppose we are given a convex real projective orbifold of form Ω/Γ for Ω a convex domain in $\mathbb{RP}^n$ and Γ a subgroup of $\mathsf{PGL}(n+1,\mathbb{R})$. We can also think of Ω as a domain in $\mathbb{S}^n$ and $\Gamma \subset \mathsf{SL}_\pm(n+1,\mathbb{R})$. We can think of them in both ways and we will use a convenient one for the purpose.

1.1.8.1. *Convex hulls.*

DEFINITION 1.1.22. Given a subset K of a convex domain Ω of an affine subspace $\mathbb{A}^n$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$), the *convex hull* $\mathscr{C}\mathscr{H}(K)$ of K is defined as the smallest convex set containing K in $\mathrm{Cl}(\Omega) \subset \mathbb{A}^n$ where we required $\mathrm{Cl}(\Omega)$ is a bounded subset of $\mathbb{A}^n$.

The convex hull is *well-defined* as long as Ω is properly convex. Otherwise, it may be not. This does not change the convex hull. (Usually it will be clear what Ω is by context but we will mention these.) For $\mathbb{RP}^n$, the convex hull depends on Ω but one can check that the convex hull is well-defined on $\mathbb{S}^n$ as long as Ω is properly convex. Also, it is commonly well-known that each point of the convex hull of a set K has a direction vector equal to a linear sum of at most $n+1$ vectors in the direction of K. Hence, the convex hull is a union of n-simplices with vertices in K. Also, if K is compact, then the convex hull is also compact (See Berger [26].)

LEMMA 1.1.23. *Let Ω be a convex open set. Let $\{K_i\}$ be a sequence for a compact set K_i in a properly convex domain for each i. Suppose that $\{K_i\}$ geometrically converges to a compact set $K \subset \Omega$, Then $\{\mathscr{C}\mathscr{H}(K_i)\} \to \mathscr{C}\mathscr{H}(K)$.*

PROOF. It is sufficient to prove for $\mathbb{S}^n$. We write each element of $\mathscr{C}\mathscr{H}(K_i)$ as a finite sum $\left(\left(\sum_{j=1}^{n+1} \lambda_{i,j}\vec{v}_{i,j}\right)\right)$ for $\vec{v}_{i,j}$ in the direction of K_i and $\lambda_{i,j} \geq 0$. Lemma 1.1.7 implies the result. [$\mathbb{S}^n$T] □

1.2. Affine orbifolds

An *affine orbifold* is an orbifold with a geometric structure modeled on $(\mathbb{A}^n, \mathbf{Aff}(\mathbb{A}^n))$. An affine orbifold has a notion of affine geodesics as given by local charts. Recall that a geodesic is *complete* in a direction if the affine geodesic parameter is infinite in the direction.

- An affine orbifold has a *parallel end* if the corresponding end has an end neighborhood foliated by properly embedded affine geodesics parallel to one another in charts and each leaf is complete in one direction. We assume that the affine geodesics are leaves assigned as above.
  - We obtain a smooth complete vector field X_E in a neighborhood of E for each end following the affine geodesics, which is affinely parallel in the flow; i.e., leaves have parallel tangent vectors. We call this an *end vector field*.

  - We denote by $X_{\mathscr{O}}$ the vector field partially defined on $\mathscr{O}$ by taking the union of vector fields defined on some mutually disjoint neighborhoods of the ends using the partition of unity.
  - The oriented direction of the parallel end is uniquely determined in the developing image of each p-end neighborhood of the universal cover of $\mathscr{O}$.
  - Finally, we put a fixed complete Riemannian metric on $\mathscr{O}$ so that for each end there is an open neighborhood where the metric is invariant under the flow generated by $X_{\mathscr{O}}$. Note that such a Riemannian metric always exists.
- An affine orbifold has a *totally geodesic end* E if each end can be completed by a totally geodesic affine hypersurface. That is, there exists a neighborhood of the end E diffeomorphic to $\Sigma_E \times [0,1)$ for an $(n-1)$-orbifold Σ_E that compactifies to an orbifold diffeomorphic to $\Sigma_E \times [0,1]$, and each point of $\Sigma_E \times \{1\}$ has a neighborhood affinely diffeomorphic to a neighborhood of a point p in ∂H for a half-space H of an affine space. This implies the fact that the corresponding p-end holonomy group $h(\pi_1(\tilde{E}))$ for a p-end $\tilde{E}$ going to E acts on a hyperspace P corresponding to $E \times \{1\}$.

Recall that an orbifold is a topological space stratified by open manifolds (See Chapter 4 of [51]). An affine or projective orbifold is *triangulated* if there is a smoothly embedded n-cycle consisting of geodesic n-simplices on the compactified orbifold relative to ends by adding an ideal point to a radial end and an ideal boundary to each totally geodesic ends. where the interiors of i-simplices in the cycle are mutually disjoint and are embedded in strata of the same or higher dimension.

1.2.1. Affine suspension constructions. The affine subspace $\mathbb{R}^{n+1}$ is a dense open subset of $\mathbb{RP}^{n+1}$ which is the complement of $(n+1)$-dimensional projective space $\mathbb{RP}^{n+1}$. Thus, an affine transformation is a restriction of a unique projective automorphism acting on $\mathbb{R}^{n+1}$. The group of affine transformations $\mathbf{Aff}(\mathbb{A}^{n+1})$ is isomorphic to the group of projective automorphisms acting on $\mathbb{R}^{n+1}$ by the restriction homomorphism.

A *dilatation* γ in an affine subspace $\mathbb{R}^{n+1}$ is a linear transformation with respect to an affine coordinate system so that all its eigenvalues have norm > 1 or < 1. Here, γ is an expanding map in the dynamical sense. A *scalar dilatation* is a dilation with a single eigenvalue.

An affine orbifold $\mathscr{O}$ is *radiant* if $h(\pi_1(\mathscr{O}))$ fixes a point in $\mathbb{R}^{n+1}$ for the holonomy homomorphism $h : \pi_1(\mathscr{O}) \to \mathbf{Aff}(\mathbb{A}^{n+1})$. A real projective orbifold $\mathscr{O}$ of dimension n has a developing map $\mathbf{dev}' : \tilde{\mathscr{O}} \to \mathbb{S}^n$ and the holonomy homomorphism $h' : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$. We regard $\mathbb{S}^n$ is embedded as a unit sphere in $\mathbb{R}^{n+1}$ temporarily. We obtain a radiant affine $(n+1)$-orbifold by taking $\tilde{\mathscr{O}}$ and $\mathbf{dev}'$ and h': Define $D'' : \tilde{\mathscr{O}} \times \mathbb{R}_+ \to \mathbb{R}^{n+1}$ by sending (x,t) to $t\mathbf{dev}'(x)$. For each element of $\gamma \in \pi_1(\mathscr{O})$, we define the transformation γ' on $\tilde{\mathscr{O}} \times \mathbb{R}_+$ from

$$
\begin{aligned}
\gamma'(x,t) =& (\gamma(x), \theta(\gamma)||h'(\gamma)(t\mathbf{dev}'(x))||) \\
& \text{for a homomorphism } \theta : \pi_1(\mathscr{O}) \to \mathbb{R}_+.
\end{aligned} \tag{1.2.1}
$$

Also, there is a transformation $S_s : \tilde{\mathscr{O}} \times \mathbb{R}_+ \to \tilde{\mathscr{O}} \times \mathbb{R}_+$ sending (x,t) to (x,st) for $s \in \mathbb{R}_+$. Thus,

$$\tilde{\mathscr{O}} \times \mathbb{R}_+/\langle S_\rho, \pi_1(\mathscr{O})\rangle, \rho \in \mathbb{R}_+, \rho > 1$$

is an affine orbifold with the fundamental group isomorphic to $\pi_1(\mathscr{O}) \times \mathbb{Z}$ where the developing map is given by D'' the holonomy homomorphism is given by h' and sending the generator of $\mathbb{Z}$ to S_ρ. We call the result the *affine suspension* of $\mathscr{O}$, which of course is radiant. The representation of $\pi_1(\mathscr{O}) \times \mathbb{Z}$ with the center $\mathbb{Z}$ mapped to a scalar dilatation is called an *affine suspension* of h. A *special affine suspension* is an affine suspension with $\theta \equiv 1$ identically.

There is a variation called *generalized affine suspension*. Here we use any γ that is a dilatation and normalizes $h'(\pi_1(\mathscr{O}))$ and we deduce that

$$\tilde{\mathscr{O}} \times \mathbb{R}_+ / \langle \gamma, \pi_1(\mathscr{O}) \rangle$$

is an affine orbifold with the fundamental group isomorphic to $\langle \pi_1(\mathscr{O}), \mathbb{Z} \rangle$. (See Sullivan-Thurston [147], Barbot [10] and Choi [47] also.)

DEFINITION 1.2.1. We denote by $C(\tilde{\mathscr{O}})$ the manifold $\tilde{\mathscr{O}} \times \mathbb{R}$ with the structure given by D'', and say that $C(\tilde{\mathscr{O}})$ is the *affine suspension* of $\tilde{\mathscr{O}}$.

Let $S_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, given by $\vec{v} \to t\vec{v}$, $t \in \mathbb{R}_+$, be a one-parameter family of dilations fixing a common point. A family of self-diffeomorphisms Ψ_t on an affine orbifold M lifting to $\hat{\Psi}_t : \widetilde{M} \to \widetilde{M}$ so that $D \circ \hat{\Psi}_t = S_{e^t} \circ D$ for $t \in \mathbb{R}$ is called a group of *radiant flow diffeomorphisms*.

LEMMA 1.2.2. *Let $\mathscr{O}$ be a strongly tame real projective n-orbifold.*

- *An affine suspension $\mathscr{O}'$ of $\mathscr{O}$ always admits a group of radiant flow diffeomorphisms. Here, $\{\Phi_t\}$ is a circle and all flow lines are closed.*
- *Conversely, if there exists a group of radiant flow diffeomorphisms where all orbits are closed and have the homology class $[[* \times \mathbb{S}^1]]$ on $\mathscr{O} \times \mathbb{S}^1$ with an affine structure, then $\mathscr{O} \times \mathbb{S}^1$ is affinely diffeomorphic to one obtained by an affine suspension construction from a real projective structure on $\mathscr{O}$.*

PROOF. The first item is clear by the above construction.

The generator of the $\pi_1(\mathbb{S}^1)$-factor goes to a scalar dilatation since it induces the identity map on the space of directions of radial segments from the global fixed point. Thus, each closed curve along $* \times \mathbb{S}^1$ gives us a nontrivial homology. The homology direction of the flow equals $[[* \times \mathbb{S}^1]] \in \mathbb{S}(H_1(\mathscr{O} \times \mathbb{S}^1; \mathbb{R}))$. By Theorem D of [79], there exists a connected cross-section homologous to

$$[\mathscr{O} \times *] \in H_n(\mathscr{O} \times \mathbb{S}^1, V \times \mathbb{S}^1; \mathbb{R}) \cong H^1(\mathscr{O} \times \mathbb{S}^1; \mathbb{R})$$

where V is the union of the disjoint end neighborhoods of product forms in $\mathscr{O}$. By Theorem C of [79], any cross-section is isotopic to $\mathscr{O} \times *$. The radial flow is transverse to the cross-section isotopic to $\mathscr{O} \times *$ and hence $\mathscr{O}$ admits a real projective structure. It follows easily now that $\mathscr{O} \times \mathbb{S}^1$ is an affine suspension. (See [10] for examples.) $\square$

An affine suspension of a horospherical orbifold is called a *suspended horoball orbifold*. An end of an affine orbifold with an end neighborhood affinely diffeomorphic to this is said to be of *suspended horoball type*. This has also a parallel end since the fixed point in the boundary of $\mathbb{R}^n$ gives a unique direction.

PROPOSITION 1.2.3. *Under the affine suspension construction, a strongly tame real projective n-orbifold has radial, totally geodesic, or horospherical ends if and only if the affine $(n+1)$-orbifold affinely suspended from it has parallel, totally geodesic, or suspended horospherical ends.*

Again affine $(n+1)$-orbifold suspended have *type $\mathscr{R}$- or $\mathscr{T}$-ends* if the corresponding real projective n-orbifold has $\mathscr{R}$- or $\mathscr{T}$-ends in correspondingly.

1.3. The needed linear algebra

Here, we will collect the linear algebra we will need in this monograph. A source is a comprehensive book by Hoffman and Kunz [101].

DEFINITION 1.3.1. Given an eigenvalue λ of an element $g \in \mathsf{SL}_\pm(n+1,\mathbb{R})$, a $\mathbb{C}$-*eigenvector* $\vec{v}$ is a nonzero vector in

$$\mathbb{R}E_\lambda(g) := \mathbb{R}^{n+1} \cap \left(\ker(g-\lambda I) + \ker(g-\bar{\lambda} I)\right), \lambda \neq 0, \Im\lambda \geq 0$$

A $\mathbb{C}$-*fixed point* is the direction of a $\mathbb{C}$-eigenvector in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$ or $\mathbb{CP}^n$).

Any element of g has a primary decomposition. (See Section 6.8 of [101].) Write the minimal polynomial of g as $\prod_{i=1}^m (x-\lambda_i)^{r_i}$ for $r_i \geq 1$ and mutually distinct complex numbers $\lambda_1, \dots, \lambda_m$. Define

$$C_{\lambda_i}(g) := \ker(g-\lambda_i \mathrm{I})^{r_i} \subset \mathbb{C}^{n+1}$$

where $r_i = r_j$ if $\lambda_i = \bar{\lambda}_j$. Then the primary decomposition theorem states

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^m C_{\lambda_i}(g),$$

which is a canonical decomposition.

A *real primary subspace* is the sum $\mathbb{R}^{n+1} \cap (C_\lambda(g) + C_{\bar{\lambda}}(g))$ for λ an eigenvalue of g.

A point $[\vec{v}], \vec{v} \in \mathbb{R}^{n+1}$, is *affiliated* with a norm μ of an eigenvalue if

$$\vec{v} \in \mathscr{R}_\mu(g) := \bigoplus_{i \in \{j \mid |\lambda_j| = \mu\}} C_{\lambda_i}(g) \cap \mathbb{R}^{n+1}. \tag{1.3.1}$$

Let $\mu_1, \dots, \mu_l$ denote the set of distinct norms of eigenvalues of g. We also have $\mathbb{R}^{n+1} = \bigoplus_{i=1}^l \mathscr{R}_{\mu_i}(g)$. Here, $\mathscr{R}_\mu(g) \neq \{0\}$ if μ equals $|\lambda_i|$ for at least one i.

PROPOSITION 1.3.2. *Let g be an element of $\mathsf{PGL}(n+1,\mathbb{R})$ (resp. $\mathsf{SL}_\pm(n+1,\mathbb{R})$) acting on $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$). Let V and W be independent complementary subspaces where g acts on. Suppose that every norm of the eigenvalue of any eigenvector in the direction of V is strictly larger than any norms of the eigenvalues of the vectors in the direction of W. Let V^S be the subspace that is the join of the $\mathbb{C}$-eigenspaces of V. Then*

- *for $x \in \mathbb{RP}^n - \Pi(V) - \Pi(W)$ (resp. $\mathbb{S}^n - \Pi'(V) - \Pi'(W)$), $\{g^n(x)\}$ accumulates to only points in $\Pi(V^S)$ (resp. $\Pi'(V^S)$) as $n \to \infty$.*
- *Let U be a neighborhood of x in $\mathbb{RP}^n - \Pi(V) - \Pi(W)$ (resp. $\mathbb{S}^n - \Pi'(V) - \Pi'(W)$). There exists an open subset U' of $\Pi(V^S)$ (resp. $\Pi'(V^S)$) where each point of U' is realized as a limit point of $\{g^n(y)\}$ as $n \to \infty$ for some y in U.*

PROOF. It is sufficient to prove for $\mathbb{C}^{n+1}$ and $\mathbb{CP}^n$. We write the minimal polynomial of g as $\prod_{i=1}^m (x-\lambda_i)^{r_i}$ for $r_i \geq 1$ and mutually distinct complex numbers $\lambda_1, \dots, \lambda_m$. Let $W^\mathbb{C}$ be the complexification of the subspace corresponding to W and $V^\mathbb{C}$ the one for V. Then $C_{\lambda_i}(g)$ is a subspace of $W^\mathbb{C}$ or $V^\mathbb{C}$ by elementary linear algebra.

Now, we write the matrix of g determined only up to $\pm\mathrm{I}$ in terms of above primary decomposition spaces. Then we write the matrix in the Jordan form in an upper triangular form. The diagonal terms of the matrix of g^n dominates nondiagonal terms in terms of ratios of the absolute values. The lemma easily follows.

The last part follows by writing x and y in terms of vectors in directions of V and W and other g-invariant subspaces. $\square$

1.3.1. Nilpotent and orthopotent groups. Let $\mathbb{U}$ denote a maximal nilpotent subgroup of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ given by upper triangular matrices with diagonal entries equal to 1. We let $\mathbb{U}_{\mathbb{C}}$ denote the group of by upper triangular matrices with diagonal entries equal to 1 in $\mathsf{SL}_{\pm}(n+1,\mathbb{C})$.

Let $\mathsf{O}(n+1)$ denote the orthogonal group of $\mathbb{R}^{n+1}$ with the standard hermitian inner product.

LEMMA 1.3.3 (Iwasawa Decomposition). *The matrix of $g \in \mathbf{Aut}(\mathbb{S}^n)$ can be written under an orthogonal coordinate system as $k(g)a(g)n(g)$ where $k(g)$ is an element of $\mathsf{O}(n+1)$, $a(g)$ is a positive diagonal element, and $n(g)$ is real unipotent. Also, diagonal elements of $a(g)$ are the norms of eigenvalues of g as elements of $\mathbf{Aut}(\mathbb{S}^n)$.*

PROOF. See Theorem 1.3 of Chapter IX of [98]. □

Recall that all maximal unipotent subgroups are conjugate to each other in $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. (See Section 21.3 of Humphreys [102].) We define

$$\mathbb{U}' := \bigcup_{k\in \mathsf{O}(n+1)} k\mathbb{U}k^{-1} = \bigcup_{k\in \mathsf{SL}_{\pm}(n+1,\mathbb{R})} k\mathbb{U}k^{-1}.$$

The second equality is explained: Each maximal unipotent subgroup is characterized by a maximal flag. Each maximal unipotent subgroup is conjugate to a standard lower triangular unipotent group by an orthogonal element in $\mathsf{O}(n+1)$ since $\mathsf{O}(n+1)$ acts transitively on the maximal flag space.

COROLLARY 1.3.4. *Suppose that we have for a positive constant C_1, and an element $g \in \mathsf{SL}_{\pm}(n+1,\mathbb{R})$,*

$$\frac{1}{C_1} \leq \lambda_{n+1}(g) \leq \lambda_1(g) \leq C_1$$

for the minimal norm $\lambda_{n+1}(g)$ of the eigenvalue of g and the maximal norm $\lambda_1(g)$ of the eigenvalues of g. Then g is in a bounded distance from $\mathbb{U}'$ with the bound depending only on C_1.

PROOF. Let us fix an Iwasawa decomposition $\mathsf{SL}_{\pm}(n+1,\mathbb{R}) = \mathsf{O}(n+1)D_{n+1}\mathbb{U}$ for a positive diagonal group D_{n+1}. By Lemma 1.3.3, we can find an element $k \in \mathsf{O}(n+1)$ so that

$$g = kk(g)k^{-1}ka(g)k^{-1}kn(g)k^{-1}$$

where $k(g) \in \mathsf{O}(n+1), a(g) \in D_n^+, n(g) \in \mathbb{U}'$. Then $kk(g)k^{-1} \in \mathsf{O}(n+1)$ and $ka(g)k^{-1}$ is uniformly bounded from I by a constant depending only on C_1 by assumption. □

A subset of a Lie group is of *polynomial growth* if the volume of the ball $B_R(\mathrm{I})$ radius R is less than or equal to a polynomial of R. As usual, the metric is given by the standard positive definite left-invariant bilinear form that is invariant under the conjugations by the compact group $\mathsf{O}(n+1)$.

LEMMA 1.3.5. *$\mathbb{U}'$ is of polynomial growth in terms of the distance from* I.

PROOF. Let $\mathbf{Aut}(\mathbb{S}^n)$ have a left-invariant Riemannian metric. Clearly $\mathbb{U}$ is of polynomial growth by Gromov [94] since $\mathbb{U}$ is nilpotent. Given fixed $g \in \mathsf{O}(n+1)$, the distance between gug^{-1} and u for $u \in \mathbb{U}'$ is proportional to a constant c_g, $c_g > 1$, multiplied by $\mathbf{d}(u,\mathrm{I})$. Choose $u \in \mathbb{U}'$ which is unipotent. We can write $u(s) = \exp(s\vec{u}), s \geq 0$ where $\vec{u}$ is a nilpotent matrix of unit norm. $g(t) := \exp(t\vec{x}), t \geq 0$ for $\vec{x}$ in the Lie algebra of $\mathsf{O}(n+1)$ of unit norm. For a family of $g(t) \in \mathsf{O}(n+1)$, we define

$$u(t,s) = g(t)u(s)g(t)^{-1} = \exp(s\mathrm{Ad}_{g(t)}\vec{u}). \tag{1.3.2}$$

We compute

$$u(t,s)^{-1}\frac{du(t,s)}{dt} = u(t,s)^{-1}(\vec{x}u(t,s) - u(t,s)\vec{x}) = (\mathrm{Ad}_{u(t,s)^{-1}} - \mathrm{I})(\vec{x}).$$

Since $\vec{u}$ is nilpotent, $\mathrm{Ad}_{u(t,s)^{-1}} - \mathrm{I}$ is a polynomial of variables t,s. The norm of $du(t,s)/dt$ is bounded above by a polynomial in s and t. The conjugation orbits of $\mathsf{O}(n+1)$ in $\mathbf{Aut}(\mathbb{S}^n)$ are compact. Also, the conjugation by $\mathsf{O}(n+1)$ preserves the distances of elements from I since the left-invariant metric μ is preserved by conjugation at I and geodesics from I go to geodesics from I of same μ-lengths under the conjugations by (1.3.2). Hence, we obtain a parametrization of $\mathbb{U}'$ by $\mathbb{U}$ and $\mathsf{O}(n+1)$ where the volume of each orbit of $\mathsf{O}(n+1)$ grows polynomially. Since $\mathbb{U}$ is of polynomial growth, $\mathbb{U}'$ is of polynomial growth in terms of the distance from I. □

THEOREM 1.3.6 (Zassenhaus [158]). *For every discrete group G of $\mathsf{GL}(n+1,\mathbb{R})$, all of which have the shape in a complex basis in $\mathbb{C}^n$*

$$\begin{pmatrix} e^{i\theta_1} & * & \cdots & * \\ 0 & e^{i\theta_2} & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_{n+1}} \end{pmatrix},$$

there exists a positive number ε, so that all the matrices A from G which satisfy the inequalities $|e^{i\theta_j} - 1| < \varepsilon$ for every $j = 1,\dots,n+1$ are contained in the radical of the group, i.e., the subgroup G_u of elements of G with only unit eigenvalues.

An element g of $\mathsf{GL}(n+1,\mathbb{R})$ (resp. $\mathsf{PGL}(n+1,\mathbb{R})$) is said to be *unit-norm-eigenvalued* if it (resp. its representative) has only eigenvalues of norm 1. A group is unit-norm-eigenvalued if all of its elements are *unit-norm-eigenvalued*.

A subgroup G of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ is *orthopotent* if there is a flag of subspaces $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = \mathbb{R}^{n+1}$ preserved by G so that G acts as an orthogonal group on Y_{j+1}/Y_j for each $j = 0,\dots,m-1$ for some choices of inner-products. (See D. Fried [80].)

THEOREM 1.3.7. *Let G be a unit-norm-eigenvalued subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})$. Then G is orthopotent, and the following hold*:

- *If G is discrete, then G is virtually unipotent.*
- *If G is a connected Lie group, then G is an extension of a solvable group by a compact group; i.e., G/S is a compact group for a normal solvable group S in G.*
- *If G is contractible, then G is a simply connected solvable Lie group.*

PROOF. By Corollary 1.3.4, G is in $\mathbb{U}'$.

Suppose that G is discrete. Then G is of polynomial growth by Lemma 1.3.5. By Gromov [94], G is virtually nilpotent.

Choose a finite-index normal nilpotent subgroup G' of G. Since G' is solvable, Theorem 3.7.3 of [150] shows that G' can be put into an upper triangular form for a complex basis. Let G'_u denote the subset of G' with only elements with all eigenvalues equal to 1. G'_u is a normal subgroup since it is in an upper triangular form. The map $G' \to G'/G'_u$ factors into a map $G' \to (\mathbb{S}^1)^n$ by taking the complex eigenvalues. By Theorem 1.3.6, the image is a discrete subgroup of $(\mathbb{S}^1)^n$. Hence, G'/G'_u is finite where G'_u is unipotent. (Another proof is given in the Remark of page 124 of Jenkins [105].)

Suppose that G is a connected Lie group. Since $a(g) = \mathrm{I}$ for $a(g)$ for all $g \in G$, Corollary 1.3.4 shows that $G \subset K\mathbb{U}K$ for a compact Lie group K. Recall that a *distal group* is a linear group whose elements do not decrease norms of vectors. Since $\mathbb{U}$ is a distal group, G is a distal group, and hence G is orthopotent by [62] or [132].

Since $G \subset K\mathbb{U}K$, it is of polynomial growth, Corollary 2.1 of Jenkins [105] implies that G is an extension of a solvable Lie group by a compact Lie group.

If G is contractible, G then can only be an extension by a finite group. Since G is determined by its Lie algebra, G must be solvable by the second item. $\square$

1.3.2. Elements of dividing groups. Suppose that Ω, $\Omega \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$), is an open domain that is properly convex but not necessarily strictly convex. Let Γ, $\Gamma \subset \mathsf{SL}_{\pm}(n+1,\mathbb{R})$ (resp. $\subset \mathsf{PGL}(n+1,\mathbb{R})$), be a discrete group acting on Ω so that Ω/Γ is compact.

An element of Γ is said to be *elliptic* if it is conjugate to an element of a compact subgroup of $\mathsf{PGL}(n+1,\mathbb{R})$ or $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$.

LEMMA 1.3.8. *Suppose that Ω is a properly convex domain in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$), and Γ is a group of projective automorphisms of Ω. Suppose that Ω/Γ is an orbifold. Then an element g of Γ is elliptic if and only if g fixes a point of Ω if and only if g is of finite order.*

PROOF. Let us assume $\Omega \subset \mathbb{S}^n$. Let g be an elliptic element of Γ. Take a point $x \in \Omega$. Let $\vec{x}$ denote a vector in a cone $C(\Omega) \subset \mathbb{R}^{n+1}$ corresponding to x. Then the orbits $\{g^n(\vec{x})|n \in \mathbb{Z}\}$ has a compact closure. There is a fixed vector in $C(\Omega)$, which corresponds to a fixed point of Ω.

If x is a point of Ω fixed by g, then it is in the stabilizer group. Since Ω/Γ is an orbifold, g is of finite order.

If g is of finite order, g is certainly elliptic. [$\mathbb{S}^n$T] $\square$

We recall the definitions of Benoist [23]: For an element g of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$, we denote by $\lambda_1(g),\dots,\lambda_{n+1}(g)$ the sequence of the norms of eigenvalues of g with repetitions by their respective multiplicities. The first one $\lambda_1(g)$ is called the *spectral radius* of g.

Assume $\lambda_1(g) \neq \lambda_{n+1}(g)$ for the following definitions.

- An element g of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ is *proximal* if $\lambda_1(g)$ has multiplicity one.
- g is *positive proximal* if g is proximal and $\lambda_1(g)$ is an eigenvalue of g.
- An element g of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ is *semi-proximal* if $\lambda_1(g)$ or $-\lambda_1(g)$ is an eigenvalue of g.
- An element g of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ is *positive semi-proximal* if $\lambda_1(g)$ is an eigenvalue of g. (Definition 3.1 of [23].)
- g is called *positive bi-proximal* if g and g^{-1} is both positive proximal.
- g is called *positive bi-semi-proximal* if g and g^{-1} is both positive semi-proximal.

Of course, the proximality is a stronger condition than semi-proximality.

Let Ω be a properly convex open domain in $\mathbb{S}^n$. For each positive bi-semi-proximal element $g \in \Gamma$ acting on Ω, we have two disjoint compact convex subspaces

$$A_g := A \cap \mathrm{Cl}(\Omega) \text{ and } R_g := R \cap \mathrm{Cl}(\Omega)$$

for the eigenspace A associated with the largest of eigenvalues of g and the eigenspace R associated with the smallest of the eigenvalues of g. Note $g|A_g$ and $g|R_g$ are both identity maps. Here, A_g is associated with $\lambda_1(g)$ and R_g is with $\lambda_n(g)$, which is an eigenvalue as well. A_g is called an *attracting fixed subset* and R_g a *repelling fixed subset*.

Let g be a positive bi-semi-proximal element. For g, $g \in \mathsf{SL}_{\pm}(n+1,\mathbb{R})$,

- we denote by $V^A_g := \ker(g-\lambda_1(g)\mathrm{I})^{m_1} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ where m_1 is the multiplicity of the eigenvalue $\lambda_1(g)$ in the characteristic polynomial of g, and

- by $V_g^R = \ker(g - \lambda_n(g)\mathrm{I})^{m_n} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ where m_n is the multiplicity of the eigenvalue $\lambda_n(g)$.

We denote $\hat{A}_g = \langle V_g^A \rangle \cap \mathrm{Cl}(\Omega)$ and $\hat{R}_g = \langle V_g^R \rangle \cap \mathrm{Cl}(\Omega)$. Clearly,

$$A_g \subset \hat{A}_g \text{ and } R_g \subset \hat{R}_g.$$

LEMMA 1.3.9 (Lemma 3.2 of [23]). *Suppose a nonidentity projective automorphism g acts on a properly convex domain. Then g is positive bi-semi-proximal.*

The following propositions are related to Section 2, 3 of [67], using somewhat different apporaches. We denote by $\|\cdot\|$ a standard Euclidean norm of a vector space over $\mathbb{R}$.

LEMMA 1.3.10. *Suppose that Ω is a properly convex domain in $\mathbb{S}^n$. Suppose that an infinite-order element g acts on Ω with only single norm of eigenvalue. Then*

$$\inf_{y\in\Omega}\{d_\Omega(y, g(y)) | y \in \Omega \cap Q\} = 0.$$

PROOF. g fixes a point x in $\mathrm{Cl}(\Omega)$ by the Brouwer-fixed-point theorem. If g fix a point in Ω, we are done. Assume $x \in \mathrm{bd}\Omega$ is a fixed point of g.

We prove by induction. When $\dim\Omega = 0, 1$, this is clearly obvious. Suppose that we proved the conclusion when $\dim\Omega = n-1$, $n \geq 2$. We now assume that $\dim\Omega = n$.

We may assume that $\Omega \subset \mathbb{A}^n$ for an affine space $\mathbb{A}^n$ since Ω is properly convex. We choose a coordinate system where x is the origin of $\mathbb{A}^n$. Then g has a form of a rational map. We denote by Dg_x the linear map that is the differential of g at x.

Let S_r denote the similarity transformation of $\mathbb{A}^n$ fixing x. Then we obtain

$$S_r \circ g \circ S_{1/r} : S_r(\Omega) \to S_r(g(\Omega)).$$

Recall the definition of the linear map $Dg_x : \mathbb{R}^n \to \mathbb{R}^n$ is one satisfying

$$\lim_{y,u\to 0} \frac{\|g(y) - g(u) - Dg_x(y-u)\|}{\|y-u\|} \to 0.$$

Hence,

$$\lim_{r\to\infty} r\left\|g(S_{1/r}(y)) - g(S_{1/r}(u)) - S_{1/r}Dg_x(y-u)\right\| \to 0.$$

Setting $u = 0$, we obtain

$$\lim_{r\to\infty} \left\|S_r \circ g \circ S_{1/r}(y) - Dg_x(y)\right\| \to 0.$$

We obtain that as $r \to \infty$, $\{S_r \circ g \circ S_{1/r}\}$ converges to Dg_x on a sufficiently small open ball around x.

Also, it is easy to show that as $r \to \infty$, $\{S_r(\Omega)\}$ geometrically converges to a cone $\Omega_{x,\infty}$ with the vertex at x on which Dg_x acts on.

Let x_n be an affine coordinate function for a sharply-supporting hyperspace of Ω taking 0 value at x. It will be specified a bit later. For now any such one will do. Let $\mathbf{x}(t)$ be a projective geodesic with $\mathbf{x}(0) = x$ at $t = 0$ and $\mathbf{x}(t) \in \Omega, x_n(\mathbf{x}(t)) = t$ for $t > 0$ and let $\vec{u} = d\mathbf{x}(t)/dt \neq 0$ at $t = 0$. We assumed in the premise that g is unit-norm-eigenvalued. Then

$$\lim_{t\to 0} d_\Omega(g(\mathbf{x}(t)), \mathbf{x}(t)) = d_{\Omega_{x,\infty}}(Dg_x(\vec{u}), \vec{u}) \tag{1.3.3}$$

considering $\vec{u}$ as an element of the cone $\Omega_{x,\infty}$: This follows from

$$d_\Omega(g \circ S_{1/r} \circ S_r(\mathbf{x}(t)), \mathbf{x}(t)) = d_{S_r(\Omega)}(S_r \circ g \circ S_{1/r}(S_r(\mathbf{x}(t))), S_r(\mathbf{x}(t)))$$

since $S_r : (\Omega, d_\Omega) \to (S_r(\Omega), d_{S_r(\Omega)})$ is an isometry. We set $\mathbf{x}(t) = \mathbf{x}(1/r)$ and obtain $S_r(\mathbf{x}(1/r)) \to \vec{u}$ as $r \to \infty$. Since $S_r(\Omega) \to \Omega_{x,\infty}$ as $r \to \infty$, (1.1.3) shows that

$$\{d_{S_r(\Omega)}(S_r \circ g \circ S_{1/r}(S_r(\mathbf{x}(1/r))), S_r(\mathbf{x}(1/r)))\} \to d_{\Omega_{x,\infty}}(Dg_x(\vec{u}), \vec{u}) \text{ as } r \to \infty. \tag{1.3.4}$$

Now, $\Omega^o_{x,\infty}$ is a convex cone of form $C(U)$ for a convex open domain U in the infinity of $\mathbb{A}^n$. The space U^* of sharply-supporting hyperspaces of $\Omega^o_{x,\infty}$ at x is a convex compact set.

Since g acts on a ball U^*, g fixes a point by the Brouwer-fixed-point theorem, which corresponds to a hyperspace. Let P be a hyperspace in $\mathbb{A}^n$ passing x sharply-supporting $\Omega^o_{x,\infty}$ invariant under Dg_x.

We now choose the affine coordinate x_n for P so that P is the zero set. There are three possiblity for $\Omega_{x,\infty}$ by Proposition 1.1.4:

- a complete affine space,
- a prroperly convex domain, or
- a convex but complete and not properly convex domain

First, suppose that $\Omega_{x,\infty}$ is a properly convex cone. Projecting $\Omega^o_{x,\infty}$ to the space $\mathbb{S}^{n-1}_x$ of rays starting from x, we obtain a properly convex open domain $\Omega_1 = R_x(\Omega)$. Here, $\dim \Omega_1 \leq n-1$.

By the induction hypothesis on dimension $\dim \Omega_1$, since the Dg_x-action has only one-norm of the eigenvalues, we can find a sequence $\{z_i\}$ in Ω_1 so that $\{d_{\Omega_1}(Dg_x(z_i), z_i)\} \to 0$. Since $\Omega_{x,\infty}$ is a proper convex cone in $\mathbb{A}^n$, we choose a sequence $u_i \in \Omega_{x,\infty}$ with $x_n(u_i) = 1$ and u_i has the direction of z_i from x. Let $\vec{u}_i$ denote the vector in the direction of $\overrightarrow{xz_i}$ on $\mathbb{A}^n$ where $x_n(\vec{u}_i) = 1$. Since g is unit-norm-eigenvalued, $x_n(Dg_x(\vec{u}_i)) = 1$ also. Hence, the geodesic to measure the Hilbert metric from $\vec{u}_i$ to $Dg_x(\vec{u}_i)$ is on $x_n = 1$. Let P_1 denote the affine subspace given by $x_n = 1$. The projection $\Omega_{x,\infty} \cap P_1 \to U$ from x is a projective diffeomorphism and hence is an isometry. Therefore, $d_{\Omega_{x,\infty}}(Dg_x(\vec{u}_i), \vec{u}_i) \to 0$.

We can find arcs $\mathbf{x}_i(t)$ with

$$x_n(\mathbf{x}_i(t)) = t \text{ and } d\mathbf{x}_i(t)/dt = \vec{u}_i \text{ at } t = 0.$$

Also, we find a sequence of points $\{\mathbf{x}_i(t_i)\}$, $t_i \to 0$, so that

$$d_\Omega(g(\mathbf{x}_i(t_i)), \mathbf{x}_i(t_i)) = d_{S_{1/t_i}(\Omega)}(S_{1/t_i} \circ g \circ S_{t_i}(S_{1/t_i}(\mathbf{x}_i(t_i))), S_{1/t_i}(\mathbf{x}_i(t_i))).$$

Since $\{S_{1/t_j}(\mathrm{Cl}(\Omega))\} \to \mathrm{Cl}(\Omega_{x,\infty})$, and $\{S_{1/t_j}(\mathbf{x}_i(t_j))\} \to \vec{u}_i$ as $j \to \infty$, we obtain

$$\{d_\Omega(g(\mathbf{x}_i(t_j)), \mathbf{x}_i(t_j))\} \to d_{\Omega_{x,\infty}}(Dg_x(\vec{u}_i), \vec{u}_i)$$

by (1.3.4).

By choosing j_i sufficiently large for each i, we obtain

$$\{d_\Omega(g(\mathbf{x}_i(t_{j_i})), \mathbf{x}_i(t_{j_i}))\} \to 0.$$

Suppose that $\Omega_{x,\infty}$ is not a properly convex cone. If Ω_1 is a complete affine space, then we can use the argument very similar to

Now, $\Omega_1 = R_x(\Omega)$ is a convex but not properly convex domain. By Proposition 1.1.4, such a set is foliated by complete affine spaces of dimension j, $0 < j < n-1$ or is a complete affine space of dimension $n-1$. The quotient space $O_x := \Omega_1/\sim$ with equivalence relationship given by complete affine subspace is a properly convex open domain of dimension $< n$. Recall that there is a pseudo-metric d_{Ω_1} on Ω_1. Suppose that $\dim O_x = 0$. Then we have we have a sequence $y_i \in \Omega_1$ so that $d_{O_x}(y_i, Dg_x(y_i)) = 0$. Suppose now that $\dim O_x \geq 1$. Note that the projection $\pi : \Omega_1 \to O_x$ is projective and $d_{\Omega_1}(y,z) = d_{O_x}(\pi(y), \pi(z))$ for all $x, y \in \Omega_1$, which is fairly easy to show. The differential

Dg_x induces a projective map $D'g_x : O_x \to O_x$. Since $\dim O_x < \dim \Omega$, we have by the induction a sequence $y_i \in O_x$ so that $d_{O_x}(y_i, D'g_x(y_i)) \to 0$ as $i \to \infty$. We take an inverse image z_i in Ω_1 of y_i. Then $d_{\Omega_1}(z_i, Dg_x(z_i)) = d_{O_x}(y_i, D'g_x(y_i)) \to 0$ as $i \to \infty$. Similarly to above, we obtain the desired result. □

We believe that these were already well known by Benoist and Cooper-Long-Tillmann [67].

PROPOSITION 1.3.11. *Let Ω be a properly convex domain in $\mathbb{S}^n$. Suppose that $\Gamma \subset \mathsf{SL}_\pm(n+1,\mathbb{R})$ is a discrete group acting on Ω so that Ω/Γ is compact and Hausdorff. Let g be a non-torsion non-identity element. Then the following hold*:

- *g has two distinct positive eigenvalues associated with Q.*
- *The largest and the smallest norms of g are realized by positive eigenvalues bigger than* 1 *and less than* 1*, and g is positive semiproximal.*
- *In particular g is not orthopotent, and, hence, g cannot be unipotent.*

PROOF. Notice it is sufficient to prove for the case of Q since we can let $Q = \mathbb{S}^n$. Suppose that g acts with a single norm of eigenvalues on a subspace Q with $Q \cap \Omega \neq \emptyset$. Applying Lemma 1.3.10 where n is replaced by the dimension of Q, we obtain 0 as the infimum of the Hilbert lengths of closed curves in a compact orbifold Ω/Γ. Since Ω/Γ is a compact orbifold, there should be a positive lower bound. This is a contradiciton.

Lemma 1.3.9 and the fact that the product of the norms of eigenvalues are 1 proved this by taking g and g^{-1}. □

We generalize Proposition 5.1 of Benoist [22]. By Theorem 1.1.19 following from Selberg's Lemma [142], there is a finite index subgroup $\Gamma' \subset \Gamma$ elements of Γ are not elliptic. (In fact a finite manifold cover is enough.)

THEOREM 1.3.12 (Benoist [23]). *Suppose that Ω is properly convex but not necessarily strictly convex in $\mathbb{S}^n$. Let Γ be a discrete group acting on Ω so that Ω/Γ is compact and Hausdorff. Let Γ' be the finite index subgroup of Γ without torsion. Then each nonidentity element g, $g \in \Gamma'$ is positive bi-semi-proximal with following properties*:

- $\lambda_1(g) > 1, \lambda_n(g) < 1,$
- $A_g, \hat{A}_g \subset \mathrm{bd}\Omega$, $R_g, \hat{R}_g \subset \mathrm{bd}\Omega$ *are properly convex subsets in the boundary.*
- $\dim A_g = \dim\ker(g - \lambda_1 \mathrm{I}) - 1$ *and* $\dim R_g = \dim\ker(g - \lambda_n \mathrm{I}) - 1$.
- *Let K be a compact set in Ω. Then $\{g^i(K)|n \geq 0\}$ has the limit set in A_g, and $\{g^i(K)|n < 0\}$ has the limit set in R_g.*

Furthermore, if Ω is strictly convex, then $A_g = \hat{A}_g$ is a point in $\mathrm{bd}\Omega$ *and $R_g = \hat{R}_g$ is a point in* $\mathrm{bd}\Omega$ *and g is positive bi-proximal.*

PROOF. By Proposition 1.3.11, every nonidentity element g of Γ' has a norm of eigenvalue > 1. By Lemma 1.3.9, g is positive bi-semi-proximal.

By Proposition 1.3.2, A_g is a limit point of $\{g^i(x)|i > 0\}$. Hence, A_g is not empty and $A_g \subset \mathrm{bd}\Omega$. Similarly R_g is not empty as well.

Then A_g equals the intersection $\langle V_1 \rangle \cap \mathrm{Cl}(\Omega)$ for the eigenspace V_1 of g corresponding to $\lambda_1(g)$. Since g fixes each point of $\langle V_1 \rangle$, it follows that A_g is a compact convex subset of $\mathrm{bd}\Omega$. Similarly, R_g is a compact convex subset of $\mathrm{bd}\Omega$.

Suppose that $\hat{A}_g \cap \Omega \neq \emptyset$. Then g acts on the open properly convex domain $\hat{A}_g \cap \Omega$ as a unit-norm-eigenvalued element. By Lemma 1.3.10 applied to $\hat{A}_g \cap \Omega$, we obtain a contradiction again by obtaining a sequence of closed curves of d_Ω-lengths in Ω/Γ converging to 0 which is impossible for a closed orbifold. Thus, $\hat{A}_g \subset \mathrm{bd}\Omega$. As above, it is a compact convex subset. Similarly, $\hat{R}_g$ is a compact convex subset of $\mathrm{bd}\Omega$.

The second item follows from the second item of Proposition 1.3.2 applied to an open subset of Ω.

Suppose that Ω is strictly convex. Then

$$\dim A_g = 0, \dim \hat{A}_g = 0, \dim R_g = 0, \dim \hat{R}_g = 0$$

by the strict convexity. Proposition 5.1 of [22] proves that g is proximal. g^{-1} is also proximal by the same proposition. These are positive proximal since g acts on a proper cone. Hence, g is positive bi-proximal. □

Note here that $\hat{A}_g$ may contain A_g properly and $\hat{R}_g$ may contain R_g properly also.

1.3.3. The higher-convergence-group. For this section, we only work with $\mathbb{S}^n$ since only this version is needed. We considering $\mathsf{SL}_\pm(n+1,\mathbb{R})$ as an open subspace of $M_{n+1}(\mathbb{R})$. We can compactify $\mathsf{SL}_\pm(n+1,\mathbb{R})$ as $\mathbb{S}(M_{n+1}(\mathbb{R}))$. Denote by $(\!(g)\!)$ the equivalence class of $g \in \mathsf{SL}_\pm(n+1,\mathbb{R})$.

THEOREM 1.3.13 (The higher-convergence-group property). *Let g_i be any unbounded sequence of projective automorphisms of a properly convex domain Ω in $\mathbb{S}^n$. We consider $g_i \in \mathsf{SL}_\pm(n+1,\mathbb{R})$ according to convention 1.1.5. Then we can choose a subsequence of $\{(\!(g_i)\!)\}$, $g_i \in \mathsf{SL}_\pm(n+1,\mathbb{R})$, converging to $(\!(g_\infty)\!)$ in $\mathbb{S}(M_{n+1}(\mathbb{R}))$ for $g_\infty \in M_{n+1}(\mathbb{R})$ where the following hold :*

- *g_∞ is undefined on $\mathbb{S}(\ker g_\infty)$ and the range is $\mathbb{S}(\mathrm{Im}\, g_\infty)$.*
- $\dim \mathbb{S}(\ker g_\infty) + \dim \mathbb{S}(\mathrm{Im}\, g_\infty) = n-1$.
- *For every compact subset K of $\mathbb{S}^n - \mathbb{S}(\ker g_\infty)$, $\{g_i(K)\} \to K_\infty$ for a subset K_∞ of $\mathbb{S}(\mathrm{Im}\, g_\infty)$.*
- *Given a convergent subsequence $\{g_i\}$ as above, $\{(\!(gg_i)\!)\}$ is also convergent to $(\!(gg_\infty)\!)$ and $\mathbb{S}(\ker gg_\infty) = \mathbb{S}(\ker g_\infty)$ and $\mathbb{S}(\mathrm{Im}\, gg_\infty) = g\mathbb{S}(\mathrm{Im}\, g_\infty)$*
- *$\{(\!(g_i g)\!)\}$ is also convergent to $(\!(g_\infty g)\!)$ and*

$$\mathbb{S}(\ker g_\infty g) = g^{-1}(\mathbb{S}(\ker g_\infty)) \text{ and } \mathbb{S}(\mathrm{Im}\, g_\infty g) = \mathbb{S}(\mathrm{Im}\, g_\infty).$$

PROOF. Since $\mathbb{S}(M_{n+1}(\mathbb{R}))$ is compact, we can find a subsequence of g_i converging to an element $(\!(g_\infty)\!)$. The second item is the consequence of the rank and nullity of g_∞. The third item follows by considering the compact open topology of maps and g_i divided by its maximal norm of the matrix entries.

The two final item are straightforward. □

LEMMA 1.3.14. *$(\!(g_\infty)\!)$ can be obtained by taking the limit of $g_i/m(g_i)$ in $M_{n+1}(\mathbb{R})$ first and then taking the direction where $m(g_i)$ is the maximal norm of elements of g_i in the matrix form of g_i.*

PROOF. This follows since $g_i/m(g_i)$ does not go to zero. □

This definition was suggested by Goldman.

DEFINITION 1.3.15. An unbounded sequence $\{g_i\}$, $g_i \in \mathsf{SL}_\pm(n+1,\mathbb{R})$, so that $\{(\!(g_i)\!)\}$ is convergent in $\mathbb{S}(M_{n+1}(\mathbb{R}))$ is called a *convseq*. In the above $g_\infty \in M_{n+1}(\mathbb{R})$ is called a *convergence limit*, determined only up to a positive constant. The element $(\!(g_\infty)\!) \in \mathbb{S}(M_{n+1}(\mathbb{R}))$ where $\{(\!(g_i)\!)\} \to (\!(g_\infty)\!)$ is called a *convergence limit*.

We may also do this for $\mathsf{PGL}(n+1,\mathbb{R})$. An unbounded sequence $\{g_i\}$, $g_i \in \mathsf{PGL}(n+1,\mathbb{R})$ so that $\{[g_i]\}$ is convergent in $\mathbb{P}(M_{n+1}(\mathbb{R}))$ is called a *convseq*. Also, the element $[g_\infty] \in \mathbb{P}(M_{n+1}(\mathbb{R}))$ is $\{[g_i]\} \to [g_\infty]$ is called called a *convergence limit*.

We have more interpretations: We use the KAK-decomposition (or polar decomposition) of Cartan for $\mathsf{SL}_\pm(n+1,\mathbb{R})$. We may write $g_i = k_i d_i \hat{k}_i^{-1}$ where $k_i, \hat{k}_i \in \mathsf{O}(n+1,\mathbb{R})$ and d_i is a positive diagonal matrix with a nonincreasing set of elements

$$a_{1,i} \geq a_{2,i} \geq \cdots \geq a_{n+1,i}.$$

Let $\mathbb{S}([1,m])$ denote the subspace spanned by $\vec{e}_1, \dots, \vec{e}_m$, and let $\mathbb{S}([m+1,n+1])$ denote the subspace spanned by $\vec{e}_{m+1}, \dots, \vec{e}_{n+1}$. We assume that $\{k_i\}$ converges to k_∞, $\{\hat{k}_i\}$ converges to $\hat{k}_\infty$, and $\{[a_{1,i}, a_{2,i}, \cdots, a_{n+1,i}]\}$ is convergent in $\mathbb{RP}^n$. We will further require this for convergence sequences.

For sequence in $\mathsf{PGL}(n+1,\mathbb{R})$, we may also write $g_i = k_i d_i \hat{k}_i^{-1}$ where d_i is represented by positive diagonal matrices as above. Then we require as above.

This of course generalized the convergence sequence ideas, without the second set of requirements above, for $\mathsf{PSL}(2,\mathbb{R})$ as given by Tukia (see [2]).

Given a convergence sequence $\{g_i\}, g_i \in \mathbf{Aut}(\mathbb{S}^n)$, we define

$$\hat{A}_*(\{g_i\}) := \mathbb{S}(\operatorname{Im} g_\infty) \tag{1.3.5}$$

$$\hat{N}_*(\{g_i\}) := \mathbb{S}(\ker g_\infty) \tag{1.3.6}$$

$$A_*(\{g_i\}) := \mathbb{S}(\operatorname{Im} g_\infty) \cap \mathrm{Cl}(\Omega) \tag{1.3.7}$$

$$N_*(\{g_i\}) := \mathbb{S}(\ker g_\infty) \cap \mathrm{Cl}(\Omega) \tag{1.3.8}$$

For a matrix A, we denote by $|A|$ the maximum of the norms of entries of A. Let U be an orthogonal matrix in $\mathsf{O}(n+1,\mathbb{R})$. Then we obtain

$$\frac{1}{n+1}|A| \leq |AU| \leq (n+1)|A| \tag{1.3.9}$$

where the second inequality follows since the entries of AU are dot products of rows of A with elements of U whose entries are bounded above by 1 and below by -1 and we can multiply U^{-1} to AU to obtain the first inequality. Hence, we obtain for $g = kD\hat{k}^{-1}$ for $k, \hat{k} \in \mathsf{O}(n+1,\mathbb{R})$ and D diagonal as above.

$$\frac{1}{(n+1)^2}|D| \leq |g| \leq (n+1)^2|D| \tag{1.3.10}$$

Recall Definition 1.3.1, we obtain

THEOREM 1.3.16. *Let $\{g_i\}, g_i \in \mathbf{Aut}(\mathbb{S}^n)$, be a convergence sequence. We consider $g_i \in \mathsf{SL}_\pm(n+1,\mathbb{R})$ according to convention 1.1.5. Then we may assume that the following holds up to a choice of subsequence of g_i:*

- *there exists m_a, $1 \leq m_a < n+1$, where $\{a_{j,i}/a_{1,i}\} \to 0$ for $j > m_a$ and $a_{j,i}/a_{1,i} > \varepsilon$ for $j \leq m_a$ for a uniform $\varepsilon > 0$.*
- *there exists m_r, $1 \leq m_a < m_r \leq n+1$, where $a_{j,i}/a_{n+1,i} < C$ for $j \geq m_r$ for a uniform $C > 1$, and $\{a_{j,i}/a_{n+1,i}\} \to \infty$ for $j < m_r$.*
- *$\hat{N}_*(\{g_i\})$ is the geometric limit of $\hat{k}_i(\mathbb{S}([m_a+1,n+1]))$.*
- *$\hat{A}_*(\{g_i\})$ is the geometric limit of $A^p(g_i) = k_i(\mathbb{S}([1,m_a]))$.*
- *$\{g_i^{-1}\}$ is also a convergent sequence up to a choice of subsequences, and $\hat{A}_*(\{g_i^{-1}\}) \subset \hat{N}_*(\{g_i\})$.*

PROOF. We choose a subsequence so that m_a and m_r are defined respectively and $\{k_i\}, \{\hat{k}_i\}$ form convergent sequences. We denote D_∞ as the limit of $\{D_i/|D_i|\}$ and k_∞ and $\hat{k}_\infty$ as the limit of $\{k_i\}$ and $\{\hat{k}_i\}$. Then we obtain by (1.3.10),

$$\frac{1}{(n+1)^3}|k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}(\vec{v})| \leq |g_\infty(\vec{v})| \leq (n+1)^3 |k_i \circ D_\infty \circ \hat{k}_i^{-1}(\vec{v})|$$

for every $\vec{v} \in \mathbb{R}^{n+1}$. Thus, $k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}(\vec{v}) = 0$ if and only if $g_\infty(\vec{v}) = 0$ and, moreover, images and null spaces of $k_\infty \circ D_\infty \circ \hat{k}_\infty^{-1}$ and g_∞ coincide. Hence, we obtain that

$$\mathbb{S}(\operatorname{Im} g_\infty) = k_\infty \mathbb{S}(\operatorname{Im} D_\infty) = k_\infty \mathbb{S}([1, m_a]) \text{ and}$$
$$\mathbb{S}(\ker g_\infty) = \hat{k}_\infty(\mathbb{S}(\ker D_\infty) = \hat{k}_\infty((\mathbb{S}([m_a+1, n+1]))).$$

Hence, the first four items follow.

The last item follows by considering the third and fourth items and the fact that $m_r \geq m_a$. □

When m_a, m_r exists for $\{g_i\}$ and $\{k_i\}$ and $\{\hat{k}_i\}$ are convergent for a convergence sequence, we say that g_i are *set-convergent*.

We define for each i,

$$F^p(g_i) := k_i \mathbb{S}([1, m_r - 1]), \text{ and } R^p(g_i) := \hat{k}_i \mathbb{S}([m_r, n+1]).$$

We define $\hat{R}_*(\{g_i\})$ as the geometric limit of $\{R^p(g_i)\}$, and $\hat{F}_*(\{g_i\})$ as the geometric limit of $\{F^p(g_i)\}$. We also define

$$R_*(\{g_i\}) := \hat{R}_*(\{g_i\}) \cap \operatorname{Cl}(\Omega), F_*(\{g_i\}) := \hat{F}_*(\{g_i\}) \cap \operatorname{Cl}(\Omega).$$

LEMMA 1.3.17. *Suppose that $\{g_i\}$ and $\{g_i^{-1}\}$ are set-convergent sequences. Then $\hat{R}_*(\{g_i\}) = \hat{A}_*(\{g_i^{-1}\})$ and $\hat{F}_*(\{g_i\}) = \hat{N}_*(\{g_i^{-1}\})$.*

PROOF. For $g_i = k_i d_i \hat{k}_i^{-1}$, we have $g_i^{-1} = \hat{k}_i^{-1} d_i^{-1} k_i$. Hence $A^p(g_i^{-1}) = \hat{k}_i \mathbb{S}([m_r, n+1]) = R^p(g_i)$. We also have $F^p(g_i^{-1}) = \hat{k}_i \mathbb{S}([m_a+1, n+1])$. The result follows. □

LEMMA 1.3.18. *We also have $\hat{A}_*(\{g^i\}) \subset \hat{A}_g$ and $\hat{R}_*(\{g^i\}) \subset \hat{R}_g$ for positive bi-semi-proximal element g with $\lambda_1(g) > 1$.*

PROOF. We consider g in $\mathbb{C}^{n+1}$. Write g in the coordinate system where the complexification is of the Jordan form. Let V_g^A denote $\mathscr{R}_{\lambda_1(g)}$ in $\mathbb{R}^{n+1}$ which is a g-invariant subspace from (1.3.1). There is a complementary g which is a direct sum of $\mathscr{R}_\mu(g)$ for $\mu < \lambda_1(g)$. Then we use Proposition 1.3.2 applied to $\Pi'(V_g^A)$ and $\Pi'(N_g^A)$.

For the second part, we use g^{-1} and argue using obvious facts $\hat{R}_g = \hat{A}_{g^{-1}}$ and $\hat{A}_*(\{g^{-i}\}) = \hat{R}_*(\{g^i\})$. □

PROPOSITION 1.3.19. *$A_*(\{g_i\})$ contains an open subset of $\hat{A}_*(\{g_i\})$ and hence $\langle A_*(\{g_i\})\rangle = \hat{A}_*(\{g_i\})$. Also, $R_*(\{g_i\})$ contains an open subset of $\hat{R}_*(\{g_i\})$ and hence $\langle R_*(\{g_i\})\rangle = \hat{R}_*(\{g_i\})$.*

PROOF. We write $g_i = k_i D_i \hat{k}_i^{-1}$. By Theorem 1.3.16, $\hat{A}_*(\{g_i\})$ is the geometric limit of $k_i(\mathbb{S}[1, m_a])$ for some m_a as above. $g_i(U) = k_i D_i(V_i)$ for an open set $U \subset \tilde{\mathscr{O}}$ and $V_i = \hat{k}_i^{-1}(U)$. Since $\hat{k}_i^{-1}$ is a $\mathbf{d}$-isometry, V_i is an open set containing a closed ball B_i of fixed radius ε. $\{D_i\}$ converges to a diagonal matrix D_∞. We may assume without loss of generality that $\{B_i\} \to B_\infty$ where B_∞ is a ball of radius ε. We may assume $B_i \cap B_\infty \supset B$

for a fixed ball of radius $\varepsilon/2$ for sufficiently large i. Then $\{D_i(B)\} \to D_\infty(B) \subset \mathbb{S}([1,m_a])$. Here, $D_\infty(B)$ is a subset of $\mathbb{S}([1,m_a])$ containing an open set. Since $\{D_i(B_i)\}$ geometrically converges to a subset containing $D_\infty(B)$, up to a choice of subsequence. Thus, $\{k_i D_i(V_i)\}$ geometrically converges to a subset containing $k_\infty D_\infty(B)$ by Lemma 1.1.8.

For the second part, we use the sequence $g_i^{-1} = \hat{k}_i D_i^{-1} k_i^{-1}$ and Lemma 1.3.17. $\square$

LEMMA 1.3.20. *Suppose that Γ acts properly discontinuously on a properly convex open domain Ω and $\{g_i\}$ is a set-convergent sequence in Γ. Suppose that $\{g_i\}$ is not bounded in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ and is a set-convergent sequence. Then the following hold*:

(i) $\hat{R}_*(\{g_i\}) \cap \Omega = \emptyset$,
(ii) $\hat{A}_*(\{g_i\}) \cap \Omega = \emptyset$,
(iii) $\hat{F}_*(\{g_i\}) \cap \Omega = \emptyset$, *and*
(iv) $\hat{N}_*(\{g_i\}) \cap \Omega = \emptyset$.

PROOF. (ii) Suppose not. Since $\hat{A}_*(\{g_i\}) \cap \Omega \neq \emptyset$, $A_*(\{g_i\})$ meets Ω. Since $A_*(\{g_i\})$ is the set of points of limits $g_i(x)$ for $x \in \Omega$, the proper discontinuity of the action of Γ shows that $A_*(\{g_i\})$ does not meet Ω.

(iv) For each x in Ω, a fixed ball B in Ω centered at x does not meet $\hat{k}_i(\mathbb{S}([m_a+1,n+1]))$ for infinitely many i. Otherwise $\{g_i^m(B)\}$ converges to a nonproperly convex set in $\mathrm{Cl}(\Omega)$ as $m \to \infty$, a contradiction. Hence, the second item follows.

The remaining items follow by changing g_i to g_i^{-1} and Lemma 1.3.17. $\square$

THEOREM 1.3.21. *Let $\{g_i\}$ be a set-convergence sequence in Γ acting properly discontinuously on a properly convex domain Ω. Then*

$$A_*(\{g_i\}) = \hat{A}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega) = \hat{A}_*(\{g_i\}) \cap \mathrm{bd}\Omega, \tag{1.3.11}$$

$$N_*(\{g_i\}) = \hat{N}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega) = \hat{N}_*(\{g_i\}) \cap \mathrm{bd}\Omega, \tag{1.3.12}$$

$$R_*(\{g_i\}) = \hat{R}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega) = \hat{R}_*(\{g_i\}) \cap \mathrm{bd}\Omega, \tag{1.3.13}$$

$$F_*(\{g_i\}) = \hat{F}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega) = \hat{F}_*(\{g_i\}) \cap \mathrm{bd}\Omega \tag{1.3.14}$$

are subsets of $\mathrm{bd}\Omega$ *and they are nonempty sets. Also, we have*

$$\hat{A}_*(\{g_i\}) \subset \hat{F}_*(\{g_i\}), \qquad A_*(\{g_i\}) \subset F_*(\{g_i\}), \tag{1.3.15}$$

$$\hat{R}_*(\{g_i\}) \subset \hat{N}_*(\{g_i\}), \qquad R_*(\{g_i\}) \subset N_*(\{g_i\}). \tag{1.3.16}$$

PROOF. By Lemma 1.3.20, we only need to show the respective sets are not empty. By the third item of Theorem 1.3.13, a point x in $\hat{A}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega)$ is a limit of $g_i(y)$ for some $y \in \Omega$. Since Γ acts properly discontinuously, $x \notin \Omega$ and $x \in \mathrm{bd}\Omega$. By taking $\{g_i^{-1}\}$, we obtain $\hat{R}_*(\{g_i\}) \cap \mathrm{Cl}(\Omega) \neq \emptyset$. Since $\hat{R}_*(\{g_i\}) \subset \hat{N}_*(\{g_i\})$ and $\hat{A}_*(\{g_i\}) \subset \hat{F}_*(\{g_i\})$, the rest follows. The last collections are from definitions. $\square$

PROPOSITION 1.3.22. *For an automorphism g of Ω, and a set-convergence sequence $\{g_i\}$, the following hold*:

$$\hat{A}_*(\{gg_i\}) = g(\hat{A}_*(\{g_i\})), \hat{A}_*(\{g_i g\}) = \hat{A}_*(\{g_i\}), \tag{1.3.17}$$

$$\hat{N}_*(\{gg_i\}) = \hat{N}_*(\{g_i\}), \hat{N}_*(\{g_i g\}) = g^{-1}(\hat{N}_*(\{g_i\})), \tag{1.3.18}$$

$$\hat{F}_*(\{gg_i\}) = g(\hat{F}_*(\{g_i\})), \hat{F}_*(\{g_i g\}) = \hat{F}_*(\{g_i\}), \tag{1.3.19}$$

$$\hat{R}_*(\{gg_i\}) = \hat{R}_*(\{g_i\}), \hat{R}_*(\{g_i g\}) = g^{-1}(\hat{R}_*(\{g_i\})). \tag{1.3.20}$$

PROOF. The fourth and fifth items of Theorem 1.3.13 imply the first and second lines here. The third line follows from the second line by Lemma 1.3.17. Also, the fourth line follows from the first line by Lemma 1.3.17. □

Of course, there are $\mathbb{RP}^n$-versions of the results here. However, we do not state these.

1.4. Convexity of real projective orbifolds

1.4.1. Convexity. An $\mathbb{RP}^n$-orbifold is *convex* if it is projectively diffeomorhic to a projective quotient of a convex domain in an open hemisphere in $\mathbb{S}^n$. (Note that this definition is more stricter than ones in [46] but conforms to definitions in many literatures.)

In the following, a zero-dimensional sphere $\mathbb{S}^0_\infty$ denotes a pair of antipodal points.

PROPOSITION 1.4.1.

- *A real projective n-orbifold is convex if and only if the developing map sends the universal cover to a convex domain in* $\mathbb{RP}^n$ (*resp.* $\mathbb{S}^n$).
- *A real projective n-orbifold is properly convex if and only if the developing map sends the universal cover to a precompact properly convex open domain in an affine patch of* $\mathbb{RP}^n$ (*resp.* $\mathbb{S}^n$).
- *If a convex real projective n-orbifold is not properly convex and not complete affine, then its holonomy is reducible in* $\mathsf{PGL}(n+1,\mathbb{R})$ (*resp.* $\mathsf{SL}_\pm(n+1,\mathbb{R})$). *In this case, $\tilde{\mathcal{O}}$ is foliated by affine subspaces l of dimension i with the common boundary* $\mathrm{Cl}(l)-l$ *equal to a fixed subspace* $\mathbb{RP}^{i-1}_\infty$ (*resp.* $\mathbb{S}^{i-1}_\infty$) *in* $\mathrm{bd}\tilde{\mathcal{O}}$. *Furthermore, this holds for any convex domain* $\mathbb{RP}^n$ (*resp.* $\mathbb{S}^n$) *and the projective group action on it.*

PROOF. We prove for $\mathbb{S}^n$ first. Since the universal cover is projectively diffeomrophc to a convex open dpmain, the developing map must be an embedding. The converse is also trivial. (See Proposition A.2 of [46].)

The second follows immediately.

For the final item, a convex subset of $\mathbb{S}^n$ is a convex subset of an affine subspace $\mathbb{A}^n$, isomorphic to an affine space, which is the interior of a hemisphere H. We may assume that $D^o \neq \emptyset$ by restricting to a spanning subspace of D in $\mathbb{S}^n$. Let D be a convex subset of H^o. If D is not properly convex, the closure $\mathrm{Cl}(D')$ must be of the form $\mathbb{S}^{i_0} * K$ for a properly convex domain by Proposition 1.1.4.

Since $\mathbb{S}^{i_0}$ must be holonomy invariant, the holonomy group is reducible.

For the $\mathbb{RP}^n$-version, we use the double covering map $p_{\mathbb{S}^n}$ mapping an open hemisphere to an affine subspace. [$\mathbb{S}^n$T] □

PROPOSITION 1.4.2. *Let Ω be a properly convex domain in $\mathbb{S}^n$. The image Ω' be the image of Ω under the double covering map $p_{\mathbb{S}^n}$. Then the restriction* $\mathrm{Cl}(\Omega) \to \mathrm{Cl}(\Omega')$ *is one-to-one and onto.*

PROOF. This follows since we can find an affine subspace $\mathbb{A}^n$ containing $\mathrm{Cl}(\Omega)$. Since the covering map restricts to a homeomorphism on $\mathbb{A}^n$, this follows. □

1.4.2. Needed convexity facts. We will use the following many times in the monograph.

LEMMA 1.4.3 (Chapter 11 of [148]). *Let K be a closed subset of a convex domain Ω in* $\mathbb{RP}^n$ (*resp.* $\mathbb{S}^n$) *so that each point of* $\mathrm{bd}K$ *has a convex neighborhood. Then K is a convex domain.*

PROOF. Assume $\Omega \subset \mathbb{S}^n$. We can connect each pair of points by a broken projective geodesics. Then local convexity shows that we can make the number of geodesic segments to go down by one using triangles. Finally, we may obtain a geodesic segment connecting the pair of points. $[\mathbb{S}^n\mathrm{S}]$ □

LEMMA 1.4.4. *Let Ω be a properly convex domain in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Let σ be a convex domain in $\mathrm{Cl}(\Omega) \cap P$ for a subspace P. Then either $\sigma \subset \mathrm{bd}\Omega$ or σ^o is in Ω.*

PROOF. Assume $\Omega \subset \mathbb{S}^n$. Suppose that σ^o meets $\mathrm{bd}\Omega$ and is not contained in it entirely. Since Ω is convex and $\mathrm{bd}\Omega$ is closed, $s \cap \Omega$ for a segment s in σ can have only one component which must be open. Since the complement of $\sigma^o \cap \mathrm{bd}\Omega$ is a relatively open set in σ^o, we can find a segment $s \subset \sigma^o$ with a point z so that a component s_1 of $s - \{z\}$ is in $\mathrm{bd}\Omega$ and the other component s_2 is disjoint from it.

We may perturb s in a 2-dimensional totally geodesic space containing s and so that the new segment $s' \subset \mathrm{Cl}(\Omega)$ meets $\mathrm{bd}\Omega$ only in its interior point. This contradicts the fact that Ω is convex by Theorem A.2 of [46]. $[\mathbb{S}^n\mathrm{P}]$ □

Where to put this? Right place?

We define $\mathbb{S}^{n*} := \mathbb{S}(\mathbb{R}^{n+1*})$, i.e, the sphericalization of the dual space $\mathbb{R}^{n+1*}$.

THEOREM 1.4.5. *Suppose that G acts on a convex domain Ω in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$), so that Ω/G is a compact orbifold. Then if G have only unit-norm eigenvalued elements, then Ω is complete affine.*

PROOF. Theorem 1.3.7 shows that G is orthopotent and has a unipotent group U of finite index. A unipotent group has a global fixed point in $\mathbb{S}^{n*}$, and so does U^*. Thus, there exists a hyperspace P in $\mathbb{S}^n$ where U acts on. $P \cap \mathrm{Cl}(\Omega) \neq \emptyset$ and $P \cap \Omega = \emptyset$ by Proposition 1.4.13. Thus, Ω is in an affine subspace $\mathbb{A}$ bounded by P. Also, G acts on $\mathbb{A}$ as a group of affine transformations since every projective action on a complete affine space is affine. (See Berger [26].) Orthopotent groups are distal. Lemma 2 of Fried [80] implies the conclusion. $[\mathbb{S}^n\mathrm{T}]$ □

1.4.3. The flexibility of boundary.

The following lemma gives us some flexibility of boundary. A smooth hypersurface embedded in a real projective manifold is called *strictly convex* if under a chart to an affine subspace, it maps to a hypersurface which is defined by a real function with positive Hessians at points of the hypersurface.

LEMMA 1.4.6. *Let M be a strongly tame properly convex real projective orbifold with strictly convex ∂M. We can modify ∂M inward M and the result bound a strongly tame or compact properly convex real projective orbifold M' with strictly convex $\partial M'$*

PROOF. Let Ω be a properly convex domain covering M. We may assume that $\Omega \subset \mathbb{S}^n$. We may modify M by pushing ∂M inward. We take an arbitrary inward vector field defined on a tubular neighborhood of ∂M. (See Section 4.4 of [51] for the definition of the tubular neighborhoods.) We use the flow defined by them to modify ∂M. By the C^2-convexity condition, for sufficiently small change the image of ∂M is still strictly convex and smooth. Let the resulting compact n-orbifold be denoted by M'. M' is covered by a subdomain Ω' in Ω.

Since M' is a compact suborbifold of M, Ω' is a properly embedded domain in Ω and thus, $\mathrm{bd}\Omega' \cap \Omega = \partial\Omega'$. $\partial\Omega'$ is a strictly convex hypersurface since so is $\partial M'$. This means that Ω' is locally convex. A locally convex closed subset of a convex domain is convex by Lemma 1.4.3.

Hence, Ω' is convex and hence is properly convex being a subset of a properly convex domain. So is M'. $[\mathbb{S}^n\mathrm{S}]$ $\square$

REMARK 1.4.7. Thus, by choosing one in the interior, we may assume without loss of generality that a strictly convex boundary component can be pushed out to a strictly convex boundary component.

1.4.4. The Benoist theory . In late 1990s, Benoist more or less completed the theory of the divisible action as started by Benzécri, Vinberg, Koszul, Vey, and so on in the series of papers [22], [21], [23], [24], [18], [17]. The purpose is to generalize these to sweeping actions with the main result Lemma 1.4.16. The comprehensive theory will aid us much in this paper.

PROPOSITION 1.4.8 (Corollary 2.13 [23]). *Suppose that a discrete subgroup Γ of $\mathsf{SL}_\pm(n,\mathbb{R})$ (resp. $\mathsf{PGL}(n,\mathbb{R})$), $n \geq 2$, acts on a properly convex $(n-1)$-dimensional open domain Ω in $\mathbb{S}^{n-1}$ (resp, $\mathbb{RP}^{n-1}$) so that Ω/Γ is a compact orbifold. Then the following statements are equivalent.*

- *Every subgroup of finite index of Γ has a finite center.*
- *Every subgroup of finite index of Γ has a trivial center.*
- *Every subgroup of finite index of Γ is irreducible in $\mathsf{SL}_\pm(n,\mathbb{R})$ (resp. in $\mathsf{PGL}(n,\mathbb{R})$). That is, Γ is strongly irreducible.*
- *The Zariski closure of Γ is semisimple. (simple?)*
- *Γ does not contain an infinite nilpotent normal subgroup.*
- *Γ does not contain an infinite abelian normal subgroup.*

PROOF. Corollary 2.13 of [23] considers $\mathsf{PGL}(n,\mathbb{R})$ and $\mathbb{RP}^{n-1}$. However, the version for $\mathbb{S}^{n-1}$ follows from this since we can always lift a properly convex domain in $\mathbb{RP}^{n-1}$ to one Ω in $\mathbb{S}^{n-1}$ and the group to one in $\mathsf{SL}_\pm(n,\mathbb{R})$ acting on Ω by Theorem 1.1.20. $\square$

The center of a group G is denoted by $\mathbb{Z}(G)$. A *virtual center* of a group G is a subgroup of G each of whose elements is centralized by a finite index subgroup of G. A group with properties above is said to be a group with a *trivial virtual center*.

A group G acts on a space X *cocompactly* if there is a compact subset Y of X so that $X = \bigcup_{g\in G} gY$.

THEOREM 1.4.9 (Theorem 1.1 of [23]). *Suppose that a virtual-center-free discrete subgroup Γ of $\mathsf{SL}_\pm(n,\mathbb{R})$ (resp. $\mathsf{PGL}(n,\mathbb{R})$), $n \geq 2$, acts on a properly convex $(n-1)$-dimensional open domain $\Omega \subset \mathbb{S}^{n-1}$ so that Ω/Γ is a compact orbifold. Then every representation of the component of* $\mathrm{Hom}(\Gamma, \mathsf{SL}_\pm(n,\mathbb{R}))$ *(resp.* $\mathrm{Hom}(\Gamma, \mathsf{PGL}(n,\mathbb{R}))$*) containing the inclusion representation also acts on a properly convex $(n-1)$-dimensional open domain cocompactly.*

When Ω/Γ admits a hyperbolic structure and $n = 3$, Inkang Kim [111] proved this simultaneously for a union of components.

PROPOSITION 1.4.10 (Theorem 1.1. of Benoist [21]). *Suppose that a discrete subgroup Γ of $\mathsf{SL}_\pm(n,\mathbb{R})$ (resp. $\mathsf{PGL}(n,\mathbb{R})$), $n \geq 2$, acts on a properly convex $(n-1)$-dimensional open domain Ω in $\mathbb{S}^{n-1}$ (resp, $\mathbb{RP}^{n-1}$) so that Ω/Γ is a compact orbifold. Then*

- *Ω is projectively diffeomorphic to the interior of a strict join $K := K_1 * \cdots * K_{l_0}$ where K_i is a properly convex open domain of dimension $n_i \geq 0$ in the subspace $\mathbb{S}^{n_i}$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^{n_i}$ in $\mathbb{RP}^n$). K_i corresponds to a convex cone $C_i \subset \mathbb{R}^{n_i+1}$ for each i.*

- *Ω is the image of $C_1 + \cdots + C_{l_0}$.*
- *Let Γ'_i be the image of Γ' to K_i for the restriction map of the subgroup Γ' of $\pi_1(\Sigma)$ acting on each K_j, $j = 1, \dots, l_0$. We denote by Γ_i an arbitrary extension of Γ'_i by requiring it to act trivially on K_j for $j \neq i$ and to have 1 as the eigenvalue associated with vectors in their directions.*
- *The subgroup corresponding to $\mathbb{R}^{l_0-1}$ acts trivially on each K_j and form a positive diagonalizable matrix group.*
- *The fundamental group $\pi_1(\Sigma)$ is virtually isomorphic to a subgroup of $\mathbb{R}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$ for $(l_0 - 1) + \sum_{i=1}^{l_0} n_i = n$.*
- *$\pi_1(\Sigma)$ acts on K^o cocompactly and discretely and in a semisimple manner (Theorem 3 of Vey [151]).*
- *The Zariski closure of Γ' equals $\mathbb{R}^{l_0-1} \times G_1 \times \cdots \times G_{l_0}$. Each Γ_j acts on K_j^o cocompactly, and G_j is an simple Lie group (Remark after Theorem 1.1 of [21]), and G_j acts trivially on K_m for $m \neq j$.*
- *A virtual center of $\pi_1(\Sigma)$ of maximal rank is isomorphic to $\mathbb{Z}^{l_0-1}$ corresponding to the subgroup of $\mathbb{R}^{l_0-1}$. (Proposition 4.4 of [21].)*

We will often indicate by $\mathbb{Z}^{l_0-1}$ the virtual center of $\pi_1(\Sigma)$. See Example 5.5.3 of Morris [135] for a group acting properly on a product of two hyperbolic spaces but restricts to a non-discrete group for each factor space.

COROLLARY 1.4.11. *Assume as in Proposition 1.4.10. Then every normal solvable subgroup of a finite-index subgroup Γ' of Γ is virtually central in Γ.*

PROOF. If Γ is virtually abelian, this is obvious.

Suppose that Ω is properly convex. Let G be a normal solvable subgroup of a finite-index subgroup Γ' of Γ. We may assume without loss of generality that Γ' acts on each K_i by taking a further finite index subgroup and replacing G by a finite index subgroup of G. Now, G is a normal solvable subgroup of the Zariski closure $\mathscr{Z}(\Gamma')$. By Theorem 1.1 of [21], $\mathscr{Z}(\Gamma')$ equals $G_1 \times \cdots \times G_l \times \mathbb{R}_+^{l-1}$ and $K = K_1 * \cdots * K_l$ where G_i is reductive and the following holds:

- if K_i is homogeneous, then G_i is simple and G_i is commensurable with $\mathbf{Aut}(K_i)$.
- Otherwise, K_i^o is divisible and G_i is a union of components of $\mathsf{SL}_\pm(V_i)$

The image of G into G_i by the restriction homomorphism to K_i is a normal solvable subgroup of G_i. Since G_i is virtually simple, the image is a finite group. Hence, G must be virtually a subgroup of the diagonalizable group $\mathbb{R}_+^{l-1}$ and hence is virtually central in Γ'. $\square$

If $l_0 = 1$, Γ is strongly irreducible as shown by Benoist. However, the images of these groups will be subgroups of $\mathsf{PGL}(m, \mathbb{R})$ and $\mathsf{SL}_\pm(m, \mathbb{R})$ for $m \leq n$. If $l_0 > 1$, we say that such an image in Γ is *virtually factorizable*. Otherwise, such an image a *non-virtually-factorizable* group.

An action of a projective group G on a properly convex domain Ω is *sweeping* if the action is cocompact but Ω/G is not required to Hausdorff. A dividing action is sweeping.

Recall the *commutant* H of a group acting on a properly convex domain is the maximal diagonalizable group commuting with the group. (See Vey [151].)

We have a useful theorem:

THEOREM 1.4.12 (Proposition 3 of Vey [151]). *Suppose that a projective group Γ acts on a properly convex open domain Ω in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$), with a sweeping action. Then Ω equals a convex hull of the orbit $\Gamma(x)$ for any $x \in \Omega$.*

We generalize Proposition 1.4.10.

PROPOSITION 1.4.13. *Suppose that a projective group G acts on an n-dimensional properly convex open domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) as a sweeping action. Then the following hold*:

- *Let L be any subspace where G acts on. Then $L \cap \mathrm{Cl}(\Omega) \neq \emptyset$ but $L \cap \Omega = \emptyset$.*
- *If G acts on a compact properly convex set K, then K must meet $\mathrm{Cl}(\Omega) \cup \mathscr{A}(\mathrm{Cl}(\Omega))$.*
- *Suppose that G is semi-simple. Then all the items up to the last one in the conclusion of Proposition 1.4.10 with G replacing $\pi_1(\tilde{E})$ without discreteness hold. In particular, $\mathrm{Cl}(\Omega) = K_1 * \cdots * K_{l_0}$ for properly convex sets $K_1, \dots, K_{l_0}$.*
- *Suppose that G is semi-simple. Then the closure of G has a virtual center containing a group of diagonalizable projective automorphisms isomorphic to $\mathbb{Z}^{l_0 - 1}$ acting trivially on each K_i.*

PROOF. Assume $\Omega \subset \mathbb{S}^n$. Suppose that $L \cap \mathrm{Cl}(\Omega) = \emptyset$. Then there is a lower bound to the **d**-distance from $\mathrm{bd}\Omega$ to L. Let $x \in \Omega$. We denote the space of oriented maximal open segments containing x and ending in L by $L_{\Omega,x}$. This is a set homeomorphic to $\mathbb{S}^{\dim L}$.

Let l_+ denote the endpoint of $L \cap l$ ahead of x. Let $l_{\Omega,0}$ denote the endpoint of $l \cap \Omega$ ahead of x, and $l_{\Omega,1}$ denote the endpoint of $l \cap \Omega$ after x. We define a function $f : \Omega \to (0, \infty)$ given by

$$f(x) = \inf\{\log(l_+, l_{\Omega,0}, x, l_{\Omega,1}) | l \in L_{\Omega,x}\}$$

where the logarithm measures the Hilbert distance between $l_{\Omega,0}$ and x on the properly convex segment with endpoints l_+ and $l_{\Omega,1}$. This is a continuous positive function. As $x \to \mathrm{bd}\Omega$, $f(x) \to 0$.

Since $f(g(x)) = f(x)$ for all $x \in \Omega$ and $g \in G$, f induces a continuous map $\bar{f} : \Omega/G \to (0, \infty)$. Here, $\bar{f}$ can take as close value to 0 as one wishes. This contradicts the compactness of Ω/G.

Suppose that $L \cap \Omega \neq \emptyset$. Then G acts on the convex domain $L \cap \Omega$ open in L. We define a function $f : \Omega \to [0, \infty)$ given by measuring the Hilbert distance from $L \cap \Omega$. Then $f(x) \to \infty$ as $x \to \mathrm{bd}\Omega - L$. Again $f(g(x)) = f(x)$ for all $x \in \Omega, g \in G$. This induces $\bar{f} : \Omega/G \to [0, \infty)$. Since $\bar{f}$ can take as large value as one wishes for, this contradicts the compactness of Ω/G.

For the second item, suppose that such a set K exists. K and $\mathscr{A}(K)$ are disjoint from $\mathrm{Cl}(\Omega)$. For $x \in \Omega$, we define $K_{\Omega,x}$ to be the space of oriented open segments containing x and ending in K and $\mathscr{A}(K)$. We define l_+ to be the first point of $K \cap l$ ahead of x. Then a similar argument to the above proof applies and we obtain a contradiction.

Now, we go to the third item. Let G have a G-invariant decomposition $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_{l_0}$ where G acts irreducibly. This item follows by Lemma 2 of [151] since any decomposition of $\mathbb{R}^n$ gives rise to a diagonalizable commutant of rank l_0.

For the fourth item, we prove for the case when G has a G-invariant decomposition $\mathbb{R}^n = V_1 \oplus V_2$. Then by the second item, G acts on $K = K_1 * K_2$ for properly convex domain $K_i \subset \mathbb{S}(V_i)$ for $i = 1, 2$. G acts cocompactly on K^o and G is a subgroup of $G_1 \times G_2 \times \mathbb{R}_+$ where G_i is isomorphic to $G|K_i$ extended to act trivially on K_{i+1} with $G_i|V_{i+1} = \mathrm{I}$ where the indices in mod 2.

The closure $\bar{G}$ of G in $\mathbf{Aut}(K)$ is a subgroup of $\bar{G}_1 \times \bar{G}_2 \times \mathbb{R}_+$ for the closure $\bar{G}_i$ of G_i in $\mathbf{Aut}(K_i)$ for $i = 1, 2$.

$$\bar{G} \subset \{(g_1, g_2, r) | g_i \in \bar{G}_i, i = 1, 2, r \in \mathbb{R}_+\}.$$

For a fixed pair (g_1, g_2), if there are more than one associated r, then we obtain by taking differences that $(\mathrm{I}, \mathrm{I}, r)$ is in the group $\bar{G}$ for $r \neq 1$. This implies that $\bar{G}$ contains a nontrivial subgroup of $\mathbb{R}_+$.

Otherwise, $\bar{G}$ is in a graph of homomorphism $\lambda : \bar{G}_1 \times \bar{G}_2 \to \mathbb{R}_+$. An orbit of an action of this on the manifold $\mathbb{F}K_1^o \times \mathbb{F}K_2^o \times \mathbb{R}_+$ is in the orbit of the image of λ. Hence, each orbit of a compact set meets $(y_1, y_2) \times (0,1)$ for $y_i \in \mathbb{F}K_i^o$, $i = 1, 2$, at a compact set. Thus, we do not have a cocompact action.

Furthermore, if we have a G-invariant decomposition $K_1 * \cdots * K_m$, we can use the decomposition $K_1 * (K_2 * \cdots * K_m)$. Now, we use the induction, to obtain the result. $[\mathbb{S}^n\mathrm{T}]$ $\square$

PROPOSITION 1.4.14. *Assume as in Proposition 1.4.10. Then K is the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g$ for the attracting limit set A_g of g. Also, for any partial join $\hat{K} := K_{i_1} * \cdots * K_{i_j}$ for a subcollection $\{i_1, \ldots, i_j\}$, the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g \cap \hat{K}$ equals $\hat{K}$.*

PROOF. We take a finite-index normal subgroup Γ' of Γ so that $\mathbb{Z}_2^{l_0 - 1}$ is the center of Γ. Using Theorem 1.1.19, we may assume that Γ' is torsion-free. Note that kA_g for any $k \in \Gamma'$ equals $A_{kgk^{-1}} = A_g$ since $kgk^{-1} = g$. Thus, Γ' acts on $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g$ since it is a Γ'-invariant set. The interior C of the convex hull of $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g$ is a subdomain in K^o. Since $C/\Gamma' \to K^o/\Gamma'$ is a homotopy equivalence of closed manifolds, we obtain $C = K^o$ and $\mathrm{Cl}(C) = K$ by Lemma 1.4.16.

For the second part, if the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g \cap \hat{K}$ is a proper subset of $\hat{K}$, then the closure of the convex hull of $\bigcup_{g \in \mathbb{Z}^{l_0 - 1}} A_g$ is a proper subset of K. This is a contradiction. $[\mathbb{S}^n\mathrm{S}]$ $\square$

THEOREM 1.4.15 (Kobayashi [112]). *Suppose that a closed real projective orbifold has a developing map into a properly convex domain D in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Then the orbifold is projectively diffeomorphic to Ω/Γ for the holonomy group Γ and for the unique, minimal Γ-invariant open convex domain Ω in D^o.*

PROOF. This follows since all maximal segments in D are of $\mathbf{d}$-length $\leq \pi - \varepsilon_0$ for a uniform $\varepsilon_0 > 0$. Hence, the Kobayashi metric is well-defined proving that the orbifold is properly convex. Hence, the developing image Ω is a convex open domain by [112]. Clearly, Ω is Γ-invariant. It is minimal since for any Γ-invariant domain in U, U/Γ is a closed orbifold whose orientable manifold finite cover is homotopy equivalent to a manifold finite cover of Ω/Γ. Thus $U = \Omega$ by a degree argument. Also, Ω is unique since it must be that $\Omega = D^o$. $\square$

LEMMA 1.4.16 (Domains for holonomy). *Suppose that Ω is an open domain in an open hemisphere in $\mathbb{S}^{n-1}$ (resp. in $\mathbb{RP}^{n-1}$) where a projective group Γ acts on so that Ω/Γ is a closed orbifold. Suppose that Γ acts on a compact properly convex domain K where $K^o \neq \emptyset$. Then*

- *$K^o = R(\Omega)$ where R is a diagonalizable projective automorphism commuting with a finite-index subgroup of Γ with eigenvalues ± 1 only and is a composition of reflections commuting with one-another.*
- *In fact $K = K_1 * \cdots * K_k$ where $K_j = K \cap P_j$, $J = 1, \ldots, k$, for a virtually invariant subspace P_j of Γ where R equals I or $\mathscr{A}$.*
- *Ω has to be properly convex.*

- *If K^o meets with Ω, then $K^o = \Omega$.*

PROOF. It is sufficient to prove for $\mathbb{S}^n$. We prove by an induction on dimension. For $\mathbb{S}^1$, this is clear.

By Theorem 1.1.19, there is a torsion-free finite-index subgroup Γ' in Γ. Suppose that $\Omega \cap K^o \neq \emptyset$. Then $(\Omega \cap K^o)/\Gamma'$ is homotopy equivalent to Ω/Γ', a closed manifold. Hence, $\Omega \cap K^o = \Omega$ and $\Omega \subset K^o$. Similarly, $K^o \subset \Omega$. We obtain $K^o = \Omega$. Also, if $\Omega \cap \mathscr{A}(K^o) \neq \emptyset$, then $\mathscr{A}(K^o) = \Omega^o$. The lemma is proved for this case.

Proposition 1.1.15 and the second part of Theorem 1.1.19 show that K^o/Γ is again a closed orbifold. Suppose that Γ is not virtually factorizable with respect to K. Then Γ is strongly irreducible by Benoist [21]. K contains the attracting fixed points of bi-semi-proximal element g of Γ. This implies that $\mathrm{Cl}(\Omega) \cap \mathrm{Cl}(K) \neq \emptyset$ or $\mathrm{Cl}(\Omega) \cap \mathscr{A}(\mathrm{Cl}(K)) \neq \emptyset$ since Ω contains a generic point of $\mathbb{S}^n$. By the above paragraph, we may suppose that the intersection is in $\mathrm{bd}\Omega \cap \mathrm{bd}K$ or $\mathrm{bd}\Omega \cap \mathscr{A}(\mathrm{bd}K)$. Then this is a compact convex set invariant under Γ. Hence, Γ is reducible, a contradiction.

Now suppose that Γ is virtually factorizable. Then there exists a diagonalizable free abelian group D of rank $k-1$ for some $k \geq 2$ in the virtual center of Γ by Proposition 1.4.10. D acts trivially on a finite set of minimal subspace $P_1, \dots, P_k$ by Proposition 1.4.10. Since Γ permutes these subspaces, a torsion-free finite-index subgroup Γ' of Γ acts on $P_1, \dots, P_k$. Let's denote $\hat{P}_j := P_1 * \cdots * P_{j-1} * P_{j+2} * \cdots * P_k$. Then Ω is disjoint from $\hat{P}_j$ for each j since otherwise $(\hat{P}_j \cap \Omega)/\Gamma' \to \Omega/\Gamma'$ is a homotopy equivalence of different dimensional manifolds.

However, $P_j \cap \mathrm{Cl}(\Omega) \neq \emptyset$ since we can choose a sequence $\{g_i\}$ of elements $g_i \in D$ so that the associated eigenvalues for P_j goes to zero and the other eigenvalues goes to infinity while their ratios are uniformly bounded as $i \to \infty$.

Again, define $K_j := K \cap P_j$. Since K is properly convex, $K_1 * \cdots * K_k \subset K$. Since the action of Γ on K^o is cocompact and proper, Proposition 1.4.10 shows that $K = K_1 * \cdots * K_k$. We have a projection for $K^o \to K_j^o$ for each j obtained from the join structure. Then the action of Γ on K_j is cocompact since otherwise K^o/Γ cannot be compact. Also, the action of Γ' on P_j is irreducible by Benoist [21]. The scond item is proved.

We can find a sequence in D converging to a projection Π_j to each P_j with the undefined space $\hat{P}^j$, We define domains $\Omega_j := \mathrm{Cl}(\Pi_j(\Omega))$ in P_j. Since Ω is in an open subset in a hemisphere, there exists a convex hull of $\mathrm{Cl}(\Omega)$, and hence so has Ω_j for each $j = 1, \dots m$. Then Ω_j is properly convex by the third item of Proposition 1.4.1 and the irreducibilty of the action in each factor K_j in Proposition 1.4.10. Hence Ω is in a properly convex domain $\Omega_1 * \cdots * \Omega_m$.

By Theorem 1.4.15 of Kobayashi, Ω equals the interior of $\Omega_1 * \cdots * \Omega_m$. Since Γ acts cocompactly on Ω, it acts on its projection Ω_j in a sweeping manner. Suppose that $\Omega_j^o \cap K_j^o \neq \emptyset$ or $\Omega^j \cap \mathscr{A}(K_j^o) \neq \emptyset$. Theorem 1.4.12 shows that $K_j^o = \Omega_j^o$ or $\mathscr{A}(K_j^o) = \Omega_j^o$ since Γ acts on a convex domain Ω_j^o as a sweeping action. Suppose that $\Omega_j^o \cap K_j^o$ or $\Omega_j^o \cap \mathscr{A}(K_j^o)$ are empty for all j. Then $\mathrm{Cl}(\Omega_j) \cap K_j \neq \emptyset$ or $\mathrm{Cl}(\Omega_j) \cap \mathscr{A}(K_j) \neq \emptyset$ by Proposition 1.4.13. Since such intersection has a unique minimal subspace containing it, this contradicts the irreducibility of Γ'-action on P_j.

Hence, it follows that $K' := (K_1' * \cdots * K_k')^o$ is a subset of Ω for $K_j' = K_j$ or $K_j' = \mathscr{A}(K_j)$. Again $K'/\Gamma' \to \Omega/\Gamma'$ is a homotopy equivalence and hence $K' = \Omega$. Hence, the first item is proved.

The action of projective automorphisms restricting to I or $\mathscr{A}$ on each P_j gives us the final part. (See Theorem 4.1 of [61] also.) [$\mathbb{S}^n$T] $\square$

We have the following useful result.

COROLLARY 1.4.17. . *Let* $\{h_i : \Gamma \to \mathsf{SL}_{\pm}(n,\mathbb{R})\}$ *be a sequence of faithful discrete representations so that* $\mathscr{O}_i := \Omega_i/h_i(\Gamma)$ *is a closed real projective orbifold for a properly convex domain* Ω_i *in* $\mathbb{S}^n$ *for each i. Suppose that* $\{h_i\} \to h_\infty$ *algebraically,* h_∞ *is faithful with discrete image, and* $\{\mathrm{Cl}(\Omega_i)\}$ *geometrically converges to a properly convex domain* $\mathrm{Cl}(\Omega_\infty)$ *with nonempty interior* Ω_∞*. Then* $h_\infty(\Gamma)$ *acts on the interior* Ω_∞ *so that the following hold*:

- $\Omega_\infty/h_\infty(\Gamma)$ *is a closed real projective orbifold.*
- $\Omega_\infty/h_\infty(\Gamma)$ *is diffeomorphic to* $\mathscr{O}_i$ *for sufficiently large i.*
- *If* U *is a properly convex domain where* $h_\infty(\Gamma)$ *acts so that* $U/h_\infty(\Gamma)$ *is an orbifold, then* $U = \Omega_\infty$ *or* $J(\Omega_\infty)$ *where* J *is a projective automorphism commuting with* $h_\infty(\Gamma)$*. In particular, if* Γ *is non-virtually-factorizable, then* $J = \mathrm{I}$ *or* $\mathscr{A}$*.*

PROOF. By Proposition 1.1.15, the quotient $\Omega_\infty/h_\infty(\Gamma)$ is an orbifold. For the second item, see the proof of Theorem 4.1 of [61]. The third item follows from Lemma 1.4.16. □

1.4.5. Technical propositions. By the following, the first assumption of Theorem 5.4.3 are needed only for the conclusion of the theorem to hold.

PROPOSITION 1.4.18. *If a group G of projective automorphisms acts on a strict join* $A = A_1$ *with* $A_1 * A_2$ *for two compact convex sets* A_1 *and* A_2 *in* $\mathbb{S}^n$ *(resp. in* $\mathbb{RP}^n$*) with* $\dim A_1 + \dim A_2 = n-1$*, then G is virtually reducible.*

PROOF. We prove for $\mathbb{S}^n$. Let $x_1,\dots,x_{n+1}$ denote the homogeneous coordinates. There is at least one set of strict join sets A_1, A_2. We choose a maximal number collection of compact convex sets $A'_1,\dots,A'_m$ so that A is a strict join $A'_1 * \cdots * A'_m$. Here, we have $A'_i \subset S_i$ for a subspace S_i corresponding to a subspace $V_i \subset \mathbb{R}^{n+1}$ that form an independent set of subspaces.

We claim that $g \in G$ permutes the collection $\{A'_1,\dots,A'_m\}$: Suppose not. We give coordinates so that for each i, there exists some index set I_i so that elements of A'_i satisfy $x_j = 0$ for $j \in I_i$ and elements of A satisfy $x_i \geq 0$. Then we form a new collection of nonempty sets

$$J' := \{A'_i \cap g(A'_j) | 0 \leq i, j \leq n, g \in G\}$$

with more elements. Since

$$A = g(A) = g(A'_1) * \cdots * g(A'_l),$$

we can show that each A'_i is a strict join of nonempty sets in

$$J'_i := \{A'_i \cap g(A'_j) | 0 \leq j \leq l, g \in G\}$$

using coordinates. A is a strict join of the collection of the sets in J', a contraction to the maximal property.

Hence, by taking a finite index subgroup G' of G acting trivially on the collection, G' is reducible. [$\mathbb{S}^n$T] □

PROPOSITION 1.4.19. *Suppose that a set G of projective automorphisms in* $\mathbb{S}^n$ *(resp. in* $\mathbb{RP}^n$*) acts on subspaces* $S_1,\dots,S_{l_0}$ *and a properly convex domain* $\Omega \subset \mathbb{S}^n$ *(resp.* $\subset \mathbb{RP}^n$*), corresponding to independent subspaces* $V_1,\dots,V_{l_0}$ *so that* $V_i \cap V_j = \{0\}$ *for* $i \neq j$ *and* $V_1 \oplus \cdots \oplus V_{l_0} = \mathbb{R}^{n+1}$*. Let* $\Omega_i := \mathrm{Cl}(\Omega) \cap S_i$ *for each i,* $i = 1,\dots,l_0$*. Let* $\lambda_i(g)$ *denote the largest norm of the eigenvalues of g restricted to* V_i*. We assume that*

- *for each* S_i*,* $G_i := \{g|S_i | g \in G\}$ *forms a bounded set of automorphisms, and*

- *for each S_i, there exists a sequence $\{g_{i,j} \in G\}$ which has the property*
$$\left\{\frac{\lambda_i(g_{i,j})}{\lambda_k(g_{i,j})}\right\} \to \infty \text{ for each } k, k \neq i \text{ as } j \to \infty.$$

Then $\mathrm{Cl}(\Omega) = \Omega_1 * \cdots * \Omega_{l_0}$ *for* $\Omega_j \neq \emptyset, j = 1, \dots, l_0$.

PROOF. First, $\Omega_i \subset \mathrm{Cl}(\Omega)$ by definition. Each element of a strict join has a vector that is a linear combination of elements of the vectors in the directions of $\Omega_1, \dots, \Omega_{l_0}$, Hence, we obtain
$$\Omega_1 * \cdots * \Omega_{l_0} \subset \mathrm{Cl}(\Omega)$$
since $\mathrm{Cl}(\Omega)$ is convex.

Let $z = [\vec{v}_z]$ for a vector $\vec{v}_z$ in $\mathbb{R}^{n+1}$. We write $\vec{v}_z = \vec{v}_1 + \cdots + \vec{v}_{l_0}$, $\vec{v}_j \in V_j$ for each j, $j = 1, \dots, l_0$, which is a unique sum. Then z determines $z_i = [\vec{v}_i]$ uniquely.

Let z be any point. We choose a subsequence of $\{g_{i,j}\}$ so that $\{g_{i,j}|S_i\}$ converges to a projective automorphism $g_{i,\infty} : S_i \to S_i$ and $\lambda_{i,j} \to \infty$ as $j \to \infty$. Then $g_{i,\infty}$ also acts on Ω_i. By Proposition 1.3.2, $\{g_{i,j}(z_i)\} \to g_{i,\infty}(z_i) = z_{i,\infty}$ for a point $z_{i,\infty} \in S_i$. We also have

$$z_i = g_{i,\infty}^{-1}(g_{i,\infty}(z_i)) = g_{i,\infty}^{-1}(\lim_j g_{i,j}(z_i)) = g_{i,\infty}^{-1}(z_{i,\infty}). \tag{1.4.1}$$

Now suppose $z \in \mathrm{Cl}(\Omega)$. We have $\{g_{i,j}(z)\} \to z_{i,\infty}$ by the eigenvalue condition. Thus, we obtain $z_{i,\infty} \in \Omega_i$ as $z_{i,\infty}$ is the limit of a sequence of orbit points of z. Hence we also obtain $z_i \in \Omega_i$ by (1.4.1). We obtain $\Omega_i \neq \emptyset$. This also shows that $\mathrm{Cl}(\Omega) = \Omega_1 * \cdots * \Omega_{l_0}$ since $z \in \{z_1\} * \cdots * \{z_{l_0}\}$.

For the $\mathbb{RP}^n$-version, we lift Ω to an open hemisphere in $\mathbb{S}^n$. Then the $\mathbb{S}^n$-version implies the $\mathbb{RP}^n$-version. [$\mathbb{S}^n$T] □

1.5. The Vinberg duality of real projective orbifolds

The duality is a natural concept in real projective geometry and it will continue to play an essential role in this theory as well.

1.5.1. The duality. We start from linear duality. Let Γ be a group of linear transformations $\mathsf{GL}(n+1, \mathbb{R})$. Let Γ^* be the *dual group* defined by $\{g^{*-1} | g \in \Gamma\}$.

Suppose that Γ acts on a properly convex cone C in $\mathbb{R}^{n+1}$ with the vertex O.

- An open convex cone C^* in $\mathbb{R}^{n+1,*}$ is *dual* to an open convex cone C in $\mathbb{R}^{n+1}$ if $C^* \subset \mathbb{R}^{n+1*}$ is the set of linear functionals taking positive values on $\mathrm{Cl}(C) - \{O\}$. C^* is a cone with the origin as the vertex again. Note $(C^*)^* = C$, and C must be properly convex since otherwise C^o cannot be open. We generalize the notion in Section 1.5.4.
- Now Γ^* will acts on C^*. A *central dilatational extension* Γ' of Γ by $\mathbb{Z}$ is given by adding a scalar dilatation by a scalar $s > 1$ for the set $\mathbb{R}_+$ of positive real numbers.
- The dual Γ'^* of Γ' is a central dilatation extension of Γ^*. Also, if Γ' is cocompact on C if and only if Γ'^* is on C^*. (See [86] for details.)
- Given a subgroup Γ in $\mathsf{PGL}(n+1, \mathbb{R})$, the dual group Γ^* is the image in $\mathsf{PGL}(n+1, \mathbb{R})$ of the dual of the inverse image of Γ in $\mathrm{SL}_\pm(n+1, \mathbb{R})$.
- We define $\mathbb{RP}^{n*}$ as $P(\mathbb{R}^{n+1,*}))$.
- A properly convex open domain Ω in $P(\mathbb{R}^{n+1})$ is *dual* to a properly convex open domain Ω^* in $P(\mathbb{R}^{n+1,*})$ if Ω corresponds to an open convex cone C and Ω^* to its dual C^*. We say that Ω^* is dual to Ω. We also have $(\Omega^*)^* = \Omega$ and Ω is properly convex if and only if so is Ω^*.

- We call Γ a *dividing group* if a central dilatational extension acts cocompactly on C with a Hausdorff quotient. Γ is dividing if and only if so is Γ^*.
- Define $\mathbb{S}^{n*} := \mathbb{S}(\mathbb{R}^{n+1*})$. For an open properly convex subset Ω in $\mathbb{S}^n$, the dual domain is defined as the quotient in $\mathbb{S}^{n*}$ of the dual cone of the cone C_Ω corresponding to Ω. The dual set Ω^* is also open and properly convex but the dimension may not change. Again, we have $(\Omega^*)^* = \Omega$.
- If Ω is a compact properly convex domain but not necessarily open, then we define Ω^* to be the closure of the dual domain of Ω^o. This definition agrees with the definition given in Section 1.5.4 for any compact convex domains since a sharply supporting hyperspace can be perturbed to a supporting hyperspace that is not sharply supporting. (See Berger [26].)
- Given a properly convex domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$), we define the *augmented boundary* of Ω

$$\begin{aligned} (1.5.1)\quad \mathrm{bd}^{\mathrm{Ag}}\Omega := \{&(x,H) | x \in \mathrm{bd}\Omega, x \in H, \\ &H \text{ is an oriented sharply supporting hyperspace of } \Omega\} \subset \mathbb{S}^n \times \mathbb{S}^{n*}. \end{aligned}$$

Define the projection

$$\Pi^{\mathrm{Ag}}_\Omega : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}\Omega$$

by $(x,H) \mapsto x$. Each $x \in \mathrm{bd}\Omega$ has at least one oriented sharply supporting hyperspace. An oriented hyperspace is an element of $\mathbb{S}^{n*}$ since it is represented as a linear functional. Conversely, an element of $\mathbb{S}^n$ represents an oriented hyperspace in $\mathbb{S}^{n*}$. (Clearly, we can do this for $\mathbb{RP}^n$ and the dual space $\mathbb{RP}^{n*}$ but we consider only nonoriented supporting hyperspaces.)

THEOREM 1.5.1. *Let A be a subset of* $\mathrm{bd}\Omega$. *Let $A' := \Pi^{\mathrm{Ag},-1}_\Omega(A)$ be the subset of* $\mathrm{bd}^{\mathrm{Ag}}(A)$. *Then $\Pi^{\mathrm{Ag}}_\Omega | A' : A' \to A$ is a quasi-fibration.*

PROOF. We take a Euclidean metric on an affine subspace containing $\mathrm{Cl}(\Omega)$. The sharply supporting hyperspaces at x can be identified with unit normal vectors at x. Each fiber $\Pi^{\mathrm{Ag},-1}_\Omega(x)$ is a properly convex compact domain in a sphere of unit vectors through x. We find a continuous section defined on $\mathrm{bd}\Omega$ by taking the center of mass of each fiber with respect to the Euclidean metric. This gives a local coordinate system on each fiber by giving the origin, and each fiber is a compact convex domain containing the origin. Then the quasi-fibration property is clear now. [$\mathbb{S}^n$T] □

REMARK 1.5.2. We notice that for properly convex open or compact domains Ω_1 and Ω_2 in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) we have

$$(1.5.2)\qquad \Omega_1 \subset \Omega_2 \text{ if and only if } \Omega_2^* \subset \Omega_1^*$$

REMARK 1.5.3. A *proper-subspace dual $K^\dagger_X$ with respect to $X = \mathbb{RP}^k$ or $\mathbb{S}^{i_0}$* of a properly convex domain K in $X = \mathbb{RP}^k$ or $\mathbb{S}^{i_0}$ is the dual domain as obtained from considering X and corresponding vector subspace only.

We are given a strict join $A * B$ for a properly convex compact k-dimensional domain A in $\mathbb{RP}^k \subset \mathbb{RP}^n$ and a properly convex compact $n-k-1$-dimensional domain B in the complementary $\mathbb{RP}^{n-k-1} \subset \mathbb{RP}^n$. Let $A^\dagger_{\mathbb{RP}^k}$ denote the proper-subspace dual in $\mathbb{RP}^{k*}$ of A in $\mathbb{RP}^k$ and $B^\dagger_{\mathbb{RP}^{n-k-1}}$ the proper-subspace dual domain in $\mathbb{RP}^{n-k-1*}$ of B in $\mathbb{RP}^{n-k-1}$. $\mathbb{RP}^{k*}$ embeds into $\mathbb{RP}^{n*}$ as $\mathbb{P}(V_1)$ for the subspace V_1 of linear functionals cancelling vectors in directions of $\mathbb{RP}^{n-k-1}$ and $\mathbb{RP}^{n-k-1*}$ embeds into $\mathbb{RP}^{n*}$ as $\mathbb{P}(V_2)$ for the subspace V_2 of linear functionals nullifying the vectors in directions of $\mathbb{RP}^k$. These will be denoted by $\mathbb{RP}^{k\dagger}$ and $\mathbb{RP}^{n-k-1\dagger}$ respectively.

Then we have

$$(A*B)^* = A^\dagger_{\mathbb{RP}^k} * B^\dagger_{\mathbb{RP}^{n-k-1}}. \tag{1.5.3}$$

This follows from the definition and realizing every linear functional as a nonnegative sum of linear functionals in the direct-sum subspaces.

Suppose that $A \subset \mathbb{S}^k$ and $B \subset \mathbb{S}^{n-k-1}$ respectively are k-dimensional and $(n-k-1)$-dimensional domains where $\mathbb{S}^k$ and $\mathbb{S}^{n-k-1}$ are complementary subspaces in $\mathbb{S}^n$. Suppose that A and B have respective dual sets $A^\dagger_{\mathbb{S}^k} \subset \mathbb{S}^{k*}, B^\dagger_{\mathbb{S}^{n-k-1}} \subset \mathbb{S}^{n-k-1*}$. We embed $\mathbb{S}^{k*}$ and $\mathbb{S}^{n-k-1*}$ to $\mathbb{S}^{n*}$ as above. We denote the images by $\mathbb{S}^{k\dagger}$ and $\mathbb{S}^{n-k-1\dagger}$ respectively. Then the above equation also holds with the subscripts exchanged appropriately.

An element (x,H) is $\mathrm{bd}^{\mathrm{Ag}}\Omega$ if and only if $x \in \mathrm{bd}\Omega$ and h is represented by a linear functional α_H so that $\alpha_H(\vec{y}) > 0$ for all $\vec{y}$ in the open cone $C(\Omega)$ corresponding to Ω and $\alpha_H(\vec{v}_x) = 0$ for a vector $\vec{v}_x$ representing x.

Let $(x,H) \in \mathrm{bd}^{\mathrm{Ag}}\Omega$. The dual cone $C(\Omega)^*$ consists of all nonzero 1-form α so that $\alpha(\vec{y}) > 0$ for all $\vec{y} \in \mathrm{Cl}(C(\Omega)) - \{O\}$. Thus, $\alpha(\vec{v}_x) > 0$ for all $\alpha \in C^*$ and $\alpha_H(\vec{v}_x) = 0$, and $\alpha_H \notin C(\Omega)^*$ since $\vec{v}_x \in \mathrm{Cl}(C(\Omega)) - \{O\}$. But $H \in \mathrm{bd}\Omega^*$ as we can perturb α_H so that it is in C^*. Thus, x is a sharply supporting hyperspace at $H \in \mathrm{bd}\Omega^*$. We define a *duality map*

$$\mathscr{D}^{\mathrm{Ag}}_\Omega : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}^{\mathrm{Ag}}\Omega^*$$

given by sending (x,H) to (H,x) for each $(x,H) \in \mathrm{bd}^{\mathrm{Ag}}\Omega$.

PROPOSITION 1.5.4. *Let Ω and Ω^* be dual open domains in $\mathbb{S}^n$ and $\mathbb{S}^{n*}$ (resp. $\mathbb{RP}^n$ and $\mathbb{RP}^{n*}$).*

- (i) *There is a proper map $\Pi^{\mathrm{Ag}} : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}\Omega$ given by sending (x,H) to x.*
- (ii) *A projective automorphism group Γ acts properly on a properly convex open domain Ω if and only if so Γ^* acts on Ω^* (Vinberg's Theorem 1.5.8).*
- (iii) *There exists a duality map*
$$\mathscr{D}^{\mathrm{Ag}}_\Omega : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}^{\mathrm{Ag}}\Omega^*$$
which is a homeomorphism.
- (iv) *Let $A \subset \mathrm{bd}^{\mathrm{Ag}}\Omega$ be a subspace and $A^* \subset \mathrm{bd}^{\mathrm{Ag}}\Omega^*$ be the corresponding dual subspace $\mathscr{D}^{\mathrm{Ag}}_\Omega(A)$. A group Γ acts on A so that A/Γ is compact if and only if Γ^* acts on A^* and A^*/Γ^* is compact.*

PROOF. We will prove for $\mathbb{S}^n$ first. (i) Each fiber is a closed set of hyperspaces at a point forming a compact set. The set of sharply supporting hyperspaces at a compact subset of $\mathrm{bd}\Omega$ is closed. The closed set of hyperspaces having a point in a compact subset of $\mathbb{S}^{n+1}$ is compact. Thus, Π_{Ag} is proper. Clearly, Π^{Ag} is continuous, and it is an open map since $\mathrm{bd}^{\mathrm{Ag}}\Omega$ is given the subspace topology from $\mathbb{S}^n \times \mathbb{S}^{n*}$ with a product topology where Π^{Ag} extends to a projection.

(ii) See Chapter 4 of [86] or Vinberg [152].

(iii) $\mathscr{D}^{\mathrm{Ag}}_\Omega$ has the inverse map $\mathscr{D}^{\mathrm{Ag}}_{\Omega^*}$.

(iv) The item is clear from (iii). [$\mathbb{S}^n$T] □

DEFINITION 1.5.5. The two subgroups G_1 of Γ and G_2 of Γ^* are *dual* if sending $g \mapsto g^{-1,T}$ gives us an isomorphism $G_1 \to G_2$. A set in $A \subset \mathrm{bd}\Omega$ is *dual* to a set $B \subset \mathrm{bd}\Omega^*$ if $\mathscr{D}^{\mathrm{Ag}}_\Omega : \Pi^{-1}_{\mathrm{Ag}}(A) \to \Pi^{-1}_{\mathrm{Ag}}(B)$ is a one-to-one and onto map.

REMARK 1.5.6. For an open subspace $A \subset \mathrm{bd}\Omega$ that is C^1 and strictly convex, $\mathscr{D}_{\Omega}^{\mathrm{Ag}}$ induces a well-defined map

$$A \subset \mathrm{bd}\Omega \to A' \subset \mathrm{bd}\Omega^*$$

since each point has a unique sharply supporting hyperspace for an open subspace A'. The image of the map A' is also smooth and strictly convex by Lemma 1.5.7. We will simply say that A' is the *image* of $\mathscr{D}$.

Let $\mathbb{RP}_x^{n-1}$ denote the space of concurrent lines to a point x where two lines are equivalent if they agree in a neighborhood of x. Now, $\mathbb{RP}_x^{n-1}$ is projectively diffeomorphic to $\mathbb{RP}^{n-1}$. The real projective transformations fixing x induce real projective transformations of $\mathbb{RP}_x^{n-1}$. Let $x \in \mathbb{S}^n$. The space $\mathbb{S}_x^{n-1}$ denotes the space of equivalence classes of concurent lines ending at x with orientation away from x where two are considered equivalent if they agree on an open subset with a common boundary point x. An equivalence class here is called a *direction from* x. Note that $\mathbb{S}_x^{n-1}$ is well-defined on $\mathbb{RP}^n$ as well for $x \in \mathbb{RP}^n$.

For a subset K in a convex domain Ω in $\mathbb{RP}^n$ or $\mathbb{S}^n$, let x be a boundary point. We define $R_x(K)$ for a subset K of Ω the space of directions of open rays from x in Ω. We defined $R_x(K) \subset \mathbb{S}_x^{n-1}$. Any projective group fixing x induces an action on $\mathbb{S}_x^{n-1}$.

We say that a two-sided open hypersurface is *convex polyhedral* if it is a union of locally finite collection of compact polytopes in hyperspaces meeting one another in strictly convex angles where the convexity is towards one-side.

LEMMA 1.5.7. *Let Ω^* be the dual of a properly convex open domain Ω in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Then*

(i) $\mathrm{bd}\Omega$ *is C^1 and strictly convex at a point $p \in \mathrm{bd}\Omega$ if and only if $\mathrm{bd}\Omega^*$ is C^1 and strictly convex at the unique corresponding point p^*.*
(ii) *Ω is an ellipsoid if and only if so is Ω^*.*
(iii) $\mathrm{bd}\Omega^*$ *contains a properly convex domain $D = P \cap \mathrm{bd}\Omega^*$ open in a totally geodesic hyperspace P if and only if $\mathrm{bd}\Omega$ contains a vertex p with $R_p(\Omega)$ a properly convex domain. In this case, $\mathscr{D}_{\Omega}^{\mathrm{Ag}}$ sends the pair of p and the associated sharply supporting hyperspace of Ω to the pairs of the totally geodesic hyperspace containing D and points of D. Moreover, D and $R_p(\Omega)$ are properly convex, and the projective dual of D is projectively diffeomorphic to $R_p(\Omega)$.*
(iv) *Let S be a convex polyhedral open subspace of $\mathrm{bd}\Omega$. Then S is dual to a convex polyhedral open subspace of $\mathrm{bd}\Omega^*$.*

PROOF. We first prove for $\mathbb{S}^n$. (i) The one-to-one map $\mathscr{D}_{\Omega}^{\mathrm{Ag}}$ sends each pair (x,H) of a point of $\mathrm{bd}\Omega$ and the sharply supporting hyperplane to a pair of (H,x) where H is a point of $\mathrm{bd}\Omega^*$ and x is a sharply supporting hyperplane at H of Ω^*.

The fact that $\mathrm{bd}\Omega$ is C^1 and strictly convex implies that for $x \in \mathrm{bd}\Omega$, H is unique, and for H, there is only one point of $\mathrm{bd}\Omega$ where H meets $\mathrm{bd}\Omega$. Also, this is equivalent to the fact that for each $H \in \mathrm{bd}\Omega^*$, the supporting hyperspace x is unique and for each x, there is one point of $\mathrm{bd}\Omega^*$ where x meets $\mathrm{bd}\Omega^*$. This shows that $\mathrm{bd}\Omega^*$ is strictly convex and C^1.

(ii) Let $\mathbb{R}^{n+1}$ have the standard Lorentz inner product B. Let C be the open positive cone. Then the space of linear functionals positive on C is in one-to-one correspondence with vectors in C using the isomorphism $C^* \to C$ given by $\phi \mapsto \vec{v}_\phi$ so that $\phi = B(\vec{v}_\phi, \cdot)$. (See [86].)

(iii) Suppose that $R_p(\Omega)$ is properly convex. We consider the set of hyperspaces sharply supporting Ω at p. This forms a properly convex domain: Let $\vec{v}$ be the vector in $\mathbb{R}^{n+1}$ in the direction of p. Then we find the set of linear functionals positive on $C(\Omega)$ but being zero on $\vec{v}$. Let $\mathbf{V}$ be a complementary space of $\vec{v}$ in $\mathbb{R}^{n+1}$. Let $\mathbb{A}$ be given as the affine subspace $\mathbf{V}+\{\vec{v}\}$ of $\mathbb{R}^{n+1}$. We choose $\mathbf{V}$ so that $C_{\vec{v}} := C(\mathrm{Cl}(\Omega))\cap\mathbb{A}$ is a bounded convex domain in $\mathbb{A}$. We give $\mathbb{A}$ a linear structure so that $\vec{v}$ corresponds to the origin. We identify this space with $\mathbf{V}$. The set of linear functionals positive on $C(\Omega)$ and 0 at $\vec{v}$ is identical with that of linear functionals on $\mathbb{R}^{n\prime}$ positive on $C_{\vec{v}}$: we define

$$C(D) := \left\{f\in\mathbb{R}^{n+1*}\middle| f|C(\mathrm{Cl}(\Omega))-\{t\vec{v}|t\geq 0\}>0, f(\vec{v})=0\right\}$$
$$\cong \widehat{C}^*_{\vec{v}} := \left\{g\in\mathbf{V}^*\middle| g|C_{\vec{v}}-\{O\}>0\right\}\subset\mathbb{R}^{n+1*}.$$

Here $\cong$ indicates a linear isomorphism, which follows by the decomposition $\mathbb{R}^{n+1}=\{t\vec{v}|t\in\mathbb{R}\}\oplus\mathbf{V}$. Define $R'_{\vec{v}}(C_{\vec{v}})$ as the equivalence classes of properly convex segments in $C_{\vec{v}}$ ending at $\vec{v}$ where two segments are equivalent if they agree in an open neighborhood of $\vec{v}$. $R_p(\Omega)$ is identical with $R'_{\vec{v}}(C_{\vec{v}})$ by the projectivization $\mathbb{S}:\mathbb{R}^{n+1}-\{O\}\to\mathbb{S}^n$. Hence $R'_{\vec{v}}(C_{\vec{v}})$ is a properly convex open domain in $\mathbb{S}(\mathbf{V})$. Since $R'_{\vec{v}}(C_{\vec{v}})$ is properly convex, the interior of the spherical projectivization $\mathbb{S}(\widehat{C}^*_{\vec{v}})\subset\mathbb{S}(\mathbf{V}^*)$ is dual to the properly convex domain $R'_{\vec{v}}(C_{\vec{v}})\subset\mathbb{S}(\mathbf{V})$.

Again we have a projection $\mathbb{S}:\mathbb{R}^{n+1*}-\{O\}\to\mathbb{S}^{n*}$. Define $D:=\mathbb{S}(C(D))\subset\mathbb{S}^{n*}$. Since $R'_{\vec{v}}(C_{\vec{v}})$ corresponds to $R_p(\Omega)$, and $\mathbb{S}(\widehat{C}^*_{\vec{v}})$ corresponds to D, the duality follows. Also, $D\subset\mathrm{bd}\Omega^*$ since points of D are oriented sharply supporting hyperspaces to Ω by Proposition 1.5.4 (iii). (Here, we can also use Proposition 5.2.2.)

(iv) From (iii) each vertex of a convex polyhedral subspace of S correspond to a compact convex polytope in the dual subspace. Also, we can check that each side of dimension i correspond to a side of dimension $n-i-1$. [$\mathbb{S}^n$T] □

1.5.2. The duality of convex real projective orbifolds with strictly convex boundary. Since $\mathscr{O}=\Omega/\Gamma$ for an open properly convex domain Ω in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) the dual orbifold $\mathscr{O}^*=\Omega^*/\Gamma^*$ is a properly convex real projective orbifold. The dual orbifold is well-defined up to projective diffeomorphisms.

THEOREM 1.5.8 (Vinberg). *Let $\mathscr{O}$ be a strongly tame properly convex real projective open or closed orbifold. The dual orbifold $\mathscr{O}^*$ is diffeomorphic to $\mathscr{O}$.*

For proof, see Thereom 4.4. 10 in Chapter 4 of [89].

The map given by Vinberg [152] is *called* the *Vinberg duality diffeomorphism*. For an orbifold $\mathscr{O}$ with boundary, the map is a diffeomorphism in the interiors $\mathscr{O}^o\to\mathscr{O}^{*o}$. Let $\tilde{\mathscr{O}}$ denote the properly convex projective domain covering$\mathscr{O}$. Also, $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}^o}$ gives us the diffeomorphism $\partial\mathscr{O}\to\partial\mathscr{O}^*$. (We conjecture that they form a diffeomorphism $\mathscr{O}\to\mathscr{O}^*$ up to isotopies. We also remark that $\mathscr{D}_{\mathscr{O}^*}\circ\mathscr{D}_{\mathscr{O}}$ may not be identity as shown by Vinberg.)

For each $p\in\Omega$, let $\vec{p}_{V,\Omega}$ denote the vector in $C(\Omega)$ with $f_V^{-1}(\vec{p}_{V,\Omega})=1$ for the Koszul-Vinberg function $f_{V,\Omega}$ for $C(\Omega)$. (See (11.2.1).) Define $\vec{p}^*_{V,\Omega}$ as the 1-form $Df_{V,\Omega}(\vec{p}_V)$, and also define p^* as $(\!(Df_{V,\Omega}(\vec{p}_{V,\Omega}))\!)$. We obtain a compactification of Ω by defining $\mathrm{Cl}^{\mathrm{Ag}}(\Omega):=\Omega\cup\mathrm{bd}^{\mathrm{Ag}}\Omega$ by defining for any sequence $p_i\in\Omega$, we form a pair $(\vec{p}_i,\vec{p}^*_{i,V})$ where $\vec{p}^*_{i,V}$ is the 1-form in $\mathbb{R}^{n*}$ given by

$$Df_V(\vec{p}_{i,V}).$$

Clearly, a limit point of $\{\vec{p}^*_{i,V,\Omega}\}$ is a supporting 1-form of $C(\Omega)$ since it supports a properly convex domain $f_V^{-1}(1,\infty)\subset C(\Omega)$. We say that p_i converges to an element of $\mathrm{bd}^{\mathrm{Ag}}$ if this augmented sequence converges to it.

THEOREM 1.5.9. *Let Ω be a properly convex domain in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$), and let Ω^* be its dual in $\mathbb{RP}^{n*}$ (resp. in $\mathbb{S}^{n*}$). Then the Vinberg duality diffeomorphism $\mathscr{D}_\Omega : \Omega \to \Omega^*$ extends to a homeomorphism $\bar{\mathscr{D}}_\Omega^{\mathrm{Ag}} : \mathrm{Cl}^{\mathrm{Ag}}(\Omega) \to \mathrm{Cl}^{\mathrm{Ag}}(\Omega)$. Moreover for any projective group Γ acting on it, $\bar{\mathscr{D}}_\Omega^{\mathrm{Ag}}$ is equivariant with respect to the duality map $\Gamma \to \Gamma^*$ given by $g \mapsto g^{*-1}$.*

PROOF. We assume $\Omega \subset \mathbb{S}^n$. The continuity follows from the paragraph above the theorem since $\mathscr{D}_{\tilde{\mathscr{O}}}$ is induced by $(p, p^*_{V,\Omega}) \to (p^*_{V,\Omega}, p)$, and $\mathscr{D}_{\tilde{\mathscr{O}}}^{\mathrm{Ag}}$ is a map switching the orders of the pairs also.

Proposition 1.5.4 shows the injectivity of $\bar{\mathscr{D}}_\Omega^{\mathrm{Ag}}$. The map is surjective since so is $\mathscr{D}_\Omega$ and $\mathscr{D}_\Omega^{\mathrm{Ag}}$.

The equivariance follows since so are $\mathscr{D}_\Omega$ and $\mathscr{D}_\Omega^{\mathrm{Ag}}$. [$\mathbb{S}^n$T] $\square$

1.5.3. Sweeping actions. The properly convex open set D in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) has a Hilbert metric. Also the group $\mathbf{Aut}(K)$ of projective automorphisms of K in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ is a locally compact closed group.

LEMMA 1.5.10. *Let D be a properly convex open domain in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) with $\mathbf{Aut}(D)$ of smooth projective automorphisms of D. Let a group G act isometrically on an open domain D faithfully with $G \to \mathbf{Aut}(D)$ is an embedding. Suppose that D/G is compact. Then the closure $\bar{G}$ of G is a Lie subgroup acting on D properly, and there exists a smooth Riemannian metric on D that is $\bar{G}$-invariant.*

PROOF. Assume $D \subset \mathbb{S}^n$. Since $\bar{G}$ is in $\mathsf{SL}_\pm(n+1,\mathbb{R})$, the closure $\bar{G}$ is a Lie subgroup acting on D properly. Suppose that $D \subset \mathbb{S}^n$.

One can construct a Riemannian metric μ with bounded entries. Let ϕ be a function supported on a compact set F so that $G(F) \supset D$ where $\phi|F > 0$. Given a bounded subset of $\bar{G}$, the elements are in a bounded subset of the projective automorphism group $\mathsf{SL}_\pm(n+1,\mathbb{R})$. A bounded subset of projective automorphisms have uniformly bounded set of derivatives on $\mathbb{S}^n$ up to the m-th order for any m. We can assume that the derivatives of the entries of $\phi\mu$ up to the m-th order are uniformly bounded above. Let $d\eta$ be the left-invariant measure on $\bar{G}$.

Then $\{g^*\phi\mu | g \in \bar{G}\}$ is an equicontinuous family on any compact subset of D^o up to any order. For $J \subset D^o$, $\mathrm{supp}(g^*\phi\mu) \cap J \neq \emptyset$ for g in a compact set of $\bar{G}$. Thus the integral

$$\int_{g \in \bar{G}} g^*\phi\mu d\eta$$

of $g^*\phi\mu$ for $g \in \bar{G}$ is a C^∞-Riemannian metric and that is positive definite. This bestows us a C^∞-Riemannian metric μ_D on D invariant under $\bar{G}$-action. [$\mathbb{S}^n$T] $\square$

By Lemma 1.5.10, there exists a Riemannian metric on a properly convex domain Ω invariant under $\mathbf{Aut}(\Omega)$. Hence, we can define a frame bundle $\mathbb{F}\Omega$ where $\mathbf{Aut}(\Omega)$ acts freely.

PROPOSITION 1.5.11 (Lemma 1 of Vey [151]). *Suppose that a projective group G acts on an $(n-1)$-dimensional properly convex open domain Ω as a sweeping action. Then the dual group G^* acts on Ω^* as a sweeping action also.*

PROOF. The Vinberg duality map in Theorem 1.5.8 is a diffeomorphism $\Omega \to \Omega^*$. This map is equivariant under the duality homomorphism $g \mapsto g^{*-1}$ for each $g \in G$. Here, G does not need to be a dividing action. $\square$

THEOREM 1.5.12 (Generalizes1.4.15). *Suppose that a projective group G sweepingly acts on a properly convex open domain D in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Then for any properly convex open domain Ω with Ω/G is compact and $\Omega \cap D \neq \emptyset$, $\Omega = D$.*

PROOF. Suppose not. Then G acts on $D \cap \Omega$ as a sweeping manner and $D \cap \Omega$ is a proper subset of D. Let $x \in D \cap \Omega$. By Theorem 1.4.12, the convex hull of Gx must equal both $D \cap \Omega$ and D. Hence, $D \subset \Omega$. The converse also holds by the same reason. $\square$

1.5.4. Extended duality. We can generalize the duality for convex domains as was done at the beginning of Section 1.5; however, we don't generalize for $\mathbb{RP}^n$. Given a closed convex cone C_1 in $\mathbb{R}^{n+1}$, consider the set of linear functionals in $\mathbb{R}^{n+1*}$ taking nonnegative values in C_1. This forms a closed convex cone. We call this the *dual cone* of C_1 and denote it by C_1^*.

A closed cone C_2 in $\mathbb{R}^{n+1*}$ is *dual* to a closed convex cone C_1 in $\mathbb{R}^{n+1}$ if C_2 is the set of linear functionals taking nonnegative values in C_1.

For a convex compact set U in $\mathbb{S}^n$, we form a corresponding convex cone $C(U)$. Then we form $C(U)^*$ and the image of its projection a convex compact set U^* in $\mathbb{S}^{n*}$. Clearly, $(U^*)^* = U$ for a compact convex set U by definition.

Also, the definition agrees with the previous definition defined for properly convex domains. This is straightforward: Functions in $C(U)^*$ can be approximated arbitrarily by functions strictly positive on $C(U)$.

Also, for subspaces such as great sphere $\mathbb{S}^{i_0}$ in $\mathbb{S}^n$ its dual in $\mathbb{S}^{n*}$ is a great sphere $\mathbb{S}^{n-i_0-1*}$. For these subspaces, we can give an orientation on $\mathbb{S}^{i_0}$ so that we can give the orientation on $\mathbb{S}^{n-i_0-1*}$ so that under a fixed metric a basis in the orientation of $\mathbb{S}i_0$ and the dual basis of $\mathbb{S}^{n-i_0-1*}$ form the orientation of $\mathbb{R}^{n+1}$ inducing the given one on $\mathbb{S}^n$. In particular, since $\mathbb{S}^0$ is a pair of antipodal points, an orientation is a choice of a point. The orientations on $\mathbb{S}^{i_0}$ and $\mathbb{S}^{n-i_0-1*}$ are said to be *dual ones*. and the oriented $\mathbb{S}^{i_0}$ is *dual* to the oriented $\mathbb{S}^{n-i_0-1*}$.

Recall the classification of compact convex sets in Proposition 1.1.4.

PROPOSITION 1.5.13.

- *Let $\mathbb{S}^{i_0}$ be a great sphere of dimension i_0.*
- *Let $\mathbb{S}^{j_0}$ be a one of dimension j_0 with $i_0 + j_0 + 1 \leq n$ independent of $\mathbb{S}^{i_0}$.*
- *We also have the join $\mathbb{S}^{i_0+j_0+1}$ of $\mathbb{S}^{i_0}$ and $\mathbb{S}^{j_0}$ and its complementary subspace $\mathbb{S}^{n-i_0-j_0-2}$.*
- *Let $\mathbb{S}^{n-i_0-1}$ be one of dimension $n-i_0-2$ complementary to $\mathbb{S}^{i_0}$ where $\mathbb{S}^{n-i_0-j_0-2} \subset \mathbb{S}^{n-i_0-1}$.*
- *Let us identify $\mathbb{S}^{i_0\dagger}$ as $\mathbb{S}^{n-i_0-1*}$ by taking restrictions of linear maps and $\mathbb{S}^{n-i_0-j_0-2\dagger}$ as $\mathbb{S}^{i_0+j_0+1*}$.*

Let U be a convex compact proper set in $\mathbb{S}^n$. Then the following hold:

- **(i):** *U is a great i_0-sphere if and only if U^* is a great $n-i_0-1$-sphere. U^* is not convex if and only if $i_0 = n-1$.*
- **(ii):** *If U is a strict join of a properly convex domain K of dimension i_0 in a great sphere $\mathbb{S}^{i_0}$ and a complementary great sphere $\mathbb{S}^{j_0}$ for $i_0, j_0 \geq 0$, then*
  - *U^* is a strict join of $\mathbb{S}^{n-i_0-j_0-2\dagger} = \mathbb{S}^{i_0+j_0+1*}$ and a properly convex domain $K^\dagger$ in $\mathbb{S}^{i_0\dagger} = \mathbb{S}^{n-i_0-1*}$ of dimension i_0 properly dual to K in $\mathbb{S}^{i_0}$ if $i_0+j_0+1 < n$.*
  - *U^* is $K^\dagger \subset \mathbb{S}^{i_0\dagger}$ if $i_0+j_0+1 = n$.*
- **(iii):** *If U is a properly convex domain K in a great sphere $\mathbb{S}^{i_0}$ of dimension $i_0, i_0 < n$, then U^* is a strict join of the proper-subspace dual $K^\dagger$ of K in $\mathbb{S}^{i_0\dagger} = \mathbb{S}^{n-i_0-1*}$*

and a great sphere $\mathbb{S}^{i_0*}$ of dimension $n-i_0-1$ for any choice of the complement $\mathbb{S}^{n-i_0-1}$ of $\mathbb{S}^{i_0}$.

(iv): *U is a properly convex compact n-dimensional domain if and only if so is U^*.*

(v): *If U is not properly convex and has a nonempty interior, then U^* has an empty interior.*

(vi): *If U has an empty interior, then U^* is not properly convex and has a nonempty interior provided U is properly convex.*

(vii): *In particular, if U is an n-hemisphere, then U^* is a point and vice versa.*

(viii): *If $U^o \neq \emptyset$ and $U^{*o} \neq \emptyset$, then U and U^* are both properly convex domains in $\mathbb{S}^n$.*

(xi): *U is contained in a hemisphere if and only if U^* is contained in a hemisphere.*

PROOF. (i) Suppose that U is a great i_0-sphere. Then $C(U)$ is a subspace of dimension i_0+1. The set of linear functionals taking 0 values on $C(U)$ form a subspace of dimension $n-i_0$. Hence, $U^* = \mathbb{S}(C(U)^*)$ is a great sphere of dimension $n-i_0-1$. The converse is also true.

(ii) Suppose that U is not a great sphere. Proposition 1.1.4 shows us that U is contained in an n-hemisphere.

Let $\mathbb{S}^{m_0}$ be the span of U. Here, $m_0 = i_0 + j_0 + 1$. Then $U = \mathbb{S}^{j_0} * K^{i_0}$ for a great sphere $\mathbb{S}^{j_0}$ and a properly convex domain K^{i_0} in a great sphere of dimension i_0 in $\mathbb{S}^{m_0}$ independent of the first one by Proposition 1.1.4.

$C(U)$ is a closed cone in the vector subspace $\mathbb{R}^{m_0+1}$. Then $C(U) = \mathbb{R}^{j_0+1} + C(K^{i_0})$ where $C(K^{i_0}) \subset \mathbb{R}^{i_0+1}$ for independent subspaces $\mathbb{R}^{j_0+1}$ and $\mathbb{R}^{i_0+1}$. Let $C(U)'$ denote the dual of $C(U)$ in $\mathbb{R}^{m_0+1*}$. For $f \in C(U)'$, $f = 0$ on $\mathbb{R}^{j_0+1}$, and $f|\mathbb{R}^{i_0+1}$ takes a value ≥ 0 in $C(K^{i_0})$. Hence,

$$f : \mathbb{R}^{m_0+1} = \mathbb{R}^{j_0+1} \oplus \mathbb{R}^{i_0+1} \to \mathbb{R} \text{ is in } \{0\} \oplus C(K^{i_0})^*.$$

Denote the projection of $C(U)'$ in $\mathbb{S}^{m_0}$ by U'.

Suppose $m_0 = n$. Then we showed the second case of (ii).

Suppose $m_0 < n$. Then decompose $\mathbb{R}^{n+1} = \mathbb{R}^{n-m_0} \oplus \mathbb{R}^{m_0+1}$. We obtain that $f \in C(U)^*$ is a sum $f_1 + f_2$ where f_1 is an element of $C(U)'$ extended by setting $f_1|\mathbb{R}^{n-m_0} \oplus \{O\} = 0$ and f_2 is any linear functional satisfying $f_2|\{O\} \oplus \mathbb{R}^{m_0+1} = 0$ where we indicate by $\{O\}$ the trivial subspaces of the complements. Hence, $(\!(f_2)\!) \in \mathbb{S}^{m_0*} = \mathbb{S}^{n-i_0-j_0-2\dagger}$. Hence, U^* is a strict join of U' and a great sphere $\mathbb{S}^{n-i_0-j_0-2\dagger}$.

(iii) This is obtained by taking the dual of the second case of (ii).

(iv) Since the definition agrees with classical one for properly convex domains, this follows. Also, one can derive this as contrapositive of (ii) and (iii) since domains and their duals not covered by (ii) and (iii) are the properly convex domains.

(v) U is as in the second case of (ii).

(vi) If U has the empty interior, then U is covered by (ii) and (iii) or is a great sphere of dimension $< n$. (iii) corresponds to the case when the dual of K has nonempty interior.

(vii) The forward part is given by (iii) where K is a singleton in $\mathbb{S}^0$ and $i_0 = 0$. The converse part is the second case of (ii) where K is of dimension zero and $j_0 = n-1$.

(viii) The item (v) shows this using $(U^*)^* = U$.

(xi) Proposition 1.1.4 shows that (ii), (iii), and (iv) cover all compact convex sets that are not great spheres. □

We also note that for any properly convex domain K, $K \subset H^k \subset \mathbb{S}^k$ for a open hemisphere H_k, and a great sphere $\mathbb{S}^j$ in an independent space, the interior of $K * \mathbb{S}^j \subset H^k * \mathbb{S}^j$ is in an affine space $H^{k+j+1} = H^k \times H^{j+1} \subset \mathbb{S}^n$. Hence, a join is really a product of a certain form. We call this an *affine form* of a strict join.

1.5.5. Duality and geometric limits. Define the *thickness* of a properly convex domain Δ is given as

$$\min\{\max\{\mathbf{d}(x, \mathrm{bd}\Delta)|x \in \Delta\}, \max\{\mathbf{d}(y, \mathrm{bd}\Delta^*)|y \in \Delta^*\}\}$$

for the dual Δ^* of Δ.

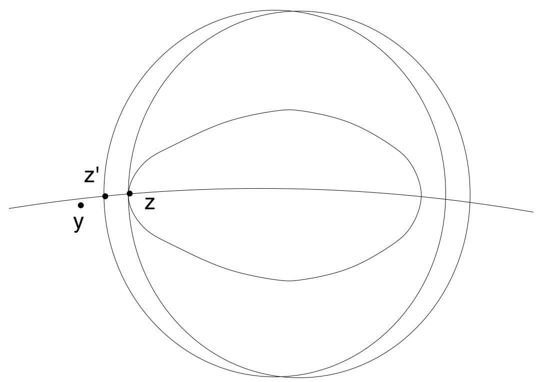

FIGURE 1. The diagram for Lemma 1.5.14.

LEMMA 1.5.14. *Let Δ be a properly convex open (resp. compact) domain in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) and its dual Δ^* in $\mathbb{RP}^{n*}$ (resp. $\mathbb{S}^{n*}$). Let ε be a positive number less than the thickness of Δ and less than $\frac{1}{2}\mathbf{d}(\Delta', \mathscr{A}(\Delta'))$ and $\frac{1}{2}\mathbf{d}(\Delta'^*, \mathscr{A}(\Delta'^*))$ for a lift Δ' of Δ to $\mathbb{S}^n$ (resp. $\Delta' = \Delta$). Then the following hold:*

- $N_\varepsilon(\Delta) \subset (\Delta^* - \mathrm{Cl}(N_\varepsilon(\mathrm{bd}\Delta^*)))^*$.
- *If two properly convex open domains Δ_1 and Δ_2 are of Hausdorff distance $< \varepsilon$ for ε less than the thickness of each Δ_1 and Δ_2, then Δ_1^* and Δ_2^* are of Hausdorff distance $< \varepsilon$.*
- *Furthermore, if $\Delta_2 \subset N_{\varepsilon'}(\Delta_1)$ and $\Delta_1 \subset N_{\varepsilon'}(\Delta_2)$ for $0 < \varepsilon' < \varepsilon$, then we have $\Delta_2^* \supset \Delta_1^* - \mathrm{Cl}(N_{\varepsilon'}(\mathrm{bd}\Delta_1^*))$ and $\Delta_1^* \supset \Delta_2^* - \mathrm{Cl}(N_{\varepsilon'}(\mathrm{bd}\Delta_2^*))$.*

PROOF. Using the double covering map $p_{\mathbb{S}^n}$ and $p_{\mathbb{S}^{n*}} : \mathbb{S}^{n*} \to \mathbb{RP}^{n*}$ of unit spheres in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1*}$, we take components of Δ and Δ^*. It is easy to show that the result for properly convex open domains in $\mathbb{S}^n$ and $\mathbb{S}^{n*}$ is sufficient.

For elements $\phi \in \mathbb{S}^{n*}$, and $x \in \mathbb{S}^n$, we say $\phi(x) < 0$ if $f(v) < 0$ for $\phi = [f], x = [\vec{v}]$ for $f \in \mathbb{R}^{n+1*}, \vec{v} \in \mathbb{R}^{n+1}$. Also, we say $\phi(x) > 0$ if $f(v) > 0$ for $\phi = [f], x = [\vec{v}]$ for $f \in \mathbb{R}^{n+1*}, \vec{v} \in \mathbb{R}^{n+1}$.

For the first item, let $y \in N_\varepsilon(\Delta)$. Suppose that $\phi(y) < 0$ for

$$\phi \in \mathrm{Cl}((\Delta^* - \mathrm{Cl}(N_\varepsilon(\mathrm{bd}\Delta^*))) \neq \emptyset.$$

Since $\phi \in \Delta^*$, the set of positive valued points of $\mathbb{S}^n$ under ϕ is an open hemisphere H containing Δ but not containing y. The boundary $\mathrm{bd}H$ of H has a closest point $z \in \mathrm{bd}\Delta$ of distance $< \varepsilon$. The closest point z' to z on $\mathrm{bd}H$ is in $N_\varepsilon(\Delta)$ since y is in $N_\varepsilon(\Delta) - H$ and z' is closest to $\mathrm{bd}\Delta$. The great circle $\mathbb{S}^1$ containing z and z' are perpendicular to $\mathrm{bd}H$ since $\overline{zz'}$ is minimizing lengths. Hence $\mathbb{S}^1$ passes the center of the hemisphere. One can push the center of the hemisphere on $\mathbb{S}^1$ until it becomes a sharply supporting hemisphere to Δ. The corresponding ϕ' is in $\mathrm{bd}\Delta^*$ and the distance between ϕ and ϕ' is less than ε. This is a contradiction. Thus, the first item holds (See Figure 1.)

For the final item, we have that

$$\Delta_2 \subset N_{\varepsilon'}(\Delta_1), \Delta_1 \subset N_{\varepsilon'}(\Delta_2) \text{ for } 0 < \varepsilon' < \varepsilon.$$

Hence, $\Delta_2 \subset (\Delta_1^* - \mathrm{Cl}(N_{\varepsilon'}(\mathrm{bd}\Delta_1^*)))^*$, and $\Delta_2^* \supset \Delta_1^* - \mathrm{Cl}(N_{\varepsilon'}(\mathrm{bd}\Delta_1^*))$ by (1.5.2), which proves the third item where we need to switch 1 and 2 also. We obtain $N_\varepsilon(\Delta_2^*) \supset \Delta_1^*$ and conversely. The second item follows. $[\mathbb{S}^n\mathrm{P}]$ $\square$

The following may not hold for $\mathbb{RP}^n$:

PROPOSITION 1.5.15. *Suppose that $\{K_i\}$ is a sequence of properly convex domains in $\mathbb{S}^n$ geometrically converging to a compact convex set K. Then $\{K_i^*\}$ geometrically converges to the compact convex set K^* dual to K.*

PROOF. Recall the compact metric space of all compact subsets of $\mathbb{S}^n$ with the Hausdorff metric $\mathbf{d}_H$. (See p.280-281 of Munkres [136].) K_i is a Cauchy sequence under the Hausdorff metric $\mathbf{d}_H$. By Lemma 1.5.14, K_i^* is also a Cauchy sequence under the Hausdorff metric of $\mathbf{d}_H$ of $\mathbb{S}^{n*}$. The Hausdorff metric of the space of all compact subsets of $\mathbb{S}^{n*}$ is a compact metric space.

Since each K_i is contained in an n-hemisphere corresponding to the linear functional ϕ_i with $\phi_i|C(K_i) \geq 0$, we deduce that K is contained in an n-hemisphere.

Let K^∞ denote the limit of the Cauchy sequence $\{K_i^*\}$. We will show $K^\infty = K^*$.

First, we show $K^\infty \subset K^*$: Let ϕ_∞ be a limit of a sequence ϕ_i for $\phi_i \in C(K_i^*)$ for each i. By Proposition 1.1.7, it will be sufficient to show $(\!(\phi_\infty)\!) \in K^*$ for every such ϕ_∞. We may assume that their Euclidean norms are 1 always with the standard Euclidean metric on $\mathbb{R}^n$. We will show that $\phi_\infty|C(K) \geq 0$.

Let $\mathbb{S}_1^n$ denote the unit sphere in $\mathbb{R}^{n+1}$ with a Fubini-Study path-metric $\mathbf{d}_1$. The projection $\mathbb{S}^n \to \mathbb{S}_1^n$ is an isometry from $\mathbf{d}$ to $\mathbf{d}_1$. Then $C(K_i) \cap \mathbb{S}_1^n \to C(K) \cap \mathbb{S}_1^n$ geometrically under the Hausdorff metric $\mathbf{d}_{H,1}$ associated with $\mathbf{d}_1$. Let $N_\varepsilon(U)$ denote the ε-neighborhood of a subset U of $\mathbb{S}_1^n$ under $\mathbf{d}_1$. Since $K_i \to K$, we find a sequence ε_i so that $N_{\varepsilon_i}(C(K_i)) \cap \mathbb{S}_1^n \supset C(K) \cap \mathbb{S}_1^n$ and $\varepsilon_i \to 0$ as $i \to \infty$.

For any ϕ in $\mathbb{R}^{n+1*}$ of unit norm, for every pair of points $x, y \in \mathbb{S}_1^n$ with $\phi(x) \geq 0$,

$$\mathbf{d}_1(x,y) \leq \delta \text{ implies } \phi(y) \geq -\delta : \tag{1.5.4}$$

This follows by integrating along the geodesic from x to y considering ϕ as a 1-form of norm 1.

Since $\min\{\phi_i|N_{\varepsilon_i}(C(K_i)) \cap \mathbb{S}_1^n\} \geq -\varepsilon_i$ by (1.5.4), we obtain $\phi_i|C(K) \cap \mathbb{S}_1^n \geq -\varepsilon_i$ for sufficiently large i. Since $\varepsilon_i \to 0$, we obtain $\phi_\infty|C(K) \cap \mathbb{S}_1^n \geq 0$, and $\phi_\infty \in C(K)^*$.

Conversely, we show $K^* \subset K^\infty$. Let $\phi \in C(K)^*$. Then $\phi|C(K) \cap \mathbb{S}_1^n \geq 0$. Define $\varepsilon_i = \min\{\phi(C(K_i) \cap \mathbb{S}_1^n)\}$. If $\varepsilon_i \geq 0$ for sufficiently large i, then $\phi \in C(K_i)^*$ for sufficiently large i and $(\!(\phi)\!) \in K^\infty$ by Proposition 1.1.7, and we are finished in this case.

Suppose $\varepsilon_i < 0$ for infinitely many i. By taking a subsequence if necessary, we assume that $\varepsilon_i < 0$ for all i. Let H_ϕ, $H_\phi \subset \mathbb{S}^n$, be the hemisphere determined by the nonnegative condition of ϕ. Then $K_i - H_\phi \neq \emptyset$ for every i. Choose a point y_i in K_i of the maximal distance from H_ϕ. Then $\mathbf{d}_1(y_i, H_\phi) \leq \delta_i$ for $0 < \delta_i \leq \pi/2$. Since $\{K_i\} \to K$, we deduce $\{\delta_i\} \to 0$ as $i \to \infty$ obviously. Assume $\delta_i < \pi/4$ without loss of generality.

Define a distance function $f_1(\cdot) := \mathbf{d}_1(\cdot, H_\phi) : \mathbb{S}^n \to \mathbb{R}_+$. Then y_i is contained in a smooth sphere S_{δ_i} at the level δ_i with a fixed center x_ϕ. Also, K_i is contained in the complement of the convex open ball B_{δ_i} bounded by S_{δ_i}.

Now, we will use convex affine geometry. Let H_i denote the hemisphere whose boundary contains y_i and is tangent to S_{δ_i} and disjoint from B_{δ_i}. Then y_i is a unique maximum point of $f_1|K_i$ since otherwise we will have a point with smaller f_1 by the convexity of K_i and B_{δ_i}. And K_i is disjoint from B_{δ_i} as y_i is the unique maximum point. Since K_i and $\mathrm{Cl}(B_{\delta_i})$ are both convex and meets only at y_i, ∂H_i supports K_i and $\mathrm{Cl}(B_{\delta_i})$ by the hyperplane separation theorem applied to $C(K_i)$ and $C(\mathrm{Cl}(B_{\delta_i}))$.

We obtained $K_i \subset H_i$. Let ϕ_i be the linear functional of unit norm corresponding to H_i. Then $\phi_i|C(K_i) \geq 0$. Let s_i be the shortest segment from y_i to ∂H_i with the other endpoint $x_i \in \partial H_i$. The center $\mathscr{A}(x_\phi)$ of H_ϕ is on the great circle $\hat{s}_i$ containing s_i. The center of H_i is on $\hat{s}_i$ and of distance δ_i from $\mathscr{A}(x_\phi)$ since $\mathbf{d}_1(y_i, x_i) = \delta_i$.

This implies that $\mathbf{d}(\phi, \phi_i) = \delta_i$. Since $\delta_i \to 0$, we obtain $\{\phi_i\} \to \phi$ and $K^* \subset K^\infty$ by Proposition 1.1.7. $\square$

CHAPTER 2

Examples of properly convex real projective orbifolds with ends: cusp openings

We give examples where our theory applies to. We explain the theory of convex projective structures on Coxeter orbifolds and the orderability theory for Coxeter orbifolds. Our work jointly done with Gye-Seon Lee and Craig Hodgson generalizing the work of Benoist and Vinberg will be discussed. We also explain the vertex orderable Coxeter orbifolds. We state the work of Heusner-Porti on projective deformations of the hyperbolic link complement and the subsequent work by Ballas. Also, we state some nice results on finite volume convex real projective structures by Cooper-Long-Tillmann and Crampon-Marquis on horospherical ends and thick and thin decomposition.

How, these examples fit into this monograph is explained in Chapter 12.

2.1. History of examples

Originally, Vinberg [153] investigated convex real projective Coxeter orbifolds as linear groups acting on convex cones. The groundbreaking work also produced many examples of real projective orbifolds and manifolds $\mathcal{O}$ suitable to our study. For example, see Kac and Vinberg [107] for the deformation of triangle groups. However, the work was reduced to studying some Cartan forms with rank equal to $n+1$ for $n = \dim \mathcal{O}$. The method turns out to be a bit hard in computing actual examples.

Later, Benoist [24] worked out some examples on prisms. Generalizing this, Choi [50] studied the orderability of Coxeter orbifolds after conversing with Kapovich about the deformability. This produced many examples of noncompact orbifolds with properly convex projective structures by the work of Vinberg. Later, Marquis [125] generalized the technique to study the convex real projective structures based on Coxeter orbifolds with truncation polytopes as base spaces. These are compact orbifolds, and so we will not mention these.

For compact hyperbolic 3-manifolds, Cooper-Long-Thistlethwaite [64] and [65] produced many examples with deformations using numerical methods. Some of these are exact computations.

We now discuss the noncompact strongly tame orbifolds with convex real projective structures.

Also, Choi, Hodgson, and Lee [58] computed the deformation spaces of convex real projective structures of some complete hyperbolic Coxeter orbifolds with or without ideal vertices, and Choi and Lee [60] showed that all compact hyperbolic weakly orderable Coxeter orbifolds have the local deformation spaces of dimension $e_+ - 3$ where e_+ is the number of ridges with order ≥ 2. These Coxeter orbifolds form a large class of Coxeter orbifolds.

We can generalize these to complete hyperbolic Coxeter orbifolds that are weakly orderable with respect to ideal vertices. Lee, Marquis, and I will prove in later papers related ideal-vertex-orderable Coxeter 3-orbifolds have smooth deformation spaces of computable dimension.

For noncompact hyperbolic 3-manifolds, Porti and Tillmann [139], Cooper-Long-Tillmann [67], and Crampon-Marquis [68] made theories where the ends were restricted to be horospherical. Ballas [4] and [5] made initial studies of deformations of complete hyperbolic 3-manifolds to convex real projective ones. Cooper, Long, and Tillmann [66] have produced a deformation theory for convex real projective manifolds parallel to ours with different types of restrictions on ends, such as requiring the end holonomy group to be abelian. They also concentrate on the openness of the deformation spaces. We will provide our theory in Part 3.

2.2. Examples and computations

We will give some series of examples due to the author and many other people. Here, we won't give compact examples since we already gave a survey in Choi-Lee-Marquis [61].

Given a polytope P, a *face* is a codimension-one side of P. A *ridge* is the codimension-two side of P. When P is 3-dimensional, a ridge is called an *edge*.

We will concentrate on n-dimensional orbifolds whose base spaces are homeomorphic to convex Euclidean polyhedrons and whose faces are silvered and each ridge is given an order. For example, a hyperbolic polyhedron with edge angles of form π/m for positive integers m will have a natural orbifold structure like this.

DEFINITION 2.2.1. A *Coxeter group* Γ is an abstract group defined by a group presentation of form

$$(R_i; (R_iR_j)^{n_{ij}}), i, j \in I$$

where I is a countable index set, $n_{ij} \in \mathbf{N}$ is symmetric for i, j and $n_{ii} = 1$, $n_{ij} \geq 2$ for $i \neq j$.

The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \dots, f, (R_iR_j)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and R_iR_j has order n_{ij} associated with the edge formed by the intersection of the i-th and j-th sides.

Let us consider only the 3-dimensional orbifolds for now. Let P be a fixed convex 3-polyhedron. Let us assign orders at each edge. We let e be the number of edges and e_2 be the numbers of edges of order-two. Let f be the number of sides.

For any vertex of P, we will remove the vertex unless the link in P form a spherical Coxeter 2-orbifold of codimension 1. This make P into a 3-dimensional orbifold.

Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from P with the above vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

In this chapter, we will exclude a *cone-type Coxeter orbifold* whose polyhedron has a side F and a vertex v where all other sides are adjacent triangles to F and contains v and all ridge orders of F are 2. Another type we will *not* study is a *product-type Coxeter orbifold* whose polyhedron is topologically a polygon times an interval and ridge orders of top and the bottom sides are all 2. These are essentially lower-dimensional orbifolds. Finally, we will *not* study Coxeter orbifolds with finite fundamental groups. If $\hat{P}$ is none of the above type, then $\hat{P}$ is said to be a *normal-type* Coxeter orbifold.

A huge class of examples are obtainable from complete hyperbolic 3-polytopes with dihedral angles that are submultiples of π. (See Andreev [3] and Roerder [140].)

DEFINITION 2.2.2. The *deformation space* $\mathfrak{D}(\hat{P})$ of projective structures on a Coxeter orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.

This definition was also used in a number of papers [50], [59], and [58]. The topology on $\mathfrak{D}(\hat{P})$ is given by as follows: $\mathfrak{D}(\hat{P})$ is a quotient space of the space of the development pairs $(\mathbf{dev}, h)$ with the compact open C^r-topology, $r \geq 2$, for the maps $\mathbf{dev} : \tilde{P} \to \mathbb{RP}^n$.

We will explain that the space is identical with $\mathrm{CDef}_{\mathscr{E}}(\hat{P})$ in Proposition 9.5.1. Also, $\mathrm{CDef}_{\mathscr{E}}(\hat{P}) = \mathrm{CDef}_{\mathscr{E},\mathrm{u,lh}}(\hat{P})$ by Corollary 12.1.5.

A point p of $\mathfrak{D}(\hat{P})$ gives a fundamental polyhedron P in $\mathbb{RP}^3$, well-defined up to projective automorphisms. By Proposition 9.5.1, $\mathfrak{D}(\hat{P})$ can be identified with $\mathrm{CDef}_{\mathscr{E}}(\hat{P})$. We concentrate on the space of $p \in \mathfrak{D}(\hat{P})$ giving a fundamental polyhedron P fixed up to projective automorphisms. This space is called the *restricted deformation space* of $\hat{P}$ and denoted by $\mathfrak{D}_P(\hat{P})$. A point t in $\mathfrak{D}_P(\hat{P})$ is said to be *hyperbolic* if a hyperbolic structure on $\hat{P}$ induces the projective structure; that is, it is projectively diffeomorphic to $\mathbb{B}/\Gamma$ for a standard unit ball $\mathbb{B}$ and a discrete group $\Gamma \subset \mathbf{Aut}(\mathbb{B})$. A point p of $\mathfrak{D}(\hat{P})$ always determines a fundamental polyhedron P up to projective automorphisms because p determines reflections corresponding to sides up to conjugations also. We wish to understand the space where the fundamental polyhedron is always projectively equivalent to P. We call this the *restricted deformation space* of $\hat{P}$ and denote it by $\mathfrak{D}_P(\hat{P})$.

The work of Vinberg [153] implies that each element of $\mathfrak{D}(\hat{P})$ gives a convex projective structure (see Theorem 2 of [50]). That is, the image of the developing map of the orbifold universal cover of $\hat{P}$ is projectively diffemorphic to a convex domain in $\mathbb{RP}^3$, and the holonomy is a discrete faithful representation.

Now, we state the key property in this chapter:

DEFINITION 2.2.3. We say that P is *orderable* if we can order the sides of P so that each side meets sides of higher index in less than or equal to 3 edges.

A pyramid with a complete hyperbolic structure and dihedral angles that are submultiples of π is an obvious example. See Proposition 4 of [50] worked out with J. R. Kim.

An example is a drum-shaped convex polyhedron which has top and bottom sides of the same polygonal type and each vertex of the bottom side is connected to two vertices on the top side and vice versa. Another example will be a convex polyhedron where the union of triangles separates each pair of the interiors of nontriangular sides. In these examples, since nontriangular sides are all separated by the union of triangular sides, the sides are either level 0 or level 1, and hence they satisfy the trivalent condition. A dodecahedron would not satisfy the condition.

If P is compact, then Marquis [125] showed that P is a truncation polytope; that is, one starts from a tetrahedron and cut a neighborhood of a vertex so as to change the combinatorial type near that vertex only. Many of these can be realized as a compact hyperbolic polytope with dihedral angle submultiples of π by the Andreev theorem [140]. If P is not compact, we do not have the classification. Also, infinitely many of these can be realized as a complete hyperbolic polytope with dihedral angles that are submultiples of π. (D. Choudhury was first to show this.)

DEFINITION 2.2.4. We denote by $k(P)$ the dimension of the projective group acting on a convex polyhedron P.

The dimension $k(P)$ of the subgroup of G acting on P equals 3 if P is a tetrahedron and equals 1 if P is a cone with base a convex polyhedron which is not a triangle. Otherwise, $k(P) = 0$.

THEOREM 2.2.5. *Let P be a convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P)$ be the dimension of the group of projective automorphisms acting on P. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.*

If we start from a complete hyperbolic polytope with dihedral angles that are submultiples of π, we know that the restricted deformation space is not empty.

If we assume that P is compact, then we refer to Marquis [125] for the complete theory. However, the topic is not within the scope of this monograph.

DEFINITION 2.2.6. Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. Suppose that t is the corresponding hyperbolic point of $\mathfrak{D}_P(\hat{P})$. We call a neighborhood of t in $\mathfrak{D}_P(\hat{P})$ the *local restricted deformation space* of P. We say that $\hat{P}$ is *projectively deformable relative to the mirrors*, or simply *deforms rel mirrors*, if the dimension of its local restricted deformation space is positive. Conversely, we say that $\hat{P}$ is *projectively rigid relative to the mirrors*, or *rigid rel mirrors*, if the dimension of its local restricted deformation space is 0.

The following theorem describes the local restricted deformation space for a class of Coxeter orbifolds arising from *ideal hyperbolic polyhedra*, i.e. polyhedra with all vertices on the sphere at infinity.

THEOREM 2.2.7 (Choi-Hodgson-Lee [58]). *Let P be an ideal 3-dimensional hyperbolic polyhedron whose dihedral angles are all equal to $\pi/3$, and suppose that $\hat{P}$ is given its Coxeter orbifold structure. If P is not a tetrahedron, then a neighborhood of the hyperbolic point in $\mathfrak{D}_P(\hat{P})$ is a smooth 6-dimensional manifold.*

The main ideas in the proof of Theorem 2.2.7 are as follows. We first show that $\mathfrak{D}_P(\hat{P})$ is isomorphic to the solution set of a system of polynomial equations following ideas of Vinberg [153] and Choi [50]. Since the faces of P are fixed, each projective reflection in a face of the polyhedron is determined by a reflection vector b_i. We then compute the Jacobian matrix of the equations for the b_i at the hyperbolic point. This reveals that the matrix has exactly the same rank as the Jacobian matrix of the equations for the Lorentzian unit normals of a hyperbolic polyhedron with the given dihedral angles. By the infinitesimal rigidity of the hyperbolic structure on $\hat{P}$, this matrix is of full rank and has the kernel of dimension six; the result then follows from the implicit function theorem. In fact, we can interpret the infinitesimal projective deformations as applying infinitesimal hyperbolic isometries to the reflection vectors

We can generalize the above theorem slightly as Hodgson pointed out.

DEFINITION 2.2.8. Given a hyperbolic n-orbifold X with totally geodesic boundary component diffeomorphic to an $(n-1)$-orbifold Σ. Let $\tilde{X}$ denote the universal cover in the Klein model $\mathbb{B}$ in $\mathbb{S}^n$. Let Γ be the group of deck transformations considered as projective automorphisms of $\mathbb{S}^n$. Then a complete hyperbolic hyperspace $\tilde{\Sigma}$ covers Σ. Every component of the inverse image of Σ is of form $g(\tilde{\Sigma})$ for $g \in \pi_1(X)$. A point $\mathrm{v}_{\tilde{\Sigma}} \in \mathbb{S}^n - \mathbb{B} - \mathscr{A}(\mathbb{B})$ is projectively dual to the hyperspace containing $\tilde{\Sigma}$ with respect to the bilinear form B. (See Section 3.1.4.) Then we form the join $C := \{\mathrm{v}_{\tilde{\Sigma}}\} * \tilde{\Sigma} - \{\mathrm{v}_{\tilde{\Sigma}}\}$. Then we form $\hat{C} := X \cup \bigcup_{g \in \Gamma} g(C)$. $\hat{C}/\Gamma$ is an n-orbifold with radial ends. We call the ends the *hyperideal ends*.

A point of $\mathfrak{D}_P(\hat{P})$ corresponding to a hyperbolic n-orbifold with hyperideal ends added will be called a *hyperbolic point* again. An 3-*dimensional hyperbolic polyhedron with possibly hyperideal vertices* is a compact convex polyhedron with vertices outside $\mathbb{B}$ removed where no interior of a 1-dimensional edge is outside $\mathbb{B}$. We will generalize this further in Section 3.2.1.

COROLLARY 2.2.9 (Choi-Hodgson-Lee). *Let P be an ideal 3-dimensional hyperbolic polyhedron with possibly hyperideal vertices whose dihedral angles are of form π/p for integers $p \geq 3$, and suppose that $\hat{P}$ is given its Coxeter orbifold structure. If P is not a tetrahedron, then a neighborhood of the hyperbolic point in $\mathfrak{D}_P(\hat{P})$ is a smooth 6-dimensional manifold.*

We did not give proof for the case when some edges orders are greater than equal to 4 in the article [58]. We can allow any of our end orbifold to be a (p,q,r)-triangle reflection orbifold for $p,q,r \geq 3$. The same proof will apply as first observed by Hodgson: We modify the proof of Theorem 1 of the article in Section 3.3 of [58]. Let $\partial_\infty \hat{P}$ denote the union of end orbifolds of $\hat{P}$ which are orbifolds based on 2-sphere with singularities admiting either a Euclidean or hyperbolic structures. Let $h: \pi_1(\hat{P}) \to \mathsf{PO}(3,1) \subset \mathsf{PGL}(4,\mathbb{R})$ denote the holonomy homomorphism associated with the convex real projective structure induced from the hyperbolic structure. We just need to show

$$H^1(\hat{P}, so(3,1)_{Ad_h}) = 0, H^1(\partial_\infty P, so(3,1)_{Ad_h}) = 0.$$

Recall that a (p,q,r)-triangle reflection orbifold for $1/p+1/q+1/r < 1$ has a rigid hyperbolic and conformal structure. By Corollary 2 of [146], the representation to $\mathsf{PO}(3,1)$ is rigid. The first part of the equation follows. The second part also follows by Corollary 2 of [146]. These examples are convex by the work of Vinberg [153]. Corollary 12.1.4 implies the proper convexity.

We comment that we are using Theorem 7 (Sullivan rigidity) of [146] as the generalization of the Garland-Raghunathan-Weil rigidity [83] [154].

2.2.1. Vertex orderable Coxeter orbifolds.

2.2.1.1. *Vinberg theory.* Let $\hat{P}$ be a Coxeter orbifold of dimension n. Let P be the fundamental convex polytope of $\hat{P}$. The reflection is given by a point, called a reflection point, and a hyperplane. Let R_i be a projective reflection on a hyperspace S_i containing a side of P. Then we can write

$$R_i := \mathrm{I} - \alpha_i \otimes \vec{v}_i$$

where α_i is zero on S_i and $\vec{v}_i$ is the reflection vector and $\alpha_i(\vec{v}_i) = 2$.

Given a reflection group Γ. We form a Cartan matrix $A(\Gamma)$ given by $a_{ij} := \alpha_i(\vec{v}_j)$. Vinberg [153] proved that the following conditions are necessary and sufficient for Γ to be a linear Coxeter group:

(C1) $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.
(C2) $a_{ii} = 2$;and
(C3) for $i \neq j$, $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2(\frac{\pi}{n_{ij}})$ an integer n_{ij}.

The Cartan matrix is a $f \times f$-matrix when P has f sides. Also, $a_{ij} = a_{ji}$ for all i,j if Γ is conjugate to a reflection group in $O^+(1,n)$. This condition is the condition of $\hat{P}$ to be a hyperbolic Coxeter orbifold.

The Cartan matrix is determined only up to an action of the group $D_{f,f}$ of nonsingular diagonal matrices:

$$A(\Gamma) \to DA(\Gamma)D^{-1} \text{ for } D \in D_{f,f}.$$

This is due to the ambiguity of choices

$$\alpha_i \mapsto c_i\alpha_i, \vec{v}_i \mapsto \frac{1}{c_i}\vec{v}_i, c_i > 0.$$

Vinberg showed that the set of all *cyclic invariants* of form $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ri_1}$ classifies the isomorphic linear Coxeter group generated by reflections up to the conjugation.

2.2.1.2. *The classification of convex real projective structures on triangular reflection orbifolds.* We will follow Kac-Vinberg [107]. Let $\hat{T}$ be a 2-dimensional Coxeter orbifold based on a triangle T. Let the edges of T be silvered. Let the vertices be given orders p,q,r where $1/p+1/q+1/r \leq 1$. If $1/p+1/q+1/r \leq 1$, then the universal cover $\tilde{T}$ of $\hat{T}$ is a properly convex domain or a complete affine plane by Vinberg [153]. We can find the topology of $\mathfrak{D}(\hat{T})$ as Goldman did in his senior thesis [85]. We may put T as a standard triangle with vertices $\vec{e}_1 := [1,0,0], \vec{e}_2 := [0,1,0], \vec{e}_3 := [0,0,1]$.

Let R_i be the reflection on a line containing $[\vec{e}_{i-1}], [\vec{e}_{i+1}]$ and with a reflection vertex $[\vec{v}_i]$. Let α_i denote the linear function on $\mathbb{R}^3$ taking zero values on $\vec{e}_{i-1}$ and $\vec{e}_{i+1}$. We choose $\vec{v}_i$ to satisfy $\alpha_i(\vec{v}_i) = 2$.

When $1/p+1/q+1/r = 1$, the triangular orbifold admits a compatible Euclidean structure. When $1/p+1/q+1/r < 1$, the triangular orbifold admits a hyperbolic structure not necessarily compatible with the real projective structure.

A linear Coxeter group Γ is hyperbolic if and only if the Cartan matrix A of Γ is indecomposable, of negative type, and equivalent to a symmetric matrix of signature $(1,n)$.

Assume that no p,q,r is 2 and $1/p+1/q+1/r < 1$. Let a_{ij} denote the entries of the Cartan matrix. It satisfies

$$a_{12}a_{21} = 4\cos^2\pi/p, a_{23}a_{32} = 4\cos^2\pi/q, a_{13}a_{31} = 4\cos^2\pi/r.$$

There are only two cyclic invariants $a_{12}a_{23}a_{31}$ and $a_{13}a_{32}a_{21}$ satisfying

$$a_{12}a_{23}a_{31}a_{13}a_{32}a_{21} = 64\cos^2\pi/p\cos^2\pi/q\cos^2\pi/r.$$

Then the *triple invariant* $a_{12}a_{23}a_{31} \in \mathbb{R}_+$ classifies the conjugacy classes of Γ. A single point of $\mathbb{R}_+$ corresponds to a hyperbolic structure. For different points, they are properly convex by [56].

Since $a_{ij} = a_{ji}$ for geometric cases, we obtain that

$$a_{12}a_{23}a_{31} = 2^3\cos(\pi/p)\cos(\pi/q)\cos(\pi/r)$$

gives the unique hyperbolic points.

We define for this orbifold $\mathfrak{D}(\hat{T}) := \mathbb{R}_+$ the space of the triple invariants. A unique point correspond to a Euclidean or hyperbolic structure.

EXAMPLE 2.2.10 (Lee's example). Consider the Coxeter orbifold $\hat{P}$ with the underlying space on a polyhedron P with the combinatorics of a cube with all sides mirrored and all edges given order 3 but with vertices removed. By the Mostow-Prasad rigidity and the Andreev theorem, the orbifold has a unique complete hyperbolic structure. There exists a six-dimensional space of real projective structures on it by Theorem 2.2.7 where one has a projectively fixed fundamental domain in the universal cover.

There are eight ideal vertices of P corresponding to eight ends of $\hat{P}$. Each end orbifold is a 2-orbifold based on a triangle with edges mirrored, and vertex orders are all 3. Each end orbifold has a real projective structure and hence is characterized by the triple invariant. Thus, each end has a neighborhood diffeomorphic to the 2-orbifold multiplied by $(0,1)$. The eight triple invariants are related when we are working on the restricted deformation space since the deformation space is only six-dimensional.

2.2.1.3. *The end mappings.* We will give some explicit conjectural class of examples where we can control the end structures. We worked this out with Greene, Gye-Seon Lee, and Marquis starting from the workshop at the ICERM in 2014.

Let $\mathscr{F}$ be the set of faces of C and give a total order $\leqslant$ on $\mathscr{F}$. A face F' is E_2-greater than F if $F < F'$ and $F \cap F'$ is an edge of label 2.

A flexible vertex of a Coxeter orbifold is a vertex of the base polytope where there is no edge of order 2 ending there. Let $\mathscr{V}_f$ be a set of flexible vertices in C, and let $\mathscr{V}$ be a subset of $\mathscr{V}_f$. A face F' is $\mathscr{V}$-greater than F if $F < F'$ and there exists a face F'' such that $F < F''$ and $F \cap F' \cap F''$ is a vertex in $\mathscr{V}$.

A combinatorial polyhedron C is $\mathscr{V}$-orderable if there is no triangular face all vertices of which are in $\mathscr{V}$ and the faces of C can be ordered so that for each face F of C, the number of faces which are E_2-greater and $\mathscr{V}$-greater than F is less than or equal to 3.

Let $\partial_{\mathscr{V}}\mathscr{O}$ denote the disjoint union of end orbifolds corresponding to the set of ideal vertices $\mathscr{V}$.

CONJECTURE 2.2.11 (Choi-Greene-Lee-Marquis [57]). Suppose that P with a set of vertices $\mathscr{V}$ is $\mathscr{V}$-orderable, and P admits a Coxeter orbifold structure with a convex real projective structure. Then the function $\mathfrak{D}(\mathscr{O}) \to \mathfrak{D}(\partial_{\mathscr{V}}\mathscr{O})$ is onto.

A combinatorial polyhedron C is *weakly $\mathscr{V}$-orderable* if there is no triangular face all vertices of which are in $\mathscr{V}$ and the faces of C can be ordered so that for each face F of C, the number of faces which are E_2-greater or $\mathscr{V}$-greater than F is less than or equal to 3. Notice we change the last "and" with "or" from the definition for $\mathscr{V}$-orderable.

CONJECTURE 2.2.12 (Choi-Greene-Lee-Marquis [57]). Suppose that P with a set of vertices V is weakly $\mathscr{V}$-orderable. Suppose P admits a Coxeter orbifold structure with an ideal or hyperideal end structure. Then the function $\mathfrak{D}(\mathscr{O}) \to \mathfrak{D}(\partial_{\mathscr{V}}\mathscr{O})$ is locally surjective at the hyperbolic point.

2.3. Some relevant results

For closed hyperbolic manifolds, the deformation spaces of convex structures on manifolds were extensively studied by Cooper-Long-Thistlethwaite [64] and [65].

2.3.1. The work of Heusener-Porti.

DEFINITION 2.3.1. Let N be a closed hyperbolic manifold of dimension equal to 3. We consider the holonomy representation of N

$$\rho : \pi_1(N) \to \mathsf{PSO}(3,1) \hookrightarrow \mathsf{PGL}(4,\mathbb{R}).$$

A closed hyperbolic three manifold N is called *infinitesimally projectively rigid* if

$$H^1(\pi_1(N), \mathfrak{sl}(4,\mathbb{R})_{\mathrm{Ad}\rho}) = 0.$$

DEFINITION 2.3.2. Let M denote a compact three-manifold with boundary a union of tori and whose interior is hyperbolic with finite volume. M is called *infinitesimally projectively rigid relative to the cusps* if the inclusion $\partial M \to M$ induces an injective homomorphism

$$H^1(\pi_1(M), \mathfrak{sl}(4,\mathbb{R})_{\mathrm{Ad}\rho}) \to H^1(\partial M, \mathfrak{sl}(4,\mathbb{R})_{\mathrm{Ad}\rho}).$$

THEOREM 2.3.3 (Heusener-Porti [99]). *Let M be an orientable 3-manifold whose interior has a complete hyperbolic metric with finite volume. If M is infinitesimally projectively rigid relative to the cusps, then infinitely many Dehn fillings on M are infinitesimally projectively rigid.*

THEOREM 2.3.4 (Heusener-Porti [99]). *Let M be an orientable 3-manifold whose interior has a complete hyperbolic metric of finite volume. If a hyperbolic Dehn filling N on M satisfies*:

(i) *N is infinitesimally projectively rigid,*
(ii) *the Dehn filling slope of N is contained in the (connected) hyperbolic Dehn filling space of M ,*

then infinitely many Dehn fillings on M are infinitesimally projectively rigid.

The complete hyperbolic manifold M that is the complement of a figure-eight knot in $\mathbb{S}^3$ is infinitesimally projectively rigid. Then infinitely many Dehn fillings on M are infinitesimally projectively rigid.

They showed the following:

- For a sufficiently large positive integer k, the homology sphere obtained by $\frac{1}{k}$–Dehn filling on the figure eight knot is infinitesimally not projectively rigid. Since the Fibonacci manifold M_k is a branched cover of $\mathbb{S}^3$ over the figure eight knot complements, for any $k \in N$, the Fibonacci manifold M_k is not projectively rigid.
- All but finitely many punctured torus bundles with tunnel number one are infinitesimally projectively rigid relative to the cusps. All but finitely many twist knots complements are infinitesimally projectively rigid relative to the cusps.

2.3.2. Ballas's work on ends. The following are from Ballas [4] and [5].

- Let M be the complement in $\mathbb{S}^3$ of 4_1 (the figure-eight knot), 5_2, 6_1, or 5_1^2 (the Whitehead link). Then M does not admit strictly convex deformations of its complete hyperbolic structure.
- Let M be the complement of a hyperbolic amphichiral knot, and suppose that M is infinitesimally projectively rigid relative to the boundary and the longitude is a rigid slope. Then for sufficiently large n, there is a one-dimensional family of strictly convex deformations of the complete hyperbolic structure on $M(m/0)$ for $m \in \mathbb{Z}$.
- Let M be the complement in $\mathbb{S}^3$ of the figure-eight knot. There exists ε such that for each $s \in (-\varepsilon, \varepsilon)$, ρ_s is the holonomy of a finite volume properly convex projective structure on M for a parameter ρ_s of representations $\pi_1(M) \to \mathsf{PGL}(4, \mathbb{R})$. Furthermore, when $s \neq 0$, this structure is not strictly convex.

We also note the excellent work of Ballas, Danciger, and Lee [6] experimenting with more of these and finding a method to glue along tori for deformed hyperbolic 3-manifolds to produce convex real projective 3-manifolds that does not admit hyperbolic structures.

2.3.3. Finite volume strictly convex real projective orbifolds with ends. We summarize the main results of two independent groups. The Hilbert metric is a complete Finsler metric on a properly convex set Ω. This is the hyperbolic metric in the Klein model when Ω is projectively diffeomorphic to a standard ball. A simplex with its Hilbert metric is isometric to a normed vector space, and appears in a natural geometry on the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$. A singular version of this metric arises in the study of certain limits of projective structures. The Hilbert metric has a Hausdorff measure and hence a notion of finite volume. (See [67].)

THEOREM 2.3.5 (Choi [45], Cooper-Long-Tillmann [67], Crampon-Marquis [68]). *For each dimension $n \geq 2$ there is a Margulis constant $\mu_n > 0$ with the following property.*

If M is a properly convex projective n-manifold and x is a point in M, then the subgroup of $\pi_1(M,x)$ generated by loops based at x of length less than μ_n is virtually nilpotent. In fact, there is a nilpotent subgroup of index bounded above by $m = m(n)$. Furthermore, if M is strictly convex and finite volume, this nilpotent subgroup is abelian. If M is strictly convex and closed, this nilpotent subgroup is trivial or infinite cyclic.

THEOREM 2.3.6 (Cooper-Long-Tillmann [67], Crampon-Marquis [68]). *Each end of a strictly convex projective manifold or orbifold of finite volume is horospherical.*

THEOREM 2.3.7 ((Relatively hyperbolic). Cooper-Long-Tillmann [67], Crampon–Marquis [68]). *Suppose that $M = \Omega/\Gamma$ is a properly convex manifold of finite volume which is the interior of a compact manifold N, and the holonomy of each component of ∂N is topologically parabolic. Then the following are equivalent*:

1 Ω *is strictly convex,*
2 $\partial\Omega$ *is* C^1,
3 $\pi_1(N)$ *is hyperbolic relative to the subgroups of the boundary components.*

Here, the definition of the term "topologically parabolic" is according to [67]. This is not a Lie group definition but a topological definition. We have found a generalization Theorem 10.3.1 and its converse Theorem 10.3.4 in Chapter 10.

Part 2

The classification of radial and totally geodesic ends.

The purpose of this part is to understand the structures of ends of real projective n-dimensional orbifolds for $n \geq 2$. In particular, we consider the radial or totally geodesic ends. Hyperbolic manifolds with cusps and hyperideal ends are examples. For this, we will study the natural conditions on eigenvalues of holonomy representations of ends when these ends are manageably understandable. This is the most technical part of the monograph containing a large number of results useful in other two parts.

We begin the study of radial ends in Chapter 3. We will divide the class of radial ends into the class of complete affine radial ends, the class of properly convex ends, and the class of convex but not properly convex and non-complete affine ends. We define lens and horospherical conditions for these ends. We give some examples of these radial ends.

In Chapter 4, we study the theory of affine actions. This is the major technical section in this part. We consider the case when there is a discrete affine action of a group Γ acting cocompactly on a properly convex domain Ω in the boundary of the affine subspace $\mathbb{A}^n$ in $\mathbb{RP}^n$ or $\mathbb{S}^n$. We study the convex domain U in an affine space $\mathbb{A}^n$ whose closure meets with $\mathrm{bd}\mathbb{A}^n$ in Ω. We can find a domain U having asymptotic hyperspaces at each point of $\mathrm{bd}\Omega$ if and only if Γ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{bd}\mathbb{A}^n$. To prove, we study the flow on the affine bundle over the unit tangent space over Ω generalizing parts of the work of Goldman-Labourie-Margulis on complete flat Lorentz 3-manifolds [91]. We end with showing that a T-end has a CA-lens neighborhood if it satisfies the uniform middle eigenvalue condition.

In Chapter 5, we study the properly convex R-end theory. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. We explain that the uniform middle eigenvalue condition implies the existence of the distanced action. The main result here is the characterization of R-ends whose end holonomy groups satisfy uniform middle eigenvalue conditions. That is, they are generalized lens-shaped R-ends. We also discuss some important properties of lens-shaped R-ends. Finally, we show that lens-shaped T-ends and lens-shaped R-ends are dual. We end with discussing the properties of T-ends as obtained by this duality.

In Chapter 6, we investigate the applications of the radial end theory such as the stability condition. We discuss the expansion and shrinking of the end neighborhoods. We will show the openness of the lens condition in Theorem 6.1.1, which is one of the central result needed in Part III. We will also prove Theorem 6.0.4, the strong irreducibility of strongly tame properly convex orbifolds with generalized-lens shaped ends or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends.

In major technical Chapter 7, we discuss the R-ends that are NPNC. First, we show that the end holonomy group for an NPNC-end E will have an exact sequence

$$1 \to N \to h(\pi_1(\tilde{E})) \longrightarrow N_K \to 1$$

where N_K is in the projective automorphism group $\mathbf{Aut}(K)$ of a properly convex compact set K, N is the normal subgroup of elements mapping to the trivial automorphism of K, and K^o/N_K is compact. We show that $\Sigma_{\tilde{E}}$ is foliated by complete affine subspaces of dimension ≥ 1. We explain that an NPNC-end satisfying the transverse weak middle eigenvalue condition for NPNC-ends is a quasi-joined R-end under some natural conditions. A quasi-joined end is an end with an end-neighborhood covered by the join of a properly convex action and a horoball action twisted by translations (see Definition 7.1.2.) For virtually abelian groups, Ballas-Cooper-Leitner [8], [9] had covered much of these material but not in our generality.

We will also classify the complete affine ends in the final chapter 8 of this part.

CHAPTER 3

Introduction to the theory of convex radial ends

In Section 3.1, we will discuss the convex radial ends of orbifolds, covering most elementary aspects of the theory. For a properly convex real projective orbifold, the space of rays for each R-end gives us a closed real projective orbifold of dimension $n-1$. The orbifold is convex. The universal cover can be a complete affine subspace (CA) or a properly convex domain (PC) or a convex domain that is neither (NPNC). We discuss objects associated with R-ends, and examples of ends; horospherical ones, totally geodesic ones, and bendings of ends to obtain more general examples of ends.

In Section 3.2, we discuss some examples of these.

3.1. End structures

3.1.1. End fundamental groups. Let $\mathscr{O}$ be a strongly tame real projective orbifold with the universal cover $\tilde{\mathscr{O}}$ and the covering map $p_{\tilde{\mathscr{O}}}$. A compact smooth orbifold $\bar{\mathscr{O}}$ whose interior is $\mathscr{O}$ is called a compactification of $\mathscr{O}$. There might be more than one compactifications. A strongly tame orbifold $\mathscr{O}$ in our paper always will come with a *compactification* $\bar{\mathscr{O}}$ which is a smooth orbifold with boundary. When we say $\mathscr{O}$, we really mean $\mathscr{O}$ with $\bar{\mathscr{O}}$. Each boundary component of $\bar{\mathscr{O}}$ is the *ideal boundary component* of $\mathscr{O}$ and is an *end* of $\mathscr{O}$.

An *end neighborhood* U of $\mathscr{O}$ is an open set U where $\Sigma_E \cup U$ forms a neighborhood of an ideal boundary component Σ_E corresponding to an end E.

Let $\hat{\mathscr{O}}$ denote the universal cover or $\bar{\mathscr{O}}$ with the covering map $\hat{p}_{\tilde{\mathscr{O}}}$. Let Γ be the deck transformation group of $\hat{\mathscr{O}} \to \bar{\mathscr{O}}$ which also restricts to the deck transformation group of $\tilde{\mathscr{O}} \to \mathscr{O}$.

Each end neighborhood U, diffeomorphic to $S_E \times (0,1)$ for an $(n-1)$-orbifold S_E, of an end E lifts to a connected open set $\tilde{U}$ in $\tilde{\mathscr{O}}$. We choose U and the diffeomorphism $f_U\ U \to S_E \times (0,1)$ that $S_E \times (0,1]$ is also diffeomorphic to a tubular neighborhood of a boundary component of $\bar{\mathscr{O}}$ corresponding to U. A subgroup $\mathbf{\Gamma}_{\tilde{U}}$ of Γ acts on $\tilde{U}$ where

$$p_{\tilde{\mathscr{O}}}^{-1}(U) = \bigcup_{g\in\pi_1(\mathscr{O})} g(\tilde{U}).$$

Each component $\tilde{U}$ is said to be a *proper pseudo-end neighborhood*.

- An *super-exiting sequence* of sets $U_1, U_2, \cdots$ in $\tilde{\mathscr{O}}$ is a sequence so that for each compact subset K of $\mathscr{O}$ there exists an integer N satisfying $p_{\mathscr{O}}^{-1}(K) \cap U_i = \emptyset$ for $i > N$.
- A *pseudo-end neighborhood sequence* is a super-exiting sequence of proper pseudo-end neighborhoods

$$\{U_i | i = 1,2,3,\dots\}, \text{ where } U_{i+1} \subset U_i \text{ for every } i.$$

- Two pseudo-end sequences $\{U_i\}$ and $\{V_j\}$ are *compatible* if for each i, there exists J such that $V_j \subset U_i$ for every j, $j > J$ and conversely for each j, there exists I such that $U_i \subset V_j$ for every i, $i > I$.
- A compatibility class of a proper pseudo-end sequence is called a *p-end* of $\tilde{\mathscr{O}}$. Each of these corresponds to an end of $\mathscr{O}$ under the universal covering map $p_{\mathscr{O}}$.
- For a pseudo-end $\tilde{E}$ of $\tilde{\mathscr{O}}$, we denote by $\mathbf{\Gamma}_{\tilde{E}}$ the subgroup $\mathbf{\Gamma}_{\tilde{U}}$ where U and $\tilde{U}$ is as above. We call $\mathbf{\Gamma}_{\tilde{E}}$ a *pseudo-end fundamental group*. We will also denote it by $\pi_1(\tilde{E})$. They are independent of the choice of U up to natural canonical inclusion homomorphisms by following Proposition 3.1.1.
- A *pseudo-end neighborhood* U of a pseudo-end $\tilde{E}$ is a $\mathbf{\Gamma}_{\tilde{E}}$-invariant open set containing a proper pseudo-end neighborhood of $\tilde{E}$. A proper pseudo-end neighborhood is an example.

(From now on, we will replace "pseudo-end" with the abbreviation "p-end".)

As a summary, the set of boundary components of $\bar{\mathscr{O}}$ has a one-to-one correspondence with the set of p-ends of $\mathscr{O}$.

PROPOSITION 3.1.1. *Let $\tilde{E}$ be a p-end of a strongly tame orbifold $\mathscr{O}$. The p-end fundamental group $\mathbf{\Gamma}_{\tilde{E}}$ of $\tilde{E}$ is independent of the choice of U.*

PROOF. Given U and U' that are end-neighborhoods for an end E, let $\tilde{U}$ and $\tilde{U}'$ be p-end neighborhoods for a p-end $\tilde{E}$ that are components of $p^{-1}(U)$ and $p^{-1}(U')$ respectively. Let $\tilde{U}''$ be the component of $p^{-1}(U'')$ that is a p-end neighborhood of $\tilde{E}$. Then $\mathbf{\Gamma}_{\tilde{U}''}$ injects into $\mathbf{\Gamma}_{\tilde{U}}$ since both are subgroups of Γ. Any $\mathscr{G}$-path in U in the sense of Bridson-Haefliger [32] is homotopic to a $\mathscr{G}$-path in U'' by a translation in the I-factor. Thus, $\pi_1(U'') \to \pi_1(U)$ is surjective. Since $\tilde{U}$ is connected, any element γ of $\mathbf{\Gamma}_{\tilde{U}}$ is represented by a $\mathscr{G}$-path connecting x_0 to $\gamma(x_0)$. (See Example 3.7 in Chapter III.$\mathscr{G}$ of [32].) Thus, $\mathbf{\Gamma}_{\tilde{U}}$ is isomorphic to the image of $\pi_1(U) \to \pi_1(\mathscr{O})$. Since $\mathbf{\Gamma}_{\tilde{U}''}$ is surjective to the image of $\pi_1(U'') \to \pi_1(\mathscr{O})$, it follows that $\mathbf{\Gamma}_{\tilde{U}''}$ is isomorphic to $\mathbf{\Gamma}_{\tilde{U}}$ and $\Gamma_{\tilde{U}'}$. $\square$

3.1.2. Totally geodesic ends. Suppose that an end E of a real projective orbifold $\mathscr{O}$ of dimension $n \geq 2$ satisfies the following:

- The end has an end neighborhood homeomorphic to a closed connected $(n-1)$-dimensional orbifold B times a half-open interval $(0,1)$.
- The end neighborhood completes to an orbifold U' diffeomorphic to $B \times (0,1]$ in the compactification orbifold $\bar{\mathscr{O}}$. Here, U' is an end-neighborhood of E *compatible* with $\bar{\mathscr{O}}$. This is the *compatiblity condition* with the compactifiction $\bar{\mathscr{O}}$. (We assumed that $\mathscr{O}$ always comes as the interior of some compact manifold $\mathscr{O}'$ with a diffeomorphism to $\bar{\mathscr{O}}$ here where the the diffeomorphism restricted to $\mathscr{O}$ is isotopic to the identity.)
- The subset of U' corresponding to $B \times \{1\}$ is the ideal boundary component.
- Each point of the added boundary component has a neighborhood projectively diffeomorphic to the quotient orbifold of an open set V in an affine half-space P so that $V \cap \partial P \neq \emptyset$ by a projective action of a finite group. This implies that the developing map extends to the universal cover of the orbifold with U' attached.

The completion is called a *compactified end neighborhood* of the end E. The boundary component S_E is called the *ideal boundary component* of the end. Such ideal boundary components may not be uniquely determined as there are two projectively nonequivalent ways to add boundary components of elementary annuli (see Section 1.4 of [44]). Two compactified end neighborhoods of an end are *equivalent* if the end neighborhood contains a common end neighborhood whose compactification projectively embed into the compactified end neighborhoods. (See Definition 9.1.1 for more detail.) The equivalence class of compactified end-neighborhoods is called a *totally geodesic end structure* (*T-end structure*) for an end E.

We also define as follows:

- The equivalence class of the chosen compactified end neighborhood is called a *totally geodesic end-structure* of the totally geodesic end. The choice of the end structure is equivalent to the choice of the ideal boundary component.
- We will also call the ideal boundary S_E the *end orbifold* (or *end ideal boundary component*) of the end.

$\mathbb{RP}^n$ has a Riemannian metric of constant curvature called the Fubini-Study metric. Recall that the universal cover $\tilde{\mathscr{O}}$ of $\mathscr{O}$ has a path-metric induced by $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{RP}^n$. We can Cauchy complete $\tilde{\mathscr{O}}$ of this path-metric. The Cauchy completion is called the *Kuiper completion* of $\tilde{\mathscr{O}}$. (See [46].) Note we may sometimes use a lift $\mathbf{dev} : \tilde{\mathscr{O}} \to \mathbb{S}^n$ lifting the developing map and use the same notation.

A *T-end* is an end equipped with a T-end structure. A *T-p-end* is a p-end $\tilde{E}$ corresponding to a T-end E. There is a totally geodesic $(n-1)$-dimensional domain $\tilde{S}_{\tilde{E}}$ in the Cauchy completion of $\tilde{\mathscr{O}}$ in the closure of a p-end neighborhood of $\tilde{E}$. Of course, $\tilde{S}_{\tilde{E}}$ covers S_E. We call $\tilde{S}_{\tilde{E}}$ the p-end ideal boundary component. We will identify it with a domain in a hyperspace in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) when $\mathbf{dev}$ is a fixed map to $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$).

DEFINITION 3.1.2. A *lens* is a properly convex domain L in $\mathbb{RP}^n$ so that ∂L is a union of two smooth strictly convex open disks. A properly convex domain L is a *generalized lens* if ∂L is a union of two open disks, one of which is strictly convex and smooth and the other is allowed to be just a topological disk. A *lens-orbifold* (or *lens*) is a compact quotient orbifold of a lens by a properly discontinuous action of a projective group Γ acting on each boundary component as well. Also, the domains or an orbifold projectively diffeomorphic to a lens or lens-orbifolds are called *lens*.

> **(Lens condition for T-ends):** The ideal boundary is identified by as a totally geodesic suborbifold in the interior of a lens-orbifold in the ambient real projective orbifold containing $\bar{\mathscr{O}}$ where f is a map from a neighborhood of the ideal boundary to a one-sided neighborhood in the lens-orbifold of the image.

If the lens condition is satisfied for a T-end, we will call it the *lens-shaped T-end*. The intersection of a lens with $\mathscr{O}$ is called a *lens end neighborhood* of the T-end. A corresponding T-p-end is said to be a *lens-shaped T-p-end*.

In these cases, $\tilde{S}_{\tilde{E}}$ is a properly convex $(n-1)$-dimensional domain, and S_E is a $(n-1)$-dimensional properly convex real projective orbifold. We will call the cover L of a lens orbifold containing S_E the CA-lens of $\tilde{S}_{\tilde{E}}$ where we assume that $\pi_1(\tilde{E})$ acts properly and cocompactly on the lens. $L \cap \tilde{\mathscr{O}}$ is said to be *lens p-end neighborhood* of $\tilde{E}$ or $\tilde{S}_{\tilde{E}}$.

We remark that for each component $\partial_i L$ for $i = 1,2$ of L, $\partial_i L/\Gamma$ is compact and both are homotopy equivalent up to a virtual manifold cover L/Γ' of L/Γ for a finite index subgroup Γ'. Also, the ideal boundary component of L/Γ' has the same homotopy type as L/Γ' and is a compact manifold. (See Selberg's Theorem 1.1.19.)

3.1.2.1. *p-end ideal boundary components.* We recall Section 3.1.2. Let E be an end of a strongly tame real projective orbifold $\mathscr{O}$. Given a totally geodesic end of $\mathscr{O}$ and an end neighborhood U diffeomorphic to $S_E \times [0,1)$ with an end-completion by a totally geodesic orbifold S_E, we take a component U_1 of $p^{-1}(U)$ and a convex domain $\tilde{S}_{\tilde{E}}$, the ideal boundary component, developing to a totally geodesic hypersurface under $\mathbf{dev}$. Here

$\tilde{E}$ is the p-end corresponding to E and U_1. There exists a subgroup $\mathbf{\Gamma}_{\tilde{E}}$ acting on $\tilde{S}_{\tilde{E}}$. Again $S_{\tilde{E}} := \tilde{S}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is projectively diffeomorphic to the *end orbifold* to be denote by S_E or $S_{\tilde{E}}$.

- We call $\tilde{S}_{\tilde{E}}$ a p-end ideal boundary component of $\tilde{\mathscr{O}}$.
- We call $S_{\tilde{E}}$ an ideal boundary component of $\mathscr{O}$.

We may regard $\tilde{S}_{\tilde{E}}$ as a domain in a hyperspace in $\mathbb{RP}^n$ or $\mathbb{S}^n$.

3.1.3. Radial ends. A *segment* is a convex arc in a 1-dimensional subspace of $\mathbb{RP}^n$ or $\mathbb{S}^n$. We will denote the closed segment by $\overline{xy}$ if x and y are endpoints. It is uniquely determined by x and y if x and y are not antipodal. In the following, all the sets are required to be inside an affine subspace $\mathbb{A}^n$ and its closure to be either in $\mathbb{RP}^n$ or $\mathbb{S}^n$.

Let $\tilde{\mathscr{O}}$ denote the universal cover of $\mathscr{O}$ with the developing map **dev**. Suppose that an end E of a real projective orbifold satisfies the following:

- The end has an end neighborhood U foliated by properly embedded projective geodesics.
- Choose any map $f : \mathbb{R}\times[0,1] \to \mathscr{O}$ so that $f|\mathbb{R}\times\{t\}$ for each t is a geodesic leaf of such a foliation of U. Then f lifts to $\tilde{f} : \mathbb{R}\times[0,1] \to \tilde{\mathscr{O}}$ where $\mathbf{dev}\circ\tilde{f}|\mathbb{R}\times\{t\}$ for each $t, t\in[0,1]$, maps to a geodesic in $\mathbb{RP}^n$ ending at a point of concurrency common for every t.

The foliation is called a *radial foliation* and leaves *radial lines* of E. Two such radial foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ of radial end neighborhoods of an end are *equivalent* if the restrictions of $\mathscr{F}_1$ and $\mathscr{F}_2$ in an end neighborhood agree. A *radial end structure* is an equivalence class of radial foliations.

Remember that $\mathscr{O}$ always comes with a smooth compact orbifold $\bar{\mathscr{O}}$ with boundary so that $\mathscr{O}$ is its boundary. We will fix a radial end structure for each end of $\mathscr{O}$ coming from a smooth foliation whose leaves end transversely to the boundary of $\bar{\mathscr{O}}$. This is the *compatibility condition* of the R-end structure to $\bar{\mathscr{O}}$.

To explain further, an end neighborhood U is *compatible* to $\bar{\mathscr{O}}$ if it a product form $\Sigma_E\times(0,1)$ where each $\Sigma_E\times\{t\}$ is transverse to the radial foliation for sufficiently small t and the diffeomorphism $f : U \to \Sigma_E\times(0,1)$ extends to U union the ideal boundary component corresponding to E as a diffeomorphism to $\Sigma_E\times(0,1]$.

A *R-end* is an end with a radial end structure. A *R-p-end* is a p-end with a p-end neighborhood covering a radial end neighborhood with induced foliation. Each lift of the radial foliation has a finite path-length induced from **dev**. A *pseudo-end* (*p-end*) vertex of a radial p-end neighborhood or a radial p-end is the common endpoint of concurrent lift of leaves of the radial foliation, which we obtain by Cauchy completion along the leaves. Note that **dev** always extends to the pseudo-end vertex. The p-end vertex is defined independently of the choice of **dev**. We will identify with a point of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) if **dev** is an embedding to $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$).

(See Definition 9.1.1 for more detail.)

REMARK 3.1.3. (End-compactification structures) If we have a compactification $\bar{\mathscr{O}}'$ of $\mathscr{O}$ not diffeomorphic to $\bar{\mathscr{O}}$, and choose $\bar{\mathscr{O}}'$ instead of $\bar{\mathscr{O}}$, all these discussions have to take place with respect to $\bar{\mathscr{O}}'$. By the s-cobordism theorem of Mazur [**128**], Barden [**12**] and Stallings and the existence theorem 11.1 of Milnor [**130**], there are tame manifolds with more than one compactifications. (This is due to Benoit Kloeckner in the mathematics overflow. See also Section 9.3.) However, notice that the radial structure determine the diffeomorphism type of $\bar{\mathscr{O}}$ since each flow line determines the unique boundary points and the set of flow lines passing a codimension-one transversal ball determines the diffeomorphism types. However, the converse is true up to isotopies by the isotopy uniqueness part of the tubular neighborhood theorem, which also holds for orbifolds (See [**?**].)

Two rays l and m with some arclength-parameterizations in Ω is *asymptotic* if

$$d_\Omega(l(t), m(t)) < C \text{ for a constant } C \text{ for all } t \geq 0$$

(See Section 3.11.3 of [74]).

problem.. I keep switching between length parameterization and otherwise.. Wierd.. CHeck this..

LEMMA 3.1.4 (Benoist [22]). *Let l be a line in a properly convex open domain Ω in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$), $n \geq 2$, ending at $x \in \mathrm{bd}\Omega$. Let m be a line ending at x also. Then for a parametrization $l(t)$ of l there is a parametrization $m(t)$ of m so that $d_\Omega(m(t), l(t)) < C$ for a constant C independent of t. Furthermore, m and l are asymptotic rays.*

PROOF. We will prove for $\mathbb{S}^n$. We choose a supporting hyperplane P at x. Then $P \cap \mathrm{Cl}(\Omega)$ is a properly convex domain. We choose a codimension-one subspace Q of P disjoint from $P \cap \mathrm{Cl}(\Omega)$ and a parameter of hyperplanes P_t passing $l(t)$ and containing Q. We denote $m(t) = m \cap P_t$. For convenience, we may suppose our interval is $[0, 1)$ and that $l(0)$ and $m(0)$ are the beginning point of l and m. Let J denote the 2-dimensional subspace containing l and m. Now, $m(t), l(t)$ are on a line $P_t \cap J$. The function $t \mapsto d_\Omega(l(t), m(t))$ is eventually decreasing by the convexity of the 2-dimensional domain $\Omega \cap J$ since we can draw four segments from x to $l(t), m(t)$ and the endpoints of $\overline{m(t)l(t)} \cap \Omega$ and the segments to the endpoints always moves outward and the the segments to $l(t), m(t)$ are constant. Hence, $d_\Omega(l(t), m(t)) \leq C' d_\Omega(l(0), m(0))$ for a constant $C' \geq 1$. (See also Section 3.2.6 and 3.2.7 of [22] For eventual decreasing property, we don't need the C^1-boundary property of Ω.)

Finally, we may use the arclength parameterizations by taking discrete equidistantly places elements in l and m respectively and a triangle inequality argument: We can show that the parameterizations are related by constants. Then we increase the intervals. [$\mathbb{S}^n$S] ☐

Following Lemma 3.1.5 gives us another characterization of R-end and the condition $R_x(\tilde{U}) = R_x(\Omega)$.

LEMMA 3.1.5. *Let Ω be a properly convex open domain in $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$), $n \geq 2$. Suppose that $\mathscr{O} = \Omega/\Gamma$ is a noncompact strongly tame orbifold. Let U be a proper end neighborhood and let $\tilde{U}$ be a connected open set in Ω covering U. Let $\Gamma_{\tilde{U}}$ denote the subgroup of Γ acting on $\tilde{U}$. Suppose that $\tilde{U}$ is foliated by segments with a common endpoint x in $\mathrm{bd}\Omega$. Suppose that $\Gamma_{\tilde{U}}$ fixes x. Then the following hold*:

- *$\Gamma_{\tilde{U}}$ acts properly on $R_x(\tilde{U})$ if and only if every radial ray in $\tilde{U}$ ending at x maps to a properly embedded arc in U.*
- *If the above item holds, then $R_x(\Omega) = R_x(\tilde{U})$ and x is an R-p-end vertex of $\tilde{\mathscr{O}}$.*

PROOF. It is sufficient to prove for $\mathbb{S}^n$. The forward direction of the first item is clear: If a leaf l does not embedd properly, then there exists a sequence $g_i \in \Gamma_{\tilde{U}}$ so that the direction of $g_i(l)$ accumulates to a point of $R_x(\tilde{U})$. The properness of the action of $\Gamma_{\tilde{U}}$ contradicts this.

For converse, suppose that $g_i(K) \cap K \neq \emptyset$ for infinitely many mutually distinct $g_i \in \Gamma_{\tilde{U}}$ for a compact set $K \subset R_x(\tilde{U})$. Then there exists a sequence $p_i \in K$ so that $g_i(p_i) \to p_\infty$, $p_\infty \in K$. We can choose a compact set $\hat{K} \subset \Omega$ so that the ray l_i ending at x in direction p_i has an endpoint $\hat{p}_i \in \hat{K}_i$ for each i.

We can choose q_i on l_i so that $g_i(q_i) \to q_\infty$ for q_∞ in Ω since the direction of $g_i(l_i)$ is in K and its endpoint is uniformly bounded away from x.

By Lemma 3.1.4, we can choose a point $\bar{q}_i$ on l_1 so that $d_\Omega(q_i, \bar{q}_i) < C$ for a uniform constant C. Thus, $d_\Omega(g_i(q_i), g_i(\bar{q}_i)) < C$. We can choose a subsequence so that $g_i(\bar{q}_i) \to q'_\infty$ for a point q'_∞ in Ω with $d_\Omega(q_\infty, q'_\infty) \leq C$. This implies that l_1 is not properly embedded in U. This is a contradiction.

For the second item, we have $R_x(\tilde{U}) \subset R_x(\Omega)$ clearly. Let l be a line from x in U, and let m be any line from x in Ω. For a parametrization of l by $[0,1)$, we obtain $d_\Omega(l(t), m(t)) < C, t \in [0,1)$, for a uniform constant $C > 0$ and a parameterization $m(t)$ of m by Lemma 3.1.4. Since l maps to a properly embedded arc in U, and $\mathrm{bd}_{\mathscr{O}} U$ is compact, it follows that $d_\Omega(l(t), \mathrm{bd}\tilde{U} \cap \Omega) \to \infty$ as $t \to 1$. This implies that $m(t) \in \tilde{U}$ for sufficiently large t. Therefore m has a direction in $R_x(\tilde{U})$. Hence, we showed $R_x(\tilde{U}) = R_x(\Omega)$. [$\mathbb{S}^n$T] □

Let Ω be a properly convex domain in $\mathbb{RP}^n$ so that $\mathscr{O} = \Omega/\Gamma$ for a discrete subgroup Γ of automorphisms of Ω. The space of radial lines in an R-end lifts to a space $R_x(\Omega)$ of lines in Ω ending at a point x of $\mathrm{bd}\Omega$. By above Lemma 3.1.5, Γ_x acts properly on $R_x(\Omega)$ since we assume that we have radial ends only. The quotient space $R_x(\Omega)/\Gamma_x$ has an $(n-1)$-orbifold structure by Lemma 3.1.5. The *end orbifold* Σ_E associated with an R-end is defined as the space of radial lines in $\mathscr{O}$. It is clear that Σ_E can be identified with $R_x(\Omega)/\Gamma_x$. By the compatibility condition from the beginning of Section 3.1.3, Σ_E is diffeomorphic to the component of $\bar{\mathscr{O}} - \mathscr{O}$ corresponding to E. The space of radial lines in an R-end has the local structure of $\mathbb{RP}^{n-1}$ since we can lift a local neighborhood to $\tilde{\bar{\mathscr{O}}}$, and these radial lines lift to lines developing into concurrent lines. The end orbifold has an induced real projective structure of one dimension lower.

shorten here...

For the following, we may assume that all subsets here are bounded subsets of an affine subspace $\mathbb{A}^n$.

- An n-dimensional submanifold L of $\mathbb{A}^n$ is said to be a *pre-horoball* if it is strictly convex, and the boundary ∂L is diffeomorphic to $\mathbb{R}^{n-1}$ and $\mathrm{bd}L - \partial L$ is a single point. The boundary ∂L is said to be a *pre-horosphere*.
- Recall that an n-dimensional subdomain L of $\mathbb{A}^n$ is a lens if L is a convex domain and ∂L is a disjoint union of two smoothly strictly convex embedded open $(n-1)$-cells $\partial_+ L$ and $\partial_- L$.
- A *cone* is a bounded domain D in an affine patch with a point in the boundary, called an *end vertex* v so that every other point $x \in D$ has an open segment $\overline{vx}^o \subset D$. A *trivial one-dimensional cone* is an open half-space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$. A cone D is a *join* $\{v\} * A$ for a subset A of D if D is a union of segments starting from v and ending at A. (See Definition 1.1.2.)
- The cone $\{p\} * L$ over a lens-shaped domain L in $\mathbb{A}^n$, $p \notin \mathrm{Cl}(L)$ is a *lens-cone* if it is a convex domain and satisfies
  - $\{p\} * L = \{p\} * \partial_+ L$ for one boundary component $\partial_+ L$ of L and
  - every segment from p to $\partial_+ L$ meets the other boundary component $\partial_- L$ of L at a unique point.
- As a consequence, each line segment from p to $\partial_+ L$ is transverse to $\partial_+ L$. L is called the *lens* of the lens-cone. (Here different lenses may give the identical lens-cone.) Also, $\{p\} * L - \{p\}$ is a manifold with boundary $\partial_+ L$.
- Each of two boundary components of L is called a *top* or *bottom* hypersurface depending on whether it is further away from p or not. The top component is denoted by $\partial_+ L$ and the bottom one by $\partial_- L$.

- An n-dimensional subdomain L of $\mathbb{A}^n$ is a *generalized lens* if L is a convex domain and ∂L is a disjoint union of a strictly convex smoothly embedded open $(n-1)$-cell $\partial_- L$ and an embedded open $(n-1)$-cell $\partial_+ L$, which is not necessarily smooth.
- A cone $\{p\} * L$ is said to be a *generalized lens-cone* if
  - $\{p\} * L = \{p\} * \partial_+ L, p \notin \mathrm{Cl}(L)$ is a convex domain for a generalized lens L, and
  - every segment from p to $\partial_+ L$ meets $\partial_- L$ at a unique point.

  A lens-cone will of course be considered a generalized lens-cone.
- We again define the top hypersurface and the bottom one as above. They are denoted by $\partial_+ L$ and $\partial_- L$ respectively. $\partial_+ L$ can be non-smooth; however, $\partial_- L$ is required to be smooth.
- A *totally-geodesic submanifold* is a convex domain in a subspace. A *cone-over* a totally-geodesic submanifold D is a union of all segments with one endpoint a point x not in the subspace spanned by D and the other endpoint in D. We denote it by $\{x\} * D$.

We apply these to ends:

DEFINITION 3.1.6.

Pre-horospherical R-end: An R-p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$ is *pre-horospherical* if it has a pre-horoball in $\tilde{\mathscr{O}}$ as a p-end neighborhood, or equivalently an open p-end neighborhood U in $\tilde{\mathscr{O}}$ so that $\mathrm{bd}U \cap \tilde{\mathscr{O}} = \mathrm{bd}U - \{v\}$ for a boundary fixed point v. $\tilde{E}$ is *pre-horospherical* if it has a pre-horoball in $\tilde{\mathscr{O}}$ as a p-end neighborhood. We require that the radial foliation of $\tilde{E}$ is the one where each leaf ends at v.

Lens-shaped R-end: An R-p-end $\tilde{E}$ is *lens-shaped* (resp. *generalized-lens-shaped*), if it has a p-end neighborhood that is projectively diffeomorphic to the interior of $L * \{v\}$ under **dev** where
- L is a lens (resp. generalized lens) and
- $h(\pi_1(\tilde{E}))$ acts properly and cocompactly on L,

and every leaf of the radial foliation of the p-end neighborhood ends corresponds to a radial segment ending at v. In this case, the image L is said to be a *CA-lens* (resp. *gCA-lens*) of such a p-end. A p-end end neighborhood of $\tilde{E}$ is (*generalized*) *lens-shaped* if it is a (*generalized*) *lens-cone* p-end neighborhood of $\tilde{E}$.

An R-end of $\mathscr{O}$ is *lens-shaped* (resp. *totally geodesic cone-shaped*, *generalized lens-shaped*) if the corresponding R-p-end is lens-shaped (resp. *totally geodesic cone-shaped*, *generalized lens-shaped*). An end neighborhood of an end E is (*generalized*) *lens-shaped* if so is a corresponding p-end neighborhood $\tilde{E}$.

An end neighborhood is *lens-shaped* if it is a lens-shaped R-end neighborhood or T-end neighborhood. A p-end neighborhood is *lens-shaped* if it is a lens-shaped R-p-end neighborhood or T-p-end neighborhood. Of course it is redundant to say that R-end or T-end satisfies the lens condition dependent on its radial or totally geodesic end structure.

DEFINITION 3.1.7. A *real projective orbifold with radial or totally geodesic ends* is a strongly tame orbifold with a real projective structure where each end is an R-end or a T-end with an end structure given for each. An end of a real projective orbifold is (resp. *generalized*) *lens-shaped* or *pre-horospherical* if it is a (resp. *generalized*) *lens-shaped* or *pre-horospherical* R-end or if it is a lens-shaped T-end.

3.1.3.1. *p-end vertices.* Let $\mathscr{O}$ be a real projective orbifold with the universal cover $\tilde{\mathscr{O}}$. We fix a developing map **dev** in this subsection and identify with its image. Given a radial end of $\mathscr{O}$ and an end neighborhood U of a product form $E \times [0,1)$ with a radial foliation, we take a component U_1 of $p^{-1}(U)$ and the lift of the radial foliation. The developing images of leaves of the foliation end at a common point x in $\mathbb{RP}^n$.

- Recall that a p-end vertex of $\tilde{\mathscr{O}}$ is the ideal point of leaves of U_1. (See Section 3.1.3.) When **dev** is fixed, we can identify it with its image under **dev**. It will be denoted by $\mathrm{v}_{\tilde{E}}$ if its p-end neighborhoods correspond to a p-end $\tilde{E}$.
- Let $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ denote the space of equivalence classes of rays from $\mathrm{v}_{\tilde{E}}$ diffeomorphic to an $(n-1)$-sphere where $\pi_1(\tilde{E})$ acts as a group of projective automorphisms. Here, $\pi_1(\tilde{E})$ acts on $\mathrm{v}_{\tilde{E}}$ and sends leaves to leaves in U_1.
- Given a p-end $\tilde{E}$ corresponding to $\mathrm{v}_{\tilde{E}}$, we define $\tilde{\Sigma}_{\tilde{E}} := R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}})$ the space of directions of developed leaves under **dev** oriented away from $\mathrm{v}_{\tilde{E}}$ into a p-dend neighborhood of $\tilde{\mathscr{O}}$ corresponding to $\tilde{E}$. The space develops to $\mathbb{S}^{n-1}_x$ by **dev** as an embedding to a convex open domain.
- Recall that $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is projectively diffeomorphic to the *end orbifold* to be denoted by Σ_E or by $\Sigma_{\tilde{E}}$. (See Lemma 3.1.5.)
- We may use the lifting of **dev** to $\mathbb{S}^n$. The endpoint x' of the lift of radial lines will be identified with the *p-end vertex* also when the lift of **dev** is fixed. Here, we can canonically identify $\mathbb{S}^{n-1}_{x'}$ and $\mathbb{S}^{n-1}_x$ and the group actions of $\mathbf{\Gamma}_{\tilde{E}}$ on them.

3.1.4. Cusp ends. A *parabolic algebra* $\mathfrak{p}$ is an algebra in a semi-simple Lie algebra $\mathfrak{g}$ whose complexification contains a maximal solvable subalgebra of $\mathfrak{g}$ (p. 279–288 of [**150**]). A *parabolic group* P of a semi-simple Lie group G is the full normalizer of a parabolic subalgebra.

An *ellipsoid* in $\mathbb{RP}^n = \mathbb{P}(\mathbb{R}^{n+1})$ (resp. in $\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1})$) is the projection $C - \{O\}$ of the null cone

$$C := \{\vec{x} \in \mathbb{R}^{n+1} | B(\vec{x},\vec{x}) = 0\}$$

for a nondegenerate symmetric bilinear form $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ of signature $(1,n)$. Ellipsoids are always equivalent by projective automorphisms of $\mathbb{RP}^n$. An *ellipsoid ball* is the closed contractible domain in an affine subspace $\mathbb{A}^n$ of $\mathbb{RP}^n$ (resp. $\mathbb{S}^n$) bounded by an ellipsoid contained in $\mathbb{A}^n$. A *horoball* is an ellipsoid ball with a point p of the boundary removed. An ellipsoid with a point p on it removed is called a *horosphere*. The *vertex* of the horosphere or the horoball is defined as p.

Let U be a horoball with a vertex p in the boundary of B. A real projective orbifold that is projectively diffeomorphic to an orbifold U/Γ_p for a discrete subgroup $\Gamma_p \subset \mathsf{PO}(1,n)$ fixing a point $p \in \mathrm{bd}B$ is called a *horoball orbifold*. A *cusp or horospherical end* is an end with an end neighborhood that is such an orbifold. A *cusp group* is a subgroup of a parabolic subgroup of an isomorphic copy of $\mathsf{PO}(1,n)$ in $\mathsf{PGL}(n+1,\mathbb{R})$ or in $\mathsf{SO}^+(1,n)$ in $\mathsf{SL}_\pm(n+1,\mathbb{R})$. A cusp group is a *unipotent cusp-group* if it is unipotent as well.

By Corollary 8.1.1, an end is pre-horospherical if and only if it is a cusp end. We will use the term interchangeably but not in Chapter 3 where we will prove this fact.

3.1.4.1. *Lie group invariant p-end neighborhoods.* We need the following lemma later.

A *p-end holonomy group* is the image of a p-end fundamental group under the holonomy homomorphism. If its universal cover $\tilde{\mathscr{O}}$ embeds to $\mathbb{S}^n$ or $\mathbb{RP}^n$, then h is injective and hence p-end holonomy group is isomorphic to the p-end fundamental group. A *end holonomy group* is the image of an end fundamental group.

LEMMA 3.1.8. *Let $\mathscr{O}$ be a convex strongly tame real projective n-orbifold, and let $\tilde{\mathscr{O}}$ be its universal cover in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Let U be a p-R-end neighborhood of a* *p-end $\tilde{E}$ in $\tilde{\mathcal{O}}$ where a p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ acts on. Let Q be a discrete subgroup of $\mathbf{\Gamma}_{\tilde{E}}$. Suppose that G is a connected Lie group virtually containing Q so that $G/G\cap Q$ is compact. Assume that G acts on the p-R-end vertex $\mathrm{v}_{\tilde{E}}$ and $\tilde{\Sigma}_{\tilde{E}}$. Then $\bigcap_{g\in G} g(U)$ contains a non-empty G-invariant p-end neighborhood of $\tilde{E}$.*

PROOF. We first assume that $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ and $Q \subset G$. Let F be the compact fundamental domain of G under $G\cap Q$. It is sufficiently to prove for the case when U is a proper p-end neighborhood since for any open set V containing U, $\bigcap_{g\in G} g(V)$ contains a p-end neighborhood $\bigcap_{g\in G} g(U)$. Hence, we assume that $\mathrm{bd}_{\tilde{\mathcal{O}}} U/\mathbf{\Gamma}_{\tilde{E}}$ is a smooth compact surface. Let F_U denote the fundamental domain of $\mathrm{bd}_{\tilde{\mathcal{O}}} U$.

Let F be a compact fundamental domain of G with respect to Q. Let L be a compact subset of $\tilde{\Sigma}_{\tilde{E}}$ and let $\hat{L}$ denote the union of all maximal open segments with endpoints $\mathrm{v}_{\tilde{E}}$ and $\mathrm{v}_{\tilde{E}-}$ in the direction of L.

We claim that $\bigcap_{g\in F} g(U) = \bigcap_{g\in G} g(U)$ contains an open set in $\hat{L}$. We show this by proving that $\bigcap_{g\in F} g(U)\cap l$ for any maximal l in $\hat{L}$ has a lower bound on its $\mathbf{d}$-length. The lower bound is uniform for L.

Suppose not. Then there exists sequence $g_i \in F$ and maximal segment l_i in $\hat{L}$ so that the sequence of $\mathbf{d}$-length of $g_i(U)\cap l_i$ from $\mathrm{v}_{\tilde{E}}$ goes to 0 as $i \to \infty$. The endpoint of $g_i(U)\cap l_i$ equals $g_i(y_i)$ for $y_i \in \mathrm{bd}_{\tilde{\mathcal{O}}} U$. This implies that $\{g_i(y_i)\} \to \mathrm{v}_{\tilde{E}}$.

Now, y_i corresponds to a direction $u_i \in \tilde{\Sigma}_{\tilde{E}}$. Since F is a compact set, u_i corresponds to a point of a compact set $F^{-1}(L)$, which corresponds to a compact set $\hat{F}_U$ of $\mathrm{bd}_{\tilde{\mathcal{O}}} U$ with directions in $F^{-1}(L)$. Hence, $y_i \in \hat{F}_U$, a compact set. Since $\mathrm{v}_{\tilde{E}}$ is a fixed point of G, and $y_i \subset \hat{F}_U$ for a compact subset $\hat{F}_U$ of $\mathbb{S}^n$ not containing $\mathrm{v}_{\tilde{E}}$, this shows that g_i form an unbounded sequence in $\mathsf{SL}_\pm(n+1,\mathbb{R})$. This is a contradiction to $g_i \in F$.

We have a nonempty set

$$\hat{U} := \bigcap_{g\in G} g(U) = \bigcap_{g\in F} g(U)$$

containing an open set in U. G acts on $\hat{U}$ clearly. We take the interior of $\hat{U}$. If G only virtually contains $\mathbf{\Gamma}_{\tilde{E}}$, we just need to add finitely many elements to the above arguments. $[\mathbb{S}^n\mathrm{S}]$ $\square$

LEMMA 3.1.9. *A p-end vertex of a horospherical p-end cannot be an endpoint of a segment in* $\mathrm{bd}\tilde{\mathcal{O}}$.

PROOF. Suppose that $\mathrm{bd}\tilde{\mathcal{O}}$ contains a segment s ending at the p-end vertex $\mathrm{v}_{\tilde{E}}$. Then s is on an invariant hyperspace of $\mathbf{\Gamma}_{\tilde{E}}$. Now conjugating $\mathbf{\Gamma}_{\tilde{E}}$ into an $(n-1)$-dimensional parabolic or cusp subgroup P of $\mathsf{SO}(n,1)$ fixing $(1,-1,0,\dots,0) \in \mathbb{R}^{n+1}$ by say an element h of $\mathsf{SL}_\pm(n+1,\mathbb{R})$. By simple computations using the matrix forms of $\mathbf{\Gamma}_{\tilde{E}}$, we can find a sequence $\{g_i\}, g_i \in h\mathbf{\Gamma}_{\tilde{E}}h^{-1} \subset P$ so that $\{g_i(h(s))\}$ geometrically converges to a great segment. Thus, for $h^{-1}g_ih \in \mathbf{\Gamma}_{\tilde{E}}$, the sequence $\{h^{-1}g_ih(s)\}$ geometrically converges to a great segment in $\mathrm{Cl}(\tilde{\mathcal{O}})$. This contradicts the proper convexity of $\tilde{\mathcal{O}}$. $[\mathbb{S}^n\mathrm{T}]$ $\square$

3.1.5. Unit-norm eigenvalued actions on ends. Here, we will collect useful results on unit-normed actions resulting in Proposition 3.1.14 and Lemma 3.1.15.

LEMMA 3.1.10.

- *Suppose that a closed connected projective group G acts properly and cocompactly on a convex domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Then G acts transitively on Ω. (Benoist [21]).*

- *Suppose that Γ is a uniform lattice in a closed connected group G acting on a convex domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Suppose that Γ acts properly and cocompactly on Ω. Then G acts transitively on Ω.*

PROOF. For the second item, we claim that G acts properly also. Let $\hat{F}$ be the fundamental domain of G with Γ action. Let $x \in \Omega$. Let F' be the image $F(x) := \{g(x)|g \in F\}$ in Ω. This is a compact set. Define

$$\Gamma_{F'} := \{g \in \Gamma | g(F(x)) \cap F(x) \neq \emptyset\}.$$

Then $\Gamma_{F'}$ is finite by the properness of the action of Γ. Since an element of G is a product of an element g' of Γ and $f \in F$, and $g' f(x) = x$, it follows that $g' F(x) \cap F(x) \neq \emptyset$ and $g' \in \Gamma_{F'}$. Hence the stabilizer G_x is a subset of $\Gamma_{F'} F$, and G_x is compact. G becomes a Riemannian isometry group with respect to a metric on Ω. The second part follows from the first part since G must act properly and cocompactly. [$\mathbb{S}^n$T] □

LEMMA 3.1.11. *Suppose that a simply connected isometry Lie group G acts smoothly on a simply connected manifold M with a metric so that each stabilizer is trivial. Suppose that $\dim G = \dim M$ and G is a closed subgroup of the isometry group of M. Then G acts transtiviely on M.*

PROOF. This follows since the orbit should be an open and a closed set. □

PROPOSITION 3.1.12. *Let N be a discrete group or an $n-1$-dimensional connected Lie group where all the elements have only eigenvalues of unit norms acting on a convex $n-1$-domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) projectively and properly and cocompactly. Then Ω is a complete affine space. If N is a connected Lie group, then N is a simply-connected orthopotent solvable group.*

PROOF. Again, we first prove for the $\mathbb{S}^n$-version. First consider when N is discrete. By Theorem 1.3.7, N is an orthopotent Lie group. Theorem 1.4.5 proved that Ω is a complete affine space.

Now, consider the case when N is a connected Lie group. By Lemma 3.1.10, N acts transitively on Ω. N has an N-invariant metric on Ω by the properness of the action. Consider an orbit map $N \to N(x)$ for $x \in \Omega$. If a stabilizer of a point x of Ω contains a group of dimension ≥ 1, then $\dim N > \dim \Omega$. The stabilizer is a finite group. Hence, N covers Ω finitely. Since Ω is contractible, the orbit map is a diffeomorphism. Hence, N is contractible.

By Theorem 1.3.7, N is a solvable Lie group.

Now, Ω cannot be properly convex: Otherwise, By Fait 1.5 of [21], N either acts irreducibly on Ω or Ω is a join of domain $\Omega_1, \dots, \Omega_n$ where N acts irreducibly on each Ω_i. Since a solvable group never acts irreducibly unless the domain is 0-dimensional by the Lie-Kolchin theorem, Ω is a simplex or a point. (See Theorem 17.6 of [102].) Then N has to be diagonalizable and this is a contradiction to the unit-norm-eigenvalued property since N acts cocompactly unless $n-1=0$. If $n-1=0$, the conclusion is true.

Now suppose that Ω is not properly convex but not complete affine. Then Ω is foliated by i_0-dimensional complete affine spaces for $i_0 < n$. The space of affine leaves is a properly convex domain K by discussions on R-ends in Section 3.1.6. Hence, N acts on K. The stabilizer N_l is i_0-dimensional since the N-action is simply transitive. Hence, N/N_l acts on a properly convex set K^o satisfying the premises. Again, this is a contradiction.

Hence, Ω is complete affine. [$\mathbb{S}^n$S] □

Given a subgroup G of an algebraic Lie group, a *syndetic hull* $\mathscr{S}(G)$ of G is a solvable Lie group with finitely many components so that $\mathscr{S}(G)/G$ is compact. (See Fried and Goldman [82] and D. Witte [156].)

LEMMA 3.1.13. *Let N be a closed orthopotent Lie group in $\mathsf{SL}_\pm(n,\mathbb{R})$ acting on $\mathbb{R}^n$ inducing a proper action on an $n-1$-dimensional affine space $\mathbb{A}^{n-1}$ that is the upper half-space of $\mathbb{R}^n$ quotient by the scalar multiplications. Suppose that N acts cocompactly on $\mathbb{A}^{n-1}$. Then there is a connected group N_u with the following properties*:

- *$N/N\cap N_u$ and $N_u/N\cap N_u$ are compact.*
- *N_u is homeomorphic to a cell,*
- *N_u acts simply transitively on $\mathbb{A}^{n-1}$,*
- *N_u is the unipotent subgroup in $\mathsf{SL}_\pm(n,\mathbb{R})$ of dimension $n-1$ of N normalized by N.*

PROOF. Since N is orthopotent, there is a flag of vector subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{R}^n$ preserved by N where N acts as an orthogonal group on V_{i+1}/V_i for each $i = 1,\dots,m-1$. Here, $\mathbb{A}^{n-1}$ is parallel to the vector subspace V_{m-1} of dimension $n-1$. (See Chapter 2 of Berger [26].)

Hence, there is a homomorphism $N \to \bigoplus_{i=0}^{m-1} \mathsf{O}(V_{i+1}/V_i)$. Let N'_u denote the kernel. Then N'_u is a unipotent group with compact N/N'_u.

We define N_u to be the Zariski closure of N'_u in $\mathsf{SL}_\pm(n,\mathbb{R})$. Now, N_u is a unipotent Lie group, and N_u/N'_u is compact by Malcev [123].

Since N_u also acts on $\mathbb{A}^{n-1}$, N_u/N'_u is compact, and N'_u acts properly, it follows that N_u acts properly on $\mathbb{A}^{n-1}$. By Lemma 3.1.10, N_u acts transitively on $\mathbb{A}^{n-1}$. The action has trivial stabilizer since N_u is unipotent. This implies N_u is homeomorphic to $\mathbb{A}^{n-1}$. $\square$

PROPOSITION 3.1.14. *Let U is in an open domain in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) radially foliated from a point $p \in \mathrm{bd}U$ with smooth $\mathrm{bd}U - \{p\}$. Suppose U is in a properly convex domain. and let N be an $n-1$-dimensional a connected Lie group with only unit norm eigenvalues acting on U and fixing p. Suppose that it acts on $R_p(U)$ properly and cocompactly. Then U is the interior of an ellipsoid and N is a unipotent cusp group and acts transitively and freely on $\mathrm{bd}U - \{p\}$*

PROOF. We first assume $U \subset \mathbb{S}^n$. By Proposition 3.1.12, $R_p(U) = \mathbb{A}$ is complete affine. N acts on $R_p(U)$ as a unipotent Lie group. Thus, N is a simply-connected unit-norm-eigenvalued solvable Lie group by Proposition 3.1.12.

By Lemma 3.1.13, there is a unipotent group N_u where $N/N\cap N_u$ and $N_u/N\cap N_u$ are compact. Since N_u is isomorphic to a unipotent subgroup, and $N\cap N_u$ is a lattice in N and one in N_u.

It follows that each element of geodesic in N passing an element of $N\cap N_u$ is also unipotent being an exponential of a nilpotent element. The compactness of $N_u/N\cap N_u$ implies that these tangent vectors form a dense set in the tangent space the identity at N_u as we can see from the central series extension by free abelian groups. It follows that $N\cap N_u = N_u$ and so $N_u \subset N$. Since they have the same dimensions and are connected, $N_u = N$.

We will now show that U is the interior of an ellipsoid. We identify p with $[1,0,\dots,0]$. Let W denote the hyperspace in $\mathbb{S}^n$ containing p sharply supporting U. Here, W corresponds to a supporting hyperspace in $\mathbb{S}^{n-1}_p$ of the set of directions of an open hemisphere $R_p(U)$ and hence is unique supporting hyperplane at p and, thus, N-invariant. Also, $W\cap \mathrm{Cl}(U)$ is a properly convex subset of W.

Let y be a point of U. Suppose that N contains a sequence $\{g_i\}$ so that

$$\{g_i(y)\} \to x_0 \in W \cap \mathrm{Cl}(\tilde{\mathscr{O}}) \text{ and } x_0 \neq p;$$

that is, x_0 in the boundary direction of A from p. Let $U_1 = \mathrm{Cl}(U) \cap W$. Let V be the smallest subspace containing p and U_1. The dimension of V is ≥ 1 as it contains x_0 and p.

Again the unipotent group N acts on V. Now, V is divided into disjoint open hemispheres of various dimensions where N acts on: By Theorem 3.5.3 of [150], N preserves a full flag structure $V_0 \subset V_1 \subset \cdots \subset V_k = V$. We take components of complement $V_i - V_{i-1}$. Let $H_V := V - V_{k-1}$.

Suppose that $\dim V = n-1$ for contradiction. Then $H_V \cap U_1$ is not empty since otherwise, we would have a smaller dimensional V. Let h_V be the component of H_V meeting U_1. Since N is unipotent, h_V has an N-invariant metric by Theorem 3 of Fried [80].

We claim that the orbit of the action of N is of dimension $n-1$ and hence locally transitive on H_V: If not, then a one-parameter subgroup N' fixes a point of h_V. This group acts trivially on h_V since the unipotent group contains a trivial orthogonal subgroup. Since N' is not trivial, it acts as a group of nontrivial translations on the affine subspace H^o. We obtain that $N'(U)$ is not properly convex. This is absurd. Hence, an orbit of N is open in h_V, and N acts locally simply-transitively without fixed points.

Since N has trivial stabilizers on h_V, there is an N-invariant Riemannian metric on h_V. The orbit of N in h_V is closed since h_V has an N-invariant metric, and N is closed in the isometry group of h_V. Thus, N acts transitively on h_V since $\dim N = \dim h_V$.

Hence, the orbit $N(y)$ of N for $y \in H_V \cap U_1$ contains a component of H_V. This contradicts the assumption that $\mathrm{Cl}(U)$ is properly convex (compare with arguments in [68].)

Suppose that the dimension of $\langle V \rangle$ is $\leq n-2$. Let J be a subspace of dimension 1 bigger than $\dim V$ and containing V and meeting U. Let $J_{\mathbb{A}}$ denote the subspace of $\mathbb{A}^{n-1}$ corresponding to the directions in J. Then $J_{\mathbb{A}}$ is sent to disjoint subspaces or to itself under N. Since N acts on $\mathbb{A}$ transitively, a nilpotent subgroup N_J of N acts on $J_{\mathbb{A}}$ transitively. Hence,

$$\dim N_J = \dim J_{\mathbb{A}} = \dim V,$$

and we are in a situation immediately above. The orbit $N_J(y)$ for a limit point $y \in H_V$ contains a component of $V - V_{k-1}$ as above. Thus, $N_J(y)$ contains the same component, an affine subspace. As above, we have a contradiction to the proper convexity since the above argument applies to N_J.

Therefore, points such as $x_0 \in W \cap \mathrm{bd}(\tilde{\mathscr{O}}) - \{p\}$ do not exist. Hence for any sequence of elements $g_i \in \mathbf{\Gamma}_{\tilde{E}}$, we have $\{g_i(y)\} \to p$. Hence,

$$\mathrm{bd} U = (\mathrm{bd} U \cap \tilde{\mathscr{O}}) \cup \{p\}.$$

Clearly, $\mathrm{bd}U$ is homeomorphic to an $(n-1)$-sphere.

Since U is radial, this means that U is a pre-horospherical p-end neighborhood. (See Definition 3.1.6.) Since N acts transitively on a complete affine space $R_p(U)$, and there is a 1 to 1 radial correspondence of $R_p(U)$ and $\mathrm{bd}U - \{p\}$, it acts so on $\mathrm{bd}U - p$. Since N is unipotent and acts transitively on $\mathrm{bd}U - \{p\}$, Lemma 7.12 of [68] shows that U is bounded by an ellipsoid. Choose $x \in U$, then $N(x) \subset U$ is an horospherical p-end neighborhood also. Since $\mathbf{Aut}(U)$ is the group of hyperbolic isometry group of U with the Hilbert metric, it follows that N is the cusp group. [$\mathbb{S}^n$T] □

LEMMA 3.1.15. *Assume that $\mathscr{O}$ is a properly convex real projective orbifold with an end E with the universal cover $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Suppose that E is a convex end with a corresponding p-end $\tilde{E}$. Suppose that eigenvalues of elements of $\mathbf{\Gamma}_{\tilde{E}}$ have unit norms only.*

Then $\mathbf{\Gamma}_{\tilde{E}}$ is conjugate to a subgroup of a parabolic subgroup in $\mathsf{SO}(n,1)$ (*resp.* $\mathsf{PO}(n,1)$), *and a finite-index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$ is unipotent and $\tilde{E}$ is horospherical, i.e., cuspidal.*

PROOF. We will assume first $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. By Theorem 1.3.7, $\mathbf{\Gamma}_{\tilde{E}}$ is virtually orthopotent. By Proposition 3.1.12, $\tilde{\Sigma}_{\tilde{E}}$ is complete affine, and $\mathbf{\Gamma}_{\tilde{E}}$ acts on it as an affine transformation group. By Theorem 3 in Fried [80], $\mathbf{\Gamma}_{\tilde{E}}$ is virtually unipotent. Since $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is a compact complete-affine manifold, a finite-index subgroup F of $\mathbf{\Gamma}_{\tilde{E}}$ is contained in a unipotent Lie subgroup acting on $\tilde{\Sigma}_{\tilde{E}}$. Now, by Malcev [123], it follows that the same group is contained in a simply connected unipotent group N acting on $\mathbb{S}^n$ since F is unipotent. The dimension of N is $n-1 = \dim \tilde{\Sigma}_{\tilde{E}}$ by Theorem 3 of [80].

Let U be a component of the inverse image of a p-end neighborhood so that $\mathrm{v}_{\tilde{E}} \in \mathrm{bd} U$. Assume that U is a radial p-end neighborhood of $\mathrm{v}_{\tilde{E}}$. The group N acts on a smaller open set covering a p-end neighborhood by Lemma 3.1.8. We let U be this open set from now on. Consequently, $\mathrm{bd} U \cap \tilde{\mathscr{O}}$ is smooth.

Now N acts transitively and properly on $\tilde{\Sigma}_{\tilde{E}}$ by Lemma 3.1.10 since F acts properly on it and N/F is compact. N acts cocompactly on $\tilde{\Sigma}_{\tilde{E}}$ since so does F, $F \subset N$.

By Proposition 3.1.14, N is a cusp group, and U is a p-end neighborhood bounded by an ellipsoid.

Since $\mathbf{\Gamma}_{\tilde{E}}$ has a finite extension of N as the Zariski closure, the connected identity component N is normalized by $\mathbf{\Gamma}_{\tilde{E}}$.

Also, for element $g \in \mathbf{\Gamma}_{\tilde{E}} - F$, suppose $g(x) \in U$. Now, $g(\mathrm{bd} U - \{\mathrm{v}_{\tilde{E}}\})$ is an orbit of $g(x)$ for $x \in \mathrm{bd} U - \{\mathrm{v}_{\tilde{E}}\})$. Hence, $g(\mathrm{bd} U - \{\mathrm{v}_{\tilde{E}}\})) \subset U$. Hence g^n is not in F for all n, a contradiction. Also, $g(x)$ cannot be outside $\mathrm{Cl}(U)$ similarly. Hence, $\mathbf{\Gamma}_{\tilde{E}}$ acts on U. Also, $\mathbf{\Gamma}_{\tilde{E}}$ is in a conjugate of a parabolic subgroup. [$\mathbb{S}^n$T] $\square$

3.1.6. R-Ends. We classify *R*-ends into three classes: complete affine ends, properly convex ends, and nonproperly convex and not complete affine ends. We also introduce T-ends.

This should be a theorem. I think...

Recall that an R-p-end $\tilde{E}$ is convex if $\tilde{\Sigma}_{\tilde{E}}$ is convex. Since $\tilde{\Sigma}_{\tilde{E}}$ is a convex open domain, it is contractible by Proposition 1.1.4, and it always lifts to $\mathbb{S}^n$ as an embedding. By Proposition 1.1.4, a convex R-end is either

(i): complete affine (CA),
(ii): properly convex (PC), or
(iii): convex but not properly convex and not complete affine (NPNC).

We follow mostly the article [52] with slight modifications.

3.2. Examples

EXAMPLE 3.2.1. The interior of a finite-volume hyperbolic n-orbifold with rank $n-1$ horospherical ends and totally geodesic boundary forms an example of a noncompact strongly tame properly convex real projective orbifold with radial or totally geodesic ends. For horospherical ends, the end orbifolds have Euclidean structures. (Also, we could allow hyperideal ends by attaching radial ends. See Section 3.2.1.)

move these examples at the end here?

EXAMPLE 3.2.2. For examples, if the end orbifold of an R-end E is a 2-orbifold based on a sphere with three singularities of order 3, then a line of singularity is a leaf of a radial foliation. End orbifolds of Porti-Tillmann orbifold [139] and the the double of a tetrahedral reflection orbifold are examples. A double orbifold of a cube with edges having orders 3 only has eight such end orbifolds. (See Proposition 4.6 of [52] and their deformations are computed in [58]. Also, see Ryan Greene [93] for the theory. These are explained again in Section 12.2.)

3.2.1. Examples of ends. We will present some examples here, which we will fully justify later.

Recall the Klein model of hyperbolic geometry: It is a pair $(\mathbb{B}, \mathbf{Aut}(\mathbb{B}))$ where $\mathbb{B}$ is the interior of an ellipsoid in $\mathbb{RP}^n$ or $\mathbb{S}^n$ and $\mathbf{Aut}(\mathbb{B})$ is the group of projective automorphisms of $\mathbb{B}$. Now, $\mathbb{B}$ has a Hilbert metric which in this case is the hyperbolic metric times a constant. Then $\mathbf{Aut}(\mathbb{B})$ is the group of isometries of $\mathbb{B}$. (See Section 1.1.6.)

From hyperbolic manifolds, we obtain some examples of ends. Let M be a complete hyperbolic manifold with cusps. M is a quotient space of the interior $\mathbb{B}$ of an ellipsoid in $\mathbb{RP}^n$ or $\mathbb{S}^n$ under the action of a discrete subgroup $\mathbf{\Gamma}$ of $\mathbf{Aut}(\mathbb{B})$. Then some horoballs are p-end neighborhoods of the horospherical R-ends.

We generalize Definition 2.2.8. Suppose that a noncompact strongly tame convex real projective orbifold M has totally geodesic embedded surfaces $S_1, .., S_m$ homotopic to the ends. Let M be covered by a properly convex domain $\tilde{M}$ in an affine subspace of $\mathbb{S}^n$.

- We remove the outside of S_js to obtain a properly convex real projective orbifold M' with totally geodesic boundary. Suppose that each S_j can be considered a lens-shaped T-end.
- Each S_i corresponds to a disjoint union of totally geodesic domains $\bigcup_{j \in J} \tilde{S}_{i,j}$ in $\tilde{M}$ for a collection J. For each $\tilde{S}_{i,j} \subset \tilde{M}$, a group $\Gamma_{i,j}$ acts on it where $\tilde{S}_{i,j}/\Gamma_{i,j}$ is a closed orbifold projectively diffeomorphic to S_i.
- Suppose that $\Gamma_{i,j}$ fixes a point $p_{i,j}$ outside $\tilde{M}$.
- Hence, we form the cone $M_{i,j} := \{p_{i,j}\} * \tilde{S}_{i,j}$.
- We obtain the quotient $M_{i,j}/\Gamma_{i,j} - \{p_{i,j}\}$ and identify $\tilde{S}_{i,j}/\Gamma_{i,j}$ to $S_{i,j}$ in M' to obtain the examples of real projective manifolds with R-ends.
- $(\{p_{i,j}\} * \tilde{S}_{i,j})^o$ is an R-p-end neighborhood and the end is a totally geodesic R-end.

The result is convex by Lemma 10.1.2 since we can think of S_j as an ideal boundary component of M' and that of $M_{i,j}/\Gamma_{i,j} - \{p_{i,j}\}$. This orbifold is called the *hyperideal extension* of the convex real projective orbifold as a convex real projective orbifold. When M is hyperbolic, each S_j is lens-shaped by Proposition 3.2.3. Hence, the hyperideal extensions of hyperbolic orbifolds are properly convex.

We will fully generalize the following in Chapter 5. We remark that Proposition 3.2.3 also follows from Lemma 12.1.2. However, we used more elementary results to prove it here.

PROPOSITION 3.2.3. *Suppose that M is a strongly tame convex real projective orbifold. Let $\tilde{E}$ be an R-p-end of M. Suppose that*

- *the p-end holonomy group of $\pi_1(\tilde{E})$*
  - *is generated by the homotopy classes of finite orders or*
  - *is simple or*
  - *satisfies the unit middle eigenvalue condition*

  and
- *$\tilde{E}$ has a $\pi_1(\tilde{E})$-invariant $n-1$-dimensional totally geodesic properly convex domain D in a p-end neighborhood and not containing the p-end vertex in the closure of D.*

Then the R-p-end $\tilde{E}$ is lens-shaped.

PROOF. Let $\tilde{M}$ be the universal cover of M in $\mathbb{S}^n$. $\tilde{E}$ is an R-p-end of M, and $\tilde{E}$ has a $\pi_1(\tilde{E})$-invariant $n-1$-dimensional totally geodesic properly convex domain D. Since $D/\pi_(\tilde{E})$ is homotopy equivalent to $R_{\vec{v}_{\tilde{E}}}(\tilde{U})$ for the end vertex $\vec{v}_{\tilde{E}}$, D cannot just project to a subspace of codimension higher than 1. Hence, D projects to an open domain. By Theorem 1.4.15, D projects onto $\tilde{\Sigma}_{\tilde{E}}$, and hence D is transverse to radial lines from $\mathrm{v}_{\tilde{E}}$.

Under the first assumption, since the end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ is generated by elements of finite order, the eigenvalues of the generators corresponding to the p-end vertex $\mathrm{v}_{\tilde{E}}$ equal 1 and hence every element of the end holonomy group has 1 as the eigenvalue at $\mathrm{v}_{\tilde{E}}$.

Now assume that the the end holonomy groups fix the p-end vertices with eigenvalues equal to 1.

Then the p-end neighborhood U can be chosen to be the open cone over the totally geodesic domain with vertex $\mathrm{v}_{\tilde{E}}$. Now, U is projectively diffeomorphic to the interior of a properly convex cone in an affine subspace $\mathbb{A}^n$. The end holonomy group acts on U as a discrete linear group of determinant 1. The theory of convex cones applies, and using the level sets of the Koszul-Vinberg function, we obtain a one-sided convex neighborhood N in U with smooth boundary (see Lemmas 4.1.5 and 4.1.6 of Goldman [86]). Let F be a fundamental domain of N with a compact closure in $\tilde{\mathscr{O}}$.

We obtain a one-sided neighborhood in the other side as follows: We take $R(N)$ for by a reflection R fixing each point of the hyperspace containing $\tilde{\Sigma}$ and the p-end vertex. Then we choose a diagonalizable transformation $\mathscr{D}$ fixing the p-end vertex and every point of $\tilde{\Sigma}$ so that the image $\mathscr{D}\circ R(F)$ is in $\tilde{\mathscr{O}}$. It follows that $\mathscr{D}\circ R(N)\subset\tilde{\mathscr{O}}$ as well. Thus, $N\cup\mathscr{D}\circ R(N)$ is the CA-lens we needed. The interior of the cone $\{\mathrm{v}_{\tilde{E}}\}*(N\cup\mathscr{D}\circ R(N))$ is the lens-cone neighborhood for $\tilde{E}$. [$\mathbb{S}^n$S] □

A more specific example is below. Let $S_{3,3,3}$ denote the 2-orbifold with base space homeomorphic to a 2-sphere and three cone-points of order 3. The 3-orbifolds satisfying the following properties are the example of Porti-Tillman [139] or the hyperbolic Coxeter 3-orbifolds based on an ideal 3-polytopes of dihedral angles $\pi/3$. (See Choi-Hodgson-Lee [58].)

The following is more specific version of Lemma 12.1.2. We give a much more elementary proof not depending on the full theory of this monograph.

PROPOSITION 3.2.4. *Let $\mathscr{O}$ be a strongly tame convex real projective 3-orbifold with R-ends where each end orbifold is diffeomorphic to a sphere $S_{3,3,3}$ or a disk with three silvered edges and three corner-reflectors of orders 3,3,3. Assume that the holonomy group of $\pi_1(\mathscr{O})$ is strongly irreducible. Then the orbifold has only lens-shaped R-ends or horospherical R-ends.*

PROOF. Again, it is sufficient to prove this for the case $\tilde{\mathscr{O}}\subset\mathbb{S}^3$. Let $\tilde{E}$ be an R-p-end corresponding to an R-end whose end orbifold is diffeomorphic to $S_{3,3,3}$. It is sufficient to consider only $S_{3,3,3}$ since it double-covers the disk orbifold. Since $\mathbf{\Gamma}_{\tilde{E}}$ is generated by finite order elements fixing a p-end vertex $\mathrm{v}_{\tilde{E}}$, every holonomy element has the eigenvalue equal to 1 at $\mathrm{v}_{\tilde{E}}$. Take a finite-index free abelian group A of rank two in $\mathbf{\Gamma}_{\tilde{E}}$. Since Σ_E is convex, a convex projective torus T^2 covers Σ_E finitely. Therefore, $\tilde{\Sigma}_{\tilde{E}}$ is projectively diffeomorphic either to

- a complete affine subspace or
- the interior of a properly convex triangle or
- a half-space

by the classification of convex tori by Nagano-Yagi [137] found in many places including [86] and [16] and Proposition 1.4.1. Since there exists a holonomy automorphism of order

3 fixing a point of $\tilde{\Sigma}_{\tilde{E}}$, it cannot be a quotient of a half-space with a distinguished foliation by lines. Thus, the end orbifold admits a complete affine structure or is a quotient of a properly convex triangle.

Suppose that $\Sigma_{\tilde{E}}$ has a complete affine structure. Since $\lambda_{\mathrm{v}_{\tilde{E}}}(g) = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$, the only possibility from Theorem 8.1.4 is when $\mathbf{\Gamma}_{\tilde{E}}$ is virtually nilpotent and we have a horospherical p-end for $\tilde{E}$.

Suppose that $\Sigma_{\tilde{E}}$ has a properly convex open triangle T' as its universal cover. A acts with an element g' with the largest eigenvalue > 1 and the smallest eigenvalue < 1 as a transformation in $\mathsf{SL}_{\pm}(3,\mathbb{R})$ the group of projective automorphisms at $\mathbb{S}^2_{\mathrm{v}_{\tilde{E}}}$. As an element of $\mathsf{SL}_{\pm}(4,\mathbb{R})$, we have $\lambda_{\mathrm{v}_{\tilde{E}}}(g') = 1$ and the product of the remaining eigenvalues is 1, the corresponding the largest and smallest eigenvalues are > 1 and < 1. Thus, an element of $\mathsf{SL}_{\pm}(4,\mathbb{R})$, g' fixes v_1 and v_2 other than $\mathrm{v}_{\tilde{E}}$ in directions of vertices of T'. Since $\mathbf{\Gamma}_{\tilde{E}}$ has an order three element exchanging the vertices of T', there are three fixed points of an element of A different from $\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}$. By commutativity, there is a properly convex compact triangle $T \subset \mathbb{S}^3$ with these three fixed points where A acts on. Hence, A is diagonalizable over the reals.

We can make any vertex of T to be an attracting fixed point of an element of A. Each element $g \in \mathbf{\Gamma}_{\tilde{E}}$ conjugates elements of A to A. Therefore g sends the attracting fixed points of elements of A to those of elements of A. Hence $g(T) = T$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$.

Each point of the edge E of $\mathrm{Cl}(T)$ is an accumulation point of an orbit of A by taking a sequence g_i so that the sequence of the largest norm of eigenvalues $\lambda_1(g_i)$ and the sequence of second largest norm of the eigenvalue $\lambda_2(g_i)$ are going to $+\infty$ while the sequence $\log|\lambda_1(g_i)/\lambda_2(g_i)|$ is bounded. Since $\lambda_{\mathrm{v}_{\tilde{E}}} = 1$, writing every vector as a linear combination of vectors in the direction of the four vectors, this follows. Hence $\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}$ and $T \subset \mathrm{Cl}(\mathscr{O})$.

If $T^o \cap \mathrm{bd}\mathscr{O} \neq \emptyset$, then $T \subset \mathrm{bd}\mathscr{O}$ by Lemma 1.4.4. Then each segment from $\mathrm{v}_{\tilde{E}}$ ending in $\mathrm{bd}\mathscr{O}$ has the direction in $\mathrm{Cl}(\Sigma_{\tilde{E}}) = T'$. It must end at a point of T. Hence, $\tilde{\mathscr{O}} = (T * \mathrm{v}_{\tilde{E}})^o$, an open tetrahedron σ. Since the holonomy group acts on it, we can take a finite-index group fixing each vertex of σ. Thus, the holonomy group is virtually reducible. This is a contradiction.

Therefore, $T \subset \mathscr{O}$ as $T \cap \mathrm{bd}\mathscr{O} = \emptyset$. We have a totally geodesic R-end, and by Proposition 3.2.3, the end is lens-shaped. (See also [37].) [$\mathbb{S}^n$S] $\square$

The following construction is called "bending" and was investigated by Johnson and Millson [106]. These give us examples of R-ends that are not totally geodesic R-ends. See Ballas and Marquis [7] for other examples.

EXAMPLE 3.2.5 (Bending). Let $\mathscr{O}$ have the usual assumptions. We will concentrate on an end and not take into consideration of the rest of the orbifold. Certainly, the deformation given here may not extend to the rest. (If the totally geodesic hypersurface exists on the orbifold, the bending does extend to the rest.)

Suppose that $\mathscr{O}$ is an oriented hyperbolic manifold with a hyperideal end E. Then E is a totally geodesic R-end with an R-p-end $\tilde{E}$. Let the associated orbifold Σ_E for E of $\mathscr{O}$ be a closed 2-orbifold and let c be a two-sided simple closed geodesic in Σ_E. Suppose that E has an open end neighborhood U in $\mathscr{O}$ diffeomorphic to $\Sigma_E \times (0,1)$ with totally geodesic boundary $\mathrm{bd}U \cap \mathscr{O}$ diffeomorphic to Σ_E. Let $\tilde{U}$ be a p-end neighborhood in $\tilde{\mathscr{O}}$ corresponding to $\tilde{E}$ bounded by $\tilde{\Sigma}_{\tilde{E}}$ covering Σ_E. Then U has a radial foliation whose leaves lift to radial lines in $\tilde{U}$ from $\mathrm{v}_{\tilde{E}}$.

Let A be an annulus in U diffeomorphic to $c\times(0,1)$, foliated by leaves of the radial foliation of U. Now a lift $\tilde{c}$ of c is in an embedded disk A', covering A. Let g_c be the deck transformation corresponding to $\tilde{c}$ and c. Suppose that g_c is orientation-preserving. Since g_c is a hyperbolic isometry of the Klein model, the holonomy g_c is conjugate to a diagonal matrix with entries $\lambda,\lambda^{-1},1,1$, where $\lambda>1$ and the last 1 corresponds to the vertex $\mathrm{v}_{\tilde{E}}$. We take an element k_b of $\mathsf{SL}_{\pm}(4,\mathbb{R})$ of form in this system of coordinates

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & b & 1 \end{pmatrix} \tag{3.2.1}$$

where $b\in\mathbb{R}$. k_b commutes with g_c. Let us just work on the end E. We can "bend" E by k_b:

Now, k_b induces a diffeomorphism $\hat{k}_b$ of an open neighborhood of A in U to another one of A since k_b commutes with g_c. We can find tubular neighborhoods N_1 of A in U and N_2 of A. We choose N_1 and N_2 so that they are diffeomorphic by a projective map $\hat{k}_b$. Then we obtain two copies A_1 and A_2 of A by completing $U-A$.

Give orientations on A and U. Let $N_{1,-}$ denote the left component of N_1-A and let $N_{2,+}$ denote the right component of N_2-A.

We take a disjoint union $(U-A)\sqcup N_1\sqcup N_2$ and

- identify the projectively diffeomorphic copy of $N_{1,-}$ in N_1 with $N_{1,-}$ in $U-A$ by the identity map and
- identify the projectively diffeomorphic copy of $N_{2,+}$ in N_2 with $N_{2,+}$ in $U-A$ by the identity also.

We glue back N_1 and N_2 by the real projective diffeomorphism $\hat{k}_b$ of a neighborhood of N_1 to that of N_2. Then $N_1-(N_{1,-}\cup A)$ is identified with $N_{2,+}$ and $N_2-(N_{2,+}\cup A)$ is identified with $N_{1,-}$. We obtain a new manifold.

For sufficiently small b, we see that the end is still lens-shaped. and it is not a totally geodesic R-end. (This follows since the condition of being a lens-shaped R-end is an open condition. See Section 11.2.)

For the same c, let k_s be given by

$$\begin{pmatrix} s & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & 0 & s & 0\\ 0 & 0 & 0 & 1/s^3 \end{pmatrix} \tag{3.2.2}$$

where $s\in\mathbb{R}_+$. These give us bendings of the second type. For s sufficiently close to 1, the property of being lens-shaped is preserved and being a totally geodesic R-end. (However, these will be understood by cohomology.)

If $s\lambda<1$ for the maximal eigenvalue λ of a closed curve c_1 meeting c odd number of times, we have that the holonomy along c_1 has the attracting fixed point at $\mathrm{v}_{\tilde{E}}$. This implies that we no longer have lens-shaped R-ends if we have started with a lens-shaped R-end.

CHAPTER 4

The affine action on a properly convex domain whose boundary is in the ideal boundary

In this chapter, we will show the asymptotic niceness of the affine actions when the affine group Γ acts on a convex domain Ω in $\mathbb{A}^n$ and a properly convex domain in the ideal boundary of $\mathbb{A}^n$. We will find a properly convex domain in $\mathbb{A}^n$ with boundary in Ω. The main tools will be Anosov flows on the affine bundles over the unit tangent bundles as in Goldman-Labourie-Margulis [91]. We will introduce a flat bundle and decompose it in an Anosov-type manner. Then we will find an invariant section. We will prove the asymptotic niceness using the sections. In Section 4.1, we will define asymptotic niceness and flow decomposition of the vector bundles over $\mathbb{U}\Omega/\Gamma$. In Section 4.2, we begin with a strictly convex domain Ω with a hyperbolic Γ and the main result Theorem 4.1.1. We define proximal flows and decompose the vector bundle flows into contracting and repelling and neutral subbundles. In Section 4.2.2, we show that contracting and expansion properties of the contracting and repelling subbundles, with a somewhat technical argument involving pulling-back. However, the neutral subbudles here are more of a generalized type than what they had. We obtain the neutralized sections as Goldman-Labourie-Margulis did. We will prove the main result for strictly convex Ω at the end of this section using the neutralized sections to obtain asymptotic hyperspaces. In Section 4.3, we will generalize these results to the case when Ω is not necessarily strictly convex. They are Theorems 4.3.1 and 4.3.8. A basic technique here is to make the unit tangent bundle larger to an augmented unit tangent bundle by blowing up using the compact sets of hyperspaces at the endpoints of geodesics. The strategy to prove the second main result is analogous to the strictly convex case for Ω. In Section 4.4, we discuss the lens condition for T-ends obtained by the uniform middle eigenvalue condition. We will end by finding strictly convex smooth hypersurfaces approximating any convex boundary components for these types of domains. Except for Section 4.4, we will work only in $\mathbb{S}^n$ for simplicity.

4.1. Affine actions

Let Γ be an affine group acting on the affine subspace $\mathbb{A}^n$ with boundary $\mathrm{bd}\mathbb{A}^n = \mathbb{S}^{n-1}_\infty$ in $\mathbb{S}^n$, $\mathbb{A}^n$ is an open n-hemisphere. Let U be a properly convex invariant Γ-invariant domain with the property in $\mathbb{A}^n$:

$$\mathrm{Cl}(U) \cap \mathrm{bd}\mathbb{A}^n = \mathrm{Cl}(\Omega) \subset \mathrm{bd}\mathbb{A}^n$$

for a properly convex open domain Ω. To begin with, we assume only that Ω is properly convex. We also assume that Ω/Γ is a closed orbifold. The action of Γ on $\mathbb{S}^n$ or $\mathbb{RP}^n$ is said to be a *properly convex affine action*. Also, (Γ, U, Ω) is said to be a *properly convex affine triple*.

A sharply supporting hyperspace P at $x \in \partial\mathrm{Cl}(\Omega)$ is *asymptotic* to U if there are no other sharply supporting hyperplane P' at x so that $P' \cap \mathbb{A}^n$ separates U and $P \cap \mathbb{A}^n$. In this case, we say that hyperspace P is *asymptotic* to U. We will use the abbreviation *AS-hyperspace* to indicate for asymptotic sharply supporting hyperspace.

Let (Γ, U, Ω) be a properly convex affine triple. A properly convex affine action of Γ is said to be *asymptotically nice* if with respect to U if Γ acts on a compact subset

$$J := \{H | H \text{ is an AS- hyperspace in } \mathbb{S}^n \text{ at } x \in \partial \mathrm{Cl}(\Omega), H \not\subset \mathbb{S}^{n-1}_\infty\}$$

where we require that every sharply supporting $(n-2)$-dimensional space of Ω in $\mathbb{S}^{n-1}_\infty$ is contained in at least one of the element of J. As a consequence, for any sharply supporting $(n-2)$-dimensional space Q of Ω, the set

$$H_Q := \{H \in J | H \supset Q\}$$

is compact and bounded away from $\mathrm{bd}\mathbb{A}^n$ in the Hausdorff metric $\mathbf{d}_H$.

THEOREM 4.1.1. *We assume that Γ is a hyperbolic group with a properly convex affine action. Let Γ have an affine action on the affine subspace $\mathbb{A}^n$, $\mathbb{A}^n \subset \mathbb{S}^n$, acting on a properly convex domain Ω in $\mathrm{bd}\mathbb{A}^n$. Suppose that Ω/Γ is a closed $n-1$-dimensional orbifold, and suppose that Γ satisfies the uniform middle-eigenvalue condition. Then Γ acts on a properly convex open domain U with following properties:*

- *(Γ, U, Ω) is a properly convex triple, and Γ is asymptotically nice with the properly convex open domain U, and*
- *if any open set U' so that (Γ, U', Ω) is a properly convex triple, then the AS-hyperspace at each point of $\partial \mathrm{Cl}(\Omega)$ exists and is the same as that of U. That is Γ is also asymptotically nice with respect to U'.*

DEFINITION 4.1.2. A subspace U of $\mathbb{R}^n$ is *expanding* under a linear map L if $\|L(u)\| \geq C\|u\|$ for every $u \in \mathbb{R}^n$ for a fixed norm $\|\cdot\|$ of $\mathbb{R}^n$ and $C > 1$.

A subspace U of $\mathbb{R}^n$ is *contracting* under a linear map L if $\|L(u)\| \leq C\|u\|$ for every $u \in \mathbb{R}^n$ for a fixed norm $\|\cdot\|$ of $\mathbb{R}^n$ and $0 < C < 1$.

The expanding condition is equivalent to the condition that all the norms of eigenvalues of $L|U$ are strictly larger than 1. (See Corollary 1.2.3 of Katok and Hasselblatt [110].)

In this section, we will work with $\mathbb{S}^n$ only, while the $\mathbb{RP}^n$ versions of the results follows from the results here in an obvious manner by Results in Section 1.1.8 and then projecting back to $\mathbb{RP}^n$.

For each element of $g \in \Gamma$,

$$h(g) = \begin{pmatrix} \frac{1}{\lambda_{\tilde{E}}(g)^{1/n}} \hat{h}(g) & \vec{b}_g \\ \vec{0} & \lambda_{\tilde{E}}(g) \end{pmatrix} \tag{4.1.1}$$

where $\vec{b}_g$ is $n \times 1$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant ± 1 and $\lambda_{\tilde{E}}(g) > 0$. In the affine coordinates, it is of the form

$$x \mapsto \frac{1}{\lambda_{\tilde{E}}(g)^{1+\frac{1}{n}}} \hat{h}(g) x + \frac{1}{\lambda_{\tilde{E}}(g)} \vec{b}_g. \tag{4.1.2}$$

Let $\lambda_1(g)$ denote the maximal norm of the eigenvalue of g, $g \in \Gamma$. If there exists a uniform constant $C > 1$ so that

$$C^{-1} \mathrm{length}_\Omega(g) \leq \log \frac{\lambda_1(g)}{\lambda_{\tilde{E}}(g)} \leq C \mathrm{length}_\Omega(g), \quad g \in \mathbf{\Gamma}_{\tilde{E}} - \{\mathrm{I}\}, \tag{4.1.3}$$

then Γ is said to satisfy the *umec* with respect to the boundary hyperspace.

By taking g^{-1} instead, we obtain the equivalent condition:

$$C^{-1}\mathrm{length}_\Omega(g) \leq \left|\log \frac{\lambda_n(g)}{\lambda_{\tilde{E}}(g)}\right| \leq C\mathrm{length}_\Omega(g), \quad g \in \mathbf{\Gamma}_{\tilde{E}} - \{\mathrm{I}\}, \tag{4.1.4}$$

This implies that

$$\lambda_1(g)/\lambda_{\tilde{E}}(g) > 1 \text{ and } \lambda_n(g)/\lambda_{\tilde{E}}(g) < 1 \tag{4.1.5}$$

as we can see by taking the inverse of g.

We denote by $\mathscr{L} : \mathbf{Aff}(\mathbb{A}^n) \to \mathsf{GL}(n,\mathbb{R})$ the homomorphism $g \mapsto M_g$ taking the *linear part* of an affine transformation $g : x \mapsto M_g x + \vec{b}_g$ to $M_g \in \mathsf{GL}(n,\mathbb{R})$.

LEMMA 4.1.3. *Let Γ be an affine group acting on the affine subspace $\mathbb{A}^n$ with boundary* $\mathrm{bd}\mathbb{A}^n$ *in $\mathbb{S}^n$ satisfying the uniform middle eigenvalue condition with respect to* $\mathrm{bd}\mathbb{A}^n$. *Then the linear part of g equal to $\frac{1}{\lambda_{\tilde{E}}(g)^{1+\frac{1}{n}}}\hat{h}(g)$ has a nonzero expanding subspace and a contracting subspace in $\mathbb{R}^n$.* □

4.1.1. Flow setup. The following flow setup will be applicable in the following. A slight modification is required later in Section 4.3.

We generalize the work of Goldman-Labourie-Margulis [91] using Anosov flows: We assume that Γ has a properly convex affine action with the triple (Γ, U, Ω) for $U \subset \mathbb{A}^n$. Since Ω is properly convex, Ω has a Hilbert metric. Let $T\Omega$ denote the tangent space of Ω. Let $\mathbb{U}\Omega$ denote the *unit tangent bundle over* Ω. This has a smooth structure as a quotient space of $T\Omega - O/\sim$ where

- O is the image of the zero-section, and
- $\vec{v} \sim \vec{w}$ if $\vec{v}$ and $\vec{w}$ are over the same point of Ω and $\vec{v} = s\vec{w}$ for a real number $s > 0$.

Let $\mathbb{A}^n$ be the n-dimensional affine subspace. Let $h : \Gamma \to \mathbf{Aff}(\mathbb{A}^n)$ denote the representation as described in (4.1.2). We form the product $\mathbb{U}\Omega \times \mathbb{A}^n$ that is an affine bundle over $\mathbb{U}\Omega$. We take the quotient $\tilde{\mathbf{A}} := \mathbb{U}\Omega \times \mathbb{A}^n$ by the diagonal action

$$g(x,\vec{u}) = (g(x), h(g)\vec{u}) \text{ for } g \in \Gamma, x \in \mathbb{U}\Omega, \vec{u} \in \mathbb{A}^n.$$

We denote the quotient by $\mathbf{A}$ which fibers over the smooth orbifold $\mathbb{U}\Omega/\Gamma$ with fiber $\mathbb{A}^n$.

Let V^n be the vector space associated with $\mathbb{A}^n$. Then we can form $\tilde{\mathbf{V}} := \mathbb{U}\Omega \times V^n$ and take the quotient under the diagonal action:

$$g(x,\vec{u}) = (g(x), \mathscr{L}(h(g))\vec{u}) \text{ for } g \in \Gamma, x \in \mathbb{U}\Omega, \vec{u} \in V^n$$

where $\mathscr{L}$ is the homomorphism taking the linear part of g. We denote by $\mathbf{V}$ the fiber bundle over $\mathbb{U}\Omega/\Gamma$ with fiber V^n.

There exists a flow $\hat{\Phi}_t : \mathbb{U}\Omega/\Gamma \to \mathbb{U}\Omega/\Gamma$ for $t \in \mathbb{R}$ given by sending $\vec{v}$ to the unit tangent vector to at $\alpha(t)$ where α is a geodesic tangent to $\vec{v}$ with $\alpha(0)$ equal to the base point of $\vec{v}$. This flow is induced from the geodesic flow $\tilde{\hat{\Phi}}_t : \mathbb{U}\Omega \to \mathbb{U}\Omega$.

We define a flow on $\tilde{\Phi}_t : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ by considering a unit-speed-geodesic flow-line $\vec{l}$ in $\mathbb{U}\Omega$ and considering $\vec{l} \times \mathbb{A}^n$ and acting trivially on the second factor as we go from $\vec{v}$ to $\hat{\Phi}_t(\vec{v})$ (See remarks in the beginning of Section 3.3 and equations in Section 4.1 of [91].) Each flow line in $\mathbb{U}\Sigma$ lifts to a flow line on $\mathbf{A}$ from every point in it. This induces a flow $\Phi_t : \mathbf{A} \to \mathbf{A}$.

We define a flow on $\mathscr{L}(\tilde{\Phi}_t) : \tilde{\mathbf{V}} \to \tilde{\mathbf{V}}$ by considering a unit-speed geodesic-flow line $\vec{l}$ in $\mathbb{U}\Omega$ and and considering $\vec{l} \times V^n$ and acting trivially on the second factor as we go from $\vec{v}$ to $\tilde{\hat{\Phi}}_t(\vec{v})$ for each t. This induces a flow $\mathscr{L}(\Phi_t) : \mathbf{V} \to \mathbf{V}$. (This generalizes the flow on [91].)

We let $\|\cdot\|_{\text{fiber}}$ denote some metric on these bundles over $\mathbb{U}\Sigma/\Gamma$ defined as a fiberwise inner product: We chose a cover of Ω/Γ by compact sets K_i and choosing a metric over $K_i \times \mathbb{A}^n$ and use the partition of unity. This induces a fiberwise metric on $\mathbf{V}$ as well. Pulling the metric back to $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{V}}$, we obtain a fiberwise metric to be denoted by $\|\cdot\|_{\text{fiber}}$.

We recall the trivial product structure. $\mathbb{U}\Omega \times \mathbb{A}^n$ is a flat $\mathbb{A}^n$-bundle over $\mathbb{U}\Omega$ with a flat affine connection $\nabla^{\tilde{\mathbf{A}}}$, and $\mathbb{U}\Omega \times V^n$ has a flat linear connection $\nabla^{\tilde{\mathbf{V}}}$. The above action preserves the connections. We have a flat affine connection $\nabla^{\mathbf{A}}$ on the bundle $\mathbf{A}$ over $\mathbb{U}\Sigma$ and a flat linear connection $\nabla^{\mathbf{V}}$ on the bundle $\mathbf{V}$ over $\mathbb{U}\Sigma$.

REMARK 4.1.4. In [91], the authors uses the term "recurrent geodesic". A geodesic is "recurrent" in their sense if it accumulates to compact subsets in both directions. They work in a compact subsurface where geodesics are recurrent in both directions. In our work, since Ω/Γ is a closed orbifold, every geodesic is recurrent in their sense. Hence, their theory generalizes here.

4.2. The proximal flow.

We will start with the case when Γ is hyperbolic and hence when Ω must be strictly convex with $\partial\mathrm{Cl}(\Omega)$ being C^1 by Theorem 1.1 of [22].

In the case when the linear part of the affine maps are unimodular, Theorem 8.2.1 of Labourie [117] shows that such a domain U exists but without showing the asymptotic niceness of the group. Also, when the linear part of Γ is a geometrically finite Kleinian group in $\mathsf{SO}(n,1)$, Barbot showed this result in Theorem 4.25 of [11] in the context of globally hyperbolic Lorentzian spacetimes. We believe our theory also generalize to the case when $\mathscr{L}(\Gamma)$ is convex cocompact. Fried also found a solution using cocyles [81] with informal notes in the same context but in the dual picture of R-ends as in Chapter 5.

The hyperbolicity of Γ shows that Ω is strictly convex by Benoist [22]. We will generalize the theorem to Theorem 4.3.1 without the hyperbolicity condition of Γ. Furthermore, we will show that the middle eigenvalue condition actually implies the existence of the properly convex domain U in Theorem 4.3.1. Also, the uniqueness of the set of asymptotic hyperspaces is given by Theorem 4.3.8.

The reason for presenting weaker Theorem 4.1.1 is to convey the basic idea of the proof of the generalized theorem.

4.2.1. The decomposition of the flow. We are assuming that Γ is hyperbolic. Since $\Sigma := \Omega/\Gamma$ is a closed strictly convex real projective orbifold, $\mathbb{U}\Sigma := \mathbb{U}\Omega/\Gamma$ is a compact smooth orbifold again. A geodesic flow on $\mathbb{U}\Omega/\Gamma$ is Anosov and hence topologically mixing. Hence, the flow is nonwandering everywhere. (See [20].) Γ acts irreducibly on Ω, and $\partial\mathrm{Cl}(\Omega)$ is C^1. Denote by $\Pi_\Omega : \mathbb{U}\Omega \to \Omega$ the projection to the base points.

We can identify $\mathrm{bd}\mathbb{A}^n = \mathbb{S}(V^n) = \mathbb{S}^{n-1}$ where g acts by $\mathscr{L}(g) \in \mathsf{GL}(n,\mathbb{R})$. We give a decomposition of $\tilde{\mathbf{V}}$ into three parts $\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_0, \tilde{\mathbf{V}}_-$:

- For each vector $\vec{u} \in \mathbb{U}\Omega$, we find the maximal oriented geodesic l ending at the backward endpoint $\partial_+ l$ and the forward endpoint $\partial_- l \in \partial\mathrm{Cl}(\Omega)$. They correspond to the 1-dimensional vector subspaces $\tilde{\mathbf{V}}_+(\vec{u})$ and $\tilde{\mathbf{V}}_-(\vec{u}) \subset V$.
- Recall that $\partial\mathrm{Cl}(\Omega)$ is C^1 since Ω is strictly convex (see [22]) There exists a unique pair of sharply supporting hyperspheres H_+ and H_- in $\mathrm{bd}\mathbb{A}^n$ at each of $\partial_+ l$ and $\partial_- l$. We denote by $H_0 = H_+ \cap H_-$. It is a codimension 2 great sphere in $\mathrm{bd}\mathbb{A}^n$ and corresponds to a vector subspace $\tilde{\mathbf{V}}_0(\vec{u})$ of codimension-two in $\tilde{\mathbf{V}}$.

- For each vector $\vec{u}$, we find the decomposition of V as $\tilde{\mathbf{V}}_+(\vec{u}) \oplus \tilde{\mathbf{V}}_0(\vec{u}) \oplus \tilde{\mathbf{V}}_-(\vec{u})$ and hence we can form the subbundles $\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_0, \tilde{\mathbf{V}}_-$ over $\mathbb{U}\Omega$ where
$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_+ \oplus \tilde{\mathbf{V}}_0 \oplus \tilde{\mathbf{V}}_-.$$

The map $\mathbb{U}\Omega \to \partial \mathrm{Cl}(\Omega)$ by sending a vector to the endpoint of the geodesic tangent to it is C^1. The map $\partial \mathrm{Cl}(\Omega) \to \mathscr{H}$ sending a boundary point to its sharply supporting hyperspace in the space $\mathscr{H}$ of hyperspaces in $\mathrm{bd}\mathbb{A}^n$ is continuous. Hence $\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_0$, and $\tilde{\mathbf{V}}_-$ are continuous bundles. Since the action preserves the decomposition of $\tilde{\mathbf{V}}$, $\mathbf{V}$ also decomposes as

$$\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_0 \oplus \mathbf{V}_-. \tag{4.2.1}$$

For each complete geodesic l in Ω, let $\vec{l}$ denote the set of unit vectors on l in one of the two directions. On $\vec{l}$, we have a decomposition

$$\tilde{\mathbf{V}}|\vec{l} = \tilde{\mathbf{V}}_+|\vec{l} \oplus \tilde{\mathbf{V}}_0|\vec{l} \oplus \tilde{\mathbf{V}}_-|\vec{l} \text{ of form}$$
$$\vec{l} \times \tilde{\mathbf{V}}_+(\vec{u}), \vec{l} \times \tilde{\mathbf{V}}_0(\vec{u}), \vec{l} \times \tilde{\mathbf{V}}_-(\vec{u}) \text{ for a vector } \vec{u} \text{ tangent to } l$$

where we recall:

- $\tilde{\mathbf{V}}_+(\vec{u})$ is the space of vectors in the direction of the backward endpoint of $\vec{l}$.
- $\tilde{\mathbf{V}}_-(\vec{u})$ is the space of vectors in the direction of the forward endpoint of $\vec{l}$.
- $\tilde{\mathbf{V}}_0(\vec{u})$ is the space vectors in directions of $H_0 = H_+ \cap H_-$ for ∂l.

That is, these bundles are constant bundles along l.

Suppose that $g \in \Gamma$ acts on a complete geodesic l with a unit vector $\vec{u}$ in the direction of the action of g. Then $\tilde{\mathbf{V}}_-(\vec{u})$ and $\tilde{\mathbf{V}}_+(\vec{u})$ corresponding to endpoints of l are respectively eigenspaces of the largest norm $\lambda_1(g)$ of the eigenvalues and the smallest norm $\lambda_n(g)$ of the eigenvalues of the linear part $\mathscr{L}(g)$ of g. Hence

- on $\tilde{\mathbf{V}}_-(\vec{u})$, g acts by expending by
$$\frac{\lambda_1(g)}{\lambda_{\tilde{E}}(g)} > 1, \tag{4.2.2}$$
and
- on $\tilde{\mathbf{V}}_+(\vec{u})$, g acts by contracting by
$$\frac{\lambda_n(g)}{\lambda_{\tilde{E}}(g)} < 1. \tag{4.2.3}$$

There exists a flow $\hat{\Phi}_t : \mathbb{U}\Omega \to \mathbb{U}\Omega$ for $t \in \mathbb{R}$ given by sending $\vec{v}$ to the unit tangent vector to at $\alpha(t)$ where α is a geodesic tangent to $\vec{v}$ with $\alpha(0)$ equal to the base point of $\vec{v}$.

We define a flow on $\tilde{\Phi}_t : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ by considering a unit-speed geodesic-flow line $\vec{l}$ in $\mathbb{U}\Omega$ and considering $\vec{l} \times \mathbb{A}^n$ and acting trivially on the second factor as we go from $\vec{v}$ to $\hat{\Phi}_t(\vec{v})$ (See remarks in the beginning of Section 3.3 and equations in Section 4.1 of [91].) Each flow line in $\mathbb{U}\Sigma$ lifts to a flow line on $\mathbf{A}$ from every point in it. This induces a flow $\Phi_t : \mathbf{A} \to \mathbf{A}$.

We defined a flow on $\tilde{\Phi}_t : \tilde{\mathbf{V}} \to \tilde{\mathbf{V}}$ by considering a unit-speed geodesic-flow line $\vec{l}$ in $\mathbb{U}\Omega$ and and considering $\vec{l} \times V$ and acting trivially on the second factor as we go from $\vec{v}$ to $\tilde{\Phi}_t(\vec{v})$ for each t. (This generalizes the flow on [91].) Also, $\mathscr{L}(\tilde{\Phi}_t)$ preserves $\tilde{\mathbf{V}}_+$, $\tilde{\mathbf{V}}_0$, and $\tilde{\mathbf{V}}_-$ since on the line l, the endpoint $\partial_\pm l$ does not change. Again, this induces a flow

$$\mathscr{L}(\Phi)_t : \mathbf{V} \to \mathbf{V}, \mathbf{V}_+ \to \mathbf{V}_+, \mathbf{V}_0 \to \mathbf{V}_0, \mathbf{V}_- \to \mathbf{V}_-.$$

We let $\|\cdot\|_S$ denote some metric on these bundles over $\mathbb{U}\Sigma/\Gamma$ defined as a fiberwise inner product: We chose a cover of Ω/Γ by compact sets K_i and choosing a metric over $K_i \times \mathbb{A}^n$ and use the partition of unity. This induces a fiberwise metric on $\mathbf{V}$ as well. Pulling the metric back to $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{V}}$, we obtain a fiberwise metrics to be denoted by $\|\cdot\|_S$.

By the uniform middle-eigenvalue condition, $\mathbf{V}$ satisfies the following properties for $\vec{u} \in \mathbb{U}\Omega/\Gamma$:

- the flat linear connection $\nabla^{\mathbf{V}}$ on $\mathbf{V}$ is bounded with respect to $\|\cdot\|_{\text{fiber}}$.
- hyperbolicity: There exists constants $C, k > 0$ so that

$$\|\mathscr{L}(\Phi_t)(\vec{v})\|_{\text{fiber}} \geq \frac{1}{C}\exp(kt)\,\|\vec{v}\|_{\text{fiber}} \text{ as } t \to \infty \tag{4.2.4}$$

for $\vec{v} \in \mathbf{V}_+$ and

$$\|\mathscr{L}(\Phi_t)(\vec{v})\|_{\text{fiber}} \leq C\exp(-kt)\,\|\vec{v}\|_{\text{fiber}} \text{ as } t \to \infty \tag{4.2.5}$$

for $\vec{v} \in \mathbf{V}_-$

Using Proposition 4.2.1, we prove this property by taking C sufficiently large according to t_1, which is a standard technique.

4.2.2. The proof of the proximal property. We may assume that Γ has no finite order elements by taking a finite index group using Theorem 1.1.19. Also, by Benoist [22], elements of Γ are positive bi-proximal. (See Theorem 1.3.12.)

We can apply this to $\mathbf{V}_-$ and $\mathbf{V}_+$ by possibly reversing the direction of the flow. The Anosov property follows from the following proposition.

Let $\mathbf{V}_{-,1}$ denote the subset of $\mathbf{V}_-$ of the unit length under $\|\cdot\|_{\text{fiber}}$.

PROPOSITION 4.2.1. *Let Ω/Γ be a closed strictly convex real projective orbifold with hyperbolic fundamental group Γ. Then there exists a constant t_1 so that*

$$\|\mathscr{L}(\Phi_t)(\vec{v})\|_{\text{fiber}} \leq \tilde{C}\,\|\vec{v}\|_{\text{fiber}}, \vec{v} \in \mathbf{V}_- \text{ and } \|\mathscr{L}(\Phi)_{-t}(\vec{v})\|_{\text{fiber}} \leq \tilde{C}\,\|\vec{v}\|_{\text{fiber}}, \vec{v} \in \mathbf{V}_+$$

for $t \geq t_1$ and a uniform $\tilde{C}$, $0 < \tilde{C} < 1$.

PROOF. It is sufficient to prove the first part of the inequalities since we can substitute $t \to -t$ and switching $\mathbf{V}_+$ with $\mathbf{V}_-$ as the direction of the vector changed to the opposite one.

By following Lemma 4.2.5, the uniform convergence implies that for given $0 < \varepsilon < 1$, for every vector $\vec{v}$ in $\mathbf{V}_{-,1}$, there exists a uniform T so that for $t > T$, $\mathscr{L}(\Phi_t)(\vec{v})$ is in an ε-neighborhood $U_\varepsilon(S_0)$ of the image S_0 of the zero section. Hence, we obtain that $\mathscr{L}(\Phi)_t$ is uniformly contracting near S_0, which implies the result. □

Now, we will prove Lemma 4.2.5; but we need some preliminary material:

REMARK 4.2.2. We need to only prove the following for a finite index group of $\mathbf{\Gamma}$ since the contracting properties are invariant under finite regular covering maps. Hence, we may assume that each element is proximal or semi-proximal by Theorem 1.3.12.

LEMMA 4.2.3. *Let Γ act properly discontinuously on a strictly convex domain Ω. Assume that g_i is a sequence of distinct positive bi-proximal elements of Γ. Suppose that the sequence of attracting fixed point a_i and that of repelling fixed point $\{r_i\}$ of g_i form sequences converging to distinct pair of points. Then*

$$\{\text{length}_\Omega(g_i)\} \to \infty \tag{4.2.6}$$

PROOF. Since $\{a_i\} \to a_*$ and $\{r_i\} \to r_*$, the segment $\overline{a_i r_i}$ passes a fixed compact domain U in Ω for sufficiently large i. Suppose that $\mathrm{length}_\Omega(g_i) < C$ for a constant C. Then $g_i(U)$ passes $\overline{a_i r_i}$ for each i. Hence, $g_i(U)$ is a subset a ball of radius $2L + C$. Since $\{g_i\}$ form a sequence of mutually distinct elements, this contradicts the proper discontinuity of the action of Γ. □

LEMMA 4.2.4. *Let Ω be strictly convex and C^1. We choose a subsequence $\{g_i\}$ of positive-bi-proximal elements of Γ so that the sequences $\{a_i\}$ and $\{r_i\}$ are convergent for the attracting fixed point $a_i \in \mathrm{Cl}(\Omega)$ and the repelling fixed point $r_i \in \mathrm{Cl}(\Omega)$ of each g_i. Suppose that*

$$\{a_i\} \to a_* \text{ and } \{r_i\} \to r_* \text{ for } a_*, r_* \in \partial \mathrm{Cl}(\Omega), a_* \neq r_*.$$

Suppose that g_i is an unbounded sequence. Then for every compact $K \subset \mathrm{Cl}(\Omega) - \{r_*\}$,

$$\{g_i(K)\} \to \{a_*\} \tag{4.2.7}$$

uniformly.

PROOF. Each g_i acts on an $(n-3)$-dimensional subspace W_{g_i} in $\mathbb{S}^{n-1}_\infty$ disjoint from Ω. Here, W_{g_i} is the intersection of two sharply supporting hyperspaces of Ω at a_i and r_i. The set $\{W_{g_i}\}$ is precompact by our condition. By the C^1-property, we may assume that $\{W_{g_i}\} \to W_*$ for an $n-3$-dimensional subspace W_* that is the intersection of two hyperspaces supported at a_* and r_*. Also, $W_{g_i} \cap \mathrm{Cl}(\Omega) = \emptyset$ by this property.

Let η_i denote the complete geodesic connecting a_i and r_i. Let η_∞ denote the one connecting a_* and r_*. Since W_* is the intersection of two sharply supporting hyperspaces of Ω at a_* and r_*, η_∞ has endpoints a_*, r_*, and Ω is strictly convex, it follows $\langle \eta_\infty \rangle \cap W_* = \emptyset$.

We call $P_i \cap \Omega$ for the $n-2$-dimensional subspace P_i containing W_{g_i} a *slice* of g_i. The closure of a component of Ω with a slice of g_i removed is called a *half-space* of g_i.

Let H_i denote the half-space of g_i containing K. Since $\{\mathrm{Cl}(\eta_i)\}$ and $\{W_{g_i}\}$ are geometrically convergent respectively, and $\langle \eta_\infty \rangle \cap W_\infty = \emptyset$, it follows that $\{g_i(P_i)\}$ geometrically converges to a hyperspace containing W_∞ passing a_*. Therefore, one deduces easily that $\{g_i(H_i)\} \to \{a_*\}$ geometrically. Since $K \subset H_i$, the lemma follows. □

The line bundle $\mathbf{V}_-$ lifts to $\tilde{\mathbf{V}}_-$ where each unit vector $\vec{u}$ on Ω one associates the line $\mathbf{V}_{-,\vec{u}}$ corresponding to the starting point in $\partial \mathrm{Cl}(\Omega)$ of the oriented geodesic l tangent to it. $\tilde{\mathbf{V}}_-|\vec{l}$ equals $\vec{l} \times \mathbf{V}_{-,\vec{u}}$. $\mathscr{L}(\Phi)_t$ lifts to a parallel translation or constant flow $\mathscr{L}(\tilde{\Phi})_t$ of form

$$(\vec{u}, \vec{v}) \to (\hat{\Phi}_t(\vec{u}), \vec{v}).$$

LEMMA 4.2.5. *Suppose that Ω is strictly convex with $\partial \mathrm{Cl}(\Omega)$ being C^1, and Γ acts properly discontinuously and cocompactly on Ω satisfying the uniform middle eigenvalue condition. Then $\{\|\mathscr{L}(\Phi)_t|\mathbf{V}_-\|_{\mathrm{fiber}}\} \to 0$ uniformly as $t \to \infty$.*

PROOF. Let F be a fundamental domain of $\mathbb{U}\Omega$ under Γ. It is sufficient to prove this for $\mathscr{L}(\tilde{\Phi})_t$ on the fibers of over F of $\mathbb{U}\Omega$ with a fiberwise metric $\|\cdot\|_{\mathrm{fiber}}$.

We choose an arbitrary sequence $\{x_i\}$, $\{x_i\} \to x$ in F. For each i, let $\vec{v}_{-,i}$ be a Euclidean unit vector in $V_{-,i} := \tilde{\mathbf{V}}_-(x_i)$ for the unit vector $x_i \in \mathbb{U}\Omega$. That is, $\vec{v}_{-,i}$ is in the 1-dimensional subspace in $\mathbb{R}^n$, corresponding to the backward endpoint of the geodesic l_i in Ω determined by x_i in $\partial \mathrm{Cl}(\Omega)$ and in a direction of $\mathrm{Cl}(\Omega)$.

We will show that

$$\left\{ \left\| \mathscr{L}(\tilde{\Phi}_{t_i})(x_i, \vec{v}_{-,i}) \right\|_{\mathrm{fiber}} \right\} \to 0 \text{ for any sequence } t_i \to \infty,$$

which is sufficient to prove the uniform convergence to 0 by the compactness of $\mathbf{V}_{-,1}$.

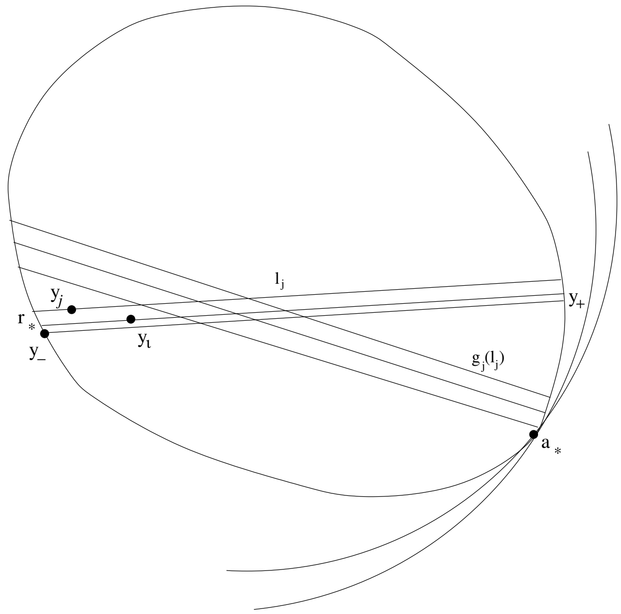

FIGURE 1. The figure for Lemma 4.2.5. Here y_i, y_j denote the images under Π_Ω the named points in the proof of Lemma 4.2.5.

It is sufficient to show that any sequence of $\{t_i\} \to \infty$ has a subsequence $\{t_j\}$ so that

$$\{\left\|\mathscr{L}(\tilde{\Phi}_{t_j})(x_i, \vec{v}_{-,j})\right\|_{\text{fiber}}\} \to 0.$$

This follows since if the uniform convergence did not hold, then we can easily find a sequence without such subsequences.

Let $y_i := \hat{\Phi}_{t_i}(x_i)$ for the lift of the flow $\hat{\Phi}$. By construction, we recall that each $\Pi_\Omega(y_i)$ is in the geodesic l_i. Since we have the sequence of vectors $\{x_i\} \to x$, $x_i, x \in F$, we obtain that $\{l_i\}$ geometrically converges to a line l_∞ passing $\Pi_\Omega(x)$ in Ω. Let y_+ and y_- be the endpoints of l_∞ where $\{\Pi_\Omega(y_i)\} \to y_-$. Hence,

$$\{(\!(\vec{v}_{+,i})\!)\} \to y_+, \{(\!(\vec{v}_{-,i})\!)\} \to y_-.$$

Find a deck transformation g_i so that $g_i(y_i) \in F$ and g_i acts on the line bundle $\tilde{\mathbf{V}}_-$ by the linearization of the matrix of form of (4.1.1):

$$\begin{aligned}
&\mathscr{L}(g_i) : \tilde{\mathbf{V}}_- \to \tilde{\mathbf{V}}_- \text{ with}\\
&(y_i, \vec{v}) \mapsto (g_i(y_i), \mathscr{L}(g_i)(\vec{v})) \text{ where}\\
&\mathscr{L}(g_i) = \frac{1}{\lambda_{\tilde{E}}(g_i)^{1+\frac{1}{n}}} \hat{h}(g_i) : \tilde{\mathbf{V}}_-(y_i) = \tilde{\mathbf{V}}_-(x_i) \to \tilde{\mathbf{V}}_-(g_i(y_i)).
\end{aligned} \tag{4.2.8}$$

(Goal): We will show $\{(g_i(y_i), \mathscr{L}(g_i)(\vec{v}_{-,i}))\} \to 0$ under $\|\cdot\|_{\text{fiber}}$. This will complete the proof since g_i acts as isometries on $\tilde{\mathbf{V}}_-$ with $\|\cdot\|_{\text{fiber}}$.

Also, we may assume that $\{g_i\}$ is a sequence of mutually distinct elements up to a choice of subsequences since $g_i^{-1}(F)$ contains y_i and y_i forms an unbounded sequence.

Since $g_i(l_i) \cap F \neq \emptyset$, we choose a subsequence of g_i and relabel it g_i so that $\{g_i(l_i)\}$ converges to a nontrivial line $\hat{l}_\infty$ in Ω.

By our choice of l_i, y_i, g_i as above, and Remark 4.2.2, we may assume without loss of generality that each g_i is positive bi-proximal since Ω is strictly convex.

We choose a subsequence of $\{g_i\}$ so that the sequences $\{a_i\}$ and $\{r_i\}$ are convergent for the attracting fixed point $a_i \in \mathrm{Cl}(\Omega)$ and the repelling fixed point $r_i \in \mathrm{Cl}(\Omega)$ of each g_i. Then

$$\{a_i\} \to a_* \text{ and } \{r_i\} \to r_* \text{ for } a_*, r_* \in \partial \mathrm{Cl}(\Omega).$$

(See Figure 1.)

Suppose that $a_* = r_*$. Then we choose an element $g \in \Gamma$ so that $g(a_*) \neq r_*$ and replace the sequence by $\{gg_i\}$ and replace F by $F \cup g(F)$. The above uniform convergence condition still holds. Then for the new attracting fixed points a_i' of gg_i, we have $\{a_i'\} \to g(a_*)$ and the sequence $\{r_i'\}$ of repelling fixed point r_i' of gg_i converges to r_* also by Lemma 5.3.8. Hence, we may assume without loss of generality that

$$a_* \neq r_*$$

by replacing our sequence g_i.

Now, Lemma 4.2.4 shows that for every compact $K \subset \mathrm{Cl}(\Omega) - \{r_*\}$,

$$\{g_i(K)\} \to \{a_*\} \qquad (4.2.9)$$

uniformly.

Suppose that both $y_+, y_- \neq r_*$. Then $\{g_i(l_i)\}$ converges to a singleton $\{a_*\}$ by (4.2.9) and this cannot be since $\hat{l}_\infty \subset \Omega$. If

$$r_* = y_+ \text{ and } y_- \in \partial \mathrm{Cl}(\Omega) - \{r_*\},$$

then $\{g_i(y_i)\} \to a_*$ by (4.2.9) again. Since $g_i(y_i) \in F$, this is a contradiction. Therefore

$$r_* = y_- \text{ and } y_+ \in \partial \mathrm{Cl}(\Omega) - \{r_*\}.$$

Let $d_i = (\!(\vec{v}_{+,i})\!)$ denote the other endpoint of l_i from $(\!(\vec{v}_{-,i})\!)$.

- Since $\{(\!(\vec{v}_{-,i})\!)\} \to y_-$ and $\{l_i\}$ converges to a nontrivial line l_∞, it follows that $\{d_i = (\!(\vec{v}_{+,i})\!)\}$ is in a compact set in $\partial \mathrm{Cl}(\Omega) - \{r_*\}$, and $\{d_i\} \to y_+$.
- Then $\{g_i(d_i)\} \to a_*$ as $\{d_i\}$ is in a compact set in $\partial \mathrm{Cl}(\Omega) - \{r_*\}$.
- Thus, $\{g_i((\!(\vec{v}_{-,i})\!))\} \to y' \in \partial \mathrm{Cl}(\Omega)$ where $a_* \neq y'$ holds since $\{g_i(l_i)\}$ converges to a nontrivial line $\hat{l}_\infty$ in Ω as we said shortly above.

Also, g_i has an invariant great sphere $\mathbb{S}^{n-2}_i \subset \mathrm{bd}\mathbb{A}^n$ containing the attracting fixed point a_i and sharply supporting Ω at a_i. Thus, r_i is uniformly bounded at a distance from $\mathbb{S}^{n-2}_i$ since $\{r_i\} \to y_- = r_*$ and $\{a_i\} \to a_*$ with $\mathbb{S}^{n-2}_i$ geometrically converging to a sharply supporting sphere $\mathbb{S}^{n-2}_*$ at a_*.

Let $\|\cdot\|_E$ denote the standard Euclidean metric of $\mathbb{R}^n$.

- Since $\{\Pi_\Omega(y_i)\} \to y_-$, $\Pi_\Omega(y_i)$ is also uniformly bounded away from a_i and the tangent sphere $\mathbb{S}^{n-1}_i$ at a_i.
- Since $\{(\!(\vec{v}_{-,i})\!)\} \to y_-$, the vector $\vec{v}_{-,i}$ has the component $\vec{v}^p_i$ parallel to r_i and the component $\vec{v}^S_i$ in the direction of $\mathbb{S}^{n-2}_i$ where $\vec{v}_{-,i} = \vec{v}^p_i + \vec{v}^S_i$.
- Since $\{r_i\} \to r_* = y_-$ and $\{(\!(\vec{v}_{-,i})\!)\} \to y_-$, we obtain $\{\|\vec{v}^S_i\|_E\} \to 0$ and that
$$\frac{1}{C} < \|\vec{v}^p_i\|_E < C$$
for some constant $C > 1$.
- g_i acts by preserving the directions of $\mathbb{S}^{n-2}_i$ and r_i.

Since $\{g_i((\!(\vec{v}_{-,i})\!))\}$ converging to y', $y' \in \partial \mathrm{Cl}(\Omega)$, is bounded away from $\mathbb{S}^{n-2}_i$ uniformly, we obtain that

- considering the homogeneous coordinates
$$\left(\!\left(\mathscr{L}(g_i)(\vec{v}_i^S) : \mathscr{L}(g_i)(\vec{v}_i^p)\right)\!\right),$$
we obtain that the Euclidean norm of
$$\frac{\mathscr{L}(g_i)(\vec{v}_i^S)}{\left\|\mathscr{L}(g_i)(\vec{v}_i^p)\right\|_E}$$
is bounded above uniformly.

Since r_i is a repelling fixed point of g_i and $\left\|\vec{v}_i^p\right\|_E$ is uniformly bounded above, $\{\mathscr{L}(g_i)(\vec{v}_i^p)\} \to 0$ by (4.1.3) and (4.2.6).

$$\{\mathscr{L}(g_i)(\vec{v}_i^p)\} \to 0 \text{ implies } \{\mathscr{L}(g_i)(\vec{v}_i^S)\} \to 0$$

for $\|\cdot\|_E$. Hence, we obtain

$$\{\|\mathscr{L}(g_i)(\vec{v}_{-,i}))\|_E\} \to 0. \tag{4.2.10}$$

Recall that $\mathscr{L}(\tilde{\Phi})_t$ is the identity map on the second factor of $\mathbb{U}\Omega \times V^n$.

$$g_i(\mathscr{L}(\tilde{\Phi})_{t_i}(x_i, \vec{v}_{-,i})) = (g_i(y_i), \mathscr{L}(g_i)(\vec{v}_{-,i})) \in F \times \mathbf{V}_-$$

is a vector over the compact fundamental domain F of $\mathbb{U}\Omega$.

$$(g_i(y_i), \mathscr{L}(g_i)(\vec{v}_{-,i}))$$

is a vector over the compact fundamental domain F of $\mathbb{U}\Omega$.

$$\left\{\left\|\mathscr{L}(g_i)(\tilde{\Phi}_{t_i})(x_i, \vec{v}_{-,i})\right\|_E\right\} \to 0 \text{ implies } \left\{\left\|\mathscr{L}(g_i)(\tilde{\Phi}_{t_i})(x_i, \vec{v}_{-,i})\right\|_{\text{fiber}}\right\} \to 0$$

since $g_i(x_i) \in F$ and $\|\cdot\|_{\text{fiber}}$ and $\|\cdot\|_E$ are compatible over points F. Since g_i is an isometry of $\|\cdot\|_{\text{fiber}}$, we conclude that $\left\{\left\|\mathscr{L}(\tilde{\Phi}_{t_i})(x, \vec{v}_{-,i})\right\|_{\text{fiber}}\right\} \to 0$: □

4.2.3. The neutralized section. We denote by $\Gamma(\mathbf{V})$ the space of sections $\mathbb{U}\Sigma \to \mathbf{V}$ and by $\Gamma(\mathbf{A})$ the space of sections $\mathbb{U}\Sigma \to \mathbf{A}$. Recall from [91] the one parameter-group of operators $D\Phi_{t,*}$ on $\Gamma(\mathbf{V})$ and $\Phi_{t,*}$ on $\Gamma(\mathbf{A})$. In our terminology $D\Phi_{t,*} = \mathscr{L}(\Phi_t)$. Recall Lemma 8.3 of [91] also. We denote by ϕ the vector field generated by this flow on $\mathbb{U}\Sigma$.

A section $s : \mathbb{U}\Sigma \to \mathbf{A}$ is *neutralized* if

$$\nabla^{\mathbf{A}}_{\phi} s \in \mathbf{V}_0. \tag{4.2.11}$$

LEMMA 4.2.6. *If $\psi \in \Gamma(\mathbf{A})$, and*

$$t \mapsto D\Phi_{t,*}(\psi)$$

is a path in $\Gamma(\mathbf{V})$ that is differentiable at $t = 0$, then

$$\left.\frac{d}{dt}\right|_{t=0} (D\Phi_t)_*(\psi) = \nabla^{\mathbf{A}}_{\phi}(\psi).$$

Recall that $\mathbb{U}\Sigma$ is a recurrent set under the geodesic flow.

LEMMA 4.2.7. *A neutralized section $s_0 : \mathbb{U}\Sigma \to \mathbf{A}$ exists. This lifts to a map $\tilde{s}_0 : \mathbb{U}\Omega \to \tilde{\mathbf{A}}$ so that $\tilde{s}_0 \circ \gamma = \gamma \circ \tilde{s}_0$ for each γ in Γ acting on $\tilde{\mathbf{A}} = \mathbb{U}\Omega \times \mathbb{A}^n$.*

PROOF. Let s be a continuous section $\mathbb{U}\Sigma \to \mathbf{A}$. We construct $\nabla^{\mathbf{A}_\pm}$ by projecting the values of ∇ to $\mathbf{V}_\pm$ and $\nabla^{\mathbf{A}_0}$ by projecting the values of ∇ to $\mathbf{V}_0$. We decompose

$$\nabla^{\mathbf{A}}_\phi(s) = \nabla^{\mathbf{A}_+}_\phi(s) + \nabla^{\mathbf{A}_0}_\phi(s) + \nabla^{\mathbf{A}_-}_\phi(s) \in \mathbf{V}$$

so that $\nabla^{\mathbf{A}_\pm}_\phi(s) \in \mathbf{V}_\pm$ and $\nabla^{\mathbf{A}_0}_\phi(s) \in \mathbf{V}_0$ hold. This can be done since along the vector field ϕ, $\mathbf{V}_\pm$ and $\mathbf{V}_0$ are constant bundles. By the uniform convergence property of (4.2.4) and (4.2.5), the following integrals converge to smooth functions over $\mathbb{U}\Sigma$. Again

$$s_0 = s + \int_0^\infty (D\Phi_t)_*(\nabla^{\mathbf{A}_-}_\phi(s))dt - \int_0^\infty (D\Phi_{-t})_*(\nabla^{\mathbf{A}_+}_\phi(s))dt$$

is a continuous section and $\nabla^{\mathbf{A}}_\phi(s_0) = \nabla^{\mathbf{A}_0}_\phi(s_0) \in \mathbf{V}_0$ as shown in Lemma 8.4 of [91].

Since $\mathbb{U}\Sigma$ is connected, there exists a fundamental domain F so that we can lift s_0 to $\tilde{s}_0'$ defined on F mapping to $\mathbf{A}$. We can extend $\tilde{s}_0'$ to $\mathbb{U}\Omega \to \mathbb{U}\Omega \times \mathbb{A}^n$. □

Let $N_2(\mathbb{A}^n)$ denote the space of codimension two affine subspaces of $\mathbb{A}^n$. We denote by $G(\Omega)$ the space of maximal oriented geodesics in Ω. We use the quotient topology on both spaces. There exists a natural action of Γ on both spaces.

For each element $g \in \Gamma - \{\mathrm{I}\}$, we define $N_2(g)$: Now, g acts on $\mathrm{bd}\mathbb{A}^n$ with invariant subspaces corresponding to invariant subspaces of the linear part $\mathscr{L}(g)$ of g. Since g and g^{-1} are positive proximal,

- a unique fixed point in $\mathrm{bd}\mathbb{A}^n$ corresponds to the largest norm eigenvector, an attracting fixed point in $\mathrm{bd}\mathbb{A}^n$, and
- a unique fixed point in $\mathrm{bd}\mathbb{A}^n$ corresponds to the smallest norm eigenvector, a repelling fixed point

by [20] or [17]. There exists an $\mathscr{L}(g)$-invariant vector subspace $\mathbf{V}^0_g$ complementary to the sum of the subspace generated by these eigenvectors. (This space equals $\mathbf{V}_0(\vec{u})$ for the unit tangent vector $\vec{u}$ tangent to the unique maximal geodesic l_g in Ω on which g acts.) It corresponds to a g-invariant subspace $M(g)$ of codimension two in $\mathrm{bd}\mathbb{A}^n$.

Let $\tilde{c}$ be the geodesic in $\mathbb{U}\Sigma$ that is g-invariant for $g \in \Gamma$. $\tilde{s}_0(\tilde{c})$ lies on a fixed affine subspace parallel to V^0_g by the neutrality, i.e., Lemma 4.2.7. There exists a unique affine subspace $N_2(g)$ of codimension two in $\mathbb{A}^n$ containing $\tilde{s}_0(\tilde{c})$. Immediate properties are $N_2(g) = N_2(g^m), m \in \mathbb{Z} - \{0\}$ and that g acts on $N_2(g)$.

DEFINITION 4.2.8. We define $S'(\partial\mathrm{Cl}(\Omega))$ the space of hyperspaces P meeting $\mathbb{A}^n$ where $P \cap \mathrm{bd}\mathbb{A}^n$ is a sharply supporting hyperspace in $\mathrm{bd}\mathbb{A}^n$ to Ω. We denote by $S(\partial\mathrm{Cl}(\Omega))$ the space of pairs (x, H) where $H \in S'(\partial\mathrm{Cl}(\Omega))$, and x is in the boundary of H and in $\partial\mathrm{Cl}(\Omega)$.

Define Δ to be the diagonal set of $\partial\mathrm{Cl}(\Omega) \times \partial\mathrm{Cl}(\Omega)$. Denote by $\Lambda^* = \partial\mathrm{Cl}(\Omega) \times \partial\mathrm{Cl}(\Omega) - \Delta$. Let $G(\Omega)$ denote the space of maximal oriented geodesics in Ω. $G(\Omega)$ is in a one-to-one correspondence with Λ^* by the map taking the maximal oriented geodesic to the ordered pair of its endpoints.

LEMMA 4.2.9. *$G(\Omega)$ is a connected subspace.*

PROOF. We obtain the proof by generalizing Lemma 1.3 of [91] using a bi-proximal element of Γ. [22]. Or one can use the fact that $\mathbb{U}\Omega$ is connected. □

PROPOSITION 4.2.10.

- *There exists a continuous function $\hat{s} : \mathbb{U}\Omega \to N_2(\mathbb{A}^n)$ equivariant with respect to Γ-actions.*

- *Given $g \in \Gamma$ and for the unique unit-speed geodesic $\vec{l}_g$ in $\mathbb{U}\Omega$ lying over a geodesic l_g where g acts on, $\hat{s}(\vec{l}_g) = N_2(g)$.*
- *This gives a continuous map*
$$\bar{s} : G(\Omega) = \partial \mathrm{Cl}(\Omega) \times \partial \mathrm{Cl}(\Omega) - \Delta \to N_2(\mathbb{A}^n)$$
again equivariant with respect to the Γ-actions. There exists a continuous function
$$\tau : \Lambda^* \to S(\partial \mathrm{Cl}(\Omega)).$$

PROOF. Given a vector $\vec{u} \in \mathbb{U}\Omega$, we find $\tilde{s}_0(\vec{u})$. There exists a lift $\tilde{\phi}_t : \mathbb{U}\Omega \to \mathbb{U}\Omega$ of the geodesic flow ϕ_t. Then $\tilde{s}_0(\tilde{\phi}_t(\vec{u}))$ is in an affine subspace H^{n-2} parallel to V_0 for $\vec{u}$ by the neutrality condition (4.2.11). We define $\hat{s}(\vec{u})$ to be this H^{n-2}.

For any unit vector $\vec{u}'$ on the maximal (oriented) geodesic in Ω determined by $\vec{u}$, we obtain $\hat{s}(\vec{u}') = H^{n-2}$. Hence, this determines the continuous map $\bar{s} : G(\Omega) \to N_2(\mathbb{A}^n)$. The Γ-equivariance comes from that of $\tilde{s}_0$.

For $g \in \Gamma$, $\vec{u}$ and $g(\vec{u})$ lie on the lift $\vec{l}'_g$ of the g-invariant geodesic $\vec{l}_g$ in $\mathbb{U}\Omega$ provided $\vec{u}$ is tangent to $\vec{l}_g$. Since $g(\tilde{s}_0(\vec{u})) = \tilde{s}_0(g(\vec{u}))$ by equivariance, $g(\tilde{s}_0(\vec{u}))$ lies on $\hat{s}(\vec{u}) = \hat{s}(g(\vec{u}))$ in $\mathbb{U}\Omega$ by the third paragraph before the proposition. We conclude $g(\bar{s}(\vec{l}'_g)) = \bar{s}(\vec{l}'_g)$, which shows $N_2(g) = \hat{s}(\vec{l}_g)$.

The map $\bar{s}$ is defined since $\partial \mathrm{Cl}(\Omega) \times \partial \mathrm{Cl}(\Omega) - \Delta$ is in one-to-one correspondence with the space $G(\Omega)$. The map τ is defined by taking for each pair $(x, y) \in \Lambda^*$

- we take the geodesic l with endpoints x and y, and
- taking the hyperspace containing $\bar{s}(l)$ and its boundary containing x.

□

4.2.4. The asymptotic niceness. We denote by $h(x,y)$ the hyperspace part in $\tau(x,y) = (x, h(x,y))$.

LEMMA 4.2.11. *Let U be a $\mathbf{\Gamma}_{\tilde{E}}$-invariant properly convex open domain in $\mathbb{R}^n$ so that* $\mathrm{bd}U \cap \mathrm{bd}\mathbb{A}^n = \mathrm{Cl}(\Omega)$. *Suppose that x and y are attracting and repelling fixed points of an element g of Γ in $\partial \mathrm{Cl}(\Omega)$. Then $h(x,y)$ is disjoint from U.*

PROOF. Suppose not. $h'(x,y) := h(x,y) \cap \mathbb{A}^n$ is a g-invariant open hemisphere, and x is an attracting fixed point of g in it. (We can choose g^{-1} if necessary.) Then $U \cap h(x,y)$ is a g-invariant properly convex open domain containing x in its boundary.

Suppose first that $h'(x,y)$ has a limit point z of $g^{-n}(u)$ for some point $u \in h'(x,y) \cap U$.

Here, y is the smallest eigenvalue of the linear part of g so that y is the attracting fixed point of a component of $\mathbb{A}^n - h'(x,y)$ containing U for g^{-1}. The antipodal point y_- is the attracting fixed point of the other component $\mathbb{A}^n - h'(x,y)$. Take a ball B in U with a center u in the convex set $U \cap h(x,y)$ Then $\{g^{-n}(u)\}$ converges to z as $n \to \infty$. Let u_1, u_2 be two nearby points in B so that $\overline{u_1 u_2}$ is separated by $h'(x,y)$ and $\overline{u_1 u_2} \cap h'(x,y) = u$. Then $\{g^{-n_i}(\overline{u_1 u_2})\}$ geometrically coverges to $\overline{yz} \cup \overline{y_- z}$ for some sequence n_i. Hence $\mathrm{Cl}(U)$ cannot be properly convex.

If the above assumption does not hold, then an orbit $g^{-n}(u)$ for $u \in U \cap h'(x,y)$ has a limit point only in the boundary of $h'(x,y)$. Since g is biproximal, x is the repelling fixed point of $h'(x,y)$ under g^{-1}. Hence, a limit point y' is never x or x_-.

Since y' is a limit point, $y' \in \mathrm{Cl}(U)$. It follows $y' \in \mathrm{Cl}(\Omega)$. Now, $x, y' \in \mathrm{Cl}(\Omega)$ implies $\overline{xy'} \subset \mathrm{bd}\mathbb{A}^n \subset \mathrm{Cl}(\Omega)$. Finally, $\overline{xy'} \subset \partial h'(x,y)$ for the sharply supporting subspace $\partial h'(x,y)$ of $\mathrm{Cl}(\Omega)$ violates the strict convexity of Ω. (See Definition 6.0.3 and Benoist [20].)

□

Lemma 4.2.12 will be generalized to Lemma 4.3.7 with a proof generalized word-for-word.

LEMMA 4.2.12. *Let U be a $\mathbf{\Gamma}_{\tilde{E}}$-invariant properly convex open domain in $\mathbb{R}^n$ so that* $\mathrm{bd}U \cap \mathrm{bd}\mathbb{A}^n = \mathrm{Cl}(\Omega)$. *Let Γ acts on strictly convex domain Ω with $\partial\Omega$ being C^1 in a cocompact manner. Let $(x,y) \in \partial\mathrm{Cl}(\Omega) \times \partial(\mathrm{Cl}(\Omega) - \Delta$. Then*

- *$\tau(x,y)$ does not depend on y and is unique for each x.*
- *$h'(x,y) := h(x,y) \cap \mathbb{A}^n$ contains $\bar{s}(\overline{xy})$ but is independent of y and $h(x,y) = h(x)$.*
- *The map $\tau' : \partial\mathrm{Cl}(\Omega) \to S(\partial\mathrm{Cl}(\Omega))$ induced from τ is continuous.*
- *if any open set U' so that (Γ, U', Ω) is a properly convex triple, then the AS-hyperspace at each point of $\partial\mathrm{Cl}(\Omega)$ exists and is the same as that of U.*

PROOF. Let l_1 be an augmented geodesic in $\mathbb{U}\Omega$ with endpoints x and z oriented towards x. Consider a connected subspace $\mathscr{L}_x$ of $\mathbb{U}\Omega$ of points of maximal augmented geodesics in Ω ending at x. The space of geodesic leaves in $\mathscr{L}_x$ is in one-to-one correspondence with $\partial\mathrm{Cl}(\Omega) - \{x\}$. We will show that τ is locally constant on $\mathscr{L}_x$ showing that it is constant.

Let $\tilde{l}_1$ denote the lift of l_1 in $\mathbb{U}\Omega$. Let S be a compact neighborhood in $\mathscr{L}_x$ of a point y of $\tilde{l}_1$ transverse to $\tilde{l}_1$. Any two rays of geodesic flow $\Phi : S \times \mathbb{R} \to \mathbb{U}\Omega$ are asymptotic on $\mathscr{L}_{(x,h_1)}$ by Lemma 3.1.4.

Let $y \in \tilde{l}_1$. Consider another point $y' \in S \subset \mathbb{U}\Omega$ with with endpoints x and z' where

$$(x,z), (x,z') \in \Lambda^*.$$

Choose a fixed fundamental domain F of $\hat{\mathbb{U}}\Omega$. Let $\{y_i = \Phi_{t_i}(y)\}, y_i \in \tilde{l}_1$, be a sequence whose projection under Π_Ω convergs to x. We use a deck transformation g_i so that $g_i(y_i) \in F$. Then $g_i(\tau(l_1)) = \tau(g_i(l_1))$ is a hyperspace containing $g_i(x)$ and $\hat{s}(g_i(\tilde{l}_1))$.

Let $\vec{v}_+$ denote a vector in the direction of the end of l_1 other than x. Equation (4.2.4) shows that $\{\|\mathscr{L}(\Phi_t)|\mathbf{V}_+\|_{\mathrm{fiber}}\} \to \infty$ as $t \to \infty$. Since g_i is isometry under $\|\cdot\|_{\mathrm{fiber}}$, and $\hat{\Phi}_{t_i}(y) = y_i$ and $g_i(y_i) \in F$, it follows that the $\tilde{\mathbf{V}}_+$-component of $g_i(y_i, \vec{v}_+)$ satisfies

$$\{\|\mathscr{L}(g_i)(\vec{v}_+)\|_{\mathrm{fiber}}\} \to \infty. \tag{4.2.12}$$

Since $g_i(y_i) \in F$ and under the Euclidean norm since over a compact set F the metrics are compatible by a uniform constant, we obtain $\{\|\mathscr{L}(g_i)(\vec{v}_+)\|_E\} \to \infty$.

Since the affine hyperplanes in $\tau((x,z))$ and $\tau((x,z'))$ contain x in their boundary, they restrict to parallel affine hyperplanes in $\mathbb{A}$. Suppose that the affine hyperspace part of $\tau(x,z)$ differs from one of $\tau((x,z')$ by a translation by a constant times $\vec{v}_+$. This implies that the sequence of the Euclidean distances between the respective affine hyperspaces corresponding to

$$g_i(\tau((x,z)) \text{ and } g_i(\tau((x,z'))$$

goes to infinity as $i \to \infty$.

Now consider $\Phi(S \times [t_i, t_i + 1]) \subset \mathbb{U}\Omega$, and we have obtained g_i so that $g_i(\Phi(S \times [t_i, t_i + 1]))$ is in a fixed compact subset $\hat{P}$ of $\mathbb{U}\Omega$ by the uniform boundedness of $\Phi(S \times [t_i, t_i + 1]))$ shown in the second paragraph of this proof. There is a map $E : \mathbb{U}\Omega \to \Lambda^*$ given by sending the vector in $\mathbb{U}\Omega$ to the ordered pair of endpoints and supporting hyperspaces of the geodesic passing the vector. Since $\hat{s}$ is continuous, $\tau \circ E|\hat{P}$ is uniformly bounded. The above paragraph shows that the sequence of the diameters of $\tau \circ E|g_i(\Phi(S \times [t_i, t_i + 1]))$ can become arbitrarily large. This is a contradiction. Hence, τ is constant on $\mathscr{L}_x$.

This proves the first two items. The third item follows since τ' is an induced map.

Define $H(x)$ to be the open n-dimensional hemisphere in $\mathbb{S}^n$ bounded by the great sphere containing the affine hyperspace $\tau'(x)$ and containing Ω. Let

$$\hat{U} := \bigcap_{x \in \partial \mathrm{Cl}(\Omega)} H(x) \cap \mathbb{A}^n.$$

We claim that for any x, y in $\partial \mathrm{Cl}(\Omega)$, $h'(x,y)$ is disjoint from U: By Theorem 1.1 of Benoist [20], the geodesic flow on Ω/Γ is Anosov, and hence the set of closed geodesics in Ω/Γ is dense in the space of geodesics by the basic property of the Anosov flow. Since the fixed points are in $\partial \mathrm{Cl}(\Omega)$, we can find sequences $\{x_i\}, \{x_i\} \to x$ and $\{y_i\}, \{y_i\} \to y$ where x_i and y_i are fixed points of an element $g_i \in \Gamma$ for each i. If $h'(x,y) \cap U \neq \emptyset$, then $h'(x_i, y_i) \cap U \neq \emptyset$ for i sufficiently large by the continuity of the map τ from Proposition 4.2.10. This is a contradiction by Lemma 4.2.11

In particular this also holds for $\hat{U}$ and U' in the premise.

Now, we show that the affine hyperspace part of $\tau(x)$ is an AS-hyperspace for U: Suppose that for $x \in \partial \mathrm{Cl}(\Omega)$, the AS-hyperspace Q with $Q \neq \tau(x)$. Then the hemisphere H_Q bounded by Q contains U. By the above disjointness of $h(x,y)$ to U, $H_Q \subset H(x)$. Then again we choose a segment l_1 ending at x. Then we choose sequences g_i as above in the proof before (4.2.12). This shows as above the sequence of the Euclidean distances between the respective affine hyperspace parts of

$$g_i(\tau(x)) \text{ and } g_i(Q)$$

goes to infinity. Proposition 4.3.6 shows that $\{g_i(\tau(x))\}$ is in the image of τ, a compact set. The set of suppprting hyperplanes of U is bounded away from $\mathrm{bd}\mathbb{A}^n$ since they have to be between those of the image of τ and U. Since $g_i(Q)$ is still a supporting hyperplane of U, the equation is a contradiction.

Now, we can replace U with U'. The existence of an AS-plane at each point follows by the argument above that $h'(x,y)$ is disjoint from U' now replacing U. The forth item follows by the paragraph above also.

□

Proof of Theorem 4.1.1. Let

$$\hat{U} := \bigcap_{x \in \partial \mathrm{Cl}(\Omega)} H(x) \cap \mathbb{A}^n.$$

Then this follows from Lemma 4.2.12.

□

4.3. Generalization to nonstrictly convex domains

4.3.1. Main argument. Now, we drop the condition of hyperbolicity on Γ. Hence, Ω, $\Omega \subset \mathrm{bd}\mathbb{A}^n$, is not necessarily strictly convex. Also, Ω is allowed to be the interior of a strict join. Here, we don't assume that Γ is not necessarily hyperbolic, and hence, it is more general. Also, we obtain an asymptotically nice properly convex domain U in $\mathbb{A}^n$ where Γ acts properly on.

THEOREM 4.3.1. *Let Γ have an affine action on the affine subspace $\mathbb{A}^n$, $\mathbb{A}^n \subset \mathbb{S}^n$, acting on a properly convex domain Ω in $\mathrm{bd}\mathbb{A}^n$. Suppose that Ω/Γ is a closed $n-1$-dimensional orbifold, and suppose that Γ satisfies the uniform middle-eigenvalue condition. Then Γ is acts on a properly convex open domain U with the following properties:*

- *(Γ, U, Ω) is a properly convex triple, and Γ is asymptotically nice with the properly convex open domain U, and*

- *if any open set U' so that (Γ, U', Ω) is a properly convex triple, then the AS-hyperspace at each pair of a point x of $\partial \mathrm{Cl}(\Omega)$ and a strictly supporting hyperplane of Ω in $\mathrm{bd}\mathbb{A}^n$ at x exists and is the same as that of U. That is U' is also asymptotically nice.*

The proof is analogous to Theorem 4.1.1. Now Ω is not strictly convex and hence for each point of $\partial \mathrm{Cl}(\Omega)$ there might be more than one sharply supporting hyperspace in $\mathrm{bd}\mathbb{A}^n$. We generalize $\mathbb{U}\Omega$ to the *augmented unit tangent bundle*

$$
\begin{aligned}
(4.3.1)\quad \mathbb{U}^{\mathrm{Ag}}\Omega := \{(\vec{x}, H_a, H_b) |\ & \vec{x} \in \mathbb{U}\Omega \text{ is a direction vector at a point} \\
& \text{of a maximal oriented geodesic } l_{\vec{x}} \text{ in } \Omega, \\
& H_a \text{ is a sharply supporting hyperspace in } \mathrm{bd}\mathbb{A}^n \text{ at the starting point of } l_{\vec{x}}, \\
& H_b \text{ is a sharply supporting hyperspace in } \mathrm{bd}\mathbb{A}^n \text{ at the ending point of } l_{\vec{x}}\}.
\end{aligned}
$$

Here, we regard $\vec{x}$ as a based vector and hence has information on where it is on l and H_a and H_b is given orientations so that Ω is in the interior direction to them. This is not a manifold but a locally compact Hausdorff space and is a metrizable space. Since the set of sharply supporting hyperspaces of Ω at a point of $\partial \mathrm{Cl}(\Omega)$ is compact, $\mathbb{U}^{\mathrm{Ag}}\Omega/\Gamma$ is a compact Hausdorff space fibering over Ω/Γ with compact fibers. The obvious metric is induced from Ω and the space $\mathbb{S}^{n*}$ of oriented hyperspaces in $\mathbb{S}^n$. We also write Π^{Ag} : $\mathbb{U}^{\mathrm{Ag}}\Omega \to \Omega$ the obvious projection $(\vec{x}, H_a, H_b) = \Pi_\Omega(\vec{x})$.

From Section 1.5, we recall the augmented boundary $\partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$. We define

$$
\Lambda^{*\mathrm{Ag}} = \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega) \times \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega) - (\Pi_{\mathrm{Ag}} \times \Pi_{\mathrm{Ag}})^{-1}(\Delta^{\mathrm{Ag}})
$$

where Δ^{Ag} is defined as the closed subset

$$
\{(x,y) | x, y \in \partial \mathrm{Cl}(\Omega), x = y \text{ or } \overline{xy} \subset \partial \mathrm{Cl}(\Omega)\}.
$$

Define $G^{\mathrm{Ag}}(\Omega)$ denote the set of oriented maximal geodesics in Ω with endpoints augmented with the sharply supporting hyperspace at each endpoint. The elements are called *augmented geodesics*. There is a one to one and onto correspondence between $\Lambda^{*\mathrm{Ag}}$ and $G^{\mathrm{Ag}}(\Omega)$. We denote by $\overline{(x,h_1)(y,h_2)}$ the complete geodesic in Ω with endpoints x, y and sharply supporting hyperspaces h_1 at x and h_2 at y.

LEMMA 4.3.2. *$G^{\mathrm{Ag}}(\Omega)$ is a connected subspace.*

PROOF. We can use the fact that $\mathbb{U}^{\mathrm{Ag}}\Omega$ is connected. □

Now, we follow Section 4.1.1 and define $\tilde{\mathbf{A}} = \mathbb{U}^{\mathrm{Ag}}\Omega \times \mathbb{A}^n$, $\tilde{\mathbf{V}} = \mathbb{U}^{\mathrm{Ag}}\Omega \times V^n$, $\mathbf{A}$ by $\mathbb{U}^{\mathrm{Ag}}\Omega \times \mathbb{A}^n/\Gamma$ and $\mathbf{V} := \mathbb{U}^{\mathrm{Ag}}\Omega \times V^n/\Gamma$ and corresponding subbundles $\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_-, \tilde{\mathbf{V}}_0$, $\mathbf{V}_+$, $\mathbf{V}_-$, and $\mathbf{V}_0$. We define the flows $\hat{\Phi}_t, \Phi_t, \tilde{\Phi}_t, \mathscr{L}(\Phi_t), \mathscr{L}(\tilde{\Phi}_t)$ by replacing $\mathbb{U}\Omega$ by $\mathbb{U}^{\mathrm{Ag}}\Omega$ and geodesics by augmented geodesics and so on in an obvious way.

For each point $\mathbf{x} = (\vec{x}, H_a, H_b)$ of $\mathbb{U}^{\mathrm{Ag}}\Omega$,

- we define $\tilde{\mathbf{V}}_+(\mathbf{x})$ to be the space of vectors in the direction of the backward endpoint of $l_{\mathbf{x}}$,
- $\tilde{\mathbf{V}}_-(\mathbf{x})$ that for the forward endpoint of $l_{\mathbf{x}}$,
- $\tilde{\mathbf{V}}_0(\mathbf{x})$ to be the space of vectors in directions of $H_a \cap H_b$.

For each $\mathbf{x} \in \mathbb{U}^{\mathrm{Ag}}\Omega$,

$$
V^n = \tilde{\mathbf{V}}_+(\mathbf{x}) \oplus \tilde{\mathbf{V}}_0(\mathbf{x}) \oplus \tilde{\mathbf{V}}_-(\mathbf{x}).
$$

This gives us a decomposition. $\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_+ \oplus \tilde{\mathbf{V}}_0 \oplus \tilde{\mathbf{V}}_-$, and $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_0 \oplus \mathbf{V}_-$. Clearly, $\mathbf{V}_+$ and $\mathbf{V}_-$ are topological line bundles since the beginning and the endpoints depend continuously on points of $\mathbb{U}^{\mathrm{Ag}}\Omega$. Also, $\tilde{\mathbf{V}}_0$ is the vector subspace of $\mathbb{R}^n$ whose directions of nonzero vectors form $H_a \cap H_b$. Since (H_a, H_b) depends continuously on points of $\mathbb{U}^{\mathrm{Ag}}\Omega$, we obtain that $\mathbf{V}_0$ is a continuous bundle on $\mathbb{U}^{\mathrm{Ag}}\Omega$.

Obviously, the geodesic flows exists on $\mathbb{U}^{\mathrm{Ag}}\Omega$ using the ordinary geodesic flow with respect to the geodesics and not considering the augmented boundary.

There exists constants $C, k > 0$ so that

$$\|\mathscr{L}(\Phi_t)(\vec{v})\|_{\text{fiber}} \geq \frac{1}{C}\exp(kt)\,\|\vec{v}\|_{\text{fiber}} \text{ as } t \to \infty \tag{4.3.2}$$

for $\vec{v} \in \mathbf{V}_+$ and

$$\|\mathscr{L}(\Phi_t)(\vec{v})\|_{\text{fiber}} \leq C\exp(-kt)\,\|\vec{v}\|_{\text{fiber}} \text{ as } t \to \infty \tag{4.3.3}$$

for $\vec{v} \in \mathbf{V}_-$.

We prove this by proving $\{\|\mathscr{L}(\Phi_t)|\mathbf{V}_-\|_{\text{fiber}}\} \to 0$ uniformly as $t \to \infty$ i.e., Proposition 4.2.1 under the more general conditions that Ω is properly convex but not necessarily strictly convex. We generalize Lemma 4.2.5. We will repeat the strategy of the proof since it is important to check. However, the proof follows the same philosophy with some technical differences.

We first need

LEMMA 4.3.3. *Let g_j be a sequence of elements of $\mathbf{\Gamma}$. Suppose that $a_n(\hat{h}(g_j)) \to 0$ as $j \to \infty$. Then $a_{n+1}(g_j)/\lambda_{\mathrm{v}_{\tilde{E}}}(g_j) \to 0$ as $j \to \infty$ provided the sequence of word-length of g_j goes to the infinity.*

PROOF. Suppose that $\mathrm{length}_\Omega(g_j)$ is bounded. Then we can conjugate g_j by an element k_j so that $k_j g_j k_j^{-1}$ sends an element of a fundamental domain F of Ω to a point of a bounded distance from it. Hence, there are only finitely element $h_1, \dots, h_l$. Hence $g_j = k_j^{-1} h_{i_j} k_j$ for $i_j = 1, \dots, l$. Then $\lambda_{\mathrm{v}_{\tilde{E}}}(g_j) = \lambda_{\mathrm{v}_{\tilde{E}}}(h_{i_j})$ taking finitely many values.

Since $a_{n+1}(g_j) = a_n(\hat{h}(g_j))/\lambda_{\mathrm{v}_{\tilde{E}}}(g_j)^{1+\frac{1}{n}}$, the conclusion follows in this condition.

It is now sufficient to consider when the sequence $\mathrm{length}_\Omega(g_j)$ is converging to infinity. Then the uniform middle eigenvalue condition implies that

$$\exp(-C\mathrm{length}_\Omega(g_j)) \leq \lambda_{n+1}(g_j)/\lambda_{\mathrm{v}_{\tilde{E}}}(g_j) \leq \exp(-C^{-1}\mathrm{length}_\Omega(g_j))$$

implies that $\lambda_{n+1}(g_j)/\lambda_{\mathrm{v}_{\tilde{E}}}(g_j) \to 0$. Let $|s|_E$ denote the length of a segment in $\mathbb{R}^{n+1}$ under the fixed Euclidean metric. We have $\lambda_{n+1}(g_j) \geq a_{n+1}(g_j)$ since we can consider segments in a eigendirection of g_j and we have at least that amount of shrinking of a segment and

$$a_{n+1}(g) = \min\left\{\frac{\|g(s)\|_E}{\|s\|_E}\middle| s \text{ is a segment in } \mathbb{R}^{n+1}\right\}.$$

Hence, the result follows. (This is related to [31], [27] and [28].) □

LEMMA 4.3.4. *Assume that Ω is properly convex and Γ acts properly discontinuously satisfying the uniform middle eigenvalue condition with respect to* $\mathrm{bd}\mathbb{A}^n$. *Then $\{\|\Phi_t|\mathbf{V}_-\|_{\text{fiber}}\} \to 0$ uniformly as $t \to \infty$.*

PROOF. We proceed as in the proof of Lemma 4.2.5. It is sufficient to prove the uniform convergence to 0 by the compactness of $\mathbf{V}_{-,1}$. Let F be a fundamental domain of $\mathbb{U}^{\mathrm{Ag}}\Omega$ under Γ. It is sufficient to prove this for $\mathscr{L}(\tilde{\Phi})_t$ on the fibers of over F of $\mathbb{U}^{\mathrm{Ag}}\Omega$ with a fiberwise metric $\|\cdot\|_{\text{fiber}}$.

We choose an arbitrary sequence $\{\mathbf{x}_i = (\vec{x}_i, H_{a_i}, H_{b_i}) \in \mathbb{U}^{\mathrm{Ag}}\Omega\}$, $\{\mathbf{x}_i\} \to \mathbf{x}$ in F where a_i, b_i are the backward and forward point of the maximal oriented geodesic passing $\vec{x}_i$ in

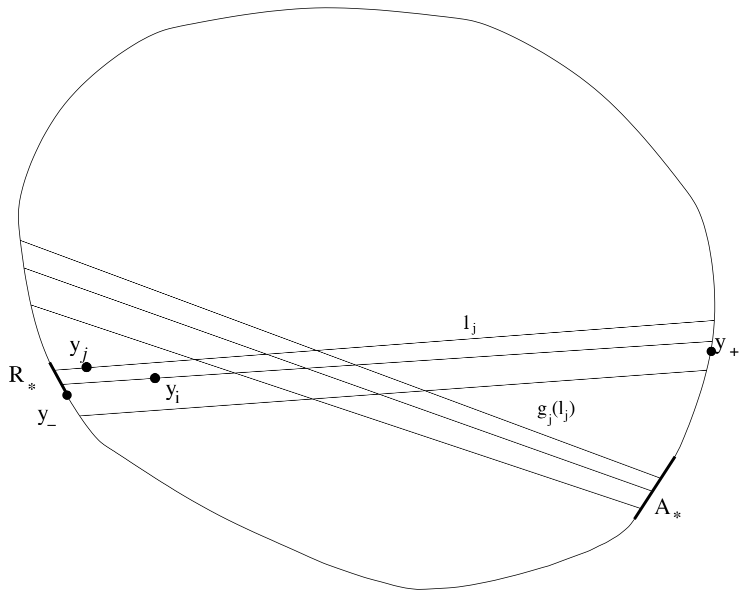

FIGURE 2. The figure for Lemma 4.3.4.

Ω. For each i, let $\vec{v}_{-,i}$ be a Euclidean unit vector in $\tilde{\mathbf{V}}_-(\mathbf{x}_i)$ for the unit vector $\mathbf{x}_i \in \mathbb{U}^{\mathrm{Ag}}\Omega$. That is, $\vec{v}_{-,i}$ is in the 1-dimensional subspace in $\mathbb{R}^n$, corresponding to the forward endpoint of the geodesic determined by $\vec{x}_i$ in $\partial \mathrm{Cl}(\Omega)$.

Let $\mathbf{x}_i$ be as above convergin to $\mathbf{x}$ in F. Here, $(\!(\vec{v}_{-,i})\!)$ is an endpoint of l_i in the direction given by $\mathbf{x}_i$. For this, we just need to show that any sequence of $\{t_i\} \to \infty$ has a subsequence $\{t_j\}$ so that $\left\{ \left\| \mathscr{L}(\tilde{\Phi}_{t_j})(\mathbf{x}_i, \vec{v}_{-,j}) \right\|_{\mathrm{fiber}} \right\} \to 0$. This follows since if the uniform convergence did not hold, then we can easily find a sequence without such subsequences.

Let $y_i := \widehat{\Phi}_{t_i}(\mathbf{x}_i)$ for the lift of the flow $\widehat{\Phi}$. By construction, we recall that each $\Pi^{\mathrm{Ag}}_{\Omega}(y_i)$ is in the geodesic l_i. Since we have the sequence $\{\mathbf{x}_i\} \to \mathbf{x}$, $\mathbf{x}_i, \mathbf{x} \in F$, we obtain that $\{l_i\}$ geometrically converges to a line l_∞ passing $\Pi^{\mathrm{Ag}}_{\Omega}(\mathbf{x})$ in Ω. Let y_+ and y_- be the endpoints of l_∞ where $\{\Pi^{\mathrm{Ag}}_{\Omega}(y_i)\} \to y_-$. Hence,

$$\{(\!(\vec{v}_{+,i})\!)\} \to y_+, \{(\!(\vec{v}_{-,i})\!)\} \to y_-.$$

(See Figure 1 for the similar situation.)

Find a deck transformation g_i so that $g_i(y_i) \in F$, and g_i acts on the line bundle $\tilde{\mathbf{V}}_-$ by the linearization of the matrix of form (4.1.1):

$$\begin{aligned}
& g_i : \tilde{\mathbf{V}}_- \to \tilde{\mathbf{V}}_- \text{ given by} \\
& (y_i, \vec{v}) \to (g_i(y_i), \mathscr{L}(g_i)(\vec{v})) \text{ where} \\
& \mathscr{L}(g_i) := \frac{1}{\lambda_{\tilde{E}}(g_i)^{1+\frac{1}{n}}} \hat{h}(g_i) : \tilde{\mathbf{V}}_-(y_i) = \tilde{\mathbf{V}}_-(x_i) \to \tilde{\mathbf{V}}_-(g_i(y_i)).
\end{aligned} \tag{4.3.4}$$

We will show $\{(g_i(y_i), \mathscr{L}(g_i)(\vec{v}_{-,i}))\} \to 0$ under $\|\cdot\|_{\mathrm{fiber}}$. This will complete the proof since g_i acts as isometries on $\tilde{\mathbf{V}}_-$ with $\|\cdot\|_{\mathrm{fiber}}$.

Since $g(y_i) \in F$ for every i, we obtain

$$g_i(l_i) \cap F \neq \emptyset.$$

Since $g_i(l_i) \cap F \neq \emptyset$, we choose a subsequence of g_i and relabel it g_i so that $\{\Pi_\Omega^{\mathrm{Ag}}(g_i(l_i))\}$ converges to a nontrivial line $\hat{l}_\infty$ in Ω.

Remark 4.2.2 shows that we may assume without loss of generality that each element of Γ is positive bi-semi-proximal.

We recall facts from Section 1.3.3. Given a generalized convergence sequence g_i, we obtain an endomorphism g_∞ in $M_n(\mathbb{R})$ so that $\{(\!(g_i)\!)\} \to (\!(g_\infty)\!) \in \mathbb{S}(M_n(\mathbb{R}))$. Recall

$$A_*(\{g_i\}) := \mathbb{S}(\operatorname{Im} g_\infty) \cap \mathrm{Cl}(\Omega) \text{ and } N_*(\{g_i\}) := \mathbb{S}(\ker g_\infty) \cap \mathrm{Cl}(\Omega).$$

We have $A_*(\{g_i\}), N_*(\{g_i\}) \subset \partial \mathrm{Cl}(\Omega)$ are both nonempty by Theorem 1.3.21.

Up to a choice of subsequence, Theorem 1.3.13 implies that for any compact subset K of $\mathrm{Cl}(\Omega) - N_*(\{g_i\})$, there is a convex compact subset K_* in A_*,

$$\{g_i(K)\} \to K_* \subset A_*. \tag{4.3.5}$$

Suppose that $y_- \in \mathrm{Cl}(\Omega) - N_*(\{g_i\})$. Then $\{g_i(y_i)\} \to \hat{y} \in A_*(\{g_i\})$ since y_i are in a compact subset of $\mathrm{Cl}(\Omega) - N_*(\{g_i\})$ and (4.3.5). This is a contradiction since $g_i(y_i) \in F$. Hence, $y_- \in N_*(\{g_i\})$.

Let $d_i = (\!(\vec{v}_{+,i})\!)$ denote the other endpoint of l_i than $(\!(\vec{v}_{-,i})\!)$ as above. Let d_∞ denote the limit of d_i in $\partial \mathrm{Cl}(\Omega)$. We deduce as above up to a choice of a subsequence:

- Since $\{(\!(\vec{v}_{-,i})\!)\} \to y_-$, $y_- \in N_*(\{g_i\})$ and $\{l_i\}$ converges to a nontrivial line $l_\infty \subset \Omega$ and $N_*(\{g_i\})$ is compact convex in $\partial \mathrm{Cl}(\Omega)$, it follows that $\{d_i\}$ is in a compact set in $\partial \mathrm{Cl}(\Omega) - N_*(\{g_i\})$.
- Then $\{g_i(d_i)\} \to a_* \in A_*(\{g_i\})$ by (4.3.5), since $\{d_i\}$ is in a compact set in $\partial \mathrm{Cl}(\Omega) - N_*(\{g_i\})$.
- Thus, $\{g_i((\!(\vec{v}_{-,i})\!))\} \to y' \in \partial \mathrm{Cl}(\Omega) - A_*(g_i)$ holds since $\{g_i(l_i)\}$ converges to a nontrivial line in Ω.

Let m_a be obtained for $\{g_i\}$ as in Theorem 1.3.16. Recall that $\|\cdot\|_E$ denote the standard Euclidean metric of $\mathbb{R}^n$. Write $g_i = k_i D_i \hat{k}_i^{-1}$ for $k_i, \hat{k}_i \in \mathsf{O}(n, \mathbb{R})$, and D_i is a positive diagonal matrix of determinant ± 1 with nonincreasing diagonal entries.

- Since $\{(\!(\vec{v}_{-,i})\!)\} \to y_-$, the vector $\vec{v}_{-,i}$ has the component $\vec{v}_i^p$ parallel to $N^p(g_i) = \hat{k}_i(\mathbb{S}([m_a+1, n]))$ and the component $\vec{v}_i^S$ in the orthogonal complement $(N^p(g_i))^\perp = \hat{k}_i(\mathbb{S}([1, m_a]))$ where $\vec{v}_{-,i} = \vec{v}_i^p + \vec{v}_i^S$. We may require $\|\vec{v}_{-,i}\|_E = 1$. (We remark

$$\{\|\vec{v}_i^p\|_E\} \to 1 \text{ and } \{\|\vec{v}_i^S\|_E\} \to 0$$

since $\{\vec{v}_{-,i}\}$ converges to a point of $N_*(\{g_i\})$.
- $\{\|\mathscr{L}(g_i)(\vec{v}_i^p)\|_E\} \to 0$ by Theorem 1.3.16 and Lemma 4.3.3 since the sequence of the word length of g_i goes to infinity while g_i moves a point far away to a point of the fundamental domain.
- Since $\{g_i((\!(\vec{v}_{-,i})\!))\}$ converges to y', $y' \in \partial \mathrm{Cl}(\Omega) - A_*(g_i)$, $\{g_i((\!(\vec{v}_{-,i})\!))\}$ is uniformly bounded away from $A_*(\{g_i\})$.

Because of the orthogonal decomposition $\hat{k}_i(\mathbb{S}([m_a+1, n])$ and $\hat{k}_i(\mathbb{S}([1, m_a]))$, and the fact that $g_i = k_i D_i \hat{k}_i^{-1}$, and $\{(\!(\mathscr{L}(g_i))\!)\} \to (\!(g_\infty)\!)$ in $\mathbb{S}(M_n(\mathbb{R}))$, it follows that $\{\mathscr{L}(g_i)(\vec{v}_i^S)\}$ either converges to zero, or $\{(\!(\mathscr{L}(g_i)(\vec{v}_i^S))\!)\}$ converges to $A_*(\{g_i\})$ by Theorem 1.3.16. We have

$$\{\mathscr{L}(g_i)(\vec{v}_i^p)\} \to 0 \text{ implies } \{\mathscr{L}(g_i)(\vec{v}_i^S)\} \to 0$$

for $\|\cdot\|_E$ since otherwise $\{(\!(\mathscr{L}(g_i)(\vec{v}_i^S))\!)\}$ converges to a point of $A_*(\{g_i\}) \subset F_*(\{g_i\})$ and hence $\{g_i((\!(\vec{v}_{-,i})\!))\}$ cannot be converging to y'.

Hence, we obtain $\{\mathscr{L}(g_i)(\vec{v}_{-,i})\} \to 0$ under $\|\cdot\|_E$. Now, we can deduce the result as in the final part of the proof of Lemma 4.2.5

□

Now, we find as in Section 4.2.3 the neutralized section $s : \mathbb{U}^{\mathrm{Ag}}\Sigma \to \mathbf{A}$ with $\nabla^{\mathbf{A}}_{\phi} s \in \mathbf{V}_0$.

Since we are looking at $\mathbb{U}^{\mathrm{Ag}}\Omega$, the section $s : \mathbb{U}^{\mathrm{Ag}}\Omega \to N_2(\mathbb{A}^n)$, we need to look at the boundary point and a sharply supporting hyperspace at the point and find the affine subspace of dimension $n-2$ in $\mathbb{R}^n$, generalizing Proposition 4.2.10. We generalize Definition 4.2.8:

DEFINITION 4.3.5. We denote by $S^{\mathrm{Ag}}(\partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega))$ the space of pairs $((x, H \cap \mathrm{bd}\mathbb{A}^n), H)$ where $H \in S'(\partial \mathrm{Cl}(\Omega))$, and x is in the boundary of H and $(x, H \cap \mathrm{bd}\mathbb{A}^n) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$.

PROPOSITION 4.3.6.

- *There exists a continuous function $\hat{s} : \mathbb{U}^{\mathrm{Ag}}\Omega \to N_2(\mathbb{A}^n)$ equivariant with respect to Γ-actions.*
- *Given $g \in \Gamma$ and for the unique unit-speed geodesic $\vec{l}_g$ in $\mathbb{U}^{\mathrm{Ag}}\Omega$ lying over an augmented geodesic l_g where g acts on, $\hat{s}(\vec{l}_g) = \{N_2(g)\}$.*
- *This gives a continuous map*
$$\bar{s}^{\mathrm{Ag}} : \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega) \times \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega) - (\Pi_{\mathrm{Ag}} \times \Pi_{\mathrm{Ag}})^{-1}(\Delta^{\mathrm{Ag}}) \to N_2(\mathbb{A}^n)$$
again equivariant with respect to the Γ-actions. There exists a continuous function
$$\tau^{\mathrm{Ag}} : \Lambda^{*\mathrm{Ag}} \to S^{\mathrm{Ag}}(\partial \mathrm{Cl}(\Omega)).$$

PROOF. The proof is entirely similar to that of Proposition 4.2.10 but using a straightforward generalization of Lemma 4.2.7. □

We generalize Proposition 4.2.12. We will define $\tau' : \mathbb{U}^{\mathrm{Ag}}\Omega \to S^{\mathrm{Ag}}(\partial \mathrm{Cl}(\Omega))$ as a composition of τ^{Ag} with the map from $\mathbb{U}^{\mathrm{Ag}}\Omega$ to $\Lambda^{*\mathrm{Ag}}$. This is a continuous map. Here, we don't assume that Γ acts on a properly convex domain in $\mathbb{A}^n$ with boundary Ω. Hence, it is more general and we need a different proof. We just need that the orbit closures are compact.

LEMMA 4.3.7. *Let an affine group Γ acts on an affine subspace $\mathbb{A}^n$ on a properly convex domain Ω in the boundary of an affine subspace $\mathbb{A}^n$. Let Γ acts on a properly convex domain Ω with a cocompact and Hausdorff quotient and satisfies the uniform middle eigenvalue condition with respect to* $\mathrm{bd}\mathbb{A}^n$. *Let $((x,h_1),(y,h_2)) \in \Lambda^{*\mathrm{Ag}}$. Then*

- *$\tau^{\mathrm{Ag}}((x,h_1),(y,h_2))$ does not depend on (y,h_2) and is unique for each (x,h_1).*
- *$h((x,h_1),(y,h_2))$ contains $\bar{s}^{\mathrm{Ag}}((x,h_1),(y,h_2))$ but is independent of (y,h_2).*
- *$h((x,h_1),(y,h_2))$ is never a hemisphere in* $\mathrm{bd}\mathbb{A}^n$ *for every $((x,h_1),(y,h_2)) \in \Lambda^{*\mathrm{Ag}}$.*
- *τ^{Ag} induces a map $\tau^{\mathrm{Ag}\prime} : \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega) \to S^{\mathrm{Ag}}(\partial \mathrm{Cl}(\Omega))$ that is continuous.*
- *There exists an asymptotically nice convex Γ-invariant open domain U in $\mathbb{A}^n$ with* $\mathrm{bd}U \cap \partial\mathbb{A}^n = \mathrm{Cl}(\Omega)$. *For every $(x,h_1) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$, $\tau(x,h_1)$ is an AS-hyperplane of U.*

PROOF. Let l_1 be an augmented geodesic in $\mathbb{U}^{\mathrm{Ag}}\Omega$ with endpoints (x,h_1) and (z,h_2) oriented towards x. Consider a connected subspace $\mathscr{L}_{(x,h_1)}$ of $\mathbb{U}^{\mathrm{Ag}}\Omega$ of points of maximal augmented geodesics in Ω ending at (x,h_1). The space of geodesic leaves in $\mathscr{L}_{(x,h_1)}$ is in one-to-one correspondence with $\mathrm{bd}^{\mathrm{Ag}}\Omega - \Pi_{\mathrm{Ag}}^{-1}(K_x)$ for the maximal flat K_x in $\partial \mathrm{Cl}(\Omega)$ containing x. We will show that τ^{Ag} is locally constant on $\mathscr{L}_{(x,h_1)}$ showing that it is constant.

Let $\tilde{l}_1$ denote the lift of l_1 in $\mathbb{U}^{\mathrm{Ag}}\Omega$. Let S be a compact neighborhood in $\mathscr{L}_{(x,h_1)}$ of a point y of $\tilde{l}_1$ transverse to $\tilde{l}_1$. Any two rays of geodesic flow $\Phi : S \times \mathbb{R} \to \mathbb{U}^{\mathrm{Ag}}\Omega$ are asymptotic on $\mathscr{L}_{(x,h_1)}$ by Lemma 3.1.4.

Let $y \in \tilde{l}_1$. Consider another point $y' \in S \subset \mathbb{U}^{\mathrm{Ag}}\Omega$ with with endpoints x and z' in a sharply supporting hyperplane h_2'. where

$$((x,h_1),(z,h_2)),((x,h_1),(z',h_2')) \in \Lambda^{*\mathrm{Ag}}.$$

Choose a fixed fundamental domain F of $\hat{\mathbb{U}}\Omega$. Let $\{y_i = \Phi_{t_i}(y)\}, y_i \in \tilde{l}_1$, be a sequence whose projection under Π_Ω convergs to x. We use a deck transformation g_i so that $g_i(y_i) \in F$. Then $g_i(\tau^{\mathrm{Ag}}(l_1)) = \tau^{\mathrm{Ag}}(g_i(l_1))$ is a hyperspace containing $g_i(x)$ and $g_i(h_1)$ and $\hat{s}(g_i(\tilde{l}_1))$.

Let $\vec{v}_+$ denote a vector in the direction of the end of l_1 other than x. Equation (4.3.2) shows that $\{\|\mathscr{L}(\Phi_t)|\mathbf{V}_+\|_{\mathrm{fiber}}\} \to \infty$ as $t \to \infty$. Since g_i is isometry under $\|\cdot\|_{\mathrm{fiber}}$, and $\hat{\Phi}_{t_i}(y) = y_i$ and $g_i(y_i) \in F$, it follows that the $\tilde{\mathbf{V}}_+$-component of $g_i(y_i, \vec{v}_+)$ satisfies

$$\{\|\mathscr{L}(g_i)(\vec{v}_+)\|_{\mathrm{fiber}}\} \to \infty. \tag{4.3.6}$$

Since $g_i(y_i) \in F$ and under the Euclidean norm since over a compact set F the metrics are compatible by a uniform constant, we obtain $\{\|\mathscr{L}(g_i)(\vec{v}_+)\|_E\} \to \infty$.

Since the affine hyperplanes in $\tau^{\mathrm{Ag}}((x,h_1),(z,h_2))$ and $\tau^{\mathrm{Ag}}((x,h_1),(z',h_2'))$ contain x and h_1 in their boundary, they restrict to parallel affine hyperplanes in $\mathbb{A}$. Suppose that the affine hyperspace part of $\tau^{\mathrm{Ag}}(((x,h_1),(z,h_2))$ differs from one of $\tau^{\mathrm{Ag}}((x,h_1),(z',h_2'))$ by a translation by a constant times $\vec{v}_+$. This implies that the sequence of the Euclidean distances between the respective affine hyperspaces corresponding to

$$g_i(\tau^{\mathrm{Ag}}((x,h_1),(z,h_2))) \text{ and } g_i(\tau^{\mathrm{Ag}}((x,h_1),(z',h_2')))$$

goes to infinity as $i \to \infty$.

Now consider $\Phi(S \times [t_i, t_i+1]) \subset \mathbb{U}^{\mathrm{Ag}}\Omega$, and we have obtained g_i so that $g_i(\Phi(S \times [t_i, t_i+1]))$ is in a fixed compact subset $\hat{P}$ of $\mathbb{U}^{\mathrm{Ag}}\Omega$ by the uniform boundedness of $\Phi(S \times [t_i, t_i+1]))$ shown in the second paragraph of this proof. There is a map $E : \mathbb{U}^{\mathrm{Ag}}\Omega \to \Lambda^{*\mathrm{Ag}}$ given by sending the vector in $\mathbb{U}^{\mathrm{Ag}}\Omega$ to the ordered pair of endpoints and supporting hyperspaces of the geodesic passing the vector. Since $\hat{s}$ is continuous, $\tau^{\mathrm{Ag}} \circ E|\hat{P}$ is uniformly bounded. The above paragraph shows that the sequence of the diameters of $\tau^{\mathrm{Ag}} \circ E|g_i(\Phi(S \times [t_i, t_i+1]))$ can become arbitrarily large. This is a contradiction. Hence, τ^{Ag} is constant on $\mathscr{L}_{(x,h_1)}$.

This proves the first two items. The fourth item follows since $\tau^{\mathrm{Ag}\prime}$ is an induced map. The image of $\tau^{\mathrm{Ag}\prime}$ is compact since $\partial \mathrm{Cl}(\Omega)$ is compact. This implies the third item.

Define $H(x,h_1)$ to be the open n-dimensional hemisphere in $\mathbb{S}^n$ bounded by the great sphere containing the affine hyperspace $\tau^{\mathrm{Ag}\prime}(x,h_1)$ and containing Ω. We define

$$U := \bigcap_{(x,h_1)\in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)} H(x,h_1) \cap \mathbb{A}^n.$$

Now, we show that the affine hyperspace part of $\tau^{\mathrm{Ag}}(x,h_1)$ is an AS-hyperspace for U: Suppose that for $(x,h_1) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$, the AS-hyperspace Q with $Q \cap \mathrm{bd}\mathbb{A}^n = h_1$ and $Q \neq \tau^{\mathrm{Ag}}(x,h_1)$. Then the hemisphere H_Q bounded by Q contains U. By definition, $H_Q \subset H(x,h_1)$. Then again we choose a segment l_1 ending at x. Then we choose sequences g_i

as above in the proof before (4.3.6). This shows as above the sequence of the Euclidean distances between the respective affine hyperspace parts of

$$g_i(\tau^{\mathrm{Ag}}(x,h_1)) \text{ and } g_i(Q)$$

goes to infinity. Proposition 4.3.6 shows that $\{g_i(\tau^{\mathrm{Ag}}(x,h_1))\}$ is in the image of τ^{Ag}, a compact set. The set of suppprting hyperplanes of U is bounded away from $\mathrm{bd}\mathbb{A}^n$ since they have to be between those of the image of τ^{Ag} and U. Since $g_i(Q)$ is still a supporting hyperplane of U, the equation is a contradiction. □

PROOF OF THEOREM 4.3.1. First, we obtain a properly convex domain where Γ acts. Since $\partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$ is compact, the image under τ^{Ag} is compact. Then $U := \bigcap_{(x,h)\in\partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)} H^o_{(x,h)} \cap \mathbb{A}^n$ contains Ω. This is an open set since the compact set of $H_{(x,h)}$, $(x,h) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$ has a lower bound on angles with $\mathrm{bd}\mathbb{A}^n$. Thus, U is asymptotically nice. Now, the proof is identical with that of Theorem 4.1.1 with Lemma 4.3.7 replacing Lemma 4.2.12.

The uniqueness part is done in Theorem 4.3.8 immediately below. □

4.3.2. Uniqueness of AS-hyperplanes. Finally, we end with some uniqueness properties.

The following generalizes Lemma 4.2.12.

THEOREM 4.3.8. *Let (Γ,U,D) be a properly convex affine triple. Suppose that Γ satisfies the uniform middle-eigenvalue condition. Then for any properly convex triple (Γ,U',D) for an open set U', the set of AS-planes for U' containing all sharply supporting hyperspaces of Ω in $\mathrm{bd}\mathbb{A}^n$ exists and is independent of the choice of U'. That is U' is also asymptotically nice.*

PROOF. For each $x, x \in \partial\mathrm{Cl}(\Omega)$, and h be a supporting hyperspace of Ω in $\mathrm{bd}\mathbb{A}^n$. let $S_{(x,h)}$ be the AS-hyperspace in $\mathbb{S}^n$ for U so that $S_{(x,h)} \cap \mathrm{bd}\mathbb{A}^n = h$. Again for each (x,h), we let $S'_{(x,h)}$ is the AS-hyperspace in $\mathbb{S}^n$ for U' if it exists so that $S'_{(x,h)} \cap \mathrm{bd}\mathbb{A}^n = h$.

If (x,h) is a fixed point of a nontrivial $g \in \Gamma$. then the same argument as in the proof of Lemma 4.2.11, show that U' and $S_{(x,h)}$ are disjoint.

Suppose that Γ is strongly-irreducible. Every extremal points of the dual domain Ω^* in $\mathbb{S}^{n-1*}_\infty$ is a limit of a sequence of fixed points of Γ^* by Proposition 5.1 of [159]. Since τ^{Ag} is continuous, U' and $S_{(x',h')}$ are disjoint if h' is an extremal point of Ω^* by the density of fixed points. Since every boundary points of Ω^* is a nonegative linear sum of extermal points of Ω^*, it follows that U' and $S_{(x',h')}$ is disjoint for any pair (x',h'). This proves that U' is also asymptotically nice.

Suppose that Γ is virtually decomposible. Then Ω is the interior of the strict join $K_1 * \cdots * K_m$. Then there are supporting hyperplanes H_j to Ω in $\mathbb{S}^{n-1}_\infty$ that contains all the factors except for K_j for each $j = 1,\dots,m$ and a supporting hyperplane to K_j in the span of K_j. Then the virtual center isomorphic to $\mathbb{Z}^{m-1}$ acts on H_j. If U' intersect with H_j, then the paragraph two above will again show the contraction. Since any supporting hyperplanes are positive linear combination of some H'_i, we have the disjointness of U' with any $S_{(x',h')}$.

Since U and U' are both asymptotically nice, the sets of AS-hyperplanes are compact. For each (x,h), $S_{(x,h)}$ and $S'_{(x,h)}$ differ by a uniformly bounded distance in $\mathbb{S}^{n*}$.

Suppose that $S_{(x,h_1)}$ is different from $S'_{(x,h_1)}$ for some $x, h_1 \in \partial\mathrm{Cl}(\Omega)$. Now, we follow the argument in the proof of Lemma 4.3.7. We again obtain a sequence $g_i \in \Gamma$ so that

$$g_i(S_{(x,h_1)} \cap \mathbb{A}^n) \text{ and } g_i(S'_{(x,h_1)} \cap \mathbb{A}^n)$$

are parallel affine planes, and the sequence of their Euclidean distances are going to ∞ as $i \to \infty$. By compactness, we know both sequences $\{g_i(S_{(x,h_1)} \cap \mathbb{A}^n)\}$ and $\{g_i(S'_{(x,h_1)} \cap \mathbb{A}^n)\}$ respectively converge to two hyperplanes up to a choice of a subsequence. This means that their Euclidean distances are uniformly bounded. Again (4.3.6) contradicts this. $\square$

REMARK 4.3.9. Theorems 4.1.1 and 4.3.1 also generalize to the case when Γ acts on Ω as convex cocompact group: i.e., there is a convex domain $C \subset \Omega$ so that C/Γ is compact but not necessarily closed. We work on the set of geodesics in C only and the set Λ of endpoints of these. In this case the limit set Λ maybe a disconnected set. The definition such as asymptotic niceness should be restricted to points of Λ only. Here we do need the connectedness of Lemmas 4.2.9 and 4.3.2 to be generalized to this case. However, the proofs indicated there will work.

4.4. Lens type T-ends

4.4.1. Existence of lens-neighborhood. The following is a consquence of Theorems 4.1.1 and 4.3.1.

THEOREM 4.4.1. *Let Γ be a discrete group in $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ (resp. $\mathsf{PGL}(n+1,\mathbb{R})$) acting on a properly convex domain Ω cocompactly and properly, $\Omega \subset \mathrm{bd}\mathbb{A}^n \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$), so that Ω/Γ is a closed n-orbifold.*

- *Suppose that Γ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{bd}\mathbb{A}^n$.*
- *Let P be the hyperspace containing Ω.*

Let U be any one-sided Γ-invariant open neighborhood of Ω. Then Γ acts on a properly convex domain L in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) with strictly convex boundary ∂L such that

$$\Omega \subset L \subset U, \partial L \subset \mathbb{S}^n - P \, (\text{resp. } \subset \mathbb{RP}^n - P).$$

Moreover, L satisfies $\mathrm{bd}_{\partial \mathrm{Cl}(L)} \partial L \subset P$ and L/Γ is a lens-orbifold.

PROOF. We prove for $\mathbb{S}^n$ first. We will just prove for the general case since the case when Ω is strictly convex and C^1, the augmented boundary is given as the set of all (x,h) where x is in $\partial \mathrm{Cl}(\Omega)$ and h is the unique supporting hyperspace of Ω at x. Assume without loss of generality that U is an asymptotically nice open domain for Γ.

For each (x,h) in the augmented boundary of Ω, define a half-space $H(x,h) \subset \mathbb{A}^n$ bounded by $\tau^{\mathrm{Ag}\prime}(x.h)$ and containing Ω in the boundary. For each $H(x,h)$, $(x,h) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)$, in the proofs of Theorems 4.1.1 and 4.3.1, an open n-hemisphere $H'(x,h) \subset \mathbb{S}^n$ satisfies $H'(x,h) \cap \mathbb{A}^n = H(x,h)$. Then we define

$$V := \bigcap_{(x,h) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega)} H'(x,h) \subset \mathbb{S}^n$$

is a convex open domain containing Ω as in the proof of Lemma 4.2.12.

Γ acts on a compact set

$$\mathscr{H} := \{h' | h' \text{ is an AS-hyperspace to } V \text{ at } (x,h) \in \partial^{\mathrm{Ag}}\mathrm{Cl}(\Omega), h' \cap \mathbb{S}^{n-1}_\infty = h\}.$$

Let $\mathscr{H}'$ denote the set of hemispheres bounded by an element of $\mathscr{H}$ and containing Ω. Then we define

$$V := \bigcap_{H \in \mathscr{H}'} H \subset \mathbb{S}^n$$

is a convex open domain containing Ω. Here again the set of AS-hyperspaces to V is closed and bounded away from $\mathbb{S}^{n-1}_\infty$.

First suppose that V is properly convex. Then V has a Γ-invariant Hilbert metric d_V that is also Finsler. (See [86] and [112].) Then

$$N_\varepsilon = \{x \in V | d_V(x,\Omega)) < \varepsilon\}$$

is a convex subset of V by Lemma 1.1.13.

A compact tubular neighborhood M of Ω/Γ in V/Γ is diffeomorphic to $\Omega/\Gamma \times [-1,1]$. (See Section 4.4.2 of [51].) We choose M in $(U \cap V)/\Gamma$. Since Ω/Γ is compact, the regular neighborhood has a compact closure. Thus, $d_V(\Omega/\Gamma, \mathrm{bd}M) > \varepsilon_0$ for some $\varepsilon_0 > 0$. If $\varepsilon < \varepsilon_0$, then $N_\varepsilon \subset M$. We obtain that $\mathrm{bd}_V N_\varepsilon/\Gamma$ is compact.

Clearly, $\mathrm{bd}_V N_\varepsilon/\Gamma$ has two components in two respective components of $(V-\Omega)/\Gamma$. Let F_1 and F_2 be compact fundamental domains of respective components of $\mathrm{bd}_V N_\varepsilon$ with respect to Γ. We procure the set $\mathscr{H}_j$ of finitely many open hemispheres H_i, $H_i \supset \Omega$, so that open sets $(\mathbb{S}^n - \mathrm{Cl}(H_i)) \cap N_\varepsilon$ cover F_j for $j = 1,2$. By Lemma 4.4.3, the following is an open set containing Ω

$$W := \bigcap_{g\in\Gamma} \bigcap_{H_i \in \mathscr{H}_1 \cup \mathscr{H}_2} g(H_i) \cap V.$$

Since any path in V from Ω to $\mathrm{bd}_V N_\varepsilon$ must meet $\mathrm{bd}_V W - P$ first, N_ε contains W and $\mathrm{bd}_V W$. A collection of compact totally geodesic polyhedrons meet in angles $< \pi$ and comprise $\mathrm{bd}_V W/\Gamma$. Let L be $\mathrm{Cl}(W) \cap \tilde{\mathscr{O}}$. Then ∂L has boundary only in $\mathrm{bd}\mathbb{A}^n$ by Lemma 4.4.2 since Γ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{bd}\mathbb{A}^n$. We can smooth $\mathrm{bd}_V W$ to a strictly convex hypersurface to obtain a lens-neighborhood $W' \subset W$ of Ω in N_ε where Γ acts cocompactly by Theorem 4.4.4

Suppose that V is not properly convex. Then $\mathrm{bd}V$ contains v, v_-. V is a convex domain of form $\{v, v_-\} * \Omega$. This follows by Proposition 1.1.4 where we take the closure of V and then the interior. We take any two open hemispheres S_1 and S_2 containing $\mathrm{Cl}(\Omega)$ so that $\{v, v_-\} \cap S_1 \cap S_2 = \emptyset$. Then $\bigcap_{g\in\Gamma} g(S_1 \cap S_2) \cap V$ is a properly convex open domain containing Ω, and we can apply the same argument as above.

To prove for $\mathbb{RP}^n$, we need to find L in a sufficiently thin neighborhood of Ω, which the theorem for $\mathbb{S}^n$ provides. Then we can project to obtained the desired set. $\square$

LEMMA 4.4.2. *Let Γ be a discrete group in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ acting on a properly convex domain Ω, $\Omega \subset \mathrm{bd}\mathbb{A}^n$, so that Ω/Γ is a closed n-orbifold. Suppose that Γ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{bd}\mathbb{A}^n$ and acts on a properly convex domain V in $\mathbb{S}^n$ so that $\mathrm{Cl}(V) \cap \mathrm{bd}\mathbb{A}^n = \mathrm{Cl}(\Omega)$ holds. Suppose that γ_i is a sequence of mutually distinct elements of Γ acting on Ω. Let J be a compact subset of V. Then $\{g_i(J)\}$ can accumulate only to a subset of $\mathrm{Cl}(\Omega)$.*

PROOF. Since Ω/Γ is compact, $\hat{h}(g_i)$ is an unbounded sequence of elements in $\mathsf{SL}_\pm(n,\mathbb{R})$. Recalling (4.1.1), we write the elements of g_i as

$$\begin{pmatrix} \frac{1}{\lambda_{\tilde{E}}(g_i)^{1/n}}\hat{h}(g_i) & \vec{b}_{g_i} \\ \vec{0} & \lambda_{\tilde{E}}(g_i) \end{pmatrix} \tag{4.4.1}$$

where $\vec{b}_{g_i}$ is an $n\times 1$-vector and $\hat{h}(g_i)$ is an $n\times n$-matrix of determinant ± 1 and $\lambda_{\tilde{E}}(g_i) > 0$. Let $m(\hat{h}(g_i))$ denote the maximal modulus of the entry of the matrix of $\hat{h}(g_i)$ in $\mathsf{SL}_\pm(n,\mathbb{R})$. We may assume without loss of generality $\{\hat{h}(g_i)/m(\hat{h}(g_i))\} \to g_{n-1,\infty}$ in $M_n(\mathbb{R})$. A matrix analysis easily tells us $\lambda_1(\hat{h}(g_i)) \leq nm(\hat{h}(g_i))$ since the later term bounds the amount of stretching of norms of vectors under the action of $\hat{h}(g_i)$. By Lemma 1.3.14, dividing by $\lambda_1(\hat{h}(g_i))$, we have

$$\hat{h}(g_i)/\lambda_1(\hat{h}(g_i)) \to A g_{n-1,\infty}$$

for $A \geq 1/n$, or the sequence is unbounded in $M_n(\mathbb{R})$.

Now, we let $\{(\!(g_i)\!)\} \to (\!(g_\infty)\!)$ where g_∞ is obtained as a limit of

$$\left(\begin{array}{cc} \frac{1}{\lambda_1(\hat{h}(g_i))}\hat{h}(g_i) & \frac{\lambda_{\tilde{E}}(g_i)^{1/n}}{\lambda_1(\hat{h}(g_i))}\vec{b}_{g_i} \\ \vec{0} & \frac{\lambda_{\tilde{E}}(g_i)^{1/n}\lambda_{\tilde{E}}(g_i)}{\lambda_1(\hat{h}(g_i))} \end{array} \right). \tag{4.4.2}$$

By the uniform middle eigenvalue condition, we obtain that $\{(\!(g_i)\!)\} \to (\!(g_\infty)\!)$ where g_∞ is of form

$$\left(\begin{array}{cc} g_{n-1,\infty} & \vec{b} \\ \vec{0} & 0 \end{array} \right). \tag{4.4.3}$$

by rescaling if necessary. Now, $R_*(\{g_i\})$ is a subset of $\mathbb{S}^{n-1}_\infty$ since the lower row is zero. Hence, $\{g_i(J)\}$ geometrically converges to $g_\infty(J) \subset \mathbb{S}^{n-1}_\infty$. Since Γ acts on $\mathrm{Cl}(V)$ and $\mathrm{Cl}(V) \cap \mathrm{bd}\mathbb{A}^n = \mathrm{Cl}(\Omega)$, we obtain $g_\infty(J) \subset \mathrm{Cl}(\Omega)$. □

LEMMA 4.4.3. *Let Γ be a discrete group of projective automorphisms of a properly convex domain V and a domain $\Omega \subset V$ of dimension $n-1$. Assume that Ω/Γ is a closed n-orbifold. Suppose that Γ satisfies the uniform middle eigenvalue condition with respect to the hyperspace containing Ω. Let P be a subspace of $\mathbb{S}^n$ so that $P \cap \mathrm{Cl}(\Omega) = \emptyset$ and $P \cap V \neq \emptyset$. Then $\{g(P) \cap V | g \in \Gamma\}$ is a locally finite collection of closed sets in V.*

PROOF. Suppose not. Then there exists a sequence $\{x_i\}$, $x_i \in g_i(P) \cap V$ and $\{g_i\}$, $g_i \in \Gamma$ so that $\{x_i\} \to x_\infty \in V$ and $\{g_i\}$ is a sequence of mutually distinct elements.

We have $x_i \in F$ for a compact set $F \subset V$. Then Lemma 4.4.2 applies. $\{g_i^{-1}(F)\}$ accumulates to $\partial\mathrm{Cl}(\Omega)$. This means that $g_i^{-1}(x_i)$ accumulates to $\partial\mathrm{Cl}(\Omega)$. Since $g_i^{-1}(x_i) \in P \cap V$, and $P \cap \mathrm{Cl}(\Omega) = \emptyset$, this is a contradiction. □

4.4.2. Approximating a convex hypersurface by strictly convex hypersurfaces. See also Chapter 9 of [70] where they obtain the C^1-property only.

THEOREM 4.4.4. *We assume that Γ is a projective group with a properly convex affine action with the triple (Γ, U, D) for $U \subset \mathbb{A}^n \subset \mathbb{S}^n$. Assume the following*:

- *U is an asymptotically nice properly convex domain closed in $\mathbb{A}^n$,*
- *the boundary $\mathrm{bd}_{\mathbb{A}^n} U = \mathrm{bd} U \cap \mathbb{A}^n$, which is an $(n-1)$-manifold, is in an asymptotically nice properly convex open domain V' where Γ acts on, and $\mathrm{Cl}(U) \cap \mathbb{A}^n \subset V'$,*
- *$\mathrm{Cl}(V') \cap \mathrm{bd}\mathbb{A}^n = D$, and*
- *$\mathrm{bd}_{\mathbb{A}^n} U/\Gamma$ is a compact convex hypersurface.*

Then there exists an asymptotically nice properly convex domain V closed in V' containing U so that $\partial V/\Gamma$ is a compact hypersurface with strictly convex smooth boundary. Furthermore, $\partial V/\Gamma$ can be chosen to be arbitrarily close to $\mathrm{bd}_{\mathbb{A}^n} U/\Gamma$ in V/Γ with any complete Riemannian metric on V'/Γ.

PROOF. Let V'' be a properly convex domain so that $\mathrm{Cl}(V'') \subset U$ and $(\mathbf{\Gamma}, V'', D)$ is a properly convex affine triple. We can construct such a domain by the proof of Theorem 4.4.1 not including the part showing that ∂L is smooth. (One has to be careful that we do not use the smoothness of ∂L where the proof uses this theorem.)

Let $\mathscr{P}$ denote the set of hyperspaces sharply supporting U at $\mathrm{bd}_{\mathbb{A}^n} U$. Let P be in $\mathscr{P}$. The dual P^* of P in $\mathbb{S}^{n*}$ is a point of the properly convex domain V''^* by (1.5.2). Hence, the set $\mathscr{P}^*$ of dual points corresponding to elements of $\mathscr{P}$ is a properly embedded hypersurface in the interior of $\mathrm{Cl}(V''^*)$ by Proposition 1.5.4 applied to $\mathrm{Cl}(U)$ and the hyperspace $\mathrm{bd}_{\mathbb{A}^n}U$ and its dual $\mathscr{P}^*$.

Also, $\mathrm{bd}_{\mathbb{A}^n}U/\mathbf{\Gamma}$ is a compact orbifold by Proposition 1.5.4(iv). By the above duality, $\mathscr{P}^*/\mathbf{\Gamma}^*$ is a compact orbifold, and so is $\mathscr{P}/\mathbf{\Gamma}$. There is a fundamental domain F of $\mathscr{P}$ under $\mathbf{\Gamma}$.

Any sequence $g_i(P)$ for $P \in F$ and unbounded sequence of $\{g_i\}$, $g_i \in \mathbf{\Gamma}$, has accumulation points only in the supporting hyperplane of V' since $\mathbf{\Gamma}$ acts properly in the interior of $\mathrm{Cl}(V''^*)$. By Theorem 4.3.8, these are supporting hyperplanes of V' since these are supporting hyperplanes of V''. Hence for each point of $\mathrm{bd}_{\mathbb{A}^n}U$, there is a neighborhood N whose closure is in V''.

Hence, $\mathscr{P}$ is a locally compact collection of elements in V''^* with accumulations only in the set of hyperplanes sharply supporting V'.

For any $\varepsilon > 0$, there exists a compact set K' so that elements of $\mathscr{P} - K'$ are ε-$\mathbf{d}_H$-close to the hyperspaces asymptotic to V' by the above paragraph. Therefore, for each $x \in \mathrm{bd}_{\mathbb{A}^n}U$, there exists a compact neighborhood $N \subset V'$ so that N intersects only compact subset of $\mathscr{P}$. Hence, Lemma 4.4.6 implies the result. □

We have a generalization:

THEOREM 4.4.5. *Assume the following*:

- Γ *is a projective discrete group acting properly on a properly convex domain* U *closed in* $\mathbb{A}^n \subset \mathbb{S}^n$,
- $\{p\} = \mathrm{Cl}(U) - U$ *is a singleton,*
- Γ *is a cusp group,*
- *the manifold boundary* $\mathrm{bd}_{\mathbb{A}^n}U$ *is in a properly convex open domain* V' *where* Γ *acts on,*
- U *is closed in* V', *and*
- $\mathrm{bd}_{\mathbb{A}^n}U/\Gamma$ *is a compact convex hypersurface.*

Then there exists an properly convex domain V *closed in* V' *containing* U *so that* $\partial V/\Gamma$ *is a compact hypersurface with strictly convex smooth boundary. Furthermore,* $\partial V/\Gamma$ *can be chosen to be arbitrarily close to* $\mathrm{bd}_{\mathbb{A}^n}U/\Gamma$ *in* V/Γ *with any complete Riemannian metric on* V'/Γ.

PROOF. Let $\mathbb{A}^n$ be the affine space bounded by a hyperspace tangent to $\mathrm{Cl}(U)$ at p. Let $\mathscr{P}$ denote the set of hyperspaces sharply supporting U. There is a unique supporting hyperplane $\partial\mathbb{A}^n$ to U at p only to which $\mathscr{P}$ can accumulate. This follows since a cusp group acts on $\mathscr{P}$ and by duality. Then the analogous proof as that of Theorem 4.4.4, we can show that for any $\varepsilon > 0$, there exists a compact set K' so that elements of $\mathscr{P} - K'$ are ε-$\mathbf{d}_H$-close to $\partial\mathbb{A}^n$. Now, Lemma 4.4.6 is applicable exactly. □

LEMMA 4.4.6. *We assume that* Γ *is a projective group acting on* $\mathbb{A}^n$ *and an open domain* $U \subset \mathbb{A}^n \subset \mathbb{S}^n$. *Suppose that* U *is a properly convex domain in an affine space* $\mathbb{A}^n$ *in* $\mathbb{S}^n$ *with* $\mathrm{bd}U \cap \mathbb{A}^n$ *an embedded hypersurface in it. Let* $\mathscr{P}$ *denote the set of sharply supporting hyperspaces of* U *meeting* ∂U. *Assume the following*:

- $\mathrm{bd}_{\mathbb{A}^n}U/\Gamma$ *is a compact orbifold.*
- $\mathrm{Cl}(U) \cap \mathbb{A}^n$ *is in a convex open domain* V' *where* Γ *acts on,*
- *Each point of* $\mathrm{bd}_{\mathbb{A}^n}U$ *has a neighborhood* $N \subset V'$ *so that* $N - \mathrm{Cl}(U)$ *has only compact set of sharpley supporting hyperspaces* P *in* $\mathscr{P}$ *with* $H_P \cap N \neq \emptyset$.

Then there exists an properly convex domain V closed in V' containing U so that $\partial V/\Gamma$ is a compact hypersurface with strictly convex smooth boundary. Furthermore, $\partial V/\Gamma$ can be chosen to be arbitrarily close to $\mathrm{bd}_{\mathbb{A}^n} U/\Gamma$ *in V/Γ with any complete Riemannian metric on V'/Γ.*

PROOF. We define an affine function f_P on $\mathbb{A}^n$ so that $f_P^{-1}(0) = P$ and $f_P > 0$ on the component H_P of $\mathbb{A}^n - P$ disjoint from U^o. It follows that if $P' = g(P)$ for $P \in \mathscr{P}$ and $g \in \Gamma$, then $f_{P'} \circ g^{-1} = f_P$.

We define a smooth function

$$g(t) = t^2 \exp(1/t^2) \text{ for } t > 0, \text{ and } g(t) = 0 \text{ for } t \leq 0.$$

We let $g_P = g \circ f_P$. Then by the premise, $g_P(x)$ for each $x \in V'$ is nonzero for only compact subset of $\mathscr{P}$.

The Γ-action on V' preserves the Hilbert metric of V' and hence the action is properly discontinuous on V'. Since the action is properly discontinuous, we can put a Γ-invariant Riemannian metric on V'. The dual $\mathscr{P}^* \subset V'^*$ is a dual set of $\mathscr{P}$ considering each hyperplane as a linear function in $\mathbb{R}^{n+1}$.

Since $\mathscr{P}$ is the boundary of the supporting hyperspaces of U, it is the boundary of U^* by duality. Hence, $\mathscr{P}^*$ can be considered a topological manifold in V'^*. Since Γ^*-action on $\mathscr{P}^* \subset V'^*$ is properly discontinuous, $\mathscr{P}^*/\Gamma^*$ is a compact topological orbifold. Also, we may assume that f_P for $P \in \mathscr{P}$ is chosen continuously with respect to P by taking the fundamental domain of $\mathscr{P}^*$ under the Γ^*-action. There is a Γ-invariant measure $d\mu$ on $\mathscr{P}^*$ compatible with a positive continuous function times a volume on each chart of $\mathscr{P}$. We define a smooth function

$$\chi_U : V' \to \mathbb{R} \text{ by } \int_{P \in \mathscr{P}} g_P d\mu.$$

Hence, χ_U is well-defined in V' by the above paragraph. Moreover,

$$\chi_U^{-1}(0) = \bigcap_{P \in \mathscr{P}} (\mathbb{A}^n - H_P) = \mathrm{Cl}(U) \cap \mathbb{A}^n.$$

The third item of the premise and the proof of Proposition 2.1 of Ghomi [84] imply that χ_U is strictly convex on $V' - U$ since only compact subset of $\mathscr{P}$ is involved in the computations for each neighborhood of the third item. By our definition, χ_U is Γ-invariant.

We give an arbitrary Riemannian metric μ' on V'/Γ. There exists a neighborhood N of $\mathrm{Cl}(U) \cap V'/\Gamma$ in V'/Γ where χ_U has a nonzero differential in $N - \mathrm{Cl}(U)/\Gamma$ as we can see from the integral $\int_{P \in \mathscr{P}} Dg_P d\mu$ where Dg_P are in a properly convex cone $C_{\mathscr{P}}$ in $\mathbb{R}^{n*}$ spanned by $\{u_P | P \in \mathscr{P}\}$ for each point of $V' - \mathrm{Cl}(U)$. Then as $\varepsilon \to 0$,

$$\Sigma_\varepsilon := \{\chi_U^{-1}(\varepsilon)\}/\Gamma \to \mathrm{bd}_{\mathbb{A}^n} U/\Gamma$$

geometrically since N has a compact closure and the gradient vectors are uniformly bounded with respect to μ' and are zero only at points of U/Γ in the closure and hence we can isotopy Σ_ε along the gradient vector field to as close to $\mathrm{bd}_{\mathbb{A}^n} U/\Gamma$ as we wish. (See Batyrev [13] and Ben-Tal [14] also.) Furthermore, since χ_U is strictly convex, $\chi_U^{-1}(\varepsilon)$ is a strictly convex smooth hypersurface on which Γ acts.

□

CHAPTER 5

Properly convex radial ends and totally geodesic ends: lens properties

We will consider properly convex ends in this chapter. In Section 5.1, we define the uniform middle eigenvalue conditions for R-ends and T-ends. We state the main results of this chapter Theorem 5.1.4: the equivalence of these conditions with the generalized lens conditions for R-ends or T-ends. The generalized lens conditions often improve to lens conditions, as shown in Theorem 5.1.5. In Section 5.2, we start to study the R-end theory. First, we discuss the holonomy representation spaces. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual. Hence, the uniform middle eigenvalue condition implies the distanced action deduced from the dual theory in Chapter 4. In Section 5.3, we prove the main results. In Section 5.3.1, we estimate the largest norm $\lambda_1(g)$ of eigenvalues in terms of word length. In Section 5.3.2, we study orbits under the action with the uniform middle eigenvalue conditions. In Section 5.3.3, we prove a minor extension of Koszul's openness for bounded manifolds, well-known to many people. In Section 5.3.4, we show how to prove the strictness of the boundary of lenses and prove our main result Theorem 5.1.4 using the orbit results and the Koszul's openness. In Section 5.3.5, we now prove major Theorem 5.1.5. In Section 5.4, we show that the lens-shaped ends have concave end-neighborhoods, and we discuss the properties of lens-shaped ends in Theorems 5.4.2 and 5.4.3. If the generalized lens-shaped end is virtually factorizable, it can be made into a lens-shaped totally-geodesic R-end, which is a surprising result. In Section 5.5, we obtain the duality between the lens-shaped T-ends and generalized lens-shaped R-ends.

The main reason that we are studying the lens-shaped ends is to use them in studying the deformations preserving the convexity properties. These objects are useful in trying to understand this phenomenon. We also remark that sometimes a lens-shaped p-end neighborhood may not exist for an R-p-end within a given convex real projective orbifold. However, a generalized lens-shaped p-end neighborhood may exist for the R-p-end.

5.1. Main results

Let $\mathscr{O}$ be a strongly tame convex real projective orbifold and let $\tilde{\mathscr{O}}$ be a convex domain in $\mathbb{S}^n$ covering $\mathscr{O}$. Let $h : \pi_1(\mathscr{O}) \to \mathrm{SL}_{\pm}(n+1,\mathbb{R})$ denote the holonomy homomorphism with its image $\mathbf{\Gamma}$. We will take $\mathbb{S}^n$ as the default place where the statements take place in this chapter. However, reader can easily modify these to $\mathbb{RP}^n$-versions by Proposition 1.4.2 results in Section 1.1.8.

DEFINITION 5.1.1. Suppose that $\tilde{E}$ is an R-p-end of generalized lens-type. Then $\tilde{E}$ have a p-end-neighborhood that is projectively diffeomorphic to the interior of $\{p\} * L^o - \{p\}$ under **dev** where $\{p\} * L$ is a generalized lens-cone over a generalized lens L where $\partial(\{p\} * L - \{p\}) = \partial_+ L$ for a boundary component $\partial_+ L$ of L, and let $h(\pi_1(\tilde{E}))$ acts on L

properly and cocompactly. A *concave pseudo-end-neighborhood* of $\tilde{E}$ is the open pseudo-end-neighborhood in $\tilde{\mathscr{O}}$ projectively diffeomorphic to $\{p\} * L - \{p\} - L$ for some choice of a lens L. A *concave end-neighborhood* of an end E an end-neighborhood covered by a concave pseudo-end-neighborhood.

5.1.1. Uniform middle eigenvalue conditions. The following applies to both R-ends and T-ends. Let $\tilde{E}$ be a p-end and $\mathbf{\Gamma}_{\tilde{E}}$ the associated p-end holonomy group. We say that $\tilde{E}$ is *non-virtually-factorizable* if any finite index subgroup has a finite center or $\mathbf{\Gamma}_{\tilde{E}}$ is virtually center-free; otherwise, $\tilde{E}$ is virtually factorizable by Theorem 1.1 of [21]. (See Section 1.4.4.)

Let $\tilde{\Sigma}_{\tilde{E}}$ denote the universal cover of the end orbifold $\Sigma_{\tilde{E}}$ associated with $\tilde{E}$. We recall Proposition 1.4.10 (Theorem 1.1 of Benoist [23]). If $\mathbf{\Gamma}_{\tilde{E}}$ is virtually factorizable, then $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the following condition:

- $\mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}}) = K_1 * \cdots * K_k$ where each K_i is properly convex or is a singleton.
- Let G_i be the restriction of the K_i-stabilizing subgroup of $\mathbf{\Gamma}_{\tilde{E}}$ to K_i. Then G_i acts on K_i^o cocompactly. (Here K_i can be a singleton, and Γ_i a trivial group.)
- A finite index subgroup G' of $\mathbf{\Gamma}_{\tilde{E}}$ is isomorphic to a cocompact subgroup of $\mathbb{Z}^{k-1} \times G_1 \times \cdots \times G_k$.
- The center $\mathbb{Z}^{k-1}$ of G' is a subgroup acting trivially on each K_i.

Note that there are examples of discrete groups of form $\mathbf{\Gamma}_{\tilde{E}}$ where G_i are non-discrete. (See also Example 5.5.3 of [135] as pointed out by M. Kapovich.)

We will use simply $\mathbb{Z}^{k-1}$ to represent the corresponding group on $\mathbf{\Gamma}_{\tilde{E}}$. Here, $\mathbb{Z}^{k-1}$ is called a virtual center of $\mathbf{\Gamma}_{\tilde{E}}$.

Let Γ be generated by finitely many elements $g_1, \dots, g_m$. Let $w(g)$ denote the minimum word length of $g \in G$ written as words of $g_1, \dots, g_m$. The *conjugate word length* $\mathrm{cwl}(g)$ of $g \in \pi_1(\tilde{E})$ is

$$\min\{w(cgc^{-1}) | c \in \pi_1(\tilde{E})\}.$$

Let d_K denote the Hilbert metric of the interior K^o of a properly convex domain K in $\mathbb{RP}^n$ or $\mathbb{S}^n$. Suppose that a projective automorphism group Γ acts on K properly. Let $\mathrm{length}_K(g)$ denote the infimum of $\{d_K(x, g(x)) | x \in K^o\}$, compatible with $\mathrm{cwl}(g)$.

DEFINITION 5.1.2. Let $\mathrm{v}_{\tilde{E}}$ be a p-end vertex of an R-p-end $\tilde{E}$. Let $K := \mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$. The p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the *umec with respect to* $\mathrm{v}_{\tilde{E}}$ *or the R-p-end structure* if the following hold:

- each $g \in \mathbf{\Gamma}_{\tilde{E}}$ satisfies for a uniform $C > 1$ independent of g

$$C^{-1}\mathrm{length}_K(g) \le \log\left(\frac{\lambda_1(g)}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)}\right) \le C\mathrm{length}_K(g), \tag{5.1.1}$$

  for the largest norm $\lambda_1(g)$ of the eigenvalues of g and the eigenvalue $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ of g at $\mathrm{v}_{\tilde{E}}$.

Of course, we choose the matrix of g so that $\lambda_{\mathrm{v}_{\tilde{E}}}(g) > 0$. See Remark 1.1.5 as we are looking for the lifting of g that acts on p-end neighborhood. We can replace $\mathrm{length}_K(g)$ with $\mathrm{cwl}(g)$ for properly convex ends by Svarc-Milnor (A. Swartz, professor at UC. Davis) or Milnor-Svarc theorem. (See Theorem 8.1 of Farb-Margalit [78].) We remark that the condition does depend on the choice of $\mathrm{v}_{\tilde{E}}$; however, the radial end structures will determine the end vertices.

The definition of course applies to the case when $\mathbf{\Gamma}_{\tilde{E}}$ has the finite-index subgroup with the above properties.

We recall a dual definition identical with the definition in Section 4.1 but adopted to T-p-ends.

DEFINITION 5.1.3. Suppose that $\tilde{E}$ is a properly convex T-p-end. Suppose that the ideal boundary component $\tilde{\Sigma}_{\tilde{E}}$ of the T-p-end is properly convex. Let $K = \mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$. Let $g^* : \mathbb{R}^{n+1*} \to \mathbb{R}^{n+1*}$ be the dual transformation of $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. The p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the *umec with respect to $\tilde{\Sigma}_{\tilde{E}}$ or the T-p-end structure*

- if each $g \in \mathbf{\Gamma}_{\tilde{E}}$ satisfies for a uniform $C > 1$ independent of g

$$C^{-1}\mathrm{length}_K(g) \leq \log\left(\frac{\lambda_1(g)}{\lambda_{K^*}(g^*)}\right) \leq C\mathrm{length}_K(g) \tag{5.1.2}$$

  for the largest norm $\lambda_1(g)$ of the eigenvalues of g and the eigenvalue $\lambda_{K^*}(g^*)$ of g^* in the vector in the direction of K^*, the point dual to the hyperspace containing K.

Again, the condition depends on the choice of the hyperspace containing $\tilde{\Sigma}_{\tilde{E}}$, i.e., the T-p-end structure. (We again lift g so that $\lambda_{K^*}(g) > 0$.)

Here $\mathbf{\Gamma}_{\tilde{E}}$ will act on a properly convex domain K^o of lower dimension, and we will apply the definition here. This condition is similar to the Anosov condition studied by Guichard and Wienhard [96], and the results also seem similar. We do not use their theories. They also use word length instead. One may look at the paper of Kassel-Potrie [109] to understand the relationship between eigenvalues and singular values. We use the eigenvalues to obtain conjugacy invariant conditions which is needed in proving the converse part of Theorem 5.1.4. Our main tools to understand these questions are in Chapter 4 which we will use here.

We will see that the condition is an open condition; and hence a "structurally stable one." (See Corollary 6.1.3.)

5.1.2. Lens and the uniform middle eigenvalue condition. As holonomy groups, the condition for being a generalized lens R-p-end and one for being a lens R-p-end are equivalent. For the following, we are not concerned with a lens-cone being in $\tilde{\mathcal{O}}$.

THEOREM 5.1.4 (Lens holonomy). *Let $\tilde{E}$ be an R-p-end of a strongly tame convex real projective orbifold. Then the holonomy group $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition for the R-p-end vertex $\mathrm{v}_{\tilde{E}}$ if and only if it acts on a lens-cone with vertex $\mathrm{v}_{\tilde{E}}$ and its lens properly and cocompactly. Moreover, in this case, the lens-cone exists in the union of great segments with the vertex $\mathrm{v}_{\tilde{E}}$ in the directions of in the direction of a properly convex domain $\Omega \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ where $h(\pi_1(\tilde{E}))$ acts properly discontinuously.*

For the following, we are concerned with a lens-cone being in $\tilde{\mathcal{O}}$.

THEOREM 5.1.5 (Theorem 5.3.21 (Actual lens-cone)). *Let $\mathcal{O}$ be a strongly tame convex real projective orbifold.*

- *Let $\tilde{E}$ be a properly convex R-p-end.*
  - *The p-end holonomy group satisfies the uniform middle-eigenvalue condition if and only if $\tilde{E}$ is a generalized lens-shaped R-p-end.*
- *Assume that the holonomy group of $\mathcal{O}$ is strongly irreducible, and $\mathcal{O}$ is properly convex. If $\mathcal{O}$ satisfies the triangle condition* (*see Definition 5.3.18*) *or $\tilde{E}$ is virtually factorizable or is a totally geodesic R-end, then we can replace the term "generalized lens-shaped" to "lens-shaped" in the above statement.*

We will prove the analogous result for totally geodesic ends in Theorem 5.5.4.

Notice that there is no condition on $\mathscr{O}$ to be properly convex.

Another main result is on the duality of lens-shaped ends: Recalling from Section 1.5.1, we have $\mathbb{RP}^{n*} = \mathbb{P}(\mathbb{R}^{n+1*})$ the dual real projective space of $\mathbb{RP}^n$. Recall also $\mathbb{S}^{n*} = \mathbb{S}(\mathbb{R}^{n+1*})$ as the dual spherical projective space of $\mathbb{S}^n$. In Section 5.2, we define the projective dual domain Ω^* in $\mathbb{RP}^{n*}$ to a properly convex domain Ω in $\mathbb{RP}^n$ where the dual group Γ^* to Γ acts on. Vinberg showed that there is a duality diffeomorphism between Ω/Γ and Ω^*/Γ^*. The ends of $\mathscr{O}$ and $\mathscr{O}^*$ are in a one-to-one correspondence. Horospherical ends are dual to themselves, i.e., "self-dual types", and properly convex R-ends and T-ends are dual to one another. (See Proposition 5.5.5.) We will see that generalized lens-shaped properly convex R-ends are always dual to lens-shaped T-ends by Corollary 5.5.7.

We mention that Fried also solved this question when the linear holonomy is in $\mathsf{SO}(2,1)$ [81]. Also, we found out later that the dual consideration of Barbot's work on the existence of globally hyperbolic spacetimes for geometrically finite linear holonomy in $\mathsf{SO}(n,1)$ also solves this problem in the setting of finding Cauchy hyperspaces in flat Lorentz spaces. (See Theorem 4.25 of [11].)

5.2. The end theory

In this section, we discuss the properties of lens-shaped radial and totally geodesic ends and their duality also.

5.2.1. The holonomy homomorphisms of the end fundamental groups: the tubes. We will discuss for $\mathbb{S}^n$ only here but the obvious $\mathbb{RP}^n$-version exists for the theory. Let $\tilde{E}$ be an R-p-end of $\tilde{\mathscr{O}}$. Let $\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$ be the subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ fixing a point $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^n$. This group can be understood as follows by letting $\mathrm{v}_{\tilde{E}} = [0,\dots,0,1]$ as a group of matrices: For $g \in \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$, we have

$$\begin{pmatrix} \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)^{1/n}}\hat{h}(g) & \vec{0} \\ \vec{v}_g & \lambda_{\mathrm{v}_{\tilde{E}}}(g) \end{pmatrix} \tag{5.2.1}$$

where $\hat{h}(g) \in \mathsf{SL}_\pm(n,\mathbb{R}), \vec{v} \in \mathbb{R}^{n*}, \lambda_{\mathrm{v}_{\tilde{E}}}(g) \in \mathbb{R}_+$, is the so-called linear part of h. Here,

$$\lambda_{\mathrm{v}_{\tilde{E}}} : g \mapsto \lambda_{\mathrm{v}_{\tilde{E}}}(g) \text{ for } g \in \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$$

is a homomorphism so it is trivial in the commutator group $[\mathbf{\Gamma}_{\tilde{E}}, \mathbf{\Gamma}_{\tilde{E}}]$. There is a group homomorphism

$$\begin{aligned} \mathscr{L}' : \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}} &\to \mathsf{SL}_\pm(n,\mathbb{R}) \times \mathbb{R}_+ \\ g &\mapsto (\hat{h}(g), \lambda_{\mathrm{v}_{\tilde{E}}}(g)) \end{aligned} \tag{5.2.2}$$

with the kernel equal to $\mathbb{R}^{n*}$, a dual space to $\mathbb{R}^n$. Thus, we obtain a diffeomorphism

$$\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}} \to \mathsf{SL}_\pm(n,\mathbb{R}) \times \mathbb{R}^{n*} \times \mathbb{R}_+.$$

We note the multiplication rules

$$(A,\vec{v},\lambda)(B,\vec{w},\mu) = (AB, \frac{1}{\mu^{1/n}}\vec{v}B + \lambda\vec{w}, \lambda\mu). \tag{5.2.3}$$

We denote by $\mathscr{L}_1 : \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}} \to \mathsf{SL}_\pm(n,\mathbb{R})$ the further projection to $\mathsf{SL}_\pm(n,\mathbb{R})$.

Let $\Sigma_{\tilde{E}}$ be the end $(n-1)$-orbifold. Given a representation

$$\hat{h} : \pi_1(\Sigma_{\tilde{E}}) \to \mathsf{SL}_\pm(n,\mathbb{R}) \text{ and a homomorphism } \lambda_{v_{\tilde{E}}} : \pi_1(\Sigma_{\tilde{E}}) \to \mathbb{R}_+,$$

we denote by $\mathbb{R}^n_{\hat{h},\lambda_{v_{\tilde{E}}}}$ the $\mathbb{R}$-module with the $\pi_1(\Sigma_{\tilde{E}})$-action given by

$$g \cdot \vec{v} = \frac{1}{\lambda_{v_{\tilde{E}}}(g)^{1/n}} \hat{h}(g)(\vec{v}).$$

And we denote by $\mathbb{R}^{n*}_{\hat{h},\lambda_{v_{\tilde{E}}}}$ the dual vector space with the right dual action given by

$$g \cdot \vec{v} = \frac{1}{\lambda_{v_{\tilde{E}}}(g)^{1/n}} \hat{h}(g)^*(\vec{v}).$$

Let $H^1(\pi_1(\tilde{E}), \mathbb{R}^{n*}_{\hat{h},\lambda_{v_{\tilde{E}}}})$ denote the cohomology space of 1-cocycles

$$\Gamma \ni g \mapsto \vec{v}(g) \in \mathbb{R}^{n*}_{\hat{h},\lambda_{v_{\tilde{E}}}}.$$

As $\mathrm{Hom}(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R}_+)$ equals $H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R})$, we obtain:

THEOREM 5.2.1. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold, and let $\tilde{\mathscr{O}}$ be its universal cover. Let $\Sigma_{\tilde{E}}$ be the end orbifold associated with an R-p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$. Then the space of representations*

$$\mathrm{Hom}(\pi_1(\Sigma_{\tilde{E}}), \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}})/\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$$

is the space mapping to

$$\mathrm{Hom}(\pi_1(\Sigma_{\tilde{E}}), \mathsf{SL}_\pm(n,\mathbb{R}))/\mathsf{SL}_\pm(n,\mathbb{R}) \times H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R})$$

with the fiber isomorphic to $H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R}^{n*}_{\hat{h},\lambda_{v_{\tilde{E}}}})$ for each $([\hat{h}], \lambda)$.

On a Zariski open subset of $\mathrm{Hom}(\pi_1(\Sigma_{\tilde{E}}), \mathsf{SL}_\pm(n,\mathbb{R}))/\mathsf{SL}_\pm(n,\mathbb{R})$, the dimensions of the fibers are constant (see Johnson-Millson [106]). A similar idea is given by Mess [129]. In fact, the dualizing these matrices gives us a representation to $\mathbf{Aff}(\mathbb{A}^n)$. (See Chapter 4.) In particular if we restrict ourselves to linear parts to be in $\mathsf{SO}(n,1)$, then we are exactly in the cases studied by Mess. (The concept of the duality is explained in Section 5.2.3)

If Σ is a closed 2-orbifold with negative Euler characteristic, one can compute the dimension of $H^1(\pi_1(\Sigma_{\tilde{E}}), \mathbb{R}^{n*}_{\hat{h},\lambda_{v_{\tilde{E}}}})$ using the twisted orbifold Euler characteristic of Porti [138] as the 0-th and 2-nd cohomology are zero.

5.2.2. Tubular actions. Let us give a pair of antipodal points v and v_-. If a group Γ of projective automorphisms fixes a pair of fixed points v and v_-, then Γ is said to be *tubular*. There is a projection $\Pi_\mathrm{v} : \mathbb{S}^n - \{\mathrm{v}, \mathrm{v}_-\} \to \mathbb{S}^{n-1}_\mathrm{v}$ given by sending every great segment with endpoints v and v_- to a point of the sphere of directions at v.

A *tube* in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) is the closure of the inverse image $\Pi_\mathrm{v}^{-1}(\Omega)$ of a domain Ω in $\mathbb{S}^{n-1}_\mathrm{v}$ (resp. in $\mathbb{RP}^{n-1}_\mathrm{v}$). We often denote the closure in $\mathbb{S}^n$ by $\mathscr{T}_\mathrm{v}(\Omega)$, and we call it a *tube domain*. Given an R-p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$, let $\mathrm{v} := \mathrm{v}_{\tilde{E}}$. The *end domain* is $R_\mathrm{v}(\tilde{\mathscr{O}})$. If an R-p-end $\tilde{E}$ has the end domain $\tilde{\Sigma}_{\tilde{E}} = R_\mathrm{v}(\tilde{\mathscr{O}})$, the group $h(\pi_1(\tilde{E}))$ acts on $\mathscr{T}_\mathrm{v}(\Omega)$.

The image of the tube domain $\mathscr{T}_\mathrm{v}(\Omega)$ in $\mathbb{RP}^n$ is still called a *tube domain* and denoted by $\mathscr{T}_{[v]}(\Omega)$ where $[v]$ is the image of v.

We will now discuss for the $\mathbb{S}^n$-version but the $\mathbb{RP}^n$ version is obviously clearly obtained from this by a minor modification.

Letting v have the coordinates $[0,\dots,0,1]$, we obtain the matrix of g of $\pi_1(\tilde{E})$ of form

$$\begin{pmatrix} \frac{1}{\lambda_\mathrm{v}(g)^{\frac{1}{n}}}\hat{h}(g) & 0 \\ \vec{b}_g & \lambda_\mathrm{v}(g) \end{pmatrix} \tag{5.2.4}$$

where $\vec{b}_g$ is an $n \times 1$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant ± 1 and $\lambda_{\mathrm{v}}(g)$ is a positive constant.

Note that the representation $\hat{h} : \pi_1(\tilde{E}) \to \mathsf{SL}_{\pm}(n, \mathbb{R})$ is given by $g \mapsto \hat{h}(g)$. Here we have $\lambda_{\mathrm{v}}(g) > 0$. If $\tilde{\Sigma}_{\tilde{E}}$ is properly convex, then the convex tubular domain and the action is said to be *properly tubular*

5.2.3. Affine actions dual to tubular actions. Let $\mathbb{S}^{n-1}$ in $\mathbb{S}^n = \mathbb{S}(\mathbb{R}^{n+1})$ be a great sphere of dimension $n-1$. A component of a component of the complement of $\mathbb{S}^{n-1}$ can be identified with an affine space $\mathbb{A}^n$. The subgroup of projective automorphisms preserving $\mathbb{S}^{n-1}$ and the components equals the affine group $\mathbf{Aff}(\mathbb{A}^n)$.

By duality, a great $(n-1)$-sphere $\mathbb{S}^{n-1}$ corresponds to a point $\mathrm{v}_{\mathbb{S}^{n-1}}$. Thus, for a group Γ in $\mathbf{Aff}(\mathbb{A}^n)$, the dual groups Γ^* acts on $\mathbb{S}^{n*} := \mathbb{S}(\mathbb{R}^{n+1,*})$ fixing $\mathrm{v}_{\mathbb{S}^{n-1}}$. (See Proposition 1.5.4 also.)

Let $\mathbb{S}^{n-1}_\infty$ denote a hyperspace in $\mathbb{S}^n$. Suppose that Γ acts on a properly convex open domain U where $\Omega := \mathrm{bd} U \cap \mathbb{S}^{n-1}_\infty$ is a properly convex domain. We recall that Γ has a properly convex affine action. Let us recall some facts from Section 1.5.4

- A great $(n-2)$-sphere $P \subset \mathbb{S}^n$ is dual to a great circle P^* in $\mathbb{S}^{n*}$ given as the set of hyperspheres containing P.
- The great sphere $\mathbb{S}^{n-1}_\infty \subset \mathbb{S}^n$ with an orientation is dual to a point $\mathrm{v} \in \mathbb{S}^{n*}$ and it with an opposite orientation is dual to $\mathrm{v}_- \in \mathbb{S}^{n*}$.
- An oriented hyperspace $P \subset \mathbb{S}^{n-1}_\infty$ of dimension $n-2$ is dual to an oriented great circle passing v and v_-, giving us an element $P^\dagger$ of the linking sphere $\mathbb{S}^{n-1*}_{\mathrm{v}}$ of rays from v in $\mathbb{S}^{n*}$.
- The space S of oriented hyperspaces in $\mathbb{S}^{n-1}_\infty$ equals $\mathbb{S}^{n-1*}_\infty$. Thus, there is a projective isomorphism

$$\mathscr{I}_2 : S = \mathbb{S}^{n-1*}_\infty \ni P \leftrightarrow P^\dagger \in \mathbb{S}^{n-1*}_{\mathrm{v}}.$$

For the following, let's use the terminology that an oriented hyperspace V in $\mathbb{S}^i$ *supports* an open submanifold A if it bounds an open i-hemisphere H in the right orientation containing A.

PROPOSITION 5.2.2. *Suppose that $\Gamma \subset \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ acts on a properly convex open domain $\Omega \subset \mathbb{S}^{n-1}_\infty$ cocompactly. Then the dual group Γ^* acts on a properly tubular domain B with vertices $\mathrm{v} := \mathrm{v}_{\mathbb{S}^{n-1}_\infty}$ and $\mathrm{v}_- := \mathrm{v}_{\mathbb{S}^{n-1}_\infty,-}$ dual to $\mathbb{S}^{n-1}_\infty$. Moreover, the domain Ω and domain $R_{\mathrm{v}}(B)$ in the linking sphere $\mathbb{S}^{n-1}_{\mathrm{v}}$ from v in the directions of B^o are projectively diffeomorphic to a pair of dual domains in $\mathbb{S}^{n-1}_\infty$ respectively.*

PROOF. Given $\Omega \subset \mathbb{S}^{n-1}_\infty$, we obtain the properly convex open dual domain Ω^* in $\mathbb{S}^{n-1*}_\infty$. An oriented $n-2$-hemisphere sharply supporting Ω in $\mathbb{S}^{n-1}_\infty$ corresponds to a point of $\mathrm{bd}\Omega^*$ and vice versa. (See Section 1.5.) An oriented great $n-1$-sphere in $\mathbb{S}^n$ supporting Ω but not containing Ω meets a great $n-2$-sphere P in $\mathbb{S}^{n-1}_\infty$ supporting Ω. The dual P^* of P is the set of hyperspaces containing P, a great circle in $\mathbb{S}^{n*}$. The set of oriented great $n-1$-spheres containing P supporting Ω but not containing Ω forms a pencil; in this case, a great open segment I_{P^*} in $\mathbb{S}^{n*}$ with endpoints v and v_-. (See Section 1.1.2 for the definition of supporting hyperspaces.) Let $P^\ddagger \in \mathbb{S}^{n-1*}_\infty$ denote the dual of P in $\mathbb{S}^{n-1}_\infty$. Then $P^\dagger := \mathscr{I}_2(P^\ddagger)$ is the direction of P^* at v as we can see from the projective isomorphism $\mathscr{I}_2$. Recall from the beginning of Section 1.5.1 P supports Ω if and only if $P^\ddagger \in \Omega^*$. Hence, there is a homeomorphism

$$I_P := \{Q|Q \text{ is an oriented great } n-1\text{-sphere supporting } \Omega, Q\cap \mathbb{S}^{n-1}_\infty = P\} \to$$
$$S_{P^*} = \{p|p \text{ is a point of a great open segment in } P^* \text{ with endpoints } \mathrm{v}, \mathrm{v}_-$$
$$\text{where the direction } P^\dagger = \mathscr{I}_2(P^\ddagger), P^\ddagger \in \Omega^*\}. \tag{5.2.5}$$

The set B of oriented hyperspaces supporting Ω possibly containing Ω meets an oriented $(n-2)$-hyperspace in $\mathbb{S}^{n-1}_\infty$ supporting Ω. Denote by α_x the great segment with vertices v and v_- in the direction of $x \in \mathbb{S}^{n-1}_{\mathrm{v}}$. Thus, we obtain

$$B^* = \bigcup_{P\in\Omega^*} S_{P^*} = \bigcup_{x\in\mathscr{I}_2(\Omega^*)} \alpha_x \subset \mathbb{S}^{n*}.$$

Let $\mathscr{T}(\Omega^*)$ denote the union of open great segments with endpoints v and v_- in direction of Ω^*. Thus, $B^* = \mathscr{T}(\Omega^*)$. Thus, there is a homeomorphism

$$I := \{Q|Q \text{ is an oriented great } n-1\text{-sphere sharply supporting } \Omega\} \to$$
$$S = \{p|p\in S_{P^*}, P^\ddagger \in \mathrm{bd}\Omega^*\} = \mathrm{bd}B^* - \{\mathrm{v}, \mathrm{v}_-\}. \tag{5.2.6}$$

Also, $R_{\mathrm{v}}(B^*) = \mathscr{I}_2(\Omega^*)$ by the above equation. Thus, Γ acts on Ω if and only if Γ acts on I if and only if Γ^* acts on S if and only if Γ^* acts on B^* and on Ω^*. □

5.2.4. Distanced tubular actions and asymptotically nice affine actions. The approach is similar to what we did in Chapter 4 but is in the dual setting.

DEFINITION 5.2.3.

Radial action: A properly tubular action of Γ is said to be *distanced* if a Γ-invariant tubular domain contains a properly convex compact Γ-invariant subset disjoint from the vertices of the tubes.

Affine action: We recall from Chapter 4. A properly convex affine action of Γ is said to be *asymptotically nice* if Γ acts on a properly convex open domain U' in $\mathbb{A}^n$ with boundary in $\Omega \subset \mathbb{S}^{n-1}_\infty$, and Γ acts on a compact subset

$$J := \{H|H \text{ is an AS-hyperspace passing } x\in \mathrm{bd}\Omega, H \not\subset \mathbb{S}^{n-1}_\infty\}$$

where we require that every sharply supporting $(n-2)$-dimensional space of Ω in $\mathbb{S}^{n-1}_\infty$ is contained in at least one of the element of J.

The following is a simple consequence of the homeomorphism given by equation (5.2.6).

PROPOSITION 5.2.4. *Let Γ and Γ^* be dual groups where Γ has an affine action on $\mathbb{A}^n$ and Γ^* is tubular with the vertex $\mathrm{v} = \mathrm{v}_{\mathbb{S}^{n-1}_\infty}$ dual to the boundary $\mathbb{S}^{n-1}_\infty$ of $\mathbb{A}^n$. Let $\Gamma = (\Gamma^*)^*$ acts on a convex open domain Ω with a closed n-orbifold Ω/Γ. Then Γ acts asymptotically nicely if and only if Γ^* acts on a properly tubular domain B and is distanced.*

PROOF. From the definition of asymptotic niceness, we can do the following: for each point x and a sharply supporting hyperspace P of $\mathrm{bd}\Omega$ passing x in $\mathbb{S}^{n-1}$, we choose a great $n-1$-sphere in $\mathbb{S}^n$ sharply supporting Ω at x containing P and uniformly bounded at a distance in the $\mathbf{d}_H$-sense from $\mathbb{S}^{n-1}_\infty$. This forms a compact Γ-invariant set J of hyperspaces.

The dual points of the supporting hyperspaces passing points of $\mathrm{bd}\Omega$ are points on $\mathrm{bd}B$ for a tube domain B with vertex v dual to $\mathbb{S}^{n-1}_\infty$ by (5.2.6) in the proof of Proposition 5.2.2. Since the hyperspaces in J sharply supporting U at $x\in \mathrm{bd}\Omega$, are bounded at a distance from $\mathbb{S}^{n-1}_\infty$ in the $\mathbf{d}_H$-sense, the dual points are uniformly bounded at a distance from the vertices v and v_-. We take the closure of the set of hyperspaces in the dual space of $\mathbb{S}^{n*}$. Let us call this compact set K. Let $\Omega^* \subset \mathbb{S}^{n-1}_{\mathrm{v}}$ be the dual domain of Ω. Then for every point of $\mathrm{bd}\Omega^*$, we have a point of K in the corresponding great segment from v to v_-. Then K is uniformly bounded at a distance from v and v_- in the $\mathbf{d}$-sense. The convex hull of K in $\mathrm{Cl}(\tilde{\mathscr{O}})$ is a compact convex set bounded at a uniform distance from v and v_- since the tube domain is properly convex. Since K is Γ^*-invariant, so is the convex hull in $\mathrm{Cl}(\tilde{\mathscr{O}})$. Therefore, Γ^* acts on B as a distanced action.

The converse is also very simple to prove by (5.2.6) in the proof of Proposition 5.2.2. We can take the intersection U of the inner components of hyperspaces involved here and obtain an open set. Also, U is not empty by an elementary geometric argument since the angles between $\mathbb{S}^{n-1}_\infty$ and the strictly supporting hyperspaces are uniformly bounded below. Hence, the asymptotic niceness is proved. ☐

THEOREM 5.2.5. *Let Γ have a nontrivial properly convex tubular action at vertex $\mathrm{v} = \mathrm{v}_{\mathbb{S}^{n-1}_\infty}$ on $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) and acts on a properly convex tube B and satisfies the uniform middle-eigenvalue conditions with respect to $\mathrm{v}_{\mathbb{S}^{n-1}_\infty}$. We assume that Γ acts on a convex open domain $\Omega \subset \mathbb{S}^{n-1}_{\mathrm{v}}$ where $B = \mathscr{T}_{\mathrm{v}}(\Omega)$ and Ω/Γ is a closed n-orbifold. Then Γ is distanced inside the tube B, and B contains a distanced Γ-invariant compact set K. Finally, we can choose the distanced set K to be in a hypersphere disjoint from* v, v_- *when Γ is virtually factorizable.*

PROOF. We will again prove for $\mathbb{S}^n$. Let Ω denote the convex domain in $\mathbb{S}^{n-1}_{\mathrm{v}}$ corresponding to B^o. By Theorems 4.1.1 and 4.3.1, Γ^* is asymptotically nice. Proposition 5.2.4 implies the result.

Now, we prove the final part to show the total geodesic property of virtually factorizable ends: Suppose that Γ acts virtually reducibly on $\mathbb{S}^{n-1}_{\mathrm{v}}$ on a properly convex domain Ω. Then Γ is virtually isomorphic to a cocompact subgroup of

$$\mathbb{Z}^{l_0-1} \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_{l_0}$$

where $\mathbf{\Gamma}_i$ is irreducible by Proposition 1.4.10. Also, Γ acts on

$$K := K_1 * \cdots * K_{l_0} = \mathrm{Cl}(\Omega) \subset \mathbb{S}^{n-1}_{\mathrm{v}}$$

where K_i denotes the properly convex compact set in $\mathbb{S}^{n-1}_{\mathrm{v}}$ where $\mathbf{\Gamma}_i$ acts on for each i. Here, K_i is 0-dimensional for $i = s+1, \dots, l_0$ for $s+1 \leq l_0$. Let B_i be the convex tube with vertices v and v_- corresponding to K_i. Each $\mathbf{\Gamma}_i$ for $i = 1, \dots, s$ acts on a nontrivial tube B_i with vertices v and v_- in a subspace.

For each i, $s+1 \leq i \leq r$, B_i is a great segment with endpoints v and v_-. A point p_i corresponds to B_i in $\mathbb{S}^{n-1}_{\mathrm{v}}$.

The virtual center isomorphic to $\mathbb{Z}^{l_0-1}$ is in the group Γ by Proposition 1.4.10. Recall that a nontrivial element g of the virtual center acts trivially on the subspace K_i of $\mathbb{S}^{n-1}_{\mathrm{v}}$; that is, g has only one associated eigenvalue in points of K_i. There exists a nontrivial element g of the virtual center with the largest norm eigenvalue in K_i for the induced g-action on $\mathbb{S}^{n-1}_{\mathrm{v}}$ since the action of Γ on Ω is cocompact. By the middle eigenvalue condition, for each i, we can find g in the center so that g has a hyperspace $K'_i \subset B_i$ with largest norm eigenvalues. The convex hull of

$$K'_1 \cup \cdots \cup K'_{l_0}$$

in $\mathrm{Cl}(B)$ is a distanced Γ-invariant compact convex set. For $(\zeta_1,\cdots,\zeta_{l_0})\in\mathbb{R}_+^{l_0}$, we define
(5.2.7)
$$\zeta(\zeta_1,\cdots,\zeta_{l_0}):=\begin{pmatrix}\zeta_1\mathrm{I}_{n_1+1} & 0 & \cdots & 0\\ 0 & \zeta_2\mathrm{I}_{n_2+1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \zeta_{l_0}\mathrm{I}_{n_{l_0}+1}\end{pmatrix}, \zeta_1^{n_1+1}\zeta_2^{n_2+1}\cdots\zeta_{l_0}^{n_{l_0}+1}=1,$$

using the coordinates where each K_i corresponds to a block.

Now, we consider the general case. The element x of $K^o\subset\mathbb{S}^{n-1}_{\mathrm{v}}$ has coordinates

$$\left(\!\left(\lambda_1,\ldots,\lambda_{l_0},\vec{x}_1,\ldots,\vec{x}_{l_0}\right)\!\right), \text{ where } \sum_{i=1}^{l_0}\lambda_i=1, x=\left(\!\left(\sum_{i=1}^{l_0}\lambda_i\vec{x}_i\right)\!\right)$$

for $\vec{x}_i$ is a unit vector in the direction of K_i^o for $i=1,\ldots,l_0$.

Let $\mathscr{Z}(G)$ for any subgroup G of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ denote the Zariski closure in $\mathsf{SL}_\pm(n+1,\mathbb{R})$.

Let Γ' denote the finite index normal subgroup acting on each of K_i in Γ. We take the Zariski closure of Γ'. It is isomorphic to

$$\mathbb{R}^{l_0-1}\times\mathscr{Z}(\mathbf{\Gamma}_1)\times\cdots\mathscr{Z}(\mathbf{\Gamma}_{l_0})$$

where $\mathscr{Z}(\mathbf{\Gamma}_i)$ is the Zariski closure of $\mathbf{\Gamma}_i$ easily derivable from Theorem 1.1 of Benoist [21] for our setting. The elements of $\mathbb{R}^{l_0}$ commute with elements of $\mathbf{\Gamma}_i$ and hence with Γ'.

There is a linear map $Z:\mathbb{Z}^{l_0-1}\to\mathbb{R}^{l_0}$ so that an isomorphism $\mathbb{Z}^{l_0-1}\to\Gamma'$ is represented by $\zeta\circ\exp\circ Z$.

Let $\log\lambda_1:\mathbb{Z}^{l_0-1}\to\mathbb{R}$ denote a map given by taking the log of the largest norm and $\log\lambda_n:\mathbb{Z}^{l_0-1}\to\mathbb{R}$ given by taking the log of the smallest norm and $\log\lambda_{\mathrm{v}}:\mathbb{Z}^{l_0-1}\to\mathbb{R}$ the log of the eigenvalue at v. Now, $\log\lambda_1$ and $\log\lambda_n$ extends to piecewise linear functions on $\mathbb{R}^{l_0-1}$ that are linear over cones with origin as the vertex.

$\log\lambda_1$ has only nonnegative values and $\log\lambda_n$ has nonpositive values. The uniform middle eigenvalue condition is equivalent to the condition that $\log\lambda_1>\log\lambda_{\mathrm{v}}>\log\lambda_n$ holds over $\mathbb{R}^n-\{O\}$.

Let B_i denote the tube $\mathscr{T}(K_i)$. We choose an element g of the virtual center having largest norm of the eigenvalue at points of K_i as an automorphism of $\mathbb{S}^{n-1}_{\mathrm{v}}$. g acts on B_i. By the uniform middle eigenvalue condition, g fixes a subspace $\hat{K}_i$ equal to $B_i\cap P_i$ for a hyperspace P_i in the span of B_i corresponding to the largest norm eigenvalue of g as an element of $\mathsf{SL}_\pm(n+1,\mathbb{R})$. By commutativity, the center also acts on $\hat{K}_i$. Define P to be the join of $P_1,\ldots,P_{l_0}$.. Hence, the center acts on the join $\hat{K}_1*\cdots*\hat{K}_{l_0}$ equals $\mathscr{T}\cap P$ for a hyperspace P. By commutativity, Γ' acts on B_i also.

Suppose that for some $g\in\Gamma-\Gamma'$, $g(P)\neq P$. Then $g(P)\cap B_j$ has a point x closer to v or v_- than $P\cap B_j$ for some j. Assume that it is closer to v without loss of generality. We find a sequence $\{\vec{k}_i\}$ so that $g_i=\zeta\circ\exp\circ Z(\vec{k}_i)$ have the largest eigenvalue at points of B_i and $\lambda_1(g_i)/\lambda_{\mathrm{v}}(g_i)\to\infty$. Since $\lambda_n(g_i^{-1})=\lambda_1(g_i)^{-1}$ and $\lambda_{\mathrm{v}}(g_i^{-1})=\lambda_{\mathrm{v}}(g_i)^{-1}$, we obtain that $\{g_i^{-1}(x)\}\to$ v as $i\to\infty$. Then we obtain that $g_i(g(P))\cap\mathscr{T}$ is not distanced. This contradicts the first paragraph of the proof.

Hence, Γ acts on the hyperspace P. Hence, letting $K=B\cap P$ completes the proof.

[$\mathbb{S}^n$T] □

5.3. The characterization of lens-shaped representations

The main purpose of this section is to characterize the lens-shaped representations in terms of eigenvalues, a major result of this monograph.

First, we prove the eigenvalue estimation in terms of lengths for non-virtually-factorizable and hyperbolic ends. We show that the uniform middle-eigenvalue condition implies the existence of limit sets. This proves Theorem 5.1.4. Finally, we prove the equivalence of the lens condition and the uniform middle-eigenvalue condition in Theorem 5.3.21 for both R-ends and T-ends under very general conditions. That is, we prove Theorem 5.1.5.

Techniques here are somewhat related to the work of Guichard-Wienhard [96] and Benoist [18].

5.3.1. The eigenvalue estimations. Let $\mathscr{O}$ be a strongly tame real projective orbifold and $\tilde{\mathscr{O}}$ be the universal cover in $\mathbb{S}^n$. Let $\tilde{E}$ be a properly convex R-p-end of $\tilde{\mathscr{O}}$, and let $\mathrm{v}_{\tilde{E}}$ be the p-end vertex. Let

$$h : \pi_1(\tilde{E}) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$$

be a homomorphism and suppose that $\pi_1(\tilde{E})$ is hyperbolic.

In this article, we assume that h satisfies the middle eigenvalue condition. We denote by the norms of eigenvalues of g by

$$\lambda_1(g),\dots,\lambda_n(g),\lambda_{\mathrm{v}_{\tilde{E}}}(g), \text{where } \lambda_1(g)\cdots\lambda_n(g)\lambda_{\mathrm{v}_{\tilde{E}}}(g) = \pm 1, \text{ and} \quad \lambda_1(g) \geq \ldots \geq \lambda_n(g), \tag{5.3.1}$$

where we allow repetitions.

Recall the linear part homomorphism $\mathscr{L}_1$ from the beginning of Section 5.2. We denote by $\hat{h} : \pi_1(\tilde{E}) \to \mathsf{SL}_{\pm}(n,\mathbb{R})$ the homomorphism $\mathscr{L}_1 \circ h$. Since $\hat{h}$ is a holonomy of a closed convex real projective $(n-1)$-orbifold, and $\Sigma_{\tilde{E}}$ is assumed to be properly convex, $\hat{h}(\pi_1(\tilde{E}))$ divides a properly convex domain $\tilde{\Sigma}_{\tilde{E}}$ in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$.

We denote by $\tilde{\lambda}_1(g),...,\tilde{\lambda}_n(g)$ the norms of eigenvalues of $\hat{h}(g)$ so that

$$\tilde{\lambda}_1(g) \geq \ldots \geq \tilde{\lambda}_n(g), \tilde{\lambda}_1(g)\ldots\tilde{\lambda}_n(g) = \pm 1 \tag{5.3.2}$$

hold. These are called the *relative norms of eigenvalues* of g. We have $\lambda_i(g) = \tilde{\lambda}_i(g)/\lambda_{\mathrm{v}_{\tilde{E}}}(g)^{1/n}$ for $i = 1,..,n$.

For each nontorsion element g, eigenvalues corresponding to

$$\lambda_1(g),\tilde{\lambda}_1(g),\lambda_n(g),\tilde{\lambda}_n(g),\lambda_{\mathrm{v}_{\tilde{E}}}(g)$$

are all positive and g is positive semi-proximal by Proposition 1.3.11. (See also Theorem 1.3.12.) We define

$$\mathrm{length}(g) := \log\left(\frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)}\right) = \log\left(\frac{\lambda_1(g)}{\lambda_n(g)}\right).$$

This equals the infimum of the Hilbert metric lengths of the associated closed curves in $\tilde{\Sigma}_{\tilde{E}}/\hat{h}(\pi_1(\tilde{E}))$ as first shown by Kuiper. (See [17] for example.)

We recall the notions in Section 1.3.2. (See [17] and [18] also.)

When $\pi_1(\tilde{E})$ is hyperbolic, all infinite order elements of $\hat{h}(\pi_1(\tilde{E}))$ are positive biproximal and a finite index subgroup has only positive biproximal elements and the identity.

Assume that $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic. Suppose that $g \in \mathbf{\Gamma}_{\tilde{E}}$ is proximal. We define

$$\alpha_g := \frac{\log\tilde{\lambda}_1(g) - \log\tilde{\lambda}_n(g)}{\log\tilde{\lambda}_1(g) - \log\tilde{\lambda}_{n-1}(g)}, \beta_g := \frac{\log\tilde{\lambda}_1(g) - \log\tilde{\lambda}_n(g)}{\log\tilde{\lambda}_1(g) - \log\tilde{\lambda}_2(g)}, \tag{5.3.3}$$

and denote by $\mathbf{\Gamma}^p_{\tilde E}$ the set of proximal elements. We define

$$\beta_{\mathbf{\Gamma}_{\tilde E}} := \sup_{g\in \mathbf{\Gamma}^p_{\tilde E}} \beta_g, \alpha_{\mathbf{\Gamma}_{\tilde E}} := \inf_{g\in \mathbf{\Gamma}^p_{\tilde E}} \alpha_g.$$

Proposition 20 of Guichard [95] shows that we have

$$1 < \alpha_{\tilde\Sigma_{\tilde E}} \leq \alpha_\Gamma \leq 2 \leq \beta_\Gamma \leq \beta_{\tilde\Sigma_{\tilde E}} < \infty \tag{5.3.4}$$

for constants $\alpha_{\tilde\Sigma_{\tilde E}}$ and $\beta_{\tilde\Sigma_{\tilde E}}$ depending only on $\tilde\Sigma_{\tilde E}$ since $\tilde\Sigma_{\tilde E}$ is properly and strictly convex.

Here, it follows that $\alpha_{\mathbf{\Gamma}_{\tilde E}}, \beta_{\mathbf{\Gamma}_{\tilde E}}$ depends on $\hat h$, and they form positive-valued functions on the union of components of

$$\mathrm{Hom}(\pi_1(\tilde E), \mathsf{SL}_\pm(n+1,\mathbb{R}))/\mathsf{SL}_\pm(n+1,\mathbb{R})$$

consisting of convex divisible representations with the algebraic convergence topology as given by Benoist [23].

THEOREM 5.3.1. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\tilde E$ be a properly convex R-p-end of the universal cover $\tilde{\mathscr{O}}$, $\tilde{\mathscr{O}} \subset \mathbb{S}^n$, $n \geq 2$. Let $\mathbf{\Gamma}_{\tilde E}$ be a hyperbolic group. Then*

$$\frac{1}{n}\left(1+\frac{n-2}{\beta_{\mathbf{\Gamma}_{\tilde E}}}\right)\mathrm{length}(g) \leq \log\tilde\lambda_1(g) \leq \frac{1}{n}\left(1+\frac{n-2}{\alpha_{\mathbf{\Gamma}_{\tilde E}}}\right)\mathrm{length}(g)$$

for every nonelliptic element $g \in \hat h(\pi_1(\tilde E))$.

PROOF. Since there is a positive bi-proximal subgroup of finite index, we concentrate on positive bi-proximal elements only. We obtain from above that

$$\frac{\log\frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)}}{\log\frac{\tilde\lambda_1(g)}{\tilde\lambda_2(g)}} \leq \beta_{\mathbf{\Gamma}_{\tilde E}}.$$

We deduce that

$$\frac{\tilde\lambda_1(g)}{\tilde\lambda_2(g)} \geq \left(\frac{\lambda_1(g)}{\lambda_n(g)}\right)^{1/\beta_{\mathbf{\Gamma}_{\tilde E}}} = \left(\frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)}\right)^{1/\beta_{\mathbf{\Gamma}_{\tilde E}}} = \exp\left(\frac{\mathrm{length}(g)}{\beta_{\mathbf{\Gamma}_{\tilde E}}}\right). \tag{5.3.5}$$

Since we have $\tilde\lambda_i \leq \tilde\lambda_2$ for $i \geq 2$, we obtain

$$\frac{\tilde\lambda_1(g)}{\tilde\lambda_i(g)} \geq \left(\frac{\lambda_1}{\lambda_n}\right)^{1/\beta_{\mathbf{\Gamma}_{\tilde E}}} \tag{5.3.6}$$

and since $\tilde\lambda_1\cdots\tilde\lambda_n = 1$, we have

$$\tilde\lambda_1(g)^n = \frac{\tilde\lambda_1(g)}{\tilde\lambda_2(g)}\cdots\frac{\tilde\lambda_1(g)}{\tilde\lambda_{n-1}(g)}\frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)} \geq \left(\frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)}\right)^{\frac{n-2}{\beta_{\mathbf{\Gamma}_{\tilde E}}}+1}.$$

We obtain

$$\log\tilde\lambda_1(g) \geq \frac{1}{n}\left(1+\frac{n-2}{\beta_{\mathbf{\Gamma}_{\tilde E}}}\right)\mathrm{length}(g). \tag{5.3.7}$$

By similar reasoning, we also obtain

$$\log\tilde\lambda_1(g) \leq \frac{1}{n}\left(1+\frac{n-2}{\alpha_{\mathbf{\Gamma}_{\tilde E}}}\right)\mathrm{length}(g).$$

□

□

REMARK 5.3.2. Under the assumption of Theorem 5.3.1, if we do not assume that $\pi_1(\tilde{E})$ is hyperbolic, then we obtain

$$\frac{1}{n}\mathrm{length}(g) \leq \log \tilde{\lambda}_1(g) \leq \frac{n-1}{n}\mathrm{length}(g) \tag{5.3.8}$$

for every semiproximal element $g \in \hat{h}(\pi_1(\tilde{E}))$.

PROOF. Let $\tilde{\lambda}_i(g)$ denote the norms of $\hat{h}(g)$ for $i = 1, 2, \dots, n$.

$$\log \tilde{\lambda}_1(g) \geq \ldots \geq \log \tilde{\lambda}_n(g), \log \tilde{\lambda}_1(g) + \cdots + \log \tilde{\lambda}_n(g) = 0$$

hold. We deduce

$$\begin{aligned}
\log \tilde{\lambda}_n(g) &= -\log \lambda_1 - \cdots - \log \tilde{\lambda}_{n-1}(g) \\
&\geq \quad -(n-1)\log \tilde{\lambda}_1 \\
\log \tilde{\lambda}_1(g) &\geq \quad -\frac{1}{n-1}\log \tilde{\lambda}_n(g) \\
\left(1+\frac{1}{n-1}\right)\log \tilde{\lambda}_1(g) &\geq \quad \frac{1}{n-1}\log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)} \\
\log \tilde{\lambda}_1(g) &\geq \quad \frac{1}{n}\mathrm{length}(g). \qquad (5.3.9)
\end{aligned}$$

We also deduce

$$\begin{aligned}
-\log \tilde{\lambda}_1(g) &= \log \tilde{\lambda}_2(g) + \cdots + \log \tilde{\lambda}_n(g) \\
&\geq \quad (n-1)\log \tilde{\lambda}_n(g) \\
-(n-1)\log \tilde{\lambda}_n(g) &\geq \quad \log \tilde{\lambda}_1(g) \\
(n-1)\log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)} &\geq \quad n \log \tilde{\lambda}_1(g) \\
\frac{n-1}{n}\mathrm{length}(g) &\geq \quad \log \tilde{\lambda}_1(g).
\end{aligned}$$

□

□

REMARK 5.3.3. We cannot show that the middle-eigenvalue condition implies the uniform middle-eigenvalue condition. This could be false. For example, we could obtain a sequence of elements $g_i \in \Gamma$ so that $\{\lambda_1(g_i)/\lambda_{v_{\tilde{E}}}(g_i)\} \to 1$ while Γ satisfies the middle-eigenvalue condition. Certainly, we could have an element g where $\lambda_1(g) = \lambda_{v_{\tilde{E}}}(g)$. However, even if there is no such element, we might still have a counter-example. For example, suppose that we might have

$$\left\{ \frac{\log\left(\frac{\lambda_1(g_i)}{\lambda_{v_{\tilde{E}}}(g_i)}\right)}{\mathrm{length}(g)} \right\} \to 0.$$

This could happen by changing $\lambda_{v_{\tilde{E}}}$ considered as a homomorphism $\pi_1(\Sigma_{\tilde{E}}) \to \mathbb{R}_+$. Such assignments are not really understood globally but see Benoist [17]. Also, an analogous phenomenon seems to happen with the Margulis space-time and diffused Margulis invariants as investigated by Charette, Drumm, Goldman, Labourie, and Margulis recently. See [91])

5.3.2. The uniform middle-eigenvalue conditions and the orbits. Let $\tilde{E}$ be a properly convex R-p-end of the universal cover $\tilde{\mathscr{O}}$ of a strongly tame properly convex real projective orbifold $\mathscr{O}$. Assume that $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition. There exists a $\mathbf{\Gamma}_{\tilde{E}}$-invariant compact set to be denoted $L_{\tilde{E}}$ distanced from $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ by Theorem 5.2.5. For the corresponding tube $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$, $L_{\tilde{E}} \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ is a compact subset distanced from $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$. Let $\mathscr{C}\mathscr{H}(L)$ be the convex hull of L in the tube $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ obtained by Theorem 5.2.5. Then $\mathscr{C}\mathscr{H}(L)$ is a $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced subset of $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$.

We of course are doing everything in $\mathbb{S}^n$ here. But $\mathbb{RP}^n$-versions are fairly clear to obtain.

DEFINITION 5.3.4. A *transversal set* is a compact subset of $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ that meets the interior of every great segment from $\mathrm{v}_{\tilde{E}}$ to $\mathrm{v}_{\tilde{E}-}$ in it. A *transversal boundary set* is a compact subset of $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ that meets the interior of every great segment from $\mathrm{v}_{\tilde{E}}$ to $\mathrm{v}_{\tilde{E}-}$ in it. We define the *limit set* $\Lambda_{\tilde{E}}$ of a properly convex R-p-end $\tilde{E}$ to be the smallest nonempty compact $\mathbf{\Gamma}_{\tilde{E}}$-invariant transversal boundary set.

Following Corollary 5.3.5 shows that the limit set is well-defined. Compare also to Definition 6.2.1.

The following main result of this subsection shows that $\Lambda_{\tilde{E}}$ is characterized.

COROLLARY 5.3.5. *A transversal boundary $\mathbf{\Gamma}_{\tilde{E}}$-invariant compact set C in $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ exists and is unique. Also, it satisfies $\mathscr{C}\mathscr{H}(C) \cap (\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}) = C$.*

PROOF. Proposition 5.3.10 will show that C is independent of the choice and meets each great segment for any distanced compact convex set L in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ at a unique point.

Since $\mathscr{C}\mathscr{H}(C) \cap (\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\})$ contains C and is also a $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced compact set, the uniqueness part of Proposition 5.3.10 shows that it equals C. $\square$

Also, $\Lambda_{\tilde{E}} \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ contains all attracting and repelling fixed points of $\gamma \in \mathbf{\Gamma}_{\tilde{E}}$ by the $\mathbf{\Gamma}_{\tilde{E}}$-invariance and the middle-eigenvalue condition.

5.3.2.1. *Hyperbolic groups.* We first consider when $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic.

LEMMA 5.3.6. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\tilde{E}$ be a properly convex R-p-end. Assume that $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic and satisfies the uniform middle eigenvalue conditions.*

- *Suppose that γ_i is a sequence of elements of $\mathbf{\Gamma}_{\tilde{E}}$ acting on $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$.*
- *The sequence of attracting fixed points a_i and the sequence of repelling fixed points b_i are so that $\{a_i\} \rightarrow a_\infty$ and $\{b_i\} \rightarrow b_\infty$ where a_∞, b_∞ are not in $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$.*
- *Suppose that the sequence $\{\lambda_i\}$ of eigenvalues where λ_i corresponds to a_i converges to $+\infty$.*

Then the point a_∞ in $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ is the geometric limit of $\{\gamma_i(K)\}$ for any compact subset $K \subset M$.

PROOF. We may assume without loss of generality that $a_\infty \neq b_\infty$ since otherwise we replace $\{g_i\}$ with $\{gg_i\}$ where $g(a_\infty) \neq b_\infty$. Proving for this case implies the general cases.

Let k_i be the inverse of the factor

$$\min\left\{\frac{\tilde{\lambda}_1(\gamma_i)}{\tilde{\lambda}_2(\gamma_i)}, \frac{\tilde{\lambda}_1(\gamma_i)}{\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma_i)^{\frac{n+1}{n}}} = \frac{\lambda_1(\gamma_i)}{\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma_i)}\right\}.$$

Then $\{k_i\} \rightarrow 0$ by the uniform middle eigenvalue condition and (5.3.5).

There exists a totally geodesic sphere $\mathbb{S}^{n-1}_i$ sharply supporting $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$ at b_i. a_i is uniformly bounded away from $\mathbb{S}^{n-1}_i$ for i sufficiently large. $\mathbb{S}^{n-1}_i$ bounds an open hemisphere H_i containing a_i where a_i is the attracting fixed point by Corollary 1.2.3 of [110] or by Proposition 1.3.2 so that for a Euclidean metric $d_{E,i}$, $\gamma_i|H_i : H_i \to H_i$ we have

$$d_{E,i}(\gamma_i(x),\gamma_i(y)) \leq k_i d_{E,i}(x,y), x,y \in H_i. \tag{5.3.10}$$

Note that $\{\mathrm{Cl}(H_i)\}$ converges geometrically to $\mathrm{Cl}(H_\infty)$ for an open hemisphere containing a in the interior.

Actually, we can choose a Euclidean metric $d_{E,i}$ on H_i^o so that $\{d_{E,i}|J\times J\}$ is uniformly convergent for any compact subset J of H_∞. Hence there exists a uniform positive constant C' so that

$$\mathbf{d}(a_i,K) < C' d_{E_i}(a_i,K). \tag{5.3.11}$$

provided $a_i, K \subset J$ and sufficiently large i.

Since $\mathbf{\Gamma}_{\tilde E}$ is hyperbolic, the domain Ω corresponding to $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$ in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$ is strictly convex. For any compact subset K of M, the equation $K \subset M$ is equivalent to

$$K \cap \mathrm{Cl}(\bigcup_{i=1}^{\infty} \overline{b_i \mathrm{v}_{\tilde E}} \cup \overline{b_i \mathrm{v}_{\tilde E -}}) = \emptyset.$$

Since the boundary sphere $\mathrm{bd}H_\infty$ meets $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$ in this set only by the strict convexity of Ω, we obtain $K \cap \mathrm{bd}H_\infty = \emptyset$. And $K \subset H_\infty$ since $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E}) \subset \mathrm{Cl}(H_\infty)$.

We have $\mathbf{d}(K,\mathrm{bd}H_\infty) > \varepsilon_0$ for $\varepsilon_0 > 0$. Thus, the distance $\mathbf{d}(K,\mathrm{bd}H_i)$ is uniformly bounded by a constant δ. $\mathbf{d}(K,\mathrm{bd}H_i) > \delta$ implies that $d_{E_i}(a_i,K) \leq C/\delta$ for a positive constant $C > 0$ Acting by g_i, we obtain $d_{E_i}(g_i(K),a_i) \leq k_i C/\delta$ by (5.3.10), which implies $\mathbf{d}(g_i(K),a_i) \leq C' k_i C/\delta$ by (5.3.11). Since $\{k_i\} \to 0$, the fact that $\{a_i\} \to a$ implies that $\{g_i(K)\}$ geometrically converges to a. $\square$

LEMMA 5.3.7. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\tilde E$ be a properly convex R-p-end. Assume that $\mathbf{\Gamma}_{\tilde E}$ is hyperbolic, and satisfies the uniform middle eigenvalue conditions. Suppose that $\{\gamma_i\}$ is sequence of elements of $\mathbf{\Gamma}_{\tilde E}$ acting on $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$ and forms a convergence sequence acting on $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$. Then any transversal boundary $\Gamma_{\tilde E}$-invariant set $L_{\tilde E}$ for $\tilde E$, contains the geometric limit of any subsequence of $\{\gamma_i(K)\}$ for any compact subset $K \subset \mathscr{T}^o_{\mathrm{v}_{\tilde E}}$. Furthermore,*

$$A_*(\{\gamma_i\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E}), R_*(\{\gamma_i\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E}) \subset L_{\tilde E}.$$

PROOF. Let $z \in \mathscr{T}^o_{\mathrm{v}_{\tilde E}}$. Let $(\!(z')\!)$ denote the element in $\Sigma_{\tilde E}$ corresponding to the ray from $\mathrm{v}_{\tilde E}$ to z. Let $\{\gamma_i\}$ be any sequence in $\mathbf{\Gamma}_{\tilde E}$ so that the corresponding sequence $\{\gamma_i((\!(z)\!))\}$ in $\Sigma_{\tilde E} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$ converges to a point z' in $\mathrm{bd}\Sigma_{\tilde E} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$.

Clearly, a fixed point of $g \in \mathbf{\Gamma}_{\tilde E} - \{\mathrm{I}\}$ in $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E}) - \{\mathrm{v}_{\tilde E}, \mathrm{v}_{\tilde E -}\}$ is in $L_{\tilde E}$ since g has at most one fixed point on each open segment in the boundary. For the attracting fixed points a_i and r_i of γ_i, we can assume that

$$\{a_i\} \to a, \{r_i\} \to r \text{ for } a_i, r_i \in L_{\tilde E}$$

where $a, r \in L_{\tilde E}$ by the $\Gamma_{\tilde E}$-invariance and the closedness of $L_{\tilde E}$. Assume $a \neq r$ first. By Lemma 5.3.6, we have $\{\gamma_i(z)\} \to a$ and hence the limit $z_\infty = a$.

However, it could be that $a = r$. In this case, we choose $\gamma_0 \in \mathbf{\Gamma}_{\tilde E}$ so that $\gamma_0(a) \neq r$. Then $\gamma_0\gamma_i$ has the attracting fixed point a'_i so that we obtain $\{a'_i\} \to \gamma_0(a)$ and repelling fixed points r'_i so that $\{r'_i\} \to r$ holds by Lemma 5.3.8. This implies the first part.

Then as above $\{\gamma_0\gamma_i(z)\} \to \gamma_0(a)$ and we need to multiply by γ_0^{-1} now to show $\{\gamma_i(z)\} \to a$. Thus, the limit set is contained in $L_{\tilde{E}}$. $\square$

LEMMA 5.3.8. *Let $\{g_i\}$ be a sequence of projective automorphisms acting on a strictly convex domain Ω in $\mathbb{S}^n$. Suppose that the sequence of attracting fixed points $\{a_i \in \mathrm{bd}\Omega\} \to a$ and the sequence of repelling fixed points $\{r_i \in \mathrm{bd}\Omega\} \to r$. Assume that the corresponding sequence of eigenvalues of a_i limits to $+\infty$ and that of r_i limits to 0. Let g be any projective automorphism of Ω. Then $\{gg_i\}$ has the sequence of attracting fixed points $\{a_i'\}$ converging to $g(a)$ and the sequence of repelling fixed points converging to r.*

PROOF. Recall that g is a quasi-isometry. Given $\varepsilon > 0$ and a compact ball B disjoint from a ball around r, we obtain that $gg_i(B)$ is in a ball of radius ε of $g(a)$ for sufficiently large i. For a choice of B and sufficiently large i, we obtain $gg_i(B) \subset B^o$. Since $gg_i(B) \subset B^o$, we obtain

$$(gg_i)^n(B) \subset (gg_i)^m(B)^o \text{ for } n > m$$

by induction, There exists an attracting fixed point a_i' of gg_i in $gg_i(B)$. Since the sequence of the diameters of sets of form $gg_i(B)$ is converging to 0, we obtain that $\{a_i'\} \to g(a)$.

Also, given $\varepsilon > 0$ and a compact ball B disjoint from a ball around $g(a)$, $g_i^{-1}g^{-1}(B)$ is in the ball of radius ε of r. Similarly to above, we obtain the needed conclusion. $\square$

5.3.2.2. *Non-hyperbolic groups.* Now, we generalize to not necessarily hyperbolic $\mathbf{\Gamma}_{\tilde{E}}$. A $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced set $L_{\tilde{E}}$ contains the attracting fixed set A_i and the repelling fixed set R_i of any $g \in \mathbf{\Gamma}_{\tilde{E}}$ by invariance and sequence arguments.

LEMMA 5.3.9. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\tilde{E}$ be a properly convex R-p-end. Assume that $\mathbf{\Gamma}_{\tilde{E}}$ is non-hyperbolic or virtually-factorizable and satisfies the uniform middle eigenvalue conditions with respect to $\mathrm{v}_{\tilde{E}}$. Suppose that $\{\gamma_i\}$ is a generalized convergence sequence of elements of $\mathbf{\Gamma}_{\tilde{E}}$ acting on $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Let $L_{\tilde{E}}$ be a transverse boundary set for $\tilde{E}$. Then $L_{\tilde{E}}$ contains the geometric limit of any subsequence of $\{\gamma_i(K)\}$ for any compact subset $K \subset \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$. Furthermore,*

$$A_*(\{\gamma_i\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}), R_*(\{\gamma_i\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) \subset L_{\tilde{E}}.$$

PROOF. Let $L = \mathscr{C}\mathscr{H}(L_{\tilde{E}}) \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Then L is a convex set uniformly bounded away from $\mathrm{v}_{\tilde{E}}$ and its antipode. by a geometric consideration.

Given any sequence g_i, we can extract a convergence sequence $\{g_i\}$ with a convergence limit g_∞.

Suppose that $L^o = \emptyset$. Then L is a convex domain on a hyperspace P disjoint from $\mathrm{v}_{\tilde{E}}$. We use a coordinate system where each $\gamma \in \Gamma$ is of form (5.2.4) where $\vec{b}_g = 0$. Dividing g_i by $\lambda_1(g_i)$ and taking a limit, we obtain that g_∞ equals

$$\begin{pmatrix} \hat{g}_\infty & 0 \\ 0 & 0 \end{pmatrix} \tag{5.3.12}$$

by the uniform middle eigenvalue condition and Lemma 1.3.14. Hence $A_*(\{g_i\}) \subset P$. By Theorem 1.3.21,

$$A_*(\{g_i\}) \subset P \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})) = L_{\tilde{E}}.$$

The remainders are simple to show.

Suppose that L^o is not empty. Then $L^o \cap N_*(\{g_i\}) = \emptyset$ by Lemma 1.3.20. Given any convergence sequence $\{g_i\}, g_i \in \Gamma$ converging to g_∞, the sequence $g_i(x)$ for $x \in L$ converges to a point of $A_*(\{g_i\})$.

By Lemma 1.3.14, $\mathrm{v}_{\tilde{E}} \in N_*(\{g_i\})$ since $\{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)/\lambda(g_i)\} \to 0$ by the uniform middle eigenvalue condition. Dividing g_i by $\lambda_1(g_i)$ and taking a limit, we obtain that g_∞ equals

$$\begin{pmatrix} \hat{g}_\infty & 0 \\ \hat{b} & 0 \end{pmatrix} \tag{5.3.13}$$

by the uniform middle eigenvalue condition and Lemma 1.3.14 dualizing the proof of Lemma 4.4.2. Here $\hat{g}_\infty$ is not zero since otherwise we have uniform convergence to $\mathrm{v}_{\tilde{E}}$ or $\mathrm{v}_{\tilde{E}-}$ for any compact set disjoint from $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ while L is invariant set, which is absurd. Since $\hat{g}_\infty \neq 0$, the image of g_∞ is now a subspace of the same dimension as $A_*(\{(\!(\hat{g}_i)\!)\})$. Actually, it is graph over $A_*(\{(\!(\hat{g}_i)\!)\})$ where the vertical direction is given by the direction to $\mathrm{v}_{\tilde{E}}$ for a linear function given by $\hat{b}$.

Since $\mathbf{\Gamma}_{\tilde{E}}$ acts on L, $g_\infty(x) \in \mathrm{Cl}(L) \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Hence, $g_\infty(L) = A_*(\{g_i\}) \subset L_{\tilde{E}}$. Using $\{g_i^{-1}\}$, we obtain $R_*(\{g_i\}) \subset L_{\tilde{E}}$.

Any element of $x \in \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ satisfies $x = (\!(\vec{v}_x)\!), \vec{v}_x = \vec{v}_L + c\vec{v}_{\tilde{E}}$ for a constant $c > 0$ and a vector $\vec{v}_L$ in the direction of a point of L and a vector $\vec{v}_{\tilde{E}}$ in direction of $\mathrm{v}_{\tilde{E}}$. Then

$$g_\infty((\!(\vec{v}_x)\!)) = (\!(g_\infty(\vec{v}_L) + cg_\infty(\vec{v}_{\tilde{E}}))\!).$$

Since $cg_\infty(\vec{v}_{\tilde{E}}) = 0$ from (5.3.13), we obtained that $g_\infty(\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})) = g_\infty(L)$. Since

$$g_\infty(\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})) \subset A_*(\{g_i\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}),$$

and $g_\infty(L) = A_*(\{g_i\})$, we obtain the result. The final statement is also proved by taking the sequence g_i^{-1}. □

For the following, $\mathbf{\Gamma}_{\tilde{E}}$ can be virtually factorizable. By following Proposition 5.3.10, $\Lambda_{\tilde{E}}$ is well-defined independent of the choice of K.

PROPOSITION 5.3.10. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\tilde{E}$ be a properly convex R-p-end. Assume that $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to the R-p-end structure. Let $\mathrm{v}_{\tilde{E}}$ be the R-end vertex and $z \in \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$. Let $L_{\tilde{E}}$ be a transversal boundary set for $\tilde{E}$, and let L be the closure of $\mathscr{C}\mathscr{H}(L_{\tilde{E}})$. Then the following properties are satisfied:*

- (i) *$L_{\tilde{E}}$ contains all the limit points of orbits of each compact subset of $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$. $L_{\tilde{E}}$ contains all attracting fixed sets of elements of $\mathbf{\Gamma}_{\tilde{E}}$. If $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic, then the set of attracting fixed point is dense in the set.*
- (ii) *For each segment s in $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ with an endpoint $\mathrm{v}_{\tilde{E}}$, the great segment containing s meets $L_{\tilde{E}}$ at a unique point other than $\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}$. That is, there is a one-to-one correspondence between $\partial\mathrm{Cl}(\Sigma_{\tilde{E}})$ and $L_{\tilde{E}}$.*
- (iii) *$L_{\tilde{E}}$ is homeomorphic to $\mathbb{S}^{n-2}$.*
- (iv) *For any $\mathbf{\Gamma}_{\tilde{E}}$-distanced compact set L' in $\mathrm{bd}\mathscr{T}_{\vec{\mathrm{v}}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ meeting every great segment in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$, we have $L_{\tilde{E}} = L'$. (uniqueness)*

PROOF. We will first prove (i),(ii),(iii) for various cases and then prove (iv) all together:

(A) Consider first when $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic. Proposition 5.3.7 proves (i) here. Let L be the closure of $\mathscr{C}\mathscr{H}(L_{\tilde{E}})$, which is $\mathbf{\Gamma}_{\tilde{E}}$-invariant. Let $K' = L \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$. Clearly $L_{\tilde{E}} \subset K'$.

Since $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic, any point y of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is a limit point of some sequence $\{g_i(x)\}$ for $x \in \tilde{\Sigma}_{\tilde{E}}$ by [22]. Thus, at least one point in the segment l_y in the direction of y, $y \in \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ with endpoints $\mathrm{v}_{\tilde{E}}$ and $\mathrm{v}_{\tilde{E}-}$ is a limit point of some subsequence of $\{g_i(x)\}$ by Lemma 5.3.6. Thus, $l_y \cap L_{\tilde{E}} \neq \emptyset$. and $l_y \cap K' \neq \emptyset$.

Let us choose a standard Euclidean metric $\|\cdot\|_E$ for $\mathbb{R}^{n+1}$. We identify $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ with a subspace V not passing $\mathrm{v}_{\tilde{E}}$ for convenience during this proof. We consider V to correspond to $\mathbb{R}^n$ and $\mathrm{v}_{\tilde{E}}$ to be the $n+1$-st unit vector.

We claim that $l_y \cap K'$ is unique: Suppose not. Let z and z' be the two points of $l_y \cap K'$. We choose a line l in $\tilde{\Sigma}_{\tilde{E}}$ ending at y. Let y_i be the sequence of points on l covering to y. We choose g_i as in the proof of Lemma 4.2.5 so that $g_i(y_i) \in F$ for a compact fundamental domain F of $\tilde{\Sigma}_{\tilde{E}}$. We assume that $\{g_i\}$ is a convergence sequence by choosing a subsequence if necessary. (Here, $(\!(\vec{v}_{-,i})\!)$ is fixed to be a single point $y = (\!(\vec{v}_-)\!)$.) Given the other endpoint z of l, we have

$$\{g_i(z)\} \to a_*(\{g_i\})$$

for an attractor of $a_*(\{g_i\})$ of $\{g_i\}$. This follows by the same reasoning the proof of Lemma 4.2.5. This means that $\{g_i(y)\}$ is uniformly bounded away from $a_*(\{g_i\})$ since $g_i(l)$ passes F with $\{g_i(z)\}$ converging to $a_*(\{g_i\})$. Since $\{g_i(y)\}$ is bounded away from $a_*(\{g_i\})$ uniformly, as in the proof of Lemma 4.2.5 using (4.2.8) and similarly to proving the conclusion of the lemma, we obtain

$$\left\{ \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)^{1+\frac{1}{n}}} \hat{h}(g_i)(\vec{v}_-) \right\} \to 0 \tag{5.3.14}$$

in the Euclidean metric. To explain more, we write $\vec{v}_-$ as a sum of $\vec{v}_i^p + \vec{v}_i^S$ as there. The rest is analogous.

Let $\mathrm{v}_{\tilde{E}}$ denote the unit vector in the direction of $\vec{v}_{\tilde{E}}$. We consider $\mathbb{R}^n$ to be a complementary subspace to this vector under the norm $\|\cdot\|$. We write the vector for z as $\vec{v}_z = \lambda \vec{v}_- + \vec{v}_{\tilde{E}}$ and the vector for z' as $\vec{v}_{z'} = \lambda' \vec{v}_- + \vec{v}_{\tilde{E}}$. Then

$$g_i(\vec{v}_z) = \lambda \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}^{\frac{1}{n}}(g_i)} \hat{h}(g_i)(\vec{v}_-) + (\lambda \vec{b}_{g_i} \cdot \vec{v}_- + \lambda_{\mathrm{v}_{\tilde{E}}}(g_i))\vec{v}_{\tilde{E}}$$

by (5.2.1). Let us denote

$$c_i := \left\| \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}^{\frac{1}{n}}(g_i)} \hat{h}(g_i)(\vec{v}_-) \right\|_E .$$

Since the direction of $g_i(\vec{v}_z)$ is bounded away from $\vec{v}_{\tilde{E}}$,

$$\left| \lambda \frac{\vec{b}_{g_i} \cdot \vec{v}_-}{c_i} + \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)}{c_i} \right|$$

is uniformly bounded. By (5.3.14), we obtain

$$\left\{ \left| \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)}{c_i} \right| \right\} \to \infty.$$

Hence,

$$\left\{ \left| \frac{\vec{b}_{g_i} \cdot \vec{v}_-}{c_i} \right| \right\} \to \infty \text{ as } i \to \infty.$$

We also have

$$g_i(\vec{v}_{z'}) = \lambda' \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}^{\frac{1}{n}}(g_i)} \hat{h}(g_i)(\vec{v}_-) + (\lambda' \vec{b}_{g_i} \cdot \vec{v}_- + \lambda_{\mathrm{v}_{\tilde{E}}}(g_i))\vec{v}_{\tilde{E}}.$$

Since $\lambda' \neq \lambda$, and $\left\{\left|\frac{\vec{b}_{g_i}\cdot\vec{v}_-}{c_i}\right|\right\} \to \infty$

$$\left\{\left|\lambda'\frac{\vec{b}_{g_i}\cdot\vec{v}_-}{c_i} + \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)\vec{v}_{\tilde{E}}}{c_i}\right| = \left|(\lambda'-\lambda)\frac{\vec{b}_{g_i}\cdot\vec{v}_-}{c_i} + \lambda\frac{\vec{b}_{g_i}\cdot\vec{v}_-}{c_i} + \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g_i)\vec{v}_{\tilde{E}}}{c_i}\right|\right\}$$

cannot be uniformly bounded. This implies that $g_i(z')$ converges to $\mathrm{v}_{\tilde{E}}$ or $\mathrm{v}_{\tilde{E}-}$. Since $z' \in K'$ and K' is $\mathbf{\Gamma}_{\tilde{E}}$-invariant, this is a contradiction.

By Lemma 5.3.6, $L_{\tilde{E}}$ meets every great segment in $\mathscr{T}$. Thus, $K' \cap l_y = L_{\tilde{E}} \cap l_y$ for every y in $\partial \mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$. Thus, $K' = L_{\tilde{E}}$, and (i) and (ii) hold for $L_{\tilde{E}}$.

(iii) Since $L_{\tilde{E}}$ is closed and compact and bounded away from $\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}$, the section $s : \partial\mathrm{Cl}(\Omega_{\tilde{E}}) \to \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ is continuous. If not, we can contradict (ii) by taking two sequences converging to distinct points in a great segment from $\vec{v}_{\tilde{E}}$ to its antipode.

(B) Now suppose that $\mathbf{\Gamma}_{\tilde{E}}$ is not virtually factorizable and is not hyperbolic. Lemma 5.3.9 proves that the orbits limit to $L_{\tilde{E}}$ only. An attracting fixed sets in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ is in $L_{\tilde{E}}$ as in case (A).

First suppose that a great segment η in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ with endpoints $\mathrm{v}_{\tilde{E}}$ and $\mathrm{v}_{\tilde{E}-}$ corresponds to an element y of $\partial\mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$. Now we take a line l in $\tilde{\Sigma}_{\tilde{E}}$ as in the hyperbolic case. Then (5.3.14) holds as above using Lemma 4.3.4 instead of Lemma 4.2.5. The identical argument will show that η^o meets with $L_{\tilde{E}}$ at a unique point. This proves (i) and (ii). (iii) follows as above.

(C) Suppose that $\mathbf{\Gamma}_E$ is virtually factorizable. We follow the proof of Theorem 5.2.5. Now, the space of open great segments with an endpoint $\mathrm{v}_{\tilde{E}}$ in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$ corresponds to a properly convex domain Ω that is the interior of the strict join $K_1 * \cdots * K_l$. Then a totally geodesic $\mathbf{\Gamma}_{\tilde{E}}$-invariant hyperspace H is disjoint from $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$ by the proof of Theorem 5.2.5. Here, we may regard $K_i \subset H$ for each $i = 1, \dots, l$. Then consider any sequence g_i so that $\{g_i(x)\} \to x_0$ for a point $x \in \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$ and $x_0 \in \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Let x' denote the corresponding point of $\tilde{\Sigma}_{\tilde{E}}$ for x. Then $\{g_i(x')\}$ converges to a point $y \in \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$. Let $\vec{x} \in \mathbb{R}^{n+1}$ be the vector in the direction of x. We write

$$\vec{x} = \vec{x}_E + \vec{x}_H$$

where $\vec{x}_H$ is in the direction of H and $\vec{x}_E$ is in the direction of $\mathrm{v}_{\tilde{E}}$. By the uniform middle eigenvalue condition and estimating the size of vectors, we obtain the same situation as in (5.3.12) and $\{g_i(x)\} \to x_0$ for $x \in L_{\tilde{E}}$ and $x_0 \in H$. Hence, $x_0 \in H \cap L_{\tilde{E}}$. Thus, every limit point of an orbit of x is in H.

If there is a point y in $L_{\tilde{E}} - H$, then there is a strict join $K_{i_1} * \cdots * K_{i_l} * \{\mathrm{v}_{\tilde{E}}\}$ for a proper collection containing y. As in the proof of Theorem 5.2.5, by Proposition 1.4.10, we can find a sequence g_i, virtually central $g_i \in \mathbf{\Gamma}_{\tilde{E}}$, so that $\{g_i | K_{i_1} * \cdots * K_{i_l}\}$ converges to the identity, and the maximal norm of g_i to be in the complementary domains. The norms of eigenvalues associated with $K_{i_1} * \cdots * K_{i_l}$ have uniformly bounded ratios with the minimal one $\lambda_n(g_i)$ in this case. The maximal norm $\bar{\lambda}_1(g_i)$ of the eigenvalue associated with $K_{i_1} * \cdots * K_{i_l}$ and $\lambda_{\tilde{E}}(g_i)$ satisfy $\{\bar{\lambda}_1(g_i)/\lambda_{\tilde{E}}(g_i)\} \to 0$ by the uniform middle eigenvalue condition. Hence $\{g_i(y)\} \to \mathrm{v}_{\tilde{E}}$. Again this is a contradiction. Hence, we obtain $L_{\tilde{E}} \subset H$. (i), (ii), (iii) follow easily now.

(iv) Suppose that we have another distanced set L'. We take a convex hull of $L_E \cup L'$ and apply the same reasoning as above. □

5.3.3. An extension of Koszul's openness. Here, we state and prove a well-known minor modification of Koszul's openness result.

A *radial affine connection* is an affine connection on $\mathbb{R}^{n+1} - \{O\}$ invariant under a scalar dilatation $S_t : \vec{v} \to t\vec{v}$ for every $t > 0$.

PROPOSITION 5.3.11 (Koszul). *Let M be a properly convex real projective compact n-orbifold with strictly convex boundary. Let*

$$h : \pi_1(M) \to \mathsf{PGL}(n+1,\mathbb{R}) \,(\textit{resp. } h : \pi_1(M) \to \mathsf{SL}_\pm(n+1,\mathbb{R}))$$

denote the holonomy homomorphism acting on a properly convex domain Ω_h in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Assume that M is projectively diffeomorphic to $\Omega_h/h(\pi_1(M))$. Then there exists a neighborhood U of h in

$$\mathrm{Hom}(\pi_1(M),\mathsf{PGL}(n+1,\mathbb{R}))\,(\textit{resp. } \mathrm{Hom}(\pi_1(M),\mathsf{SL}_\pm(n+1,\mathbb{R})))$$

so that every $h' \in U$ acts on a properly convex domain $\Omega_{h'}$ so that $\Omega_{h'}/h'(\pi_1(M))$ is a compact properly convex real projective n-orbifold with strictly convex boundary. Also, $\Omega_{h'}/h'(\pi_1(M))$ is diffeomorphic to M.

PROOF. We prove for $\mathbb{S}^n$. Let Ω_h be a properly convex domain covering M. We may modify M by pushing ∂M inward: Let Ω'_h be the inverse image of M' in M. Then M' and Ω'_h are properly convex by Lemma 1.4.6.

The linear cone $C(\Omega_h^o) \subset \mathbb{R}^{n+1} = \Pi^{-1}(\Omega_h^o)$ over Ω_h^o has a smooth strictly convex Hessian function V by Vey [151] or Vinberg [107]. Let $C(\Omega'_h)$ denote the linear cone over Ω'_h. We extend the group $\mu(\pi_1(M))$ by adding a transformation $\gamma : \vec{v} \mapsto 2\vec{v}$ to $C(\Omega_h^o)$. For the fundamental domain F' of $C(\Omega'_h)$ under this group, the Hessian matrix of V restricted to $F \cap C(\Omega'_h)$ has a lower bound. Also, the boundary $\mathrm{bd}C(\Omega'_h)$ is strictly convex in any affine coordinates in any transverse subspace to the radial directions at any point.

Let N' be a compact orbifold $C(\Omega'_h)/\langle\mu(\pi_1(\tilde{E})),\gamma\rangle$ with a flat affine structure. Note that S_t, $t \in \mathbb{R}_+$, becomes an action of a circle on M. The change of representation h to $h' : \pi_1(M) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$ is realized by a change of holonomy representations of M and hence by a change of affine connections on $C(\Omega'_h)$. Since S_t commutes with the images of h and h', S_t still gives us a circle action on N' with a different affine connection. We may assume without loss of generality that the circle action is fixed and N' is invariant under this action.

Thus, N' is a union of $B_1,\dots,B_{m_0}$ that are the products of n-balls by intervals foliated by connected arcs in circles that are flow arcs of S_t. We can change the affine structure on N' to one with the holonomy group $\langle h'(\pi_1(\tilde{E})),\gamma\rangle$ by local regluing $B_1,\dots,B_{m_0}$ as in [49]. We reglue using maps that preserve the leaves for which we need to find maps commuting with the γ-action. We assume that S_t still gives us a circle affine action since γ is not changed. We may assume that N' and $\partial N'$ are foliated by circles that are flow curves of the circle action. The change corresponds to a sufficiently small C^r-change in the affine connection for $r \geq 2$ as we can see from [49]. Now, the strict positivity of the Hessian of V in the fundamental domain and the boundary convexity are preserved. Let $C(\Omega''_h)$ denote the universal cover of N' with the new affine connection. Thus, $C(\Omega''_h)$ is also a properly convex affine cone by Koszul's work [114]. Also, it is a cone over a properly convex domain Ω''_h in $\mathbb{S}^n$. [$\mathbb{S}^n$T] □

We denote by $\mathsf{PGL}(n+1,\mathbb{R})_v$ the subgroup of $\mathsf{PGL}(n+1,\mathbb{R})$ fixing a point $v, v \in \mathbb{RP}^n$. and denote by $\mathsf{SL}_\pm(n+1,\mathbb{R})_v$ the subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ fixing a point v, $v \in \mathbb{S}^n$. Let $\mathrm{Hom}_C(\Gamma,\mathsf{PGL}(n+1,\mathbb{R})_v)$ denote the space of representations acting cocompactly and discretely on a properly convex domain in $\mathbb{S}^{n-1}_v$. Respectivley, we define $\mathrm{Hom}_C(\pi_1(M),\mathsf{SL}_\pm(n+1,\mathbb{R})_v)$ similarly.

PROPOSITION 5.3.12. *Let T be a tube domain over a properly convex domain $\Omega \subset \mathbb{RP}^{n-1}$ (resp. $\subset \mathbb{S}^{n-1}$). Let B be a strictly convex hypersurface bounding a properly convex domain in a tube domain T. Let v be a vertex of T. B meets each radial ray in T from v transversely. Assume that a projective group Γ acts on Ω properly discontinuously and cocompactly. Then there exists a neighborhood of the inclusion map in*

$$\mathrm{Hom}_C(\Gamma, \mathsf{PGL}(n+1,\mathbb{R})_v)\,(resp.\ \mathrm{Hom}_C(\pi_1(M), \mathsf{SL}_\pm(n+1,\mathbb{R})_v))$$

where every element h acts on a strictly convex hypersurface B_h in a tube domain $\mathscr{T}_h$ meeting each radial ray at a unique point and bounding a properly convex domain in $\mathscr{T}_h$.

PROOF. We assume first that $B, T \subset \mathbb{S}^n$. For sufficiently small neighborhood V of h in $\mathrm{Hom}_C(\Gamma, \mathsf{SL}_\pm(n+1,\mathbb{R})_v)$, $h(\Gamma)$, $h \in V$ acts on a properly convex domain Ω_h properly discontinuously and cocompactly by Theorem 4.1 of [49] (see Koszul [114]). A large compact subset K of Ω flows to a compact subset K_h by a diffeomorphism by a method of Section 5 of [49]. Let $\mathscr{T}_h$ denote the tube over Ω_h. Since B/Γ is a compact orbifold, we choose $V' \subset V$ so that for the projective connections on a compact neighborhood of B/Γ corresponding to elements of V', B/Γ is still strictly convex and transverse to radial lines. For each $h \in V'$, we obtain an immersion to a strictly convex domain $\iota_h : B \to \mathscr{T}_h$ transverse to radial lines since we can think of the change of holonomy as small C^1-change of connections. (Or we can use the method described in Section 5 of [49].) Let $p_{\mathscr{T}_h} : \mathscr{T}_h \to \Omega_h$ denote the projection with fibers equal to the radial lines. Also, in this way of viewing as the connection change, $p_{\mathscr{T}_h} \circ \iota_h$ is a proper immersion to Ω_h, it is a diffeomorphism to $B \to \Omega_h$. (Here again we can use Section 5 of [49].) Each point of B is transverse to a radial segment from v. By considering the compact fundamental domains of B, we see that same holds for B_h for h sufficiently near I. Also, B_h is strictly convex and smooth. By Proposition 5.3.11, the conclusion follows. [$\mathbb{S}^n$T] ☐

5.3.4. Convex cocompact actions of the p-end holonomy groups.

DEFINITION 5.3.13. In $\mathbb{S}^n$, a (resp. generalized) lens-shaped R-p-end with the p-end vertex $\mathrm{v}_{\tilde{E}}$ in $\mathbb{S}^n$ is *strictly* (*resp. generalized*) lens-shaped if we can choose a (resp. generalized) CA-lens domain D in $\mathbb{S}^n$ so that the interior of $D * \mathrm{v}_{\tilde{E}}$ is a p-end neighborhood with the top hypersurfaces A and the bottom one B so that each great open segment in $\mathbb{S}^n$ from $\mathrm{v}_{\tilde{E}}$ in the direction of $\partial \mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$ meets $\mathrm{Cl}(D) - A - B$ at a unique point. In $\mathbb{RP}^n$, such an p-end vertex $\mathrm{v}_{\tilde{E}}$ is strict one if its lift is one in $\mathbb{S}^n$.

A (resp. generalized) lens L is called *strict lens* if the following hold:

$$\partial \mathrm{Cl}(A) = \mathrm{Cl}(A) - A = \partial \mathrm{Cl}(B) = \mathrm{Cl}(B) - B, A \cup B = \partial L, \text{ and } \mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \partial \mathrm{Cl}(L).$$

Recall that in order that L is to be a lens, we assume that $\pi_1(\tilde{E})$ acts cocompactly on L. Also, $\mathrm{Cl}(A) - A$ must equal the limit set $\Lambda_{\tilde{E}}$ of $\tilde{E}$ by Corollary 5.3.5.

Also, the images of these under $p_{\mathbb{S}^n}$ are called by the same names respectively.

Obviously, a lens of a lens-shaped R-p-end is strict if and only if the R-p-end is strictly lens-shaped.

In this section, we will prove Proposition 5.3.14 obtaining a lens.

For the following, $\mathscr{O}$ needs not be properly convex but merely convex.

PROPOSITION 5.3.14. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold where $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\mathbb{RP}^n$).*

- *Let $\mathbf{\Gamma}_{\tilde{E}}$ be the holonomy group of a properly convex R-p-end $\tilde{E}$.*
- *Let $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ be an open tube corresponding to $R(\mathrm{v}_{\tilde{E}})$.*

- *Suppose that $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to the R-p-end structure, and acts on a distanced compact convex set K in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ where $K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) \subset \tilde{\mathscr{O}}$.*

Then any connected open p-end-neighborhood U containing a lift to $\tilde{\tilde{\mathscr{O}}}$ of $K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ contains a lens L' and a lens-cone p-end-neighborhood $L' * \{\mathrm{v}_{\tilde{E}}\} - \{\mathrm{v}_{\tilde{E}}\}$ of the R-p-end $\tilde{E}$. We can choose the lens L' in U so that $\mathrm{bd}L' \cap \mathscr{T}^o = A \cup B$ for strictly convex smooth connected hypersurfaces A and B. Furthermore, every lens of the cone is a strict lens.

PROOF. First suppose that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. We may assume that U embeds to a neighborhood of L under a developing map by taking U sufficiently small. We denote by U the image again. the projection of K to $\tilde{\Sigma}_{\tilde{E}}$ must be onto since it must be a $\Gamma_{\tilde{E}}$-invariant convex set. So, K must meet each great segment with endpoints $\mathrm{v}_{\tilde{E}}$ in the directions of $\tilde{\Sigma}_{\tilde{E}}$. Hence, $K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ is a separating set in $\tilde{\mathscr{O}}$, and $U - K$ has two components since the boundary of K has two components in $\tilde{\tilde{\mathscr{O}}}$.

Let $\Lambda_{\tilde{E}}$ denote $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) \cap K$. Let us choose finitely many points $z_1, \dots, z_m \in U - K$ in the two components of $U - K$.

Proposition 5.3.10 shows that the orbits of z_i for each i accumulate to points of $\Lambda_{\tilde{E}}$ only. Hence, a totally geodesic hypersphere separates $\mathrm{v}_{\tilde{E}}$ with these orbit points and another one separates $\mathrm{v}_{\tilde{E}-}$ and the orbit points. Define the convex hull $C_1 := \mathscr{C}\mathscr{H}(\mathbf{\Gamma}_{\tilde{E}}(\{z_1, \dots, z_m\}) \cup K)$. Thus, C_1 is a compact convex set disjoint from $\mathrm{v}_{\tilde{E}}$ and $\mathrm{v}_{\tilde{E}-}$ and $C_1 \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) = \Lambda_{\tilde{E}}$. (See Definition 1.1.22.)

We need the following lemma:

□

LEMMA 5.3.15. *We continue to assume as in Proposition 5.3.14. Then we can choose $z_1, \dots, z_m$ in U so that for $C_1 := \mathscr{C}\mathscr{H}(\mathbf{\Gamma}_{\tilde{E}}(\{z_1, \dots, z_m\}) \cup K)$, $\mathrm{bd}C_1 \cap \tilde{\mathscr{O}}$ is disjoint from K and $C_1 \subset U$.*

PROOF. First, suppose that K is not in a hyperspace. Then $(\mathrm{bd}K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}))/\mathbf{\Gamma}_{\tilde{E}}$ is diffeomorphic to a disjoint union of two copies of $\Sigma_{\tilde{E}}$. We can cover a compact fundamental domain of $\mathrm{bd}K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ by the interior of n-balls in $\tilde{\mathscr{O}}$ that are convex hulls of finite sets of points in U. Since $L/\mathbf{\Gamma}_{\tilde{E}}$ is compact for a lens L containing $K \cap \tilde{\mathscr{O}}$, so is $(K \cap \tilde{\mathscr{O}})/\mathbf{\Gamma}_{\tilde{E}}$, and there exists a positive lower bound of $\{d_{\tilde{\mathscr{O}}}(x, \mathrm{bd}U \cap \tilde{\mathscr{O}}) | x \in K\}$. Let F denote the union of these finite sets. We can choose $\varepsilon > 0$ so that the ε-$d_{\tilde{\mathscr{O}}}$-neighborhood U' of K in $\tilde{\mathscr{O}}$ is a subset of U. Moreover U' is convex by Lemma 1.1.13 following [67]. We let $z_1, \dots, z_m$ denote the points of F. If we choose F to be in U', then C_1 is in U' since U' is convex.

The disjointedness of $\mathrm{bd}C_1$ from $K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ follows since the $\mathbf{\Gamma}_{\tilde{E}}$-orbits of above balls cover $\mathrm{bd}K \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$.

If K is in a hyperspace, the reasoning is similar to the above. □

We continue:

LEMMA 5.3.16. *Let C be a $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced compact convex set with boundary in where $(C \cap \mathscr{T}^o_{\tilde{E}})/\mathbf{\Gamma}_{\tilde{E}}$ is compact. There are two connected hypersurfaces A and B of $\mathrm{bd}C \cap \mathscr{T}^o_{\tilde{E}}$ meeting every great segment in $\mathscr{T}^o_{\tilde{E}}$. Suppose that A and B are disjoint from another C' $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced compact convex set with boundary in where $(C' \cap \mathscr{T}^o_{\tilde{E}})/\mathbf{\Gamma}_{\tilde{E}}$ is compact. Then A (resp. B) contains no line ending in $\mathrm{bd}\tilde{\mathscr{O}}$.*

PROOF. It is enough to prove for A. Suppose that there exists a line l in A ending at a point of $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Assume $l \subset A$. The line l projects to a line l' in $\tilde{E}$.

Let $C_1 = C \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$. Since $A/\mathbf{\Gamma}_{\tilde E}$ and $B/\mathbf{\Gamma}_{\tilde E}$ are both compact, and there exists a fibration $C_1/\mathbf{\Gamma}_{\tilde E} \to A/\mathbf{\Gamma}_{\tilde E}$ induced from $C_1 \to A$ using the foliation by great segments with endpoints $\mathrm{v}_{\tilde E}, \mathrm{v}_{\tilde E-}$.

Since $A/\mathbf{\Gamma}_{\tilde E}$ is compact, we choose a compact fundamental domain F in A and choose a sequence $\{x_i \in l\}$ whose image sequence in l' converges to the endpoint of l' in $\partial \mathrm{Cl}(\tilde\Sigma_{\tilde E})$. We choose $\gamma_i \in \mathbf{\Gamma}_{\mathrm{v}_{\tilde E}}$ so that $\gamma_i(x_i) \in F$ where $\{\gamma_i(\mathrm{Cl}(l'))\}$ geometrically converges to a segment l'_∞ with both endpoints in $\partial \mathrm{Cl}(\tilde\Sigma_{\tilde E})$. Hence, $\{\gamma_i(\mathrm{Cl}(l))\}$ geometrically converges to a segment l_∞ in A. We can assume that for the endpoint z of l in A, $\{\gamma_i(z)\}$ converges to the endpoint p_1. Proposition 5.3.10 implies that the endpoint p_1 of l_∞ is in $L_{\tilde E} := L \cap \mathrm{bd}\mathscr{T}_{\tilde E}$. Let t be the endpoint of l not equal to z. Then $t \in A$. Since γ_i is not a bounded sequence, $\gamma_i(t)$ converges to a point of $\Lambda_{\tilde E}$. Thus, both endpoints of l_∞ are in $\Lambda_{\tilde E}$ and hence $l^o_\infty \subset C'$ by the convexity of C'. However, $l \subset A$ implies that $l^o_\infty \subset A$. As A is disjoint from C', this is a contradiction. The similar conclusion holds for B. $\square$

PROOF OF PROPOSITION 5.3.14 CONTINUED. We will denote by C_1 the compact convex subset $C_1 = C \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$ for C obtained by Lemma 5.3.15. Since C_1 meets in a compact segment any great segment in $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$, it follows that $\mathrm{bd}C_1 \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})^o$ is a union of two hypersurfaces A and B. Since C is contructed by taking the convex hull of $(\mathbf{\Gamma}_{\tilde E}(\{z_1,\dots,z_m\}) \cup K)$, and balls that are convex hulls of some points of $z_1,\dots,z_m$ and their images cover $\mathrm{bd}K \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})^o$, it follows that the extreme points of A or B must be vertices of the images of $z_1,\dots,z_m$ and points of $K \cap \mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})$. Since the ball cover $\mathrm{bd}K \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})^o$, A and B are disjoint from K.

Since A and analogously B do not contain any geodesic ending at $\mathrm{bd}\tilde{\mathscr{O}}$, by Lemma 5.3.16, $A \cup B = \mathrm{bd}C_1 \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})^o$ is a union of compact $n-1$-dimensional polytopes meeting one another in strictly convex dihedral angles.

Immediately following Proposition 5.3.17 completes the proof of Proposition 5.3.14.

For $\mathbb{RP}^n$ version, we can argue by projecting by $p_{\mathbb{S}^n}$ and Proposition 1.4.2. $\square$

PROPOSITION 5.3.17. *Assume the premise of Proposition 5.3.14. Suppose that a lens cone $L_1 * \{\vec{v}_{\tilde E}\} - \{\mathrm{v}_{\tilde E}\}$ is in a convex p-end neighborhood U of a p-end $\tilde E$ a p-end neighborhood of $\tilde E$. Suppose that L_1 contains a lens L in its interior where $L * \{\vec{v}_{\tilde E}\} - \{\mathrm{v}_{\tilde E}\}$ is again. Suppose that L_1 is bounded by two connected convex polyhedral hypersurfaces. Then there exists a lens L_2 bounded by two connected strictly convex hypersurfaces so that $L_2 \subset U$ and $L \subset L_2^o, L_2 \subset L_1^o$.*

PROOF. First, assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let us take the dual domain U_L of $(L * \{\mathrm{v}_{\tilde E}\})^o$. The dual U_1 of $(L_1 * \{\mathrm{v}_{\tilde E}\})^o$ is an open subset of U_L by (1.5.2). By Proposition 5.2.4, the dual action is asymptotically nice. By the uniqueness parts of Theorems 4.1.1 and 4.3.1, U_L and U_1 are asymptotically-nice properly convex domains. By Lemma 1.5.7 (iii), the hyperplanes sharply supporting $(L_1 * \{\mathrm{v}_{\tilde E}\})^o$ at $\mathrm{v}_{\tilde E}$ correspond to points of a totally geodesic domain D,

$$D = \mathrm{Cl}(U_1) \cap P = \mathrm{Cl}(U_L) \cap P$$

for a $\mathbf{\Gamma}^*_{\tilde E}$-invariant hyperplane P. Hence, U_L and U_1 are asymptotically nice domains with respect to D^o. (See Section 4.1.)

By the premise, we have a connected convex polyhedral open subspace

$$S_1 := \mathrm{bd}(L_1 * \{\mathrm{v}_{\tilde E}\}) \cap \mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E})^o \subset \mathrm{bd}(L_1 * \{\mathrm{v}_{\tilde E}\}).$$

By Lemma 1.5.7 (iv), S_1 corresponds to a connected convex polyhedral hypersurface

$$S_1^* \subset \mathrm{bd}U_1$$

by $\mathscr{D}^{\mathrm{Ag}}_{(L_1 * \{\mathrm{v}_{\tilde{E}}\})^o}$. Since S_1 is disjoint from L by the premise, it follows $S_1^* \subset \mathrm{bd} U_1 \cap U_L$ by (1.5.2). Since $S_1/\mathbf{\Gamma}_{\tilde{E}}$ is compact, so is $S_1^*/\mathbf{\Gamma}^*_{\tilde{E}}$ by Proposition 1.5.4. Theorem 4.4.1 shows that $D \cup S_1' = \mathrm{bd} U_2$. Hence, $\mathrm{bd} U_1 \cap U_L = S_1^*$.

By Theorem 4.4.4, we obtain an asymptotically nice closed domain U_2 with connected strictly convex smooth hypersurface boundary S_2 in U_L with $U_1 \cup S_1 \subset U_2^o$. The dual U_2^* of U_2 has a connected strictly convex smooth hypersurface boundary S_2^* in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$ disjoint from $(L * \{\mathrm{v}_{\tilde{E}}\})^o$ and inside $(L_1 * \{\mathrm{v}_{\tilde{E}}\})^o$ by (1.5.2). This is what we wanted.

Also, considering $(L * \{\mathrm{v}_{\tilde{E}-}\})^o$ and $(L_1 * \{\mathrm{v}_{\tilde{E}-}\})^o$, we obtain a connected strictly convex smooth hypersurface in the other component of $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o - L$ in U. The union of the two hypersurfaces bounds a lens L_2 in U. (See Section 1.5.1.)

Let F denote the compact fundamental domain of the boundary of the lens. The strictness of the lens follows from Proposition 5.3.10 since the boundary of the lens is a union of orbits of F and the limit points are only in $\Lambda_{\tilde{E}}$.

Again Proposition 1.4.2 completes the proof for $\mathbb{RP}^n$. $\square$

Proof of Theorem 5.1.4. Proposition 5.3.14 is the forward direction using $\tilde{\mathscr{O}} := \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$.

Now, we show the converse. It is sufficient to prove for the case $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let L be a CA-lens of the lens-cone where $\mathbf{\Gamma}_{\tilde{E}}$ acts cocompactly on. Let $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ be the tube corresponding to L.

We will denote by $h : \pi_1(\tilde{E}) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})$ denote the holonomy homomorphism of the end fundamental group with image $\mathbf{\Gamma}_{\tilde{E}}$. We assume that the image of h are matrices of form (5.2.1).

There is an abelianization map

$$A : \pi_1(\tilde{E}) \to H_1(\pi_1(\tilde{E}),\mathbb{R})$$

obtained by taking a homology class. The above map $g \to \log \lambda_{\mathrm{v}_{\tilde{E}}}(h(g))$ induces homomorphism

$$\Lambda'^h : H_1(\pi_1(\tilde{E}),\mathbb{R}) \to \mathbb{R}$$

that depends on the holonomy homomorphism h.

Let us give an arbitrary Riemannian metric μ on $\Sigma_{\tilde{E}}$. Recall that a *current* is a transverse measure on a partial foliation by 1-dimensional subspaces in the compact space $\mathbb{U}\Sigma_{\tilde{E}}$ on the transverse measure. (See [141].) These are not necessarily geodesic currents as in Bonahon [29]. The space of currents is denoted by $\mathscr{C}(\mathbb{U}\Sigma_{\tilde{E}})$ which is given a weak topology.

The abelianization map $\pi_1(\Sigma_{\tilde{E}}) \to H_1(\pi_1(\tilde{E}),\mathbb{R})$ can be understood as sending a closed curve to a current the corresponding homology class. This map extends to $\mathscr{C}(\mathbb{U}\Sigma_{\tilde{E}}) \to H_1(\pi_1(\tilde{E}),\mathbb{R})$. (See Proposition 1 of [141] and Theorem 14 of [73].) Also, $\Lambda^h : \pi_1(\tilde{E}) \to \mathbb{R}$ gives rise to the continuous map $\hat{\Lambda}^h : \mathscr{C}(\mathbb{U}\Sigma_{\tilde{E}}) \to \mathbb{R}$ which restricts to Λ^h on the image currents of $\pi_1(\Sigma_{\tilde{E}})$: Λ^h is given by integrating a 1-form on $\Sigma_{\tilde{E}}$ along the closed curve representing $\pi_1(\Sigma_{\tilde{E}})$ since $\mathrm{Hom}(H_1(\pi_1(\tilde{E}),\mathbb{R}),\mathbb{R}) = H^1(\pi_1(\tilde{E}),\mathbb{R})$. Since the integration along currents are well-defined, we are done.

Let $\lambda_1^{\mathrm{ul}}(h(g))$ denote the maximal norm of the eigenvalues $h(g)$ of the upper-left corner of h in (5.2.1). Obviously, $\lambda_1^{\mathrm{ul}}(h(g)) \geq \lambda_{\mathrm{v}_{\tilde{E}}}(h(g))$ for each g, $g \in \pi_1(\tilde{E})$: If the eigenvalue of the upper-left corner matrix of $h(g)$ is strictly smaller than $\lambda_1^{\mathrm{ul}}(h(g))$, Proposition 1.3.2 shows that the closure of L contains $\mathrm{v}_{\tilde{E}}$ or $\mathrm{v}_{\tilde{E}-}$ considering the orbit of $\{g^n\}$, a contradiction.

Let $g \in \mathbf{\Gamma}_{\tilde{E}}$. Let $[g]$ denote the current supported on a closed curve c_g on $\Sigma_{\tilde{E}}$ corresponding to g lifted to $\mathbb{U}\Sigma_{\tilde{E}}$. Define $\mathrm{length}_\mu(g)$ to be the infimum of the μ-length of such closed curves corresponding to g. Suppose that $\mathbf{\Gamma}_{\tilde{E}}$ does not satisfy

$$C^{-1}\mathrm{length}_\mu(g) \leq \log \frac{\lambda_1^{\mathrm{ul}}(h(g))}{\lambda_{\mathrm{v}_{\tilde{E}}}(h(g))} \leq C\mathrm{length}_\mu(g)$$

for a uniform constant $C > 1$. Then there exists a sequence g_i of elements of $\mathbf{\Gamma}_{\tilde{E}}$ so that

$$\left\{ \frac{\log\left(\frac{\lambda_1^{\mathrm{ul}}(h(g_i))}{\lambda_{\mathrm{v}_{\tilde{E}}}(h(g_i))}\right)}{\mathrm{length}_\mu(g_i)} \right\} \to 0 \text{ as } i \to \infty.$$

Let $[g_\infty]$ denote a limit point of $\{[g_i]/\mathrm{length}_\mu(g_i)\}$ in the space of currents on $\mathbb{U}\Sigma_{\tilde{E}}$. Since $\mathbb{U}\Sigma_{\tilde{E}}$ is compact, a limit point exists. We may modify h by changing the homomorphism $g \mapsto \lambda_{\mathrm{v}_{\tilde{E}}}(h(g))$ only; that is, we only modify the $(n+1)\times(n+1)$-entry of the matrices form (5.2.1) with corresponding changes. Proposition 5.3.12 implies that the perturbed CA-lens L' is still a properly convex domain with the same tube domain whose closure does not contain $\mathrm{v}_{\tilde{E}}$. By considering the image of $[g_\infty]$ in $H_1(\Sigma_{\tilde{E}}, \mathbb{R})$, we can make a sufficiently small change of h to h' in this way so that $\Lambda^{h'}([g_\infty]) > \Lambda^h([g_\infty])$. From this, we obtain that

$$\log\left(\frac{\lambda_1^{\mathrm{ul}}(h'(g_i))}{\lambda_{\mathrm{v}_{\tilde{E}}}(h'(g_i))}\right) < 0 \text{ for sufficiently large } i. \tag{5.3.15}$$

By (5.3.15), we obtain that $\lambda_1^{\mathrm{ul}}(h'(g)) < \lambda_{\mathrm{v}_{\tilde{E}}}(h('g))$ for some g and that $\lambda_{\mathrm{v}_{\tilde{E}}}(h'(g))$ at $\mathrm{v}_{\tilde{E}}$. Hence, we can decompose $\mathbb{S}^n$ into a hyperspace S' and the complementary $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$. The norms of eigenvalues associated with S' are strictly less than that of $\mathrm{v}_{\tilde{E}}$. Proposition 1.3.2 shows that the closure of L contains $\mathrm{v}_{\tilde{E}}$ or $\mathrm{v}_{\tilde{E}-}$ by considering the orbits under $\{g^i\}$. □

5.3.5. The uniform middle-eigenvalue conditions and the lens-shaped ends. Now, we aim to prove Theorem 5.1.5 restated as Theorem 5.3.21. A *radially foliated end-neighborhood system* of $\mathscr{O}$ is a collection of end-neighborhoods of $\mathscr{O}$ that is radially foliated and outside a compact suborbifold of $\mathscr{O}$ whose interior is isotopic to $\mathscr{O}$.

DEFINITION 5.3.18. We say that a strongly tame properly convex $\mathscr{O}$ with $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$) satisfies the *triangle condition* if for any fixed end-neighborhood system of $\mathscr{O}$, every triangle $T \subset \mathrm{Cl}(\tilde{\mathscr{O}})$, if $\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}, T^o \subset \tilde{\mathscr{O}}$, and $\partial T \cap \mathrm{Cl}(U) \neq \emptyset$ for a radial p-end neighborhood U, then ∂T is a subset of $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$.

For example, by Corollary 6.3.3, strongly tame strict SPC-orbifolds with generalized lens-shaped or horospherical ends satisfy this condition. The converse is not necessarily true.

A *minimal* $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced compact set is the smallest compact $\mathbf{\Gamma}_{\tilde{E}}$-invariant distanced set in $\mathscr{T}_{\tilde{E}}$.

LEMMA 5.3.19. *Suppose that $\mathscr{O}$ is a strongly tame properly convex real projective orbifold and satisfies the triangle condition. Then no triangle T with $T^o \subset \tilde{\mathscr{O}}, \partial T \subset \mathrm{bd}\tilde{\mathscr{O}}$ has a vertex equal to an R-p-end vertex.*

PROOF. Assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let $\mathrm{v}_{\tilde{E}}$ be a p-end vertex. Choose a fixed radially foliated p-end-neighborhood system. Suppose that a triangle T with $\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}$ contains a vertex equal to a p-end vertex. Let U be a component of the inverse image of a radially foliated end-neighborhood in the end-neighborhood system, and be a p-end neighborhood of a p-end $\tilde{E}$ with a p-end vertex $\mathrm{v}_{\tilde{E}}$. By the triangle condition, $\partial T \subset \mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$.

Since U is foliated by radial lines from $\mathrm{v}_{\tilde{E}}$, we choose U so that $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ covers a compact hypersurface in $\mathscr{O}$. Let $\mathscr{U}$ denote the set of segments in $\mathrm{Cl}(U)$ from $\mathrm{v}_{\tilde{E}}$. Every segment in $\mathscr{U}$ in the direction of $\tilde{\Sigma}_{\tilde{E}}$ ends in $\mathrm{bd}U \cap \tilde{\mathscr{O}}$. Also, the segments $\mathscr{U}$ in directions of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}}$ are in $\mathrm{bd}U \cap \mathrm{bd}\tilde{\mathscr{O}}$ by the definition of $\tilde{\Sigma}_{\tilde{E}}$. Also, $\mathrm{Cl}(U)$ is a union of segments in $\mathscr{U}$. Thus, $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ is a union of segments in directions of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}}$.

Since $T^o \subset \tilde{\mathscr{O}}$, each segment in $\mathscr{U}$ with interior in T^o is not in directions of $\partial \mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$. Let w be the endpoint of the maximal extension in $\tilde{\mathscr{O}}$ of such a segment. Then w is not in $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ by the conclusion of the above paragraph. This contradicts $\partial T \subset \mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$.

The proof for $\mathbb{RP}^n$ case follows by Proposition 1.4.2. [$\mathbb{S}^n$P] $\square$

LEMMA 5.3.20. *Suppose that $\mathscr{O}$ is a strongly tame properly convex real projective orbifold and satisfies the triangle condition or, alternatively, assume that an R-p-end $\tilde{E}$ is virtually factorizable. Suppose that the holonomy group $\mathbf{\Gamma}$ is strongly irreducible. Then the R-p-end $\tilde{E}$ is generalized lens-shaped if and only if it is lens-shaped.*

PROOF. Again, we prove for $\mathbb{S}^n$. If $\tilde{E}$ is virtually factorizable, this follows by Theorem 5.4.3.

Suppose that $\tilde{E}$ is not virtually factorizable. Now assume the triangle condition. Given a generalized CA-lens L, let L^b denote $\mathrm{Cl}(L) \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. We obtain the convex hull M of L^b. M is a subset of $\mathrm{Cl}(L)$. The lower boundary component of L is a smooth strictly convex hypersurface.

Let M_1 be the outer component of $\mathrm{bd}M \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Suppose that M_1 meets $\mathrm{bd}\tilde{\mathscr{O}}$. M_1 is a union of the interior of simplices. By Lemma 1.4.4, either a simplex σ in $\mathrm{Cl}(\tilde{\mathscr{O}})$ is in $\mathrm{bd}\tilde{\mathscr{O}}$ or its interior σ^o is disjoint from it. Hence, there is a simplex σ in $M_1 \cap \mathrm{bd}\tilde{\mathscr{O}}$. Taking the convex hull of $\mathrm{v}_{\tilde{E}}$ and an edge in σ, we obtain a triangle T with $\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}$ and $T^o \subset \tilde{\mathscr{O}}$. This contradicts the triangle condition by Lemma 5.3.19. Thus, $M_1 \subset \tilde{\mathscr{O}}$. By Theorem 5.3.21, the end satisfies the uniform middle eigenvalue condition. By Proposition 5.3.14, we obtain a lens-cone in $\tilde{\mathscr{O}}$. [$\mathbb{S}^n$S] $\square$

THEOREM 5.3.21. *Let $\mathscr{O}$ be a strongly tame convex real projective orbifold. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the holonomy group of a properly convex R-end $\tilde{E}$ and the end vertex $\mathrm{v}_{\tilde{E}}$. Then the following are equivalent*:

(i) *$\tilde{E}$ is a generalized lens-shaped R-end.*
(ii) *$\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$.*

Assume that the holonomy group of $\pi_1(\mathscr{O})$ is strongly irreducible, and $\mathscr{O}$ is properly convex. If $\mathscr{O}$ furthermore satisfies the triangle condition or, alternatively, assume that $\tilde{E}$ is virtually factorizable, then the following holds:

- *$\mathbf{\Gamma}_{\tilde{E}}$ is lens-shaped if and only if $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle-eigenvalue condition.*

PROOF. Assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. (ii) $\Rightarrow$ (i): This follows from Theorem 5.1.4 since we can intersect the lens with $\tilde{\mathscr{O}}$ to obtain a generalized lens and a generalized lens-cone from it. (Here, of course $\pi_1(\tilde{E})$ acts cocompactly on the generalized lens.)

(i) $\Rightarrow$ (ii): Let L be a generalized CA-lens in the generalized lens cone $L * \mathrm{v}_{\tilde{E}}$. Let B be the lower boundary component of L in the tube $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$. Since B is strictly convex, the upper component of $\mathscr{T}_{\mathrm{v}_{\tilde E}}(\tilde\Sigma_{\tilde E}) - B$ is a properly convex domain, which we denote by U. Let l_x denote the maximal segment from $\mathrm{v}_{\tilde E}$ passing x for $x \in U - L$.

We define a function $f : U - L \to \mathbb{R}$ given by $f(x)$ to be the Hilbert distance on line l_x from x to $L \cap l_x$. Then a level set of f is always strictly convex: This follows by taking a 2-plane P containing $\mathrm{v}_{\tilde E}$ passing L. Let x, y be a points of $f^{-1}(c)$ for a constant $c > 0$. Let x' be the point of $\mathrm{Cl}(L) \cap l_x$ closest to x and y' be one of $\mathrm{Cl}(L) \cap l_y$ closest to y. Let x'' be one of $\mathrm{Cl}(L) \cap l_x$ furthest from x. Let y'' be one of $\mathrm{Cl}(L) \cap l_y$ furthest from x. Since $f(x) = f(y)$, a cross-ratio argument shows that the lines extending $\overline{xy}, \overline{x'y'}$ and $\overline{x''y''}$ are concurrent outside $U \cap P$. The strict convexity of B and Lemma 1.8 of [67] shows that $f(z) < \varepsilon$ for $z \in \overline{xy}^o$.

We can approximate each level set by a convex polyhedral hypersurface in $U - L$ by taking vertices in the level set and taking the convex hull using the strict convexity of the level set. Then we can smooth it to be a strictly convex hypersurface by Proposition 5.3.17. Let V denote the domain bounded by this and B. Then V has strictly convex smooth boundary in U. Theorem 5.1.4 immediately below implies (ii).

The final part follows by Lemma 5.3.20. [$\mathbb{S}^n$S] □

5.4. The properties of lens-shaped ends.

LEMMA 5.4.1. *Let $\mathscr{O}$ be a strongly tame properly convex orbifold. Then given any end-neighborhood, there is a concave end-neighborhood in it. Furthermore, the $d_{\mathscr{O}}$-diameter of the boundary of a concave end-neighborhood of an R-end E is bounded by the Hilbert diameter of the end orbifold Σ_E of E.*

PROOF. It is sufficient to prove for the case $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Suppose that we have a generalized lens-cone V that is a p-end-neighborhood equal to the interior of $L * v_{\tilde E}$ where L is a generalized CA-lens bounded away from $v_{\tilde E}$.

Now take a p-end neighborhood U'. We assume without loss of generality that U' covers a product end-neighborhood with compact boundary.

By taking smaller U' if necessary, we may assume that U' and L are disjoint. Since $(\mathrm{bd} U' \cap \tilde{\mathscr{O}})/h(\pi_1(\tilde E))$ and $L/h(\pi_1(\tilde E))$ are compact, $\varepsilon > 0$. Let

$$L' := \{x \in V | d_V(x, L) \leq \varepsilon\}.$$

Since a lower component of ∂L is strictly convex, we can show that L' can be polyhedrally approximated and smoothed to be a CA-lens by Proposition 5.3.17.

Clearly, $h(\pi_1(\tilde E))$ acts on L'.

We choose sufficiently large ε' so that $\mathrm{bd} U \cap \tilde{\mathscr{O}} \subset L'$, and hence $V - L' \subset U$ form a concave p-end-neighborhood as above.

Let $\tilde E$ be a p-end corresponding to E. Let U be a concave p-end neighborhood of $\tilde E$ that is a cone: U is the interior of $\{\mathrm{v}\} * L - L$ for a generalized CA-lens L and the p-end vertex v corresponding to U. Let T denote the tube with vertex v in the direction of L. Then $B := \mathrm{bd} U \cap \tilde{\mathscr{O}}$ is a smooth lower boundary component of L.

Any tangent hyperspace P at a point of B meets $\mathrm{bd} T$ in a sphere of dimension $n-2$. By convexity of L and the strict convexity B, it follows that $P \cap L$ is a point. We claim that $P \cap \mathrm{bd}\tilde{\mathscr{O}} \subset P \cap \mathrm{bd} T$: We put T into an affine space $\mathbb{A}^n$ with vertices in $\mathrm{bd}\mathbb{A}^n$. Then T is foliated by parallel complete affine lines. Consider these as vertical lines. B is a strictly convex hypersurface meeting these vertical lines transversely. Then the property of P becomes clear now.

Thus any maximal segment in $\mathscr{O}$ tangent to B at x must end in $\mathrm{bd} T \cap \mathrm{bd}\tilde{\mathscr{O}}$. There is a projection $\Pi_{\mathrm{v}_{\tilde E}} : B \to \tilde\Sigma_{\tilde E}$ that is a diffeomorphism. Hence, the maximal segment is sent to a maximal segment of $\tilde{\Sigma}_{\tilde{E}}$ under Π_{v}, which forms an isometry on the segment with the Hilbert metrics. Moreover this is a Finsler isometry by considering the Finsler metric restricted to the tangent space to B at x to that of the tangent space to $\tilde{\Sigma}_{\tilde{E}}$ at $\Pi(x)$ and the Finsler metrics. The conclusion follows. [$\mathbb{S}^n$S] $\square$

5.4.1. The properties for a lens-cone in non-virtually-factorizable cases. Recall that each infinite-order $g \in \mathbf{\Gamma}_{\tilde{E}}$ is positive bi-semi-proximal by Proposition 1.3.11.

THEOREM 5.4.2. *Let $\mathscr{O}$ be a strongly tame convex real projective n-orbifold. Let $\tilde{E}$ be an R-p-end of $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) with a generalized lens p-end-neighborhood. Let $\mathrm{v}_{\tilde{E}}$ be the p-end vertex. Assume that $\pi_1(\tilde{E})$ is non-virtually-factorizable. Then $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$, and there exists a generalized CA-lens D disjoint from $\mathrm{v}_{\tilde{E}}$ with the following properties*:

- (i)
  - $\mathrm{bd} D - \partial D = \Lambda_{\tilde{E}}$ *is independent of the choice of D where $\Lambda_{\tilde{E}}$ is from Proposition 5.3.10.*
  - *D is strictly generalized lens-shaped.*
  - *Each nontorsion nonidentity element $g \in \mathbf{\Gamma}_{\tilde{E}}$ has an attracting fixed set in* $\mathrm{bd} D$ *intersected with the union of some great segments from $\mathrm{v}_{\tilde{E}}$ in the directions in* $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}}$.
  - *The closure of the union of attracting fixed set is a subset of* $\mathrm{bd} D - A - B$ *for the top and the bottom hypersurfaces A and B. The closure equals* $\mathrm{bd} D - A - B$ *if $\mathbf{\Gamma}_{\tilde{E}}$ is hyperbolic.*
- (ii)
  - *Let l be a segment $l \subset \mathrm{bd}\tilde{\mathscr{O}}$ with $l^o \cap \mathrm{Cl}(U) \neq \emptyset$ for any concave p-end-neighborhood U of $\mathrm{v}_{\tilde{E}}$. Then l is in $\bigcup S(\mathrm{v}_{\tilde{E}})$ and in the closure in $\mathrm{Cl}(V)$ of any concave or proper p-end-neighborhood V of $\mathrm{v}_{\tilde{E}}$.*
  - *The set $S(\mathrm{v}_{\tilde{E}})$ of maximal segments from $\mathrm{v}_{\tilde{E}}$ in $\mathrm{Cl}(V)$ is independent of a concave or proper p-end neighborhood V (in fact it is the set of maximal segments from $\mathrm{v}_{\tilde{E}}$ ending in* $\mathrm{bd} D - A - B$).
  - $$\bigcup S(\mathrm{v}_{\tilde{E}}) = \mathrm{Cl}(V) \cap \mathrm{bd}\tilde{\mathscr{O}}.$$
- (iii) *$S(g(\mathrm{v}_{\tilde{E}})) = g(S(\mathrm{v}_{\tilde{E}}))$ for $g \in \pi_1(\tilde{E})$.*
- (iv) *Given $g \in \pi_1(\mathscr{O})$, we have*

$$\left(\bigcup S(g(\mathrm{v}_{\tilde{E}}))\right)^o \cap \bigcup S(\mathrm{v}_{\tilde{E}}) = \emptyset \text{ or else } \bigcup S(g(\mathrm{v}_{\tilde{E}})) = \bigcup S(\mathrm{v}_{\tilde{E}}) \text{ with } g \in \mathbf{\Gamma}_{\tilde{E}}. \tag{5.4.1}$$

- (v) *A concave p-end neighborhood is a proper p-end neighborhood.*
- (vi) *Assume that $\vec{v}_{\tilde{E}'}$ is the p-end vertex of an R-p-end $\tilde{E}'$. We can choose mutually disjoint concave p-end neighborhoods for every R-p-ends. Then*

$$\left(\bigcup S(\mathrm{v}_{\tilde{E}})\right)^o \cap \bigcup S(\mathrm{v}_{\tilde{E}'}) = \emptyset \text{ or } \bigcup S(\mathrm{v}_{\tilde{E}}) = \bigcup S(\mathrm{v}_{\tilde{E}'}) \text{ with } \mathrm{v}_{\tilde{E}} = \mathrm{v}_{\tilde{E}'}, \tilde{E} = \tilde{E}'$$

for an R-p-end vertice $\mathrm{v}_{\tilde{E}}$. (This is a sharping of (iv).)

PROOF. Suppose first $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Theorem 5.3.21 implies the uniform middle eigenvalue condition.

(i) Let U_1 be a concave end neighborhood. Since $\mathbf{\Gamma}_{\tilde{E}}$ acts on U_1, U_1 is a component of the complement of a generalized lens D in a generalized R-end of form $D * \{\mathrm{v}_{\tilde{E}}\}$ by definition. The action on D is cocompact and proper since we can use a foliation by great segments in a tube corresponding to $\tilde{E}$.

Proposition 5.3.10 implies that the lens is a strict one. This implies (i).

(ii) Consider any segment l in $\mathrm{bd}\tilde{\mathscr{O}}$ with l^o meeting $\mathrm{Cl}(U_1)$ for a concave p-end-neighborhood U_1 of $\mathrm{v}_{\tilde{E}}$. Here, the generalized lens D has boundary components A and B where B is also a boundary component of U_1 in $\tilde{\mathscr{O}}$. Let T be the open tube corresponding to $\tilde{\Sigma}_{\tilde{E}}$. Then $\tilde{\mathscr{O}} \subset T$ since $\tilde{\Sigma}_{\tilde{E}}$ is the direction of all segments in $\tilde{\mathscr{O}}$ starting from $\mathrm{v}_{\tilde{E}}$. Let T_1 be a component of $\mathrm{bd}T - \partial_1 B$ containing $\mathrm{v}_{\tilde{E}}$. Then $T_1 \subset \mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}$ by the definition of concave p-end neighborhoods. In the closure of U_1, an endpoint of l is in T_1. Then $l^o \subset \mathrm{bd}T$ since l^o is tangent to $\mathrm{bd}T - \{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$. Any convex segment s from $\mathrm{v}_{\tilde{E}}$ to any point of l must be in $\mathrm{bd}T$. By the convexity of $\mathrm{Cl}(\tilde{\mathscr{O}})$, we have $s \subset \mathrm{Cl}(\tilde{\mathscr{O}})$. Thus, s is in $\mathrm{bd}\tilde{\mathscr{O}}$ since $\mathrm{bd}T \cap \mathrm{Cl}(\tilde{\mathscr{O}}) \subset \mathrm{bd}\tilde{\mathscr{O}}$. Therefore, the segment l is contained in the union of segments in $\mathrm{bd}\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}}$.

We now suppose that l is a segment from $\mathrm{v}_{\tilde{E}}$ containing a segment l_0 in $\mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}}$, and we will show that l is in $\mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}$, which will be sufficient to prove (ii). l^o contains a point p of $\mathrm{bd}D - A - B$, which is a subset of $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) \cap D$. Since $l \subset \mathrm{Cl}(\tilde{\mathscr{O}})$, we obtain $\bigcup_{g \in \mathbf{\Gamma}_{\tilde{E}}} g(l) \subset \mathrm{Cl}(\tilde{\mathscr{O}})$, a properly convex subset. Hence, $\bigcup_{g \in \mathbf{\Gamma}_{\tilde{E}}} g(l) - U_1$ is a distanced set, and has a distanced compact closure. Then the convex hull of the closure meets $\mathrm{bd}\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ in a way contradicting Proposition 5.3.10 (ii) where D is $\Lambda_{\tilde{E}}$ in the proposition. Thus, l^o does not meet $\mathrm{bd}D - A - B$. Thus,

$$l \subset \mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}.$$

We define $S(\mathrm{v}_{\tilde{E}})$ as the set of maximal segments in $\mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}$. Such a maximal segment is also maximal in $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ by the above paragraph. Hence, we can characterize $S(\mathrm{v}_{\tilde{E}})$ as the set of maximal segments in $\mathrm{bd}\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}}$ ending at points of $\mathrm{bd}D - A - B$. Also, $\bigcup S(\mathrm{v}_{\tilde{E}}) = \mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}}$.

For any other concave affine neighborhood U_2 of U_1, we have

$$U_2 = \{\mathrm{v}_{\tilde{E}}\} * D_2 - D_2 - \{\mathrm{v}_{\tilde{E}}\}$$

for a generalized CA-lens D_2. Since $\mathrm{Cl}(D_2) - \partial D_2$ equals $\mathrm{Cl}(D) - \partial D$, we obtain that $\mathrm{Cl}(U_2) \cap \mathrm{bd}\tilde{\mathscr{O}} = \mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}} = \bigcup S(\mathrm{v}_{\tilde{E}})$.

Let U' be any proper p-end-neighborhood associated with $\mathrm{v}_{\tilde{E}}$. $U_1 \subset U'$ for a concave p-end neighborhood U_1 by Lemma 5.4.1. Again, $U_1 = \{\mathrm{v}_{\tilde{E}}\} * D - D - \{\mathrm{v}_{\tilde{E}}\}$ for a generalized CA-lens D where $\mathrm{v}_{\tilde{E}} \notin \mathrm{Cl}(D)$. Hence, $\mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathscr{O}} \subset \mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathscr{O}}$. Moreover, every maximal segment in $S(\mathrm{v}_{\tilde{E}})$ is in $\mathrm{Cl}(U')$.

We can form $S'(\mathrm{v}_{\tilde{E}})$ as the set of maximal segments from $\mathrm{v}_{\tilde{E}}$ in $\mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathscr{O}}$. Then no segment l in $S'(\mathrm{v}_{\tilde{E}})$ has interior points in $\mathrm{bd}D - A - B$ as above. Thus,

$$S(\mathrm{v}_{\tilde{E}}) = S'(\mathrm{v}_{\tilde{E}}).$$

Also, since every points of $\mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathscr{O}}$ has a segment in the direction of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}}$, we obtain

$$\bigcup S(\mathrm{v}_{\tilde{E}}) = \mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathscr{O}}.$$

(iii) Since $g(D)$ is the generalized CA-lens for the the generalized lens neighborhood $g(U)$ of $g(\mathrm{v}_{\tilde{E}})$, we obtain $g(S(\mathrm{v}_{\tilde{E}})) = S(g(\mathrm{v}_{\tilde{E}}))$ for any p-end vertex $\mathrm{v}_{\tilde{E}}$.

(iv) Choose a proper p-end neighborhood U of $\tilde{E}$ covering an end-neighborhood of product form with compact boundary. We choose a generalized CA-lens L of a generalized lens-cone so that $C_{\tilde{E}} := \{\mathrm{v}_{\tilde{E}}\} * L - L - \{\mathrm{v}_{\tilde{E}}\}$ is in U by Lemma 5.4.1. We can choose $C_{\tilde{E}}$ to be a proper concave p-end neighborhood since can choose one in a proper p-end neighborhood. The properness of U shows that

$$g(C_{\tilde{E}}) = C_{\tilde{E}} \text{ for } g \in \mathbf{\Gamma}_{\tilde{E}}, \text{ or else } g(C_{\tilde{E}}) \cap C_{\tilde{E}} = \emptyset \text{ for every } g \in \mathbf{\Gamma}_{\tilde{E}}. \tag{5.4.2}$$

Let B_L denote the boundary component of L meeting the closure of $C_{\tilde{E}}$. Now, $\bigcup S(\mathrm{v}_{\tilde{E}})^o$ has an open neighborhood of form $C_{\tilde{E}} \cup \bigcup S(\mathrm{v}_{\tilde{E}})^o$ in $\mathscr{O}$ since B_L is separating hypersurface in $\tilde{\mathscr{O}}$. We obtain the conclusion since the intersection of the two sets implies the intersections of the neighborhoods of the sets.

(v) Let $C_{\tilde{E}}$ be a concave p-end neighborhood $\{\mathrm{v}_{\tilde{E}}\} * L - L - \{\mathrm{v}_{\tilde{E}}\}$ for a lens L. We will now show that $C_{\tilde{E}}$ is a proper p-end neighborhood. Suppose for contradiction that

$$g(C_{\tilde{E}}) \cap C_{\tilde{E}} \neq \emptyset \text{ and } g(C_{\tilde{E}}) \neq C_{\tilde{E}}.$$

Since $C_{\tilde{E}}$ is concave, each point x of $\mathrm{bd} C_{\tilde{E}} \cap \tilde{\mathscr{O}}$ is contained in a sharply supporting hyperspace H so that

- a component C of $C_{\tilde{E}} - H$ is in $C_{\tilde{E}}$ where
- $\mathrm{Cl}(C) \ni \mathrm{v}_{C_{\tilde{E}}}$ for the p-end vertex $\mathrm{v}_{C_{\tilde{E}}}$ of $C_{\tilde{E}}$.

Similar statements hold for $g(C_{\tilde{E}})$.

Since $g(C_{\tilde{E}}) \cap C_{\tilde{E}} \neq \emptyset$ and $g(C_{\tilde{E}}) \neq C_{\tilde{E}}$, it follows that

$$\mathrm{bd} g(C_{\tilde{E}}) \cap C_{\tilde{E}} \neq \emptyset \text{ or } g(C_{\tilde{E}}) \cap \mathrm{bd} C_{\tilde{E}} \neq \emptyset.$$

Assume the second case without the loss of generality. Let $x \in \mathrm{bd} C_E$ in $g(C_E)$ and choose H, C as above. Let $\mathrm{Cl}(C)$ be the closure containing $\mathrm{v}_{\tilde{E}}$ of a component C of $\mathrm{Cl}(\tilde{\mathscr{O}}) - H$ for a separating hyperspace H. $C \cap \mathrm{bd}\tilde{\mathscr{O}}$ is a union of lines in $S(\mathrm{v}_{\tilde{E}})$. Now, $H \cap g(C_{\tilde{E}})$ contains an open neighborhood in H of x.

Since H contains a point of a concave p-end neighborhood $g(C_{\tilde{E}})$ of $g(\tilde{E})$, it meets a points of $\{g(\mathrm{v}_{\tilde{E}})\} * g(D) - g(D) - \{g(\mathrm{v}_{\tilde{E}})\}$ for a lens D of $\tilde{E}$ and a ray from $g(\mathrm{v}_{\tilde{E}})$ in $g(C_{\tilde{E}})$. We deduce that $H \cap g(C_{\tilde{E}})$ separates $g(C_{\tilde{E}})$ into two open sets C_1 and C_2 in the direction of one of the sides of H where $\mathrm{Cl}(C_1) - H$ and $\mathrm{Cl}(C_1) - H$ meet $g\left(\bigcup S(\mathrm{v}_{\tilde{E}})\right)^o$ at nonempty sets. One of C_1 and C_2 is in C since C is a component of $\tilde{\mathscr{O}} - H$. Also, $\mathrm{Cl}(C) - H$ meets the set at $(\mathrm{Cl}(C) - H) \cap \mathrm{bd}\mathscr{O} \subset \bigcup S(\mathrm{v}_{\tilde{E}})$. Hence, this implies

$$g\left(\bigcup S(\mathrm{v}_{\tilde{E}})\right)^o \cap \bigcup S(\mathrm{v}_{\tilde{E}}) \neq \emptyset.$$

By (iv), this means $g \in \pi_1(\tilde{E})$. Hence, $g(C_{\tilde{E}}) = C_{\tilde{E}}$ and this is absurd. We have

$$g(C_{\tilde{E}}) \cap C_{\tilde{E}} = \emptyset \text{ or } g(C_{\tilde{E}}) = C_{\tilde{E}} \text{ for } g \in \pi_1(\mathscr{O}).$$

Since g acts on $C_{\tilde{E}}$ and the maximal segments in $S(\mathrm{v}_{\tilde{E}})$ must go to maximal segments, and the interior points of maximal segments cannot be an image of $\mathrm{v}_{\tilde{E}}$, we must have $g(\mathrm{v}_{\tilde{E}}) = \mathrm{v}_{\tilde{E}}$. Hence, $g(U) \cap U \neq \emptyset$ for any proper p-end neighborhood of $\tilde{E}$, and $g \in \pi_1(\tilde{E})$.

(vi) Suppose that $S(\mathrm{v}_{\tilde{E}})^o \cap S(\mathrm{v}_{\tilde{E}'}) \neq \emptyset$. Then $S(\mathrm{v}_{\tilde{E}})^o \cup C_{\tilde{E}}$ is a neighborhood of $S(\mathrm{v}_{\tilde{E}})^o$ for a proper concave p-end neighborhood $C_{\tilde{E}}$ of $\tilde{E}$. Also, $S(\mathrm{v}_{\tilde{E}'})^o \cup C_{\tilde{E}}$ is a neighborhood of $S(\mathrm{v}_{\tilde{E}'})^o$ for a proper concave p-end neighborhood of $C_{\tilde{E}'}$ of $\tilde{E}'$.

The above argument in (iv) applies within this situation to show that $\tilde{E} = \tilde{E}'$ and $\mathrm{v}_{\tilde{E}} = \mathrm{v}_{\tilde{E}'}$.

Proposition 1.4.2 implies the version for $\mathbb{RP}^n$. $[\mathbb{S}^n\mathrm{T}]$ □

5.4.2. The properties of lens-cones for factorizable case. Recall that a group G divides an open domain Ω if Ω/G is compact. For virtually factorizable ends, we have more results. We don't require that the quotient is Hausdorff.

THEOREM 5.4.3. *Let $\mathscr{O}$ be a strongly tame properly convex real projective n-orbifold. Suppose that*

- $\mathrm{Cl}(\mathscr{O})$ *is not of form* $\mathrm{v}_{\tilde{E}} * D$ *for a totally geodesic properly convex domain D, or*
- *the holonomy group* $\mathbf{\Gamma}$ *is strongly irreducible.*

Let $\tilde{E}$ be an R-p-end of the universal cover $\tilde{\mathscr{O}}$, $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$) with a generalized lens p-end-neighborhood. Let $\mathrm{v}_{\tilde{E}}$ be the p-end vertex, and the p-end orbifold $\Sigma_{\tilde{E}}$ of $\tilde{E}$. Suppose that the p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ is virtually factorizable. Then $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$, and the following statements hold:

- (i) *The R-p-end is totally geodesic. $D_i \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is projectively diffeomorphic by the projection $\Pi_{\mathrm{v}_{\tilde{E}}}$ to totally geodesic convex domain D'_i in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) disjoint from $\mathrm{v}_{\tilde{E}}$. Moreover, $\mathbf{\Gamma}_{\tilde{E}}$ is virtually a cocompact subgroup of $\mathbb{R}^{l_0-1} \times \prod_{i=1}^{l_0} \mathbf{\Gamma}_i$ where $\mathbf{\Gamma}_i$ acts on D'_i irreducibly and trivially on D'_j for $j \neq i$, and $\mathbb{R}^{l_0-1}$ acts trivially on D'_j for every $j = 1, \ldots, l_0$.*
- (ii) *The R-p-end is strictly lens-shaped, and each C'_i corresponds to a cone $C^*_i = \mathrm{v}_{\tilde{E}} * D'_i$. The R-p-end has a p-end-neighborhood equal to the interior of*
$$\{\mathrm{v}_{\tilde{E}}\} * D \text{ for } D := \mathrm{Cl}(D'_1) * \cdots * \mathrm{Cl}(D'_{l_0})$$
where the interior of D forms the boundary of the p-end neighborhood in $\tilde{\mathscr{O}}$.
- (iii) *The set $S(\mathrm{v}_{\tilde{E}})$ of maximal segments in $\mathrm{bd}\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}}$ in the closure of a p-end-neighborhood of $\mathrm{v}_{\tilde{E}}$ is independent of the p-end-neighborhood.*

$$\bigcup S(\mathrm{v}_{\tilde{E}}) = \bigcup_{i=1}^{l_0} \{\mathrm{v}_{\tilde{E}}\} * \mathrm{Cl}(D'_1) * \cdots * \mathrm{Cl}(D'_{i-1}) * \partial \mathrm{Cl}(D'_i) * \mathrm{Cl}(D'_{i+1}) * \cdots * \mathrm{Cl}(D'_{l_0}).$$

Finally, the statements (i), (ii), (iii), (iv), (v) *and* (vi) *of Theorem 5.4.2 also hold.*

PROOF. Again the $\mathbb{S}^n$-version is enough by Proposition 1.4.2. Theorem 5.3.21 implies the uniform middle eigenvalue condition. (i) This follows by Proposition 1.4.10 (see Benoist [22]).

As in the proof of Theorem 5.4.2, Theorem 5.3.21 implies that $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition. Proposition 5.3.14 implies that the CA-lens is a strict one. Theorem 5.2.5 implies that the distanced $\mathbf{\Gamma}_{\tilde{E}}$-invariant set is contained in a hyperspace P disjoint from $\mathrm{v}_{\tilde{E}}$.

(i) By the uniform middle eigenvalue condition, the largest norm of the eigenvalue $\lambda_1(g)$ is strictly bigger than $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$.

By Proposition 1.4.10, Γ is a virtually a subgroup of $\mathbb{R}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$ with $\mathbb{R}^{l_0-1}$ acting as a diagonalizable group, and there are subspaces $\hat{\mathbb{S}}_j$, $j = 1, \ldots, l_0$, in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ where the factor groups $\mathbf{\Gamma}_1, ..., \mathbf{\Gamma}_{l_0}$ act irreducibly by Benoist [22]. Let $\mathbb{S}_j$, $j = 1, \ldots, l_0$, be the projective subspaces in general position meeting only at the p-end vertex $\mathrm{v}_{\tilde{E}}$ which goes to $\hat{S}_j$ under $\Pi_{\mathrm{v}_{\tilde{E}}}$. Now, $\mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}}) \cap \mathbb{S}_j$ is a properly convex domain K_i by Benoist [22]. Let C_i denote the union of great segments from $\mathrm{v}_{\tilde{E}}$ with directions in K_i in $\mathbb{S}_i$ for each i. The abelian center isomorphic to $\mathbb{Z}^{l_0-1}$ acts as the identity on the subspace corresponding to C_i in the projective space $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$.

We denote by $D'_i := C_i \cap P$. We denote by $D = D'_1 * \cdots * D'_{l_0} \subset P$. Also, the interior of $\mathrm{v}_{\tilde{E}} * D$ is a p-end neighborhood of $\tilde{E}$. This proves (i).

Let U be the p-end-neighborhood of $\mathrm{v}_{\tilde{E}}$ obtained in (iv). $\mathbf{\Gamma}_{\tilde{E}}$ acts on $\mathrm{v}_{\tilde{E}}$ and $D'_1, \ldots, D'_{l_0}$.

Recall that the virtual center of $\mathbf{\Gamma}_{\tilde{E}}$ isomorphic to $\mathbb{Z}^{l_0-1} \subset \mathbb{R}^{l_0-1}$ has diagonalizable matrices acting trivially on $\mathbb{S}_j$ for $j = 1, \ldots, l_0$. For all C_i, every nonidentity g in the virtual center acts as nonidentity now by the uniform middle eigenvalue condition.

For each i, we can find a sequence g_j in the virtual center of $\mathbf{\Gamma}_{\tilde{E}}$ so that the premise of Proposition 1.4.19 are satisfied, and $\mathrm{Cl}(D'_i) \subset \mathrm{Cl}(\tilde{\mathscr{O}})$. By Proposition 1.4.19, (ii) follows.

Therefore, we obtain

$$\mathrm{v}_{\tilde{E}} * \mathrm{Cl}(D'_1) * \cdots * \mathrm{Cl}(D'_{i-1}) * \partial \mathrm{Cl}(D'_i) * \mathrm{Cl}(D'_{i+1}) * \cdots * \mathrm{Cl}(D'_{l_0}) = \mathrm{bd}\tilde{\mathscr{O}} \cap \mathrm{Cl}(U)$$

by the middle eigenvalue conditions. (iii) follows.

(ii) We need to show $D^o \subset \tilde{\mathscr{O}}$. By Lemma 1.4.4, we have either $D^o \subset \tilde{\mathscr{O}}$ or $D \subset \mathrm{bd}\tilde{\mathscr{O}}$. In the second case, $\mathrm{Cl}(\tilde{\mathscr{O}}) = \{\mathrm{v}_{\tilde{E}}\} * D$ since $S(\mathrm{v}_{\tilde{E}}) \subset \mathrm{bd}\tilde{\mathscr{O}}$ and $D \subset \mathrm{bd}\tilde{\mathscr{O}}$. This contradicts the premise.

If $\boldsymbol{\Gamma}$ is strongly irreducible, $\tilde{\mathscr{O}}$ cannot be a strict join by Proposition 1.4.18. Thus, this completes the proof.

We can prove the strictness of the lens and the final part by generalizing the proof of Theorem 5.4.2 to this situation. The proof statements do not change. [$\mathbb{S}^n$T] □

5.4.3. Uniqueness of vertices outside the lens. We will need this later in Chapter 11.

PROPOSITION 5.4.4. *Suppose that $h : \pi_1(E) \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ is a holonomy representation of end fundamental group $\pi_1(E)$ of a strongly tame convex real projective orbifold. Let $h(\pi_1(E))$ act on a generalized lens-cone $\{v\} * L$ with vertex v, acting on a generalized lens L properly and cocompactly, or act on a horosphere as a lattice in a cusp group. Then the following hold*:

- *the vertex of the lens-cone is determined up to the antipodal map.*
- *If the lens-cone is given an outward direction, then the vertex of any lens-cone where $h(\pi_1(E))$ acts on as a p-end neighborhood equals the vertex of the lens-cone and is uniquely determined.*
- *The vertex of the horospherical end is uniquely determined.*

PROOF. The horospherical case can be understood from the horopherical action acting on a ball of a Klein model where fixed points form a pair of antipodal points.

Suppose that $h(\pi_1(E))$ acts on a generalized lens cone $\{v\} * L$ for a generalized lens L as in the premise and a vertex v. By Theorem 5.1.4, $h(\pi_1(E))$ satisfies the uniform middle eigenvalue condition with respect to v. Suppose that there exists another point w fixed by $h(\pi_1(E))$ so that $\{w\} * L'$ is a generalized lens cone for another generalized lens L' properly and cocompactly. Let $\overrightarrow{vw}$ denote a vector tangent to $\overline{vw}$ oriented away from v. Then $\overline{vw}$ goes to a point $(\!(\overrightarrow{vw})\!)$ on $\mathbb{S}^{n-1}_v$ which $h(\pi_1(E))$ fixes. Hence, $\pi_1(E)$ acts reducibly on $\mathbb{S}^{n-1}_v$, and $h(\pi_1(E))$ is virtually factorizable. Thus, $h(\pi_1(E))$ acts on a hyperspace S disjoint from v by Theorem 5.4.3. There is a properly convex domain D in S where $h(\pi_1(E))$ acts properly discontinuously. Also, $\mathrm{Cl}(D) = K_1 * \cdots * K_m$ for properly convex domain K_j where $h(\pi_1(E))$ acts irreducibly by Proposition 1.4.10.

Suppose that $w \in S$. Then $\{w\} = K_j$ or its antipode K_{j-} for some j. The uniform middle eigenvalue condition with respect to v implies that $\pi_1(E)$ does not have the same property with respect to w. Hence, $w \notin S$.

Hence $\pi_1(E)$ acts on a domain Ω equal to the interior of $K := K_1 * \cdots * K_m$ where K_j is compact and convex and a finite-index subgroup Γ'_E of $\pi_1(E)$ acts on each K_j irreducibly. The great segment from v containing w meets S in a point w'. There exists a virtual-center diagonalizable group D acting on each K_j as the identity by Proposition 1.4.10 (more precisely Proposition 4.4 of [21]). Hence $\{w'\}$ must be one of K_k or its anitpode K_{k-} since otherwise we can find an element of D not fixing w'.

Since the action is cocompact on Ω, there must be an element of D acting with largest norm eigenvalue on K_k.

Since $h(\pi_1(E))$ acts on $\{w\} * L'$ with a compact set L' disjoint from w. We construct a tube domain T and $L' \cap T$ gives us a CA-lens in the tube. Hence, by Theorem 5.1.4 $\lambda_w(g)$ satisfies the uniform middle eigenvalue condition. We choose $g \in D$ with a largest norm eigenvalue at K_k. Since v, w, w' are distinct points in a properly convex segment and are fixed points of g, it follows that $\lambda_1(g) = \lambda_w(g) = \lambda_v(g)$. This contradicts the uniform middle eigenvalue condition for v under $\pi_1(E)$. Thus, we obtain $v = w$. $\square$

PROPOSITION 5.4.5. *Suppose that $h : \pi_1(E) \to \mathsf{SL}_\pm(n+1, \mathbb{R})$ is a representation*

- *acting properly and cocompactly on a lens neighborhood of a totally geodesic $(n-1)$-dimensional domain Ω and $\Omega/h(\pi_1(E))$ is a compact orbifold or*
- *acting on a horosphere as a cocompact cusp group.*

Then $h(\pi_1(E))$ uniquely determines the hyperplane P with one of the following properties:

- *P meets a lens domain L' with the property that $(L' \cap P)/h(\pi_1(E))$ is a compact orbifold with $L' \cap P = L'^o \cap P$.*
- *P is tangent to the $h(\pi_1(E))$-invariant horosphere at the cusp point of the horosphere.*

PROOF. The duality will prove this by Proposition 5.5.5 and Corollary 5.5.1. The vertex and the hyperspace exchanges the roles. $\square$

5.5. Duality and lens-shaped T-ends

We first discuss the duality map. We show a lens-cone p-end neighborhood of an R-p-end is dual to a lens p-end neighborhood of a T-p-end. Using this we prove Theorem 5.5.4 dual to Theorem 5.3.21, i.e., Theorem 5.1.5.

5.5.1. Duality map. The Vinberg duality diffeomorphism induces a one-to-one correspondence between p-ends of $\tilde{\mathscr{O}}$ and $\tilde{\mathscr{O}}^*$ by considering the dual relationship $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathbf{\Gamma}^*_{\tilde{E}'}$ for each pair of p-ends $\tilde{E}$ and $\tilde{E}'$ with dual p-end holonomy groups. (See Section 1.5.)

Given a properly convex domain Ω in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$), we recall the augmented boundary of Ω

$$\begin{aligned} \mathrm{bd}^{\mathrm{Ag}}\Omega := \{(x,H) | x \in \mathrm{bd}\Omega, x \in H, \\ H \text{ is an oriented sharply supporting hyperspace of } \Omega\} \subset \mathbb{S}^n \times \mathbb{S}^{n*}. \end{aligned} \tag{5.5.1}$$

This is a closed subspace. Each $x \in \mathrm{bd}\Omega$ is contained in at least one sharply supporting hyperspace oriented towards Ω. and an element of $\mathbb{S}^n$ represent an oriented hyperspace in $\mathbb{S}^{n*}$.

We recall a duality map.

$$\mathscr{D}^{\mathrm{Ag}}_{\Omega} : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}^{\mathrm{Ag}}\Omega^* \tag{5.5.2}$$

given by sending (x,H) to (H,x) for each $(x,H) \in \mathrm{bd}^{\mathrm{Ag}}\Omega$. This is a diffeomorphism since $\mathscr{D}^{\mathrm{Ag}}_{\Omega}$ has an inverse given by switching factors by Proposition 1.5.4 (iii).

We will need the corollary about the duality of lens-cone and lens-neighborhoods. Recall that given a properly convex domain D in $\mathbb{S}^n$ or $\mathbb{RP}^n$, the dual domain is the closure of the open set given by the collection of (oriented) hyperspaces in $\mathbb{S}^n$ or $\mathbb{RP}^n$ not meeting $\mathrm{Cl}(D)$. Let Ω be a properly convex domain. We need to recall the duality from Section 1.5.1 with the projection map

$$\Pi^{\mathrm{Ag}}_{\Omega} : \mathrm{bd}^{\mathrm{Ag}}\Omega \to \mathrm{bd}\Omega$$

sending each pair (x,h) of a point x, $x \in \mathrm{bd}\Omega$, and sharply supporting hyperplane h at x.

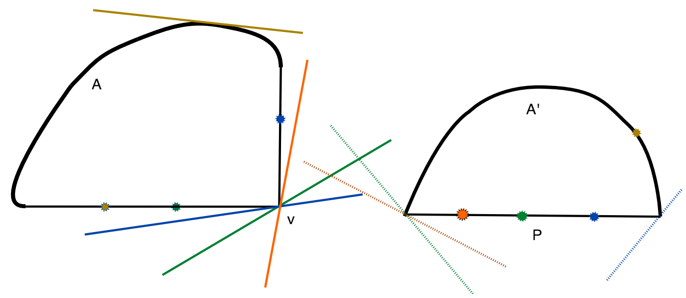

FIGURE 1. The figure for Corollary 5.5.1. Lines passing v in the left figure correspond to points on P in the left. The line passing a point of A corresponds to a point on A'. Lines in the right figure correspond to points in the left figure.

COROLLARY 5.5.1. *The following hold in* $\mathbb{S}^n$:

- *Let* L *be a lens and* $\{\mathrm{v}\} \not\in \mathrm{Cl}(L)$ *so that* $\mathrm{v} * L$ *is a properly convex lens-cone. Suppose that the smooth strictly convex boundary component* A *of* L *is tangent to a segment from* v *at each point of* $\partial \mathrm{Cl}(A)$ *and* $\{\mathrm{v}\} * L = \{\mathrm{v}\} * A$. *Then the following hold*:
  - *the dual domain of* $\mathrm{Cl}(\{\mathrm{v}\} * L)$ *is the closure of a component* L_1 *of* $L' - P$ *where* L' *is a lens and* P *is a hyperspace meeting* L'^o.
  - A *corresponds to a hypersurface* $A' \subset \mathrm{bd} L'$ *under the duality* (5.5.2).
  - $A' \cup D$ *is the boundary of* L_1 *for a totally geodesic properly convex* $(n-1)$-*dimensional compact domain* D *dual to* $R_{\mathrm{v}}(\{\mathrm{v}\} * L)$ *where* D *is given by* $\Pi^{\mathrm{Ag}}_{\{\mathrm{v}\}*L} \circ \mathscr{D}_{\{\mathrm{v}\}*L}((\Pi^{\mathrm{Ag}})^{-1}_{\{\mathrm{v}\}*L}(\{\mathrm{v}\}))$.
- *Conversely, we are given a lens* L' *and* P *is a hyperspace meeting* L'^o *but not meeting the boundary of* L'. *Let* L_1 *be a component of* $L' - P$ *with smooth strictly convex boundary* ∂L_1 *so that* $\partial \mathrm{Cl}(\partial L_1) \subset P$. *Here, we assume* ∂L_1 *is an open hypersurface. Then the following hold*:
  - *The dual of the closure of a component* L_1 *of* $L' - P$ *is the closure of* $\mathrm{v} * L$ *for a lens* L *and* $\mathrm{v} \not\in L$ *so that* $\mathrm{v} * L$ *is a properly convex lens-cone. Here,* $\{\mathrm{v}\} = \Pi^{\mathrm{Ag}}_{\mathrm{Cl}(L_1)} \circ \mathscr{D}_{\mathrm{Cl}(L_1)}(P)$.
  - *The outer boundary component* A *of* L *is tangent to a segment from* v *at each point of* $\partial \mathrm{Cl}(A)$.
- *In the above, the vertex denoted by* v *corresponds to a hyperplane denoted by* P *uniquely.*

PROOF. In the proof all hyperspaces are oriented so that L^o is in its interior direction. Let A denote the boundary component of L so that $\{\mathrm{v}\} * L = \{\mathrm{v}\} * A$. We will determine the dual domain $(\mathrm{Cl}(\{\mathrm{v}\} * L))^*$ of $\{\mathrm{v}\} * L$ by finding the boundary of D using the duality map $\mathscr{D}_{\{\mathrm{v}\}*L}$. The set of hyperspaces sharply supporting $\mathrm{Cl}(\{\mathrm{v}\} * L)$ at v forms a properly totally geodesic domain D in $\mathbb{S}^{n*}$ contained in a hyperspace P dual to v by Lemma 1.5.7. Also the set of hyperspaces sharply supporting $\mathrm{Cl}(\{\mathrm{v}\} * L)$ at points of A goes to the strictly convex hypersurface A' in $\mathrm{bd}(\mathrm{v} * L)^*$ by Lemma 1.5.7 since $\mathscr{D}_{\{\mathrm{v}\}*L}$ is a diffeomorphism. (See

Remark 1.5.6 and Figure 1.) The subspace $S := \mathrm{bd}(\{\mathrm{v}\} * A) - A$ is a union of segments from v. The sharply supporting hyperspaces containing these segments go to points in ∂D. Each point of $\mathrm{Cl}(A') - A'$ is a limit of a sequence $\{p_i\}$ of points of A', corresponding to a sequence of sharply supporting hyperspheres $\{h_i\}$ to A. The tangency condition of A at $\partial \mathrm{Cl}(A)$ implies that the limit hypersphere contains the segment in S from v. We obtain that $\mathrm{Cl}(A') - A'$ equals the set of hyperspheres containing the segments in S from v, and they go to points of ∂D with $\partial \mathrm{Cl}(A') = \partial D$. We conclude

$$\Pi^{\mathrm{Ag}}_{\{\mathrm{v}\}*L} \circ \mathscr{D}_{\{\mathrm{v}\}*L}(\mathrm{bd}(\{\mathrm{v}\} * L)) = A' \cup D.$$

Let P be the unique hyperspace containing D. Then each point of $\partial \mathrm{Cl}(A)$ goes to a sharply supporting hyperspace at a point of $\partial \mathrm{Cl}(A')$ distinct from P. Let L^* denote the dual domain of $\mathrm{Cl}(L)$. Since $\mathrm{Cl}(L) \subset \mathrm{Cl}(\{\mathrm{v}\} * L)$, we obtain $(\mathrm{Cl}(\{\mathrm{v}\} * L))^* \subset (\mathrm{Cl}(L))^*$ by (1.5.2). Since $A \subset \mathrm{bd}L$, we obtain

$$\Pi^{\mathrm{Ag}}_{\{\mathrm{v}\}*L} \circ \mathscr{D}_{\{\mathrm{v}\}*L}(\mathrm{bd}(\{\mathrm{v}\} * L)) \subset A' \cup P, \text{ and } A' \subset \mathrm{bd}L^*.$$

Proposition 1.5.4 implies that $((\{\mathrm{v}\} * L)^o)^*$ is a component L_1 of $(L^o)^* - P$ since the first domain can have boundary points in $A' \cup P$ only and cannot have points outside the component. Hence, the dual of $\mathrm{Cl}(\{\mathrm{v}\} * L)$ is $\mathrm{Cl}(L_1)$. Moreover, $A' \subset \mathrm{bd}L_1$ since A' is a strictly convex hypersurface with boundary in P.

The second item is proved similarly to the first. Now hyperspaces are oriented so that L_1^o is in its interior. Then ∂L_1 goes to a hypersurface A in the boundary of the dual domain L_1^* of L_1 under $\mathscr{D}_{L_1}$. Again A is a smooth strictly convex boundary component. Since $\partial \mathrm{Cl}(L_1) \subset P$ and L_1 is a component of $L' - P$, we obtain $\mathrm{bd}L_1 - \partial L_1 = \mathrm{Cl}(L_1) \cap P$. This is a totally geodesic properly convex domain D in P.

Suppose that $l \subset P$ be an $n-2$-dimensional space disjoint from L_1^o. Then a space of oriented hyperspaces containing l bounding an open hemisphere containing L_1^o forms a parameter dual to a convex projective geodesic in $\mathbb{S}^{n*}$. An L_1-*pencil* P_t with ends P_0, P_1 is a parameter satisfying

$$P_t \cap P = P_0 \cap P, P_t \cap L_1^o = \emptyset \text{ for all } t \in [0,1]$$

where P_t is oriented so that it bounds a open hemisphere containing L_1^o.

There is a one-to-one correspondence

$$\{P' | P' \text{ is an oriented hyperspace that supports } L_1 \text{ at points of } \partial D\} \leftrightarrow \{\mathrm{v}\} * \partial \mathrm{Cl}(A) :$$

Every supporting hyperspace P' to L_1 at points of ∂D is contained in an L_1-parameter P_t with $P_0 = P', P_1 = P$. v is the dual to P in $\mathbb{S}^{n*}$. Each of the path P_t is a segment in $\mathbb{S}^{n*}$ with an endpoint v.

Under the duality map $\Pi^{\mathrm{Ag}}_{\mathrm{Cl}(L_1)} \circ \mathscr{D}_{\mathrm{Cl}(L_1)}$, the image of $\mathrm{bd}L_1$ is a union of A and the union of these segments. Given any hyperspace P' disjoint from L_1^o, we find a one-parameter family of hyperspaces containing $P' \cap P$. Thus, we find an L_1-pencil family P_t with $P_0 = P', P_1 = P$. We can extend the L_1-pencil so that the ending hyperspace P'' of the L_1-pencil meets ∂L_1 tangentially or tangent to ∂L_1 and $P'' \cap P$ is a supporting hyperspace of D in P. Since the hyperspaces are disjoint from L_1, the segment is in L_1^*. Since L_1 is a properly convex domain, we can deduce that $(\mathrm{Cl}(L_1))^*$ is the closure of the cone $\{\mathrm{v}\} * A$.

Let L'' be the dual domain of $\mathrm{Cl}(L')$. Since $L' \supset L_1$, we obtain $L'' \subset (\mathrm{Cl}(L_1))^*$ by (1.5.2). Since $\partial L_1 \subset \mathrm{bd}L'$, we obtain $A \subset \mathrm{bd}L''$ by the duality map $\mathscr{D}_{\mathrm{Cl}(L_1)}$. We obtain that $L'' \cup A \subset \mathrm{Cl}(\{\mathrm{v}\} * A)$.

Let B be the image of the other boundary component B' of L' under $\mathscr{D}_{L'}$. We take a sharply supporting hyperspace P_y at $y \in B'$. Then $P_y \cap P$ is disjoint from $\mathrm{Cl}(D)$ by the strict convexity of B'. We find an L_1-pencil P_t of hyperspaces containing $P_y \cap P$ with $P_0 = P_y, P_1 = P$. This L_1-pencil goes into the segment from v to a point of B under the duality. We can extend the L_1-pencil so that the ending hyperspace meets ∂L_1 tangentially. The dual pencil is a segment from v to a point of A. Thus, each segment from v to a point of A meets B. Thus, $L''^o \cup A \cup B$ is a lens of the lens cone $\{\mathrm{v}\} * A$. This completes the proof. □

5.5.2. The duality of T-ends and properly convex R-ends. Let Ω be the properly convex domain covering $\mathscr{O}$. For a T-end E, the totally geodesic ideal boundary S_E of E is covered by a properly convex open domain in $\mathrm{bd}\Omega$ corresponding to a T-p-end $\tilde{E}$. We denote it by $\tilde{S}_{\tilde{E}}$.

Recall R-end structure from Section 3.1.3 (see Definition 9.1.1 also).

LEMMA 5.5.2. *Let $\mathscr{O}$ be a convex real projective strong tame orbifold with ends where $\mathscr{O} = \tilde{\mathscr{O}}/\Gamma$ for a properly convex domain $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ and a discrete projective group Γ. Let $p_{\mathscr{O}} : \tilde{\mathscr{O}} \to \mathscr{O}$ denote the covering map. Suppose that a p-end fundamental group $\Gamma_{\tilde{E}}$ for a p-end $\tilde{E}$ acts on a connected hypersurface $\tilde{\Sigma}$ in $\tilde{\mathscr{O}}^o$. Then a component U' of $\tilde{\mathscr{O}} - \tilde{\Sigma}$ is a p-end neighborhood of $\tilde{E}$. Furthermore the following hold*:

- *Suppose that U' is a horoball so that $\mathrm{bd}U' \cap \tilde{\mathscr{O}} = \mathrm{bd}U' - \{p\} = \Sigma$ for the common fixed point p of $\Gamma_{\tilde{E}}$. Then $p_{\mathscr{O}}(U')$ can be given the structure of a horospherical R-end neighborhood of $\tilde{E}$ and $p = \mathrm{v}_{\tilde{E}}$.*
- *Suppose that U' equals $U_L := L * \{p\} - L$ for a lens L where*
  - *$h(\pi_1(\tilde{E}))$ acts properly and cocompactly on U',*
  - *we have a lens-cone $L * \{p\}$ for a common fixed point p of $\Gamma_{\tilde{E}}$, and*
  - $\mathrm{bd}U_L \cap \tilde{\mathscr{O}} = \tilde{\Sigma}$.

  Then U_L can be given the structure of a concave p-R-end neighborhood of $\tilde{E}$ and $p = \mathrm{v}_{\tilde{E}}$.
- *Suppose that U' equals a component L_1 of $L - P$ for a lens L where*
  - *$h(\pi_1(\tilde{E}))$ acts properly and cocompactly on L,*
  - *$h(\pi_1(\tilde{E}))$ acts on the hyperplane P,*
  - *$L_1 \subset \tilde{\mathscr{O}}$, and*
  - $\mathrm{bd}L_1 \cap \tilde{\mathscr{O}} = \tilde{\Sigma}$.

  Then L_1^o can be given the structure of a p-T-end neighborhood of $\tilde{E}$ and $\tilde{\Sigma}_{\tilde{E}}$ can be identified with $L \cap P$.

Moreover the corresponding end completions gives us a compact smooth orbifold $\bar{\mathscr{O}}$ whose interior is $\mathscr{O}$.

PROOF. It is sufficient to prove for the case when the orbifold is orientable and without singular points since we can take a finite quotient by Theorem 1.1.19. Let U be a proper p-end neighborhood of $\tilde{E}$. We take $\tilde{\mathscr{O}}/\Gamma_{\tilde{E}}$ which is diffeomorphic to $A := B \times \mathbb{R}$ where $B := (\mathrm{bd}U \cap \tilde{\mathscr{O}})/\Gamma_{\tilde{E}}$ is a closed submanifold of codimension-one. We can take an exiting sequence U_i of p-end neighborhoods in U. Then $C := \Sigma/\Gamma_{\tilde{E}}$ is a closed submanifold freely homotopic to the above one. Hence, one U_B of the two components $\tilde{\mathscr{O}}/\Gamma_{\tilde{E}} - B$ contains U_i for sufficiently large i. Hence, a component of the inverse image of U_B which is a component of $\tilde{\mathscr{O}} - \Sigma$ is a p-end neighborhood, perhaps not a proper one.

In the first case, we have a horoball U' inside $\tilde{\mathscr{O}}^o$ and in H since the sharply supporting hyperspaces at the vertex of H must coincide by the invariance under $h(\pi_1(\tilde{E}))$ by a limiting argument. By above, one of the two component of $\tilde{\mathscr{O}}^o - \mathrm{bd}U'$ is a p-end neighborhood. One cannot put the outside component into U' by an element of $h(\pi_1(\mathscr{O}))$. Hence, U is a horospherical p-end neighborhood of $\tilde{E}$.

For the second item, Lemma 3.1.5 implies that U_L is lines in $\mathscr{O}$. Thus, U_L is the p-end neighborhood of $\tilde{E}$, and $\tilde{E}$ has a radially foliated p-end neighborhood with $\mathrm{v}_{\tilde{E}}$ as the p-end vertex.

In the third case, let D denote $\mathrm{Cl}(L_1) \cap P$, a properly convex domain. By premise, $D^o/h(\pi_1(\tilde{E}))$ is a closed orbifold of codimension-one. Then one of the two components of $\tilde{\mathscr{O}} - \mathrm{bd}L_1$ is a p-end neighborhood of $\tilde{E}$ by above. D^o is totally geodesic and $\mathrm{bd}L_1 \cap \tilde{\mathscr{O}}$ is not. Hence, L_1^o is a p-end neighborhood of $\tilde{E}$. By premise, $h(\pi_1(\tilde{E}))$ acts properly on $L_1 \cup D^o$. The T-end structure is given by $(D^o \cup L_1)/h(\pi_1(\tilde{E}))$ which is the completion of the end neighborhood $p(L_1^o)$ projectively diffeomorphic to $L_1^o/h(\pi_1(\tilde{E}))$. (See 3.1.2.) $\square$

PROPOSITION 5.5.3. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold. The following conditions are equivalent*:

(i) *A properly convex R-end E of $\mathscr{O}$ satisfies the uniform middle-eigenvalue condition.*

(ii) *The corresponding T-end E^* of $\mathscr{O}^*$ satisfies this condition with the correspondence of the vertex of the p-end $\tilde{E}$ of E to the hyperplane of p-end $\tilde{E}^*$ is given as the unique hyperplane containing $\Pi^{\mathrm{Ag}}_{\mathrm{Cl}(\tilde{\mathscr{O}})} \circ \mathscr{D}_{\tilde{\mathscr{O}}}((\Pi^{\mathrm{Ag}})^{-1}_{\mathrm{Cl}(\tilde{\mathscr{O}})}(\mathrm{v}_{\tilde{E}}))$.*

PROOF. The items (i) and (ii) are equivalent by considering (5.1.1) and (5.1.2). Proposition 1.4.2 implies the $\mathbb{RP}^n$-version. [$\mathbb{S}^n$T] $\square$

We now prove the dual to Theorem 5.3.21. For this we do not need the triangle condition or the reducibility of the end.

THEOREM 5.5.4. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold. Let $\tilde{S}_{\tilde{E}}$ be a totally geodesic ideal boundary component of a T-p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$. Then the following conditions are equivalent*:

(i) *The end holonomy group of $\tilde{E}$ satisfies the uniform middle-eigenvalue condition with respect to the T-p-end structure of $\tilde{E}$.*

(ii) *$\tilde{S}_{\tilde{E}}$ has a lens neighborhood in an ambient open manifold containing $\tilde{\mathscr{O}}$ with cocompact action of $\pi_1(\tilde{E})$, and hence $\tilde{E}$ has a lens-shaped p-end-neighborhood in $\tilde{\mathscr{O}}$.*

PROOF. We prove for the $\mathbb{S}^n$-version. Assuming (i), we can deduce the existence of a lens neighborhood from Theorem 4.4.1 and Lemma 5.5.2.

Assuming (ii), we obtain a totally geodesic $(n-1)$-dimensional properly convex domain $\tilde{S}_{\tilde{E}}$ in a subspace $\mathbb{S}^{n-1}$ on which $\mathbf{\Gamma}_{\tilde{E}}$ acts. Let U be a lens-neighborhood of it on which $\mathbf{\Gamma}_{\tilde{E}}$ acts. Then since U is a neighborhood, the sharply supporting hemisphere at each point of $\mathrm{Cl}(\tilde{S}_{\tilde{E}}) - \tilde{S}_{\tilde{E}}$ is now transverse to $\mathbb{S}^{n-1}$. Let P be the hyperspace containing $\tilde{S}_{\tilde{E}}$, and let U_1 be the component of $U - P$. Then the dual U_1^* is a lens-cone by the second part of Corollary 5.5.1 where P corresponds to a vertex of the lens-cone. The dual U^* of U is aa lens contained in a lens-cone U_1^* where $\mathbf{\Gamma}_E$ acts on U^*. We apply the part (i) $\Rightarrow$ (ii) of Theorem 5.3.21. By Proposition 5.5.3, we are done. Proposition 1.4.2 implies for $\mathbb{RP}^n$. [$\mathbb{S}^n$T] $\square$

PROPOSITION 5.5.5. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with R-ends or T-ends with universal covering domain Ω. Let $\mathscr{O}^*$ be the dual orbifolds with a universal covering domain $\tilde{\mathscr{O}}^*$. Then $\mathscr{O}^*$ is also strongly tame and can be given R-end and T-end structures to the ends where the following hold*:

- *there exists a one-to-one correspondence $\mathscr{C}$ between the set of p-ends of $\tilde{\mathscr{O}}$ and the set of p-ends of $\tilde{\mathscr{O}}^*$ by sending a p-end neighborhood to a p-end neighborhood using the Vinberg diffeomorphism of Theorem 1.5.8.*
- *$\mathscr{C}$ restricts to such a one between the subset of horospherical p-ends of $\tilde{\mathscr{O}}$ and the subset of horospherical ones of $\tilde{\mathscr{O}}^*$. Also, the augmented Vinberg duality homeomorphism $\bar{\mathscr{D}}^{\mathrm{Ag}}$ send the p-end vertex to the p-end vertex of the dual p-end.*
- *$\mathscr{C}$ restricts to such a one between the subset of all generalized lens-shaped R-ends of $\mathscr{O}$ and the subset of all lens-shaped T-ends of $\mathscr{O}^*$. Also, $\tilde{\Sigma}_{\tilde{E}}$ of an R-p-end is projectively dual to the ideal boundary component $\tilde{S}_{\tilde{E}^*}$ for the corresponding dual T-p-end $\tilde{E}^*$ of $\tilde{E}$. Also, $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$ gives a one to one correspondence between $\Pi^{\mathrm{Ag}-1}_{\tilde{\mathscr{O}}}(\bigcup S(\mathrm{v}_{\tilde{E}}))$ in $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathscr{O}}$ to $\Pi^{\mathrm{Ag}-1}_{\tilde{\mathscr{O}}^*}(\mathrm{Cl}(\tilde{S}_{\tilde{E}^*}))$ of $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathscr{O}}^*$.*
- *$\mathscr{C}$ restricts to such a one between the set of lens-shaped T-p-ends of $\tilde{\mathscr{O}}$ with the set of p-ends of generalized lens-shaped R-p-ends of $\tilde{\mathscr{O}}^*$. The ideal boundary component $\tilde{S}_{\tilde{E}}$ for a T-p-end $\tilde{E}$ is projectively diffeomorphic to the properly convex open domain dual to the domain $\tilde{\Sigma}_{\tilde{E}^*}$ for the corresponding R-p-end $\tilde{E}^*$ of $\tilde{E}$. Also, $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$ gives one to one correspondence between $\Pi^{\mathrm{Ag}-1}_{\tilde{\mathscr{O}}}(\mathrm{Cl}(\tilde{S}_{\tilde{E}}))$ in $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathscr{O}}$ to $\Pi^{\mathrm{Ag}-1}_{\tilde{\mathscr{O}}^*}(\bigcup S(\tilde{E}^*))$ of $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathscr{O}}^*$.*

PROOF. We prove for the $\mathbb{S}^n$-version first. Let $\tilde{\mathscr{O}}$ be the universal cover of $\mathscr{O}$. Let $\tilde{\mathscr{O}}^*$ be the dual domain. By the Vinberg duality diffeomorphism of Theorem 1.5.8, $\mathscr{O}^* := \tilde{\mathscr{O}}^*/\Gamma^*$ is also strongly tame for the dual group Γ^*. The first item follows by the fact that this diffeomorphism sends p-end neighborhoods to p-end neighborhoods.

Let $\tilde{E}$ be a horospherical R-p-end with x as the end vertex. Since there is a subgroup $\Gamma_{\tilde{E}}$ of a cusp group acting on $\mathrm{Cl}(\tilde{\mathscr{O}})$ with a unique fixed point, the intersection of the unique sharply supporting hyperspace h with $\mathrm{Cl}(\tilde{\mathscr{O}})$ at x is a singleton $\{x\}$. (See Theorem 8.1.3.) The dual subgroup $\Gamma^*_{\tilde{E}}$ is also a cusp group and acts on $\mathrm{Cl}(\tilde{\mathscr{O}}^*)$ with h fixed. So the corresponding $\tilde{\mathscr{O}}^*$ has the dual hyperspace x^* of x as the unique intersection at h^* dual to h at $\mathrm{Cl}(\tilde{\mathscr{O}}^*)$. There is a horosphere S where the end fundamental group $\Gamma^*_{\tilde{E}}$ acts on. By Lemma 5.5.2, S bounds a horospherical p-end neighborhood of $\tilde{E}$. Hence x^* is the vertex of a horospherical end. $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}(x) = x^*$ since $\Gamma_{\tilde{E}} \to \Gamma^*_{\tilde{E}}$ and these are unique fixed points.

An R-p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$ has a p-end vertex $\mathrm{v}_{\tilde{E}}$. $\tilde{\Sigma}_{\tilde{E}}$ is a properly convex domain in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$. The space of sharply supporting hyperspaces of $\tilde{\mathscr{O}}$ at $\mathrm{v}_{\tilde{E}}$ forms a properly convex domain of dimension $n-1$ since they correspond to hyperspaces in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ not intersecting $\tilde{\Sigma}_{\tilde{E}}$. Under the duality map $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$ in Proposition 1.5.4, $(\mathrm{v}_{\tilde{E}}, h)$ for a sharply supporting hyperspace h is sent to $(h^*, \mathrm{v}^*_{\tilde{E}})$ for a point h^* and a hyperspace $\mathrm{v}^*_{\tilde{E}}$. Lemma 1.5.7 shows that h^* is a point in a properly convex $n-1$-dimensional domain $D := \mathrm{bd}\tilde{\mathscr{O}}^* \cap P$ for $P = \mathrm{v}^*_{\tilde{E}}$, a hyperspace.

Corollary 5.5.1 implies the fact about $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$.

Since D is a properly convex domain with a Hilbert metric, $\pi_1(\tilde{E})$ acts properly on D^o. The n-orbifold $(\tilde{\mathscr{O}} \cup D^o)/\pi_1(\tilde{E})$ has closed-orbifold boundary $D^o/\pi_1(\tilde{E})$. There is a Riemannian metric on the n-orbifold so that $D^o/\pi_1(\tilde{E})$ is totally geodesic. Using the exponential map, we obtain a tubular neighborhood of $D^o/\pi_1(\tilde{E})$. Hence, $\tilde{\mathscr{O}}$ has a p-end neighborhood corresponding to $\pi_1(\tilde{E})$ containing D^o in the boundary. The dual group $\mathbf{\Gamma}^*_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition since $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the condition. By Theorem 4.4.1 and Lemma 5.5.2, we can find a p-end neighborhood U in $\tilde{\mathscr{O}}^*$ bounded by a strictly convex hypersurface $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ where $\mathrm{Cl}(\mathrm{bd}U \cap \tilde{\mathscr{O}}) - (\mathrm{bd}U \cap \tilde{\mathscr{O}}) \subset \partial D$.

By Lemma 5.5.2, $\tilde S_{\tilde E^*} \subset \mathrm{bd}\Omega^*$, and $\tilde E^*$ is a totally geodesic end with $\tilde S_{\tilde E^*}$ dual to $\tilde\Sigma_{\tilde E}$. This proves the third item.

The fourth item follows similarly. Take a T-p-end $\tilde E$. We take the ideal p-end boundary $\tilde\Sigma_{\tilde E}$. The map $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$ sends P to a singleton P^* in $\mathrm{bd}\tilde{\mathscr{O}}^*$ and points of $\mathrm{Cl}(\tilde S_{\tilde E}))$ go to hyperspaces supporting $\tilde{\mathscr{O}}^*$ at P^*. Since $\mathbf{\Gamma}^*_{\tilde E}$ satisfies the uniform middle eigenvalue condition with respect to P^*, Theorem 5.1.4 shows that there exists a lens-cone where $\mathbf{\Gamma}^*_{\tilde E}$ acts on. Also, $\mathbf{\Gamma}_{\tilde E}$ acts on a tube domain $\mathscr{T}_{P^*}(D^o)$ for a properly convex domain D^o. Then $\mathrm{Cl}(\tilde{\mathscr{O}}^*)\cap \mathrm{bd}\mathscr{T}_{P^*}(D^o)$ is a $\mathbf{\Gamma}^*_{\tilde E}$-invariant closed set. Also, this set is the image under $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$ of all hyperspaces supporting $\tilde{\mathscr{O}}$ at points of $\mathrm{Cl}(\tilde\Sigma_{\tilde E})$ by Corollary 5.5.1. Hence, $R_{P^*}(\tilde{\mathscr{O}}^*) = D^o$ by convexity. Since D^o is properly convex, $\mathbf{\Gamma}^*_{\tilde E}$ acts properly on it. By Lemma 3.1.5, P^* is a p-end vertex of a p-end neighborhood. There is a lens L so that $U_L := P^* * L - L$ is a $\mathbf{\Gamma}$-invariant. There is a boundary component $\partial_- L$ of this U_L in $\tilde{\mathscr{O}}^*$. By Lemma 5.5.2, this implies that U_L is a p-end neighborhood corresponding to $\mathbf{\Gamma}_{\tilde E}$. Corollary 5.5.1 implies the fact about $\mathscr{D}^{\mathrm{Ag}}_{\tilde{\mathscr{O}}}$.

The proof for $\mathbb{RP}^n$-version follows by Proposition 1.4.2. [$\mathbb{S}^n$T] □

REMARK 5.5.6. We also remark that the map induced on the limit points of p-end neighborhoods of Ω to that of Ω^* by $\bar{\mathscr{D}}^{\mathrm{Ag}}_{\Omega}$ is compatible with the Vinberg diffeomorphism by the continuity part of Theorem 1.5.9. That is the limit points of $\mathrm{bd}^{\mathrm{Ag}}\Omega$ of a p-end neighborhood of a p-end $\tilde E$ goes to the limit points of $\mathrm{bd}^{\mathrm{Ag}}\Omega^*$ a p-end neighborhood of a dual p-end $\tilde E^*$ of $\tilde E$ by $\bar{\mathscr{D}}^{\mathrm{Ag}}_{\Omega}$.

$\mathscr{C}$ restricts to a correspondence between the lens-shaped R-ends with lens-shaped T-ends. See Corollary 5.5.7 for detail.

Theorems 5.3.21 and 5.5.4 and Propositions 5.5.5 and 5.5.3 imply

COROLLARY 5.5.7. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold and let $\mathscr{O}^*$ be its dual orbifold. Then we can give the structure of R-ends and T-ends to ends of $\mathscr{O}$ and $\mathscr{O}^*$ so that dual end correspondence $\mathscr{C}$ restricts to a correspondence between the generalized lens-shaped R-ends with lens-shaped T-ends and horospherical ends to themselves. If $\mathscr{O}$ satisfies the triangle condition or every end is virtually factorizable, $\mathscr{C}$ restricts to a correspondence between the lens-shaped R-ends with lens-shaped T-ends and horospherical ends to themselves.*

COROLLARY 5.5.8. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold. Let $\tilde E$ be a lens-shaped p-end. Then for a lens-cone p-end neighborhood U of form $\{\mathrm{v}_{\tilde E}\} * L - \{\mathrm{v}_{\tilde E}\}$ for lens L, we have the upper boundary component A is tangent to radial rays from $\mathrm{v}_{\tilde E}$ at $\mathrm{bd}A$.*

PROOF. By Corollary 5.5.7, $\tilde E$ corresponds to a T-p-end $\tilde E^*$ of $\mathscr{O}^*$ of lens type. Now, we take the dual domain L_1' of $\{\mathrm{v}_{\tilde E}\} * L$ of $\tilde E^*$. The second part of Corollary 5.5.1 applied to L_1' gives us result for $\{\mathrm{v}_{\tilde E}\} * L$.

Proposition 1.4.2 finishes the proof. [$\mathbb{S}^n$T] □

CHAPTER 6

Application: The openness of the lens properties, and expansion and shrinking of end neighborhoods

This chapter lists applications of the main theory of Part 2, except for Chapter 7, which are results we need in Part 3. In Section 6.1, we show that the lens-shaped property is stable under the change of holonomy representations. In Section 6.2, we will define limits sets of ends and discuss the properties. We obtain the exhaustion of $\tilde{\mathscr{O}}$ by a sequence of p-end-neighborhoods of $\tilde{\mathscr{O}}$, we show that any end-neighborhood contains horospherical or concave end-neighborhood, and we discuss on maximal concave end-neighborhoods. In Section 6.3, Corollary 6.3.1 shows that the closures of p-end neighborhoods are disjoint in the closures of the universal cover in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$). We prove from this the strong irreducibility of $\mathscr{O}$, Theorem 6.0.4 under the conditions (IE) and (NA).

For results in this chapter, we don't necessarily assume that the holonomy group of $\pi_1(\mathscr{O})$ is strongly irreducible. Also, we will not explicitly mention Proposition 1.4.2 since its usage is well-established.

6.0.1. SPC-structures and its properties.

DEFINITION 6.0.1. For a strongly tame orbifold $\mathscr{O}$,

(IE) $\mathscr{O}$ or $\pi_1(\mathscr{O})$ satisfies *infinite end index condition IE* if $[\pi_1(\mathscr{O}) : \pi_1(E)] = \infty$ for the end fundamental group $\pi_1(E)$ of each end E.

(NA) $\mathscr{O}$ or $\pi_1(\mathscr{O})$ satisfies the *nonparallel end condition NA* if

$$\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$$

is finite for two distinct p-ends $\tilde{E}_1, \tilde{E}_2$ of $\mathscr{O}$.

(NA) implies that $\pi_1(E)$ contains every element $g \in \pi_1(\mathscr{O})$ normalizing $\langle h \rangle$ for an infinite order $h \in \pi_1(E)$ for an end fundamental group $\pi_1(E)$ of an end E. These conditions are satisfied by complete hyperbolic manifolds with cusps. These are group-theoretical properties with respect to the end groups.

DEFINITION 6.0.2 (Definition 6.2.3). An *SPC* on an n-orbifold is the structure of a properly convex real projective orbifold with a stable and irreducible holonomy group.

DEFINITION 6.0.3 (Definition 6.2.4). Suppose that $\mathscr{O}$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{\mathscr{O}} \subset \mathbb{RP}^n$ of the union U of some choice of a collection of disjoint end neighborhoods of $\mathscr{O}$. If every straight arc and every non-C^1-point in $\mathrm{bd}\tilde{\mathscr{O}}$ are contained in the closure of a component of $\tilde{U}$, then $\mathscr{O}$ is said to be *strictly convex* with respect to the collection of the ends. And $\mathscr{O}$ is also said to have a *sSPC* with respect to the collection of ends.

By a strongly tame orbifold with real projective structures *with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends*, we mean one with a real projective structure that has $\mathscr{R}$-type or $\mathscr{T}$-type assigned for each end and each $\mathscr{R}$-end is either generalized lens-shaped or horospherical and each $\mathscr{T}$-end is lens-shaped or horospherical.

Notice that the definition depends on the choice of U. However, we will show that if each end is required to be a lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-end, then we show that the definition is independent of U in Corollary 6.2.2.

We will prove the following in Section 6.3. The significance is that topological conditions imply the stability:

THEOREM 6.0.4. *Let $\mathscr{O}$ be a noncompact strongly tame properly convex real projective n-orbifold, $n \geq 2$, with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*). *Then the holonomy group is strongly irreducible and is not contained in a proper parabolic subgroup of* $\mathsf{PGL}(n+1,\mathbb{R})$ (*resp.* $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$). *That is, the holonomy is stable.*

6.1. The openness of lens properties.

As conditions on representations of $\pi_1(\tilde{E})$, the condition for generalized lens-shaped ends and one for lens-shaped ends are the same. Given a holonomy group of $\pi_1(\tilde{E})$ acting on a generalized lens-shaped cone p-end neighborhood, the holonomy group satisfies the uniform middle eigenvalue condition by Theorem 5.3.21. We can find a lens-cone by choosing our orbifold to be $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o/\pi_1(\tilde{E})$ by Proposition 5.3.14.

Let

$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}),\mathsf{SL}_{\pm}(n+1,\mathbb{R}))\,(\text{resp. } \mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}),\mathsf{PGL}(n+1,\mathbb{R})))$$

denote the space of representations of the fundamental group of an $(n-1)$-orbifold $\Sigma_{\tilde{E}}$.

Recall Definition 5.3.13 for strictly generalized lens-shaped R-ends. A (resp. *generalized*) *lens-shaped* representation for an R-end fundamental group is a representation acting on a (resp. generalized) lens-cone as a p-end neighborhood.

THEOREM 6.1.1. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold. Assume that the universal cover $\tilde{\mathscr{O}}$ is a subset of $\mathbb{S}^n$* (*resp.* $\mathbb{RP}^n$). *Let $\tilde{E}$ be a properly convex R-p-end of the universal cover $\tilde{\mathscr{O}}$. Then*

(i) *$\tilde{E}$ is a generalized lens-shaped R-end if and only if $\tilde{E}$ is a strictly generalized lens-shaped R-end.*

(ii) *The subspace of generalized lens-shaped representations of an R-end is open in*

$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}),\mathsf{SL}_{\pm}(n+1,\mathbb{R}))\,(\textit{resp. } \mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}),\mathsf{PGL}(n+1,\mathbb{R}))).$$

Finally, if $\mathscr{O}$ is properly convex and satisfies the triangle condition or $\tilde{E}$ is virtually factorizable, then we can replace the term "generalized lens-shaped" to "lens-shaped" in each of the above statements.

PROOF. We will assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ first. (i) If $\pi_1(\tilde{E})$ is non-virtually-factorizable, then the equivalence is given in Theorem 5.4.2 (i), and if $\pi_1(\tilde{E})$ is virtually factorizable, then it is in Theorem 5.4.3 (ii). The converse is obvious.

(ii) Let μ be a representation $\pi_1(\tilde{E}) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})$ associated with a generalized lens-cone. By Theorem 5.1.5, we obtain that $\pi_1(\tilde{E})$ satisfies the uniform middle eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$. By Theorem 5.1.4, we obtain a lens K in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ with smooth convex boundary components $A \cup B$ since $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ itself satisfies the triangle condition although it is not properly convex. (Note we don't need K to be in $\tilde{\mathscr{O}}$ for the proof.)

$K/\mu(\pi_1(\tilde{E}))$ is a compact orbifold whose boundary is the union of two closed n-orbifold components $A/\mu(\pi_1(\tilde{E})) \cup B/\mu(\pi_1(\tilde{E}))$. Suppose that μ' is sufficiently near μ. We may assume that $\mathrm{v}_{\tilde{E}}$ is fixed by conjugating μ' by a bounded projective transformation. By considering the radial segments in K, we obtain a foliation by radial lines in $\{\mathrm{v}_{\tilde{E}}\} * K$ also. By Proposition 5.3.11, applying Proposition 5.3.12 to the both boundary components of the lens, we obtain a lens-cone in a tube domain $\mathscr{T}'_{\mathrm{v}_{\tilde{E}}}$ in general different from the original one. This implies that the sufficiently small change of holonomy keeps $\tilde{E}$ to have a concave p-end neighborhood. This completes the proof of (ii).

The final statement follows by Lemma 5.3.20. [$\mathbb{S}^n$T] $\square$

THEOREM 6.1.2. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold. Assume that the universal cover $\tilde{\mathscr{O}}$ is a subset of $\mathbb{S}^n$ (resp. of $\mathbb{RP}^n$). Let $\tilde{E}$ be a T-p-end of the universal cover $\tilde{\mathscr{O}}$. Let $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$ (resp. $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R}))$) be the space of representations of the fundamental group of an $n-1$-orbifold $\Sigma_{\tilde{E}}$. Then the subspace of lens-shaped representations of a T-p-end is open.*

PROOF. By Theorem 5.5.4, the condition of the lens T-p-end is equivalent to the uniform middle eigenvalue condition for the end. Proposition 5.5.3 and Theorems 5.1.5 and 6.1.1 complete the proof. [$\mathbb{S}^n$T] $\square$

COROLLARY 6.1.3. *We are given a properly convex R- or T-end $\tilde{E}$ of a strongly tame convex orbifold $\mathscr{O}$. Assume that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\tilde{\mathscr{O}} \subset \mathbb{RP}^n$). Then the subset of*

$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))\ (\textit{resp.}\ \mathrm{Hom}_{\mathscr{E}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R})))$$

consisting of representations satisfying the uniform middle-eigenvalue condition with respect to some choices of fixed points or fixed hyperplanes of the holonomy group is open.

PROOF. For R-p-ends, this follows by Theorems 5.3.21 and 6.1.1. For T-p-ends, this follows by dual results: Theorem 5.5.4 and Theorems 6.1.2. [$\mathbb{S}^n$T] $\square$

6.2. The end and the limit sets.

DEFINITION 6.2.1.

- Define the *limit set* $\Lambda(\tilde{E})$ of an R-p-end $\tilde{E}$ with a generalized p-end-neighborhood to be $\mathrm{bd}D - \partial D$ for a generalized lens D of $\tilde{E}$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$). This is identical with the set $\Lambda_{\tilde{E}}$ in Definition 5.3.4 by Corollary 5.3.5.
- The *limit set* $\Lambda(\tilde{E})$ of a lens-shaped T-p-end $\tilde{E}$ to be $\mathrm{Cl}(\tilde{S}_{\tilde{E}}) - \tilde{S}_{\tilde{E}}$ for the ideal boundary component $\tilde{S}_{\tilde{E}}$ of $\tilde{E}$.
- The limit set of a horospherical end is the set of the end vertex.

The definition does depend on whether we work on $\mathbb{S}^n$ or $\mathbb{RP}^n$. However, by Proposition 1.4.2, there are always straightforward one-to-one correspondences. We remark that this may not equal to the closure of the union of the attracting fixed set for some cases.

COROLLARY 6.2.2. *Let $\mathscr{O}$ be a strongly tame convex real projective n-orbifold where $\tilde{\mathscr{O}} \subset \mathbb{S}^n$* (resp. $\subset \mathbb{RP}^n$). *Let U be a p-end-neighborhood of $\tilde{E}$ where $\tilde{E}$ is a lens-shaped T-p-end or a generalized lens-shaped or lens-shaped or horospherical R-p-end. Then $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ equals $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$ or $\mathrm{Cl}(\bigcup S(\mathrm{v}_{\tilde{E}}))$ or $\{\mathrm{v}_{\tilde{E}}\}$ depending on whether $\tilde{E}$ is a lens-shaped T-p-end or a generalized lens-shaped or horospherical R-p-end. Furthermore, this set is independent of the choice of U and so is the limit set $\Lambda(\tilde{E})$ of $\tilde{E}$.*

PROOF. We first assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let $\tilde{E}$ be a generalized lens-shaped R-p-end. Then by Theorem 5.3.21, $\tilde{E}$ satisfies the uniform middle eigenvalue condition. Suppose that $\pi_1(\tilde{E})$ is not virtually factorizable. Let L^b denote $\partial \mathscr{T}_{v_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}}) \cap L$ for a distanced compact convex set L where $\mathbf{\Gamma}_{\tilde{E}}$ acts on. We have $L^b = \Lambda(\tilde{E})$ by Proposition 5.3.10. Since $S(v_{\tilde{E}})$ is an $h(\pi_1(\tilde{E}))$-invariant set, and the convex hull of $\mathrm{bd}\bigcup S(v_{\tilde{E}})$ is a distanced compact convex set by the proper convexity of $\tilde{\Sigma}_{\tilde{E}}$, Theorems 5.4.2 and 5.4.3 show that the limit set is determined by the set L^b in $\bigcup S(v_{\tilde{E}})$, and $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}} = \bigcup S(v_{\tilde{E}})$.

Suppose now that $\pi_1(\tilde{E})$ is virtually factorizable. Then by Theorem 5.4.3, $\tilde{E}$ is a totally geodesic R-p-end. Proposition 5.3.10 and Theorem 5.4.3 again imply the result.

Let $\tilde{E}$ be a T-p-end. Theorems 5.5.4 and 4.4.1 imply

$$\mathrm{Cl}(A) - A \subset \mathrm{Cl}(\tilde{S}_{\tilde{E}}) \text{ for } A = \mathrm{bd}L \cap \tilde{\mathscr{O}}$$

for a CA-lens neighborhood L by the strictness of the lens. Thus, $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ equals $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$.

For horospherical ones, we simply use the definition to obtain the result. [$\mathbb{S}^n$T] □

DEFINITION 6.2.3. An *SPC-structure* or a *stable properly-convex real projective structure* on an n-orbifold is a convex real projective structure so that the orbifold has a stable irreducible holonomy group. That is, it is projectively diffeomorphic to a quotient orbifold of a properly convex domain in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) by a discrete group of projective automorphisms that is stable and irreducible.

DEFINITION 6.2.4. Suppose that $\mathscr{O}$ has an SPC-structure. Let $\tilde{U}$ be the inverse image in $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$) of the union U of some choice of a collection of mutually disjoint end neighborhoods of $\mathscr{O}$. If every straight arc in the boundary of the domain $\tilde{\mathscr{O}}$ and every non-C^1-point is contained in the closure of a component of $\tilde{U}$ for some choice of U, then $\mathscr{O}$ is said to be *strictly convex* with respect to the collection of the ends. And $\mathscr{O}$ is also said to have a *strict SPC-structure* with respect to the collection of ends.

Proposition 1.4.2 shows that this definition is equvalent to Definition 6.0.3. Corollary 6.2.5 shows the independence of the definition with respect to the choice of the end-neighborhoods when the ends are generalized lens-type $\mathscr{R}$-end or lens-shaped $\mathscr{T}$-ends. We conjecture that this holds also for the ends of four types given by Ballas-Cooper-Leitner [8].

COROLLARY 6.2.5. *Suppose that $\mathscr{O}$ is a strongly tame strictly SPC-orbifold with generalized lens-shaped R-ends or lens-shaped T-ends or horospherical ends. Let $\tilde{\mathscr{O}}$ is a properly convex domain in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) covering $\mathscr{O}$. Choose any disjoint collection of end neighborhoods in $\mathscr{O}$. Let U denote their union. Let $p_{\mathscr{O}} : \tilde{\mathscr{O}} \to \mathscr{O}$ denote the universal cover. Then any segment or a non-C^1-point of* $\mathrm{bd}\tilde{\mathscr{O}}$ *is contained in the closure of a component of $p_{\mathscr{O}}^{-1}(U)$ for any choice of U.*

PROOF. We first assume $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. By the definition of a strict SPC-orbifold, any segment or a non-C^1-point has to be in the closure of a p-end neighborhood. Corollary 6.2.2 proves the claim. [$\mathbb{S}^n$T] □

6.2.1. Convex hulls of ends. We will sharpen Corollary 6.2.2 and the convex hull part in Lemma 6.2.8. Again, these sets are all defined in $\mathbb{S}^n$ and we define the corresponding objects for $\mathbb{RP}^n$ by their images under $\mathbb{RP}^n$ by Proposition 1.4.2.

One can associate a *convex hull $I(\tilde{E})$ of a p-end $\tilde{E}$* of $\tilde{\mathscr{O}}$ as follows:

- For horospherical p-ends, the convex hull of each is defined to be the set of the end vertex actually.
- The convex hull of a lens-shaped totally geodesic p-end $\tilde{E}$ is the closure $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$ the totally geodesic ideal boundary component $\tilde{S}_{\tilde{E}}$ corresponding to $\tilde{E}$.
- For a generalized lens-shaped p-end $\tilde{E}$, the convex hull of $\tilde{E}$ is the convex hull of $\bigcup S(\mathrm{v}_{\tilde{E}})$ in $\mathrm{Cl}(\tilde{\mathscr{O}})$; that is,

$$I(\tilde{E}) := \mathscr{C}\mathscr{H}(\bigcup S(\mathrm{v}_{\tilde{E}})).$$

The first two equal $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ for any p-end neighborhood U of $\tilde{E}$ by Corollary 6.2.2. Corollary 6.2.2 and Proposition 6.2.7 imply that the convex hull of an end is well-defined. We can also characterize as the intersection

$$I(\tilde{E}) = \bigcap_{U_1 \in \mathscr{U}} \mathscr{C}\mathscr{H}(\mathrm{Cl}(U_1))$$

for the collection $\mathscr{U}$ of p-end neighborhoods U_1 of $\mathrm{v}_{\tilde{E}}$ by Proposition 6.2.7.

We define $\partial_S I(\tilde{E})$ as the set of endpoints of maximal rays from $\mathrm{v}_{\tilde{E}}$ ending at $\mathrm{bd}I(\tilde{E})$ and in the directions of $\tilde{\Sigma}_{\tilde{E}}$. It is homeomorphic to $\tilde{\Sigma}_{\tilde{E}}$ by the rays and has a compact quotient under $\Gamma_{\tilde{E}}$. Since the convex hull of $\bigcup S(\mathrm{v}_{\tilde{E}})$ is a subset of the tube with a vertex $\mathrm{v}_{\tilde{E}}$ in the directions of elements of $\mathrm{Cl}(\Pi_{\mathrm{v}_{\tilde{E}}}(R_{\mathrm{v}_{\tilde{E}}}(\Sigma_{\tilde{E}})))$, we obtain

$$\mathrm{bd}I(\tilde{E}) = \partial_S I(\tilde{E}) \cup \bigcup S(\mathrm{v}_{\tilde{E}}). \tag{6.2.1}$$

LEMMA 6.2.6. *If $\sigma \in S_i$ meets $\partial_S I(\tilde{E})$, then $\sigma^o \subset \partial_S I(\tilde{E})$ and the vertices of σ are endpoints of maximal segments in $S(\mathrm{v}_{\tilde{E}})$.*

PROOF. Suppose that $x \in \partial_S I(\tilde{E})$ and $x \in \sigma^o$ for a simplex $\sigma \in S_i$ for minimal i. The vertices are in $\bigcup S(\mathrm{v}_{\tilde{E}})$. If at least one vertex v_1 is in the interior point of a segment in $S(\mathrm{v}_{\tilde{E}})$, then by taking points in the neighborhood of x in $\bigcup S(\mathrm{v}_{\tilde{E}})$, we can deduce that σ^o is not in the boundary of the convex hull. Moreover, σ^o is a subset of $\partial I_{\tilde{E}}$ by Lemma 1.4.4. Hence, the vertices are endpoints of the maximal segments in $S(\mathrm{v}_{\tilde{E}})$.

Suppose that a point of σ^o is in a segment in $S(\mathrm{v}_{\tilde{E}})$. Then an interior point of $\Pi_{\mathrm{v}_{\tilde{E}}}(\sigma)$ meets the boundary of $\mathrm{Cl}(R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}}))$. By Lemma 1.4.4, $\Pi_{\mathrm{v}_{\tilde{E}}}(\sigma) \subset \mathrm{bd}(R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}}))$. Thus, σ is in a union of segments from $\mathrm{v}_{\tilde{E}}$ in the directions of $\mathrm{bd}(R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}}))$. By Theorems 5.4.2 and 5.4.3, such segments are contained in $\bigcup S(\mathrm{v}_{\tilde{E}})$. We obtain $\sigma \subset \bigcup S(\mathrm{v}_{\tilde{E}})$. This is a contradiction. By (6.2.1), $\sigma^o \subset \partial_S I_{\tilde{E}}$. □

A *topological orbifold* is one where we are allowed to use continuous maps as charts. We say that two very good topological orbifolds $\mathscr{O}_1$ and $\mathscr{O}_2$ are *homeomorphic* if there is a homeomorphism of the base spaces and that is induced from a homeomorphism $f : M_1 \to M_2$ of respective very good manifold covers M_1 and M_2 where f induces the isomorphism of the deck transformation group of $M_1 \to \mathscr{O}_1$ to that of $M_2 \to \mathscr{O}_2$.

PROPOSITION 6.2.7. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with radial ends or lens-shaped totally geodesic ends and satisfy* (IE) *and* (NA). *Let $\tilde{E}$ be a generalized lens-shaped R-p-end and $\mathrm{v}_{\tilde{E}}$ an associated p-end vertex. Let $I(\tilde{E})$ be the convex hull of $\tilde{E}$.*

(i) *Suppose that $\tilde{E}$ is a lens-shaped radial p-end. Then $\partial_S I(\tilde{E}) = \mathrm{bd}I(\tilde{E}) \cap \tilde{\mathscr{O}}$, and $\partial_S I(\tilde{E})$ is contained in the lens in a lens-shaped p-end neighborhood.*

(ii) *$I(\tilde{E})$ contains any concave p-end-neighborhood of $\tilde{E}$ and*

$$I(\tilde{E}) = \mathcal{CH}(\mathrm{Cl}(U))$$
$$I(\tilde{E}) \cap \tilde{\mathcal{O}} = \mathcal{CH}(\mathrm{Cl}(U)) \cap \tilde{\mathcal{O}}$$

for a concave p-end neighborhood U of $\tilde{E}$. Thus, $I(\tilde{E})$ has a nonempty interior.

(iii) *Each segment from $\mathrm{v}_{\tilde{E}}$ maximal in $\tilde{\mathcal{O}}$ meets the set $\partial_S I(\tilde{E})$ exactly once and $\partial_S I(\tilde{E})/\mathbf{\Gamma}_{\tilde{E}}$ is homeomorphic to Σ_E for very good covers.*

(iv) *There exists a nonempty interior of the convex hull $I(\tilde{E})$ of $\tilde{E}$ where $\mathbf{\Gamma}_{\tilde{E}}$ acts so that $I(\tilde{E})^o/\mathbf{\Gamma}_{\tilde{E}}$ is homeomorphic to the end orbifold times an interval.*

PROOF. Assume first that $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. (i) Suppose that $\tilde{E}$ is lens-shaped. We define S_1 as the set of 1-simplices with endpoints in segments in $\bigcup S(\mathrm{v}_{\tilde{E}})$ and we inductively define S_i to be the set of i-simplices with faces in S_{i-1}. Then

$$I(\tilde{E}) = \bigcup_{\sigma \in S_1 \cup S_2 \cup \cdots \cup S_m} \sigma.$$

Since any point of $\partial_S I(\tilde{E})$ is in some simplex σ, $\sigma \in S_i$, we obtain that $\partial_S I(\tilde{E})$ is the union

$$\bigcup_{\sigma \in S_1 \cup S_2 \cup \cdots \cup S_m, \sigma^o \subset \partial_S I(\tilde{E})} \sigma^o$$

by Lemma 6.2.6.

Suppose that $\sigma \in S_i$ with $\sigma^o \subset \partial_S I(\tilde{E})$. Then each of its vertices must be in an endpoint of a segment in $S(\mathrm{v}_{\tilde{E}})$ by Lemma 6.2.6. By Theorems 5.4.2 and 5.4.3, the endpoints of the segments in $S(\mathrm{v}_{\tilde{E}})$ are in $\Lambda(\tilde{E})$. Hence, σ^o is contained in a CA-lens-shaped domain L as the vertices of σ is in $\mathrm{bd}L - \partial L = \Lambda(\tilde{E})$ by the convexity of L.

Thus, each point of $\partial_S I(\tilde{E})$ is in $L^o \subset \tilde{\Sigma}$. Hence $\partial_S I(\tilde{E}) \subset \mathrm{bd}I(\tilde{E}) \cap \tilde{\mathcal{O}}$. Conversely, a point of $\partial I(\tilde{E}) \cap \tilde{\mathcal{O}}$ is an endpoint of a maximal segment in a direction of $\tilde{\Sigma}_{\tilde{E}}$. By (6.2.1), we obtain $\partial_S I(\tilde{E}) = \partial I(\tilde{E}) \cap \tilde{\mathcal{O}}$.

(ii) Since $I(\tilde{E})$ contains the segments in $S(\mathrm{v}_{\tilde{E}})$ and is convex, and so does a concave p-end neighborhood U, we obtain $\mathrm{bd}U \subset I(\tilde{E})$: Otherwise, let x be a point of $\mathrm{bd}U \cap \mathrm{bd}I(\tilde{E}) \cap \tilde{\mathcal{O}}$ where some neighborhood in $\mathrm{bd}U$ is not in $I(\tilde{E})$. Then since $\mathrm{bd}U$ is a union of a strictly convex hypersurface $\mathrm{bd}U \cap \tilde{\mathcal{O}}$ and $S(\mathrm{v}_{\tilde{E}})$, each sharply supporting hyperspace at x of the convex set $\mathrm{bd}U \cap \tilde{\mathcal{O}}$ meets a segment in $S(\mathrm{v}_{\tilde{E}})$ in its interior: consider the lens L so that one of the boundary components is $\mathrm{bd}U$. The supporting hyperspace at the boundary component cannot meet the closure of L in other points by the strict convexity.

This is a contradiction since x must be then in $I(\tilde{E})^o$. Thus, $U \subset I(\tilde{E})$. Thus,

$$\mathcal{CH}(\mathrm{Cl}(U)) \subset I(\tilde{E}).$$

Conversely, since $\mathrm{Cl}(U) \supset \bigcup S(\mathrm{v}_{\tilde{E}})$ by Theorems 5.4.2 and 5.4.3, we obtain that

$$\mathcal{CH}(\mathrm{Cl}(U)) \supset I(\tilde{E}).$$

(iii) We again use Proposition 1.1.19. It is sufficient to prove the result by taking a very good cover of $\mathcal{O}$ and considering the corresponding end to $\tilde{E}$. Each point of it meets a maximal segment from $\mathrm{v}_{\tilde{E}}$ in the end but not in $S(\mathrm{v}_{\tilde{E}})$ at exactly one point since a maximal segment must leave the lens cone eventually. Thus $\partial_S I(\tilde{E})$ is homeomorphic to an $(n-1)$-cell and the result follows.

(iv) This follows from (iii) since we can use rays from x meeting $\mathrm{bd}I(\tilde{E}) \cap \tilde{\mathcal{O}}$ at unique points and use them as leaves of a fibration. [$\mathbb{S}^n$P] $\square$

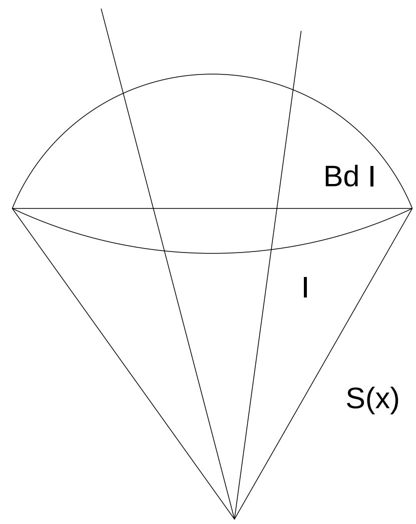

FIGURE 1. The structure of a lens-shaped p-end.

6.2.2. Expansion of lens or horospherical p-end-neighborhoods.

LEMMA 6.2.8. *Let $\mathscr{O}$ have a strongly tame properly convex real projective structure* μ.

- *Let U_1 be a p-end neighborhood of a horospherical or a lens-shaped R-p-end $\tilde{E}$ with the p-end vertex $\mathrm{v}_{\tilde{E}}$; or*
- *Let U_1 be a lens-shaped p-end neighborhood of a T-p-end $\tilde{E}$.*

Let $\mathbf{\Gamma}_{\tilde{E}}$ denote the p-end holonomy group corresponding to $\tilde{E}$. Then we can construct a sequence of lens-cone or lens p-end neighborhoods U_i, $i = 1, 2, \ldots$, satisfying $U_i \subset U_j \subset \tilde{\mathscr{O}}$ for every pair i, j, $i > j$ where the following hold:

- *Given a compact subset of $\tilde{\mathscr{O}}$, there exists an integer i_0 such that U_i for $i > i_0$ contains it.*
- *The Hausdorff distance between U_i and $\tilde{\mathscr{O}}$ can be made as small as possible, i.e.,*

$$\forall \varepsilon > 0, \exists J, J > 0, \text{ so that } \mathbf{d}_H(U_i, \tilde{\mathscr{O}}) < \varepsilon \text{ for } i > J.$$

- *There exists a sequence of convex open p-end neighborhoods U_i of $\tilde{E}$ in $\tilde{\mathscr{O}}$ so that $(U_i - U_j)/\mathbf{\Gamma}_{\tilde{E}}$ for a fixed j and $i > j$ is diffeomorphic to a product of an open interval with the end orbifold.*
- *We can choose U_i so that $\mathrm{bd}U_i \cap \tilde{\mathscr{O}}$ is smoothly embedded and strictly convex with $\mathrm{Cl}(\mathrm{bd}U_i) - \tilde{\mathscr{O}} \subset \Lambda(\tilde{E})$.*

PROOF. Suppose that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ first. Suppose that $\tilde{E}$ is a lens-shaped R-end first. Let U_1 be a lens-cone. Take a union of finitely many geodesic leaves L from $\mathrm{v}_{\tilde{E}}$ in $\tilde{\mathscr{O}}$ of $d_{\tilde{\mathscr{O}}}$-length t outside the lens-cone U_1 and take the convex hull of U_1 and $\mathbf{\Gamma}_{\tilde{E}}(L)$ in $\tilde{\mathscr{O}}$. Denote the result by Ω_t. Thus, the endpoints of L not equal to $\mathrm{v}_{\tilde{E}}$ are in $\tilde{\mathscr{O}}$.

We claim that

- $\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}}$ is a connected $(n-1)$-cell,
- $(\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}})/\mathbf{\Gamma}_{\tilde{E}}$ is a compact $(n-1)$-orbifold diffeomorphic to $\Sigma_{\tilde{E}}$, and

- $\mathrm{bd} U_1 \cap \tilde{\mathscr{O}}$ bounds a compact orbifold diffeomorphic to the product of a closed interval with $(\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}})/\mathbf{\Gamma}_{\tilde{E}}$:

First, each leaf of $g(l), g \in \mathbf{\Gamma}_{\tilde{E}}$ for l in L is so that any converging subsequence of $\{g_i(l)\}, g_i \in \mathbf{\Gamma}_{\tilde{E}}$, converges to a segment in $S(\mathrm{v}_{\tilde{E}})$ for a sequence $\{g_i\}$ of mutually distinct elements. This follows since a limit is a segment in $\mathrm{bd}\tilde{\mathscr{O}}$ with an endpoint $\mathrm{v}_{\tilde{E}}$ and must belong to $S(\mathrm{v}_{\tilde{E}})$ by Theorems 5.4.2 and 5.4.3.

Let S_1 be the set of segments with endpoints in $\mathbf{\Gamma}_{\tilde{E}}(L) \cup \bigcup S(\mathrm{v}_{\tilde{E}})$. We define inductively S_i to be the set of simplices with sides in S_{i-1}. Then the convex hull of $\mathbf{\Gamma}_{\tilde{E}}(L)$ in $\mathrm{Cl}(\tilde{\mathscr{O}})$ is a union of $S_1 \cup \cdots \cup S_m$.

We claim that for each maximal segment s in $\mathrm{Cl}(\tilde{\mathscr{O}})$ from $\mathrm{v}_{\tilde{E}}$ not in $S(\mathrm{v}_{\tilde{E}})$, s^o meets $\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}}$ at a unique point: Suppose not. Then let v' be its other endpoint of s in $\mathrm{bd}\tilde{\mathscr{O}}$ with $s^o \cap \mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}} = \emptyset$. Thus, $v' \in \mathrm{bd}\Omega_t$.

Now, v' is contained in the interior of a simplex σ in S_i for some i. Since $\sigma^o \cap \mathrm{bd}\tilde{\mathscr{O}} \neq \emptyset$, $\sigma \subset \mathrm{bd}\tilde{\mathscr{O}}$ by Lemma 1.4.4. Since the endpoints $\mathbf{\Gamma}_{\tilde{E}}(L)$ are in $\tilde{\mathscr{O}}$, the only possibility is that the vertices of σ are in $\bigcup S(\mathrm{v}_{\tilde{E}})$. Also, σ^o is transverse to radial rays since otherwise v' is not in $\mathrm{bd}\tilde{\mathscr{O}}$. Thus, σ^o projects to an open simplex of same dimension in $\tilde{\Sigma}_{\tilde{E}}$. Since U_1 is convex and contains $\bigcup S(\mathrm{v}_{\tilde{E}})$ in its boundary, σ is in the lens-cone $\mathrm{Cl}(U_1)$. Since a lens-cone has boundary a union of a strictly convex open hypersurface A and $\bigcup S(\mathrm{v}_{\tilde{E}})$, and σ^o cannot meet A tangentially, it follows that σ^o is in the interior of the lens-cone. and no interior point of σ is in $\mathrm{bd}\tilde{\mathscr{O}}$, a contradiction. Therefore, each maximal segment s from $\mathrm{v}_{\tilde{E}}$ meets the boundary $\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}}$ exactly once.

As in Lemma 5.3.16, $\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}}$ contains no segment ending in $\mathrm{bd}\tilde{\mathscr{O}}$. The strictness of convexity of $\mathrm{bd}\Omega_t \cap \tilde{\mathscr{O}}$ follows by smoothing as in the proof of Proposition 5.3.14. By taking sufficiently many leaves for L with sufficiently large $d_{\tilde{\mathscr{O}}}$-lengths t_i, we can show that any compact subset is inside Ω_t. Choose some sequence $\{t_i\}$ so that $\{t_i\} \to \infty$ as $i \to \infty$. Now, let $U_i := \Omega_{t_i}$. From this, the final item follows. The first three items now follow if $\tilde{E}$ is an R-end.

Suppose now that $\tilde{E}$ is horospherical and U_1 is a horospherical p-end neighborhood. We can smooth the boundary to be strictly convex. $\mathbf{\Gamma}_{\tilde{E}}$ is in a parabolic or cusp subgroup of a conjugate of $\mathsf{SO}(n,1)$ by Theorem 8.1.4. By taking L sufficiently densely, we can choose similarly to above a sequence Ω_i of polyhedral convex horospherical open sets at $\mathrm{v}_{\tilde{E}}$ so that eventually any compact subset of $\tilde{\mathscr{O}}$ is in it for sufficiently large i. Theorem 4.4.5 gives us a smooth strictly convex horospherical p-end neighborhood U_i.

Suppose now that $\tilde{E}$ is totally geodesic. Now we use the dual domain $\tilde{\mathscr{O}}^*$ and the group $\mathbf{\Gamma}^*_{\tilde{E}}$. Let $\mathrm{v}_{\tilde{E}^*}$ denote the vertex dual to the hyperspace containing $\tilde{S}_{\tilde{E}}$. By the diffeomorphism induced by great segments with the common endpoint $\mathrm{v}^*_{\tilde{E}}$, we obtain an orbifold homeomorphism

$$(\mathrm{bd}\tilde{\mathscr{O}}^* - \bigcup S(\mathrm{v}_{\tilde{E}^*}))/\mathbf{\Gamma}^*_{\tilde{E}} \cong \Sigma_{\tilde{E}^*}/\mathbf{\Gamma}^*_{\tilde{E}},$$

a compact orbifold. Then we obtain U_i containing $\tilde{\mathscr{O}}^*$ in $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})$ by taking finitely many hyperspace F_i disjoint from $\tilde{\mathscr{O}}^*$ but meeting $\mathscr{T}_{\mathrm{v}_{\tilde{E}}}(\tilde{\Sigma}_{\tilde{E}})^o$. Let H_i be the open hemisphere containing $\tilde{\mathscr{O}}^*$ bounded by F_i. Then we form $U_1 := \bigcap_{g \in \mathbf{\Gamma}_{\tilde{E}}} g(H_i)$. By taking more hyperspaces, we obtain a sequence

$$U_1 \supset U_2 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots \supset \tilde{\mathscr{O}}^*$$

so that $\mathrm{Cl}(U_{i+1}) \subset U_i$ and

$$\bigcap_i \mathrm{Cl}(U_i) = \mathrm{Cl}(\tilde{\mathscr{O}}^*).$$

That is, by using sufficiently many hyperspaces, we can make U_i disjoint from any compact subset disjoint from $\mathrm{Cl}(\tilde{\mathscr{O}}^*)$. Now taking the dual U_i^* of U_i and by equation (1.5.2) we obtain

$$U_1^* \subset U_2^* \subset \cdots \subset U_i^* \subset U_{i+1}^* \subset \cdots \subset \tilde{\mathscr{O}}.$$

Then $U_i^* \subset \tilde{\mathscr{O}}$ is an increasing sequence eventually containing all compact subset of $\tilde{\mathscr{O}}$ by duality from the above disjointness. This completes the proof for the first three items.

The fourth item follows from Corollary 6.2.2. $[\mathbb{S}^n\mathrm{P}]$ $\square$

6.2.3. Shrinking of lens and horospherical p-end-neighborhoods. We now discuss the "shrinking" of p-end-neighborhoods. These repeat some results.

COROLLARY 6.2.9. *Suppose that $\mathscr{O}$ is a strongly tame properly convex real projective orbifold and let $\tilde{\mathscr{O}}$ be a properly convex domain in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) covering $\mathscr{O}$. Then the following statements hold:*

(i) *If $\tilde{E}$ is a horospherical R-p-end, every p-end-neighborhood of $\tilde{E}$ contains a horospherical p-end-neighborhood.*
(ii) *Suppose that $\tilde{E}$ is a generalized lens-shaped or lens-shaped R-p-end. Let $I(\tilde{E})$ be the convex hull of $\bigcup S(\mathrm{v}_{\tilde{E}})$, and let V be a p-end-neighborhood V where $(\mathrm{bd}V \cap \tilde{\mathscr{O}})/\pi_1(\tilde{E})$ is a compact orbifold. If $V^o \supset I(\tilde{E}) \cap \tilde{\mathscr{O}}$, then V contains a lens-cone p-end neighborhood of $\tilde{E}$.*
(iii) *If $\tilde{E}$ is a generalized lens-shaped R-p-end or satisfies the uniform middle eigenvalue condition, every p-end-neighborhood of $\tilde{E}$ contains a concave p-end-neighborhood.*
(iv) *Suppose that $\tilde{E}$ is a lens-shaped T-p-end or satisfies the uniform middle eigenvalue condition. Then every p-end-neighborhood contains a lens p-end-neighborhood L with strictly convex boundary in $\tilde{\mathscr{O}}$.*
(v) *We can choose a collection of mutually disjoint end neighborhoods for all ends that are lens-shaped T-end neighborhood, concave R-end neighborhood or a hospherical ones.*

PROOF. Suppose that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ first.

(i) Let $\mathrm{v}_{\tilde{E}}$ denote the R-p-end vertex corresponding to $\tilde{E}$. By Theorem 8.1.4, we obtain a conjugate G of a subgroup of a parabolic or cusp subgroup of $\mathsf{SO}(n,1)$ as the finite index subgroup of $h(\pi_1(\tilde{E}))$ acting on U, a p-end-neighborhood of $\tilde{E}$. We can choose a G-invariant ellipsoid of $\mathbf{d}$-diameter $\leq \varepsilon$ for any $\varepsilon > 0$ in U containing $\mathrm{v}_{\tilde{E}}$.

(ii) This follows from Proposition 5.3.14 since the convex hull of $\bigcup S(\mathrm{v}_{\tilde{E}})$ contains a generalized lens with the right properties.

(iii) This was proved in Proposition 5.4.1.

(iv) The existence of a lens-shaped p-end neighborhood of $\tilde{S}_{\tilde{E}}$ follows from Theorem 4.4.1.

(v) We choose a mutually disjoint end neighborhoods for all ends. Then we choose the desired ones by the above. $[\mathbb{S}^n\mathrm{T}]$ $\square$

6.2.4. The mc-p-end neighborhoods. The mc-p-end neighborhood will be useful in other papers.

DEFINITION 6.2.10. Let $\tilde{E}$ be a lens-shaped R-end of a strongly tame convex projective orbifold $\mathscr{O}$ with the universal cover $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\mathbb{RP}^n$). Let $\mathscr{CH}(\Lambda(\tilde{E}))$ denote the convex hull of $\Lambda(\tilde{E})$. Let U' be any p-end neighborhood of $\tilde{E}$ containing $\mathscr{CH}(\Lambda(\tilde{E})) \cap \tilde{\mathscr{O}}$. We define a *maximal concave p-end neighborhood* or *mc-p-end-neighborhood* U to be one of the components of $U' - \mathscr{CH}(\Lambda(\tilde{E}))$ containing a p-end neighborhood of $\tilde{E}$. The *closed maximal concave p-end neighborhood* is $\mathrm{Cl}(U)\cap\tilde{\mathscr{O}}$. An ε-$d_{\tilde{\mathscr{O}}}$-neighborhood U'' of a maximal concave p-end neighborhood is called an *ε-mc-p-end-neighborhood*.

In fact, these are independent of choices of U'. Note that a maximal concave p-end neighborhood U is uniquely determined since so is $\Lambda(\tilde{E})$.

Each radial segment s in $\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}}$ meets $\mathrm{bd}U\cap\tilde{\mathscr{O}}$ at a unique point since the point $s\cap\mathrm{bd}U$ is in an $n-1$-dimensional ball $D = P\cap U$ for a hyperspace P sharply supporting $\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))$ with $\partial\mathrm{Cl}(D)\subset\bigcup S(\mathrm{v}_{\tilde{E}})$.

LEMMA 6.2.11. *Let D be an i-dimensional totally geodesic compact convex domain, $i\geq 1$. Let $\tilde{E}$ be a generalized lens-shaped R-p-end with the p-end vertex $\mathrm{v}_{\tilde{E}}$. Suppose $\partial D\subset\bigcup S(\mathrm{v}_{\tilde{E}})$. Then $D^o\subset V$ for a maximal concave p-end neighborhood V, and for sufficiently small $\varepsilon>0$, an ε-$d_{\tilde{\mathscr{O}}}$-neighborhood of D^o is contained in V' for any ε-mc-p-end neighborhood V'.*

PROOF. Suppose that $\tilde{\mathscr{O}}\subset\mathbb{S}^n$ first. Assume that U is a generalized lens-cone of $\mathrm{v}_{\tilde{E}}$. Then $\Lambda(\tilde{E})$ is the set of endpoints of segments in $S(\mathrm{v}_{\tilde{E}})$ which are not $\mathrm{v}_{\tilde{E}}$ by Theorems 5.4.2 and 5.4.3. Let P be the subspace spanned by $D\cup\{\mathrm{v}_{\tilde{E}}\}$. Since

$$\partial D, \Lambda(\tilde{E})\cap P\subset\bigcup S(\mathrm{v}_{\tilde{E}})\cap P,$$

and $\partial D\cap P$ is closer than $\Lambda(\tilde{E})\cap P$ from $\mathrm{v}_{\tilde{E}}$, it follows that $P\cap\mathrm{Cl}(U)-D$ has a component C_1 containing $\mathrm{v}_{\tilde{E}}$ and $\mathrm{Cl}(P\cap\mathrm{Cl}(U)-C_1)$ contains $\Lambda(\tilde{E})\cap P$. Hence

$$\mathrm{Cl}(P\cap\mathrm{Cl}(U)-C_1)\supset\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))\cap P$$

by the convexity of $\mathrm{Cl}(P\cap\mathrm{Cl}(U)-C_1)$. Since $\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))\cap P$ is a convex set in P, we have only two possibilities:

- D is disjoint from $\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))^o$ or
- D contains $\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))\cap P$.

Let V be an mc-p-end neighborhood of U. Since $\mathrm{Cl}(V)$ includes the closure of the component of $U-\mathscr{C}\mathscr{H}(\Lambda(\tilde{E}))$ with a limit point $\mathrm{v}_{\tilde{E}}$, it follows that $\mathrm{Cl}(V)$ includes D.

Since D is in $\mathrm{Cl}(V)$, the boundary $\mathrm{bd}V'\cap\tilde{\mathscr{O}}$ of the ε-mc-p-end neighborhood V' do not meet D. Hence $D^o\subset V'$. [$\mathbb{S}^n$T] □

The following gives us a characterization of ε-mc-p-end neighborhoods of $\tilde{E}$.

COROLLARY 6.2.12. *Let $\mathscr{O}$ be a properly convex real projective orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE). Let $\tilde{E}$ be a generalized lens-shaped R-end. Then the following statements hold*:

(i) *A concave p-end neighborhood of $\tilde{E}$ is always a subset of an mc-p-end-neighborhood of the same R-p-end.*
(ii) *The closed mc-p-end-neighborhood of $\tilde{E}$ is the closure in $\tilde{\mathscr{O}}$ of a union of all concave end neighborhoods of $\tilde{E}$.*
(iii) *The mc-p-end-neighborhood V of $\tilde{E}$ is a proper p-end neighborhood, and covers an end-neighborhood with compact boundary in $\mathscr{O}$.*
(iv) *An ε-mc-p-end-neighborhood of $\tilde{E}$ for sufficiently small $\varepsilon>0$ is a proper p-end neighborhood.*
(v) *For sufficiently small $\varepsilon>0$, the image end-neighborhoods in $\mathscr{O}$ of ε-mc-p-end neighborhoods of R-p-ends are mutually disjoint or identical.*

PROOF. Suppose first that $\tilde{\mathscr{O}}\subset\mathbb{S}^n$. (i) Since the limit set $\Lambda(\tilde{E})$ is in any generalized CA-lens L by Corollary 6.2.2, a generalized lens-cone p-end neighborhood U of $\tilde{E}$

contains $\mathscr{C}\mathscr{H}(\Lambda) \cap \tilde{\mathscr{O}}$. Hence, a concave end neighborhood is contained in an mc-p-end-neighborhood.

(ii) Let V be an mc-p-end neighborhood of $\tilde{E}$. Then define S to be the subspace of end-points in $\mathrm{Cl}(\tilde{\mathscr{O}})$ of maximal segments in V from $\mathrm{v}_{\tilde{E}}$ in directions of $\tilde{\Sigma}_{\tilde{E}}$. Then S is homeomorphic to $\tilde{\Sigma}_{\tilde{E}}$ by the map induced by radial segments as shown in the paragraph before. Thus, $S/\pi_1(\tilde{E})$ is a compact set since S is contractible and $\tilde{\Sigma}_{\tilde{E}}/\pi_1(\tilde{E})$ is a $K(\pi_1(\tilde{E}))$-space up to taking a torsion-free finite-index subgroup by Theorem 1.1.19 (Selberg's lemma). We can approximate S in the $d_{\tilde{\mathscr{O}}}$-sense by smooth convex boundary component S_ε outwards of a generalized CA-lens by Theorem 4.4.4 since $\tilde{E}$ satisfies the uniform middle-eigenvalue condition. A component $U - S_\varepsilon$ is a concave p-end neighborhood. (ii) follows from this.

(iii) Since a concave p-end neighborhood is a proper p-end neighborhood by Theorems 5.4.2(iv) and 5.4.3, and V is a union of concave p-end neighborhoods, we obtain

$$g(V) \cap V = \emptyset \text{ or } g(V) = V \text{ for } g \in \pi_1(\mathscr{O}) \text{ by (ii).}$$

Suppose that $g(\mathrm{Cl}(V) \cap \tilde{\mathscr{O}}) \cap \mathrm{Cl}(V) \neq \emptyset$. Then $g(V) = V$ and $g \in \pi_1(\tilde{E})$: Otherwise, $g(V) \cap V = \emptyset$, and $g(\mathrm{Cl}(V) \cap \tilde{\mathscr{O}})$ meets $\mathrm{Cl}(V)$ in a totally geodesic hypersurface S equal to $\mathscr{C}\mathscr{H}(\Lambda)^o$ by the concavity of V. Furthermore, for every $g \in \pi_1(\mathscr{O})$, $g(S) = S$, since S is a maximal totally geodesic hypersurface in $\tilde{\mathscr{O}}$. Hence, $g(V) \cup S \cup V = \tilde{\mathscr{O}}$ since these are subsets of a properly convex domain $\tilde{\mathscr{O}}$, the boundary of V and $g(V)$ are in S, and S is now in the interior of $\tilde{\mathscr{O}}$. Then $\pi_1(\mathscr{O})$ acts on S, and S/G is homotopy equivalent to $\tilde{\mathscr{O}}/G$ for a finite-index torsion-free subgroup G of $\pi_1(\mathscr{O})$ by Theorem 1.1.19 (Selberg's lemma). This contradicts the condition (IE).

Hence, only possibility is that $\mathrm{Cl}(V) \cap \tilde{\mathscr{O}} = V \cup S$ for a hypersurface S and

$$g(V \cup S) \cap V \cup S = \emptyset \text{ or } g(V \cup S) = V \cup S \text{ for } g \in \pi_1(\mathscr{O}).$$

Now suppose that $S \cap \mathrm{bd}\tilde{\mathscr{O}} \neq \emptyset$. Let S' be a maximal totally geodesic domain in $\mathrm{Cl}(V)$ containing S. Then $S' \subset \mathrm{bd}\tilde{\mathscr{O}}$ by convexity and Lemma 1.4.4, meaning that $S' = S \subset \mathrm{bd}\tilde{\mathscr{O}}$. In this case, $\tilde{\mathscr{O}}$ is a cone over S and the end vertex $\mathrm{v}_{\tilde{E}}$ of $\tilde{E}$. For each $g \in \pi_1(\mathscr{O})$,

$$g(V) \cap V \neq \emptyset \text{ implies } g(V) = V$$

since $g(\mathrm{v}_{\tilde{E}})$ is on $\mathrm{Cl}(S)$. Thus, $\pi_1(\mathscr{O}) = \pi_1(\tilde{E})$. This contradicts the condition (IE) of $\pi_1(\tilde{E})$.

We showed that $\mathrm{Cl}(V) \cap \tilde{\mathscr{O}} = V \cup S$ for a hypersurface S and covers a submanifold in $\mathscr{O}$ which is a closure of an end-neighborhood covered by V. Thus, an mc-p-end-neighborhood $\mathrm{Cl}(V) \cap \tilde{\mathscr{O}}$ is a proper end neighborhood of $\tilde{E}$ with compact embedded boundary $S/\pi_1(\tilde{E})$.

(iv) Obviously, we can choose positive ε so that an ε-mc-p-end-neighborhood is a proper p-end neighborhood also.

(v) For two mc-p-end neighborhoods U and V for different R-p-ends, we have $U \cap V = \emptyset$ by reasoning as in (iii) replacing $g(V)$ with U: We showed that $\mathrm{Cl}(V) \cap \tilde{\mathscr{O}}$ for an mc-p-end-neighborhood V covers an end neighborhood in $\mathscr{O}$.

Suppose that U is another mc-p-end neighborhood different from V. We claim that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) \cap \tilde{\mathscr{O}} = \emptyset$: Suppose not. $g(\mathrm{Cl}(V))$ for $g \notin \mathbf{\Gamma}_{\tilde{E}}$ must be a subset of U since otherwise we have a situation of (iii) for V and $g(V)$. Since the preimage of the end neighborhoods are disjoint, $g(V)$ is a p-end neighborhood of the same end as U. Since both are ε-mc-p-end-neighborhood which are canonically defined, we obtain $U = g(V)$. This was ruled out in (iii). $\square$

6.3. The strong irreducibility of the real projective orbifolds.

The main purpose of this section is to prove Theorem 6.0.4, the strong irreducibility result. But we will discuss the convex hull of the ends first. We show that the closure of convex hulls of p-end neighborhoods are disjoint in $\mathrm{bd}\tilde{\mathscr{O}}$. The infinity of the number of these will show the strong irreducibility.

For the following, we need a stronger condition of lens-shaped ends, and not just the generalized lens-shaped property, to obtain the disjointedness of the closures of p-end neighborhoods. For now, we cannot do the following for the generalized lens.

COROLLARY 6.3.1. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfy* (*NA*)*. Let $\mathscr{U}$ be the collection of the components of the inverse image in $\tilde{\mathscr{O}}$ of the union of disjoint collection of end neighborhoods of $\mathscr{O}$. Now replace each R-p-end neighborhood of collection $\mathscr{U}$ by a concave p-end neighborhood by Corollary 6.2.9* (iii)*. Then the following statements hold*:

(i) *Given horospherical, concave, or one-sided lens p-end-neighborhoods U_1 and U_2 contained in $\bigcup\mathscr{U}$, we have $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$.*
(ii) *Let U_1 and U_2 be in $\mathscr{U}$. Then $\mathrm{Cl}(U_1)\cap\mathrm{Cl}(U_2)\cap\mathrm{bd}\tilde{\mathscr{O}} = \emptyset$ or $U_1 = U_2$ holds.*

PROOF. Suppose first that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Suppose that $\mathscr{O}$ has end $E_1,\dots,E_m$. Since the neighborhoods in $\mathscr{U}$ are mutually disjoint,

- $\mathrm{Cl}(U_1'')\cap\mathrm{Cl}(U_2'')\cap\tilde{\mathscr{O}} = \emptyset$ or
- $U_1'' = U_2''$ for any pair $U_1'', U_2'' \in \mathscr{U}$.

(i) Assume without loss of generality that E_1 and E_2 are R-ends. Suppose that U_1 and U_2 are concave p-end neighborhoods of R-p-ends $\tilde{E}_1$ and $\tilde{E}_2$ respectively. Let U_1' be the interior of the associated generalized lens-cone of U_1 in $\mathrm{Cl}(\tilde{\mathscr{O}})$ and U_2' be that of U_2. Let U_i'' be the concave p-end-neighborhood of U_i' for $i = 1,2$ by Corollary 6.2.9 (iii) that cover respective end neighborhoods in $\mathscr{O}$.

Assume that $U_i'' \in \mathscr{U}$, $i = 1,2$, and $U_1'' \neq U_2''$. Suppose that the closures of U_1'' and U_2'' intersect in $\mathrm{bd}\tilde{\mathscr{O}}$. Suppose that they are both R-p-end neighborhoods. Then the respective closures of convex hulls I_1 and I_2 as obtained by Proposition 6.2.7 intersect as well. Take a point $z \in \mathrm{Cl}(U_1'')\cap\mathrm{Cl}(U_2'')\cap\mathrm{bd}\tilde{\mathscr{O}}$. Suppose that we choose $p_i = \mathrm{v}_{\tilde{E}_i}$, $i = 1,2$, for each p-end $\tilde{E}_i$.

Suppose that $\overline{p_1p_2}^o \subset \mathrm{bd}\tilde{\mathscr{O}}$. Then there exists a segment in $\mathrm{bd}\tilde{\mathscr{O}}$ from $\mathrm{v}_{\tilde{E}_1}$ to $\mathrm{v}_{\tilde{E}_2}$. By Theorems 5.4.2 and 5.4.3, these vertices are in the closures of p-end neighborhoods of one other. Since $\mathrm{Cl}(U_1'')$ and $\mathrm{Cl}(U_2'')$ contains some open metric ball neighborhoods of p_1 and p_2 respectively for the metric $\mathbf{d}$ restricted to $\mathrm{Cl}(\tilde{\mathscr{O}})$, and U_j'' is dense in $\mathrm{Cl}(U_j'')$, $j = 1,2$, we obtain $U_1''\cap U_2'' \neq \emptyset$. This is a contradiction.

Hence, $\overline{p_1p_2}^o \subset \tilde{\mathscr{O}}$ holds. Then $\overline{p_1z} \subset S(\mathrm{v}_{\tilde{E}_2})$ and $\overline{p_2z} \subset S(\mathrm{v}_{\tilde{E}_1})$ and these segments are maximal since otherwise $U_1''\cap U_2'' \neq \emptyset$. The segments intersect transversely at z since otherwise we violated the maximality in Theorems 5.4.2 and 5.4.3. We obtain a triangle $\triangle(p_1p_2z)$ in $\mathrm{Cl}(\tilde{\mathscr{O}})$ with vertices p_1, p_2, z.

There is a lens L_i for each $\tilde{E}_i$ so that $L_i * p_i$ is a p-end neighborhood of $\tilde{E}_i$. Proposition 5.5.8 contradicts the existence of above triangle because of the uniqueness of the supporting hyperspace at z which is in both of $\mathrm{Cl}(L_i) - L_i$ for $i = 1,2$.

Now, consider when U_1 is a one-sided lens-neighborhood of a T-p-end and let U_2 be a concave R-p-end neighborhood of an R-p-end of $\tilde{\mathscr{O}}$. Let z be the intersection point in $\mathrm{Cl}(U_1)\cap\mathrm{Cl}(U_2)$. Suppose that the hyperspace containing the ideal boundary of U_1 is transversal to the line $\overline{(z\vec{v}_{\tilde{E}_2})}$. We can use the same reasoning as above by choosing any p_1

in $\tilde\Sigma_{\tilde E_1}$ so that $\overline{p_1 z}$ passes the interior of $\tilde E_1$. Let p_2 be the R-p-end vertex of U_2. Now we obtain the triangle with vertices p_1, p_2, and z as above. Proposition 5.5.8 again contradicts.

Suppose that the hyperspace containing the ideal boundary of U_1 is tangent to the line $\overline(z\vec{v}_{\tilde E_2})$. We take the convex hull of z with the convex domain $\tilde\Sigma_{\tilde E_2}$ in the hyperspace S containing them. The convex hull is in $\mathrm{Cl}(\tilde{\mathcal{O}}) \cap S$ where $\pi_1(\tilde E_2)$ acts on. The uniqueness part of Theorem 1.4.15 shows that the interior of the convex hull is still $\tilde\Sigma_{\tilde E_2}$. This is a contradiction.

Next, consider when U_1 and U_2 are one-sided lens-neighborhoods of T-p-ends respectively. If the hyperspaces S_i containing the ideal boundary $\tilde\Sigma_{\tilde E_i}$ for $i = 1, 2$, are the same, then we can again take the convex hull as in the above paragraph and obtain the contradiction.

Suppose that these hyperspaces S_1 and S_2 are transversal. Let z be a point of intersection of $\mathrm{Cl}(\Omega_1)$ and $\mathrm{Cl}(\Omega_2)$. Then S_i must be an asymptotic hyperspace of Ω_j at z for $i \neq j, i, j = 1, 2$. clear?

We can see that $\mathrm{Cl}(\tilde{\mathcal{O}})$ meets S_i at $\mathrm{Cl}(\tilde\Sigma_{\tilde E_i})$ again by the uniqueness part of Theorem 1.4.15. Hence, $(\pi_1(\tilde E_i), \tilde\Sigma_{\tilde E_i}, \tilde{\mathcal{O}}^o)$is a properly convex triple.

Let H_i denote the open half-space containing $\tilde{\mathcal{O}}^o$ bounded by S_i. Let $B = H_1 \cap H_2$ be the convex domain containing $\tilde{\mathcal{O}}^o$.

Let $L = S_1 \cap S_2$. Then L is sharply supporting Ω_i at z in S_i. Let A_i denote the asymptotic hyperspace to U_i at z containing L. We use Lemma 4.2.11 as in the proof of Theorem 4.3.8 to show that $\tilde{\mathcal{O}}^o$ cannot intersect the supporting hyperplanes A_1 and A_2.

Since A_i is asymptotic to U_i, S_j, $j \neq i$, it follows that $S_i \cap B$ cannot separate B. Since S_i must meet the boundary of $\tilde{\mathcal{O}}$, the only possibility is $S_1 = A_2$ and $S_2 = A_1$. This proves our claim.

This means that $L_1^o \cap L_2^o \neq \emptyset$ by the asympototpicity of A_1 and A_2. Moreover for any choices of lens L_1' and L_2' their interiors meet. This contradicts that there is some p-end neighborhoods of lens type in any proper end-neighborhoods by Theorem 4.4.1.

We finally consider when U is a horospherical R-p-end. Since $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathcal{O}}$ is a unique point, Lemma 3.1.9. implies the result. $[\mathbb{S}^n\mathrm{P}]$ $\square$

We modify Theorem 5.4.3 by replacing some conditions. In particular, we don't assume that $h(\pi_1(\mathcal{O}))$ is strongly irreducible.

LEMMA 6.3.2. *Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold and satisfy* (*IE*)*. Let $\tilde E$ be a virtually factorizable R-p-end of $\tilde{\mathcal{O}}$ of generalized lens-shaped. Then*

- *there exists a totally geodesic hyperspace P on which $h(\pi_1(\tilde E))$ acts,*
- *$D := P \cap \tilde{\mathcal{O}}$ is a properly convex domain,*
- *$D^o \subset \tilde{\mathcal{O}}$, and*
- *$D^o/\pi_1(\tilde E)$ is a compact orbifold.*

PROOF. Assume first that $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. The proof of Theorem 5.4.3 shows that

- either $\mathrm{Cl}(\tilde{\mathcal{O}})$ is a strict join $\{\mathrm{v}_{\tilde E}\} * D'$ for a properly convex domain D in a hyperspace, or
- the conclusion of Theorem 5.4.3 holds.

In both cases, $\pi_1(\tilde E)$ acts on a totally geodesic convex compact domain D of codimension 1. D is the intersection $P_{\tilde E} \cap \mathrm{Cl}(\tilde{\mathcal{O}})$ for a $\pi_1(\tilde E)$-invariant subspace $P_{\tilde E}$. Suppose that D^o is not a subset of $\tilde{\mathcal{O}}$. Then by Lemma 1.4.4, $D \subset \mathrm{bd}\tilde{\mathcal{O}}$.

In the former case, $\mathrm{Cl}(\tilde{\mathscr{O}})$ is the join $\mathrm{v}_{\tilde{E}} * D$. For each $g \in \pi_1(\tilde{E})$ satisfying $g(\{\mathrm{v}_{\tilde{E}}\}) \neq \mathrm{v}_{\tilde{E}}$, we have $g(D) \neq D$ since $g(\mathrm{v}_{\tilde{E}}) * g(D) = \{\mathrm{v}_{\tilde{E}}\} * D$. The intersection $g(D) \cap D$ is a proper compact convex subset of D and $g(D)$. Moreover,

$$\mathrm{Cl}(\tilde{\mathscr{O}}) = \{\mathrm{v}_{\tilde{E}}\} * g(\{\mathrm{v}_{\tilde{E}}\}) * (D \cap g(D)).$$

We can continue as many times as there is a mutually distinct collection of vertices of form $g(\mathrm{v}_{\tilde{E}})$. Since this process must stop, we have a contradiction since by Condition (IE), there are infinitely many distinct end vertices of form $g(\mathrm{v}_{\tilde{E}})$ for $g \in \pi_1(\mathscr{O})$.

Now, we go to the alternative $D^o \subset \tilde{\mathscr{O}}$ where $D^o/\mathbf{\Gamma}_{\tilde{E}}$ is projectively diffeomorphic to $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$. [$\mathbb{S}^n$S] □

Proof of Theorem 6.0.4. It is sufficient to prove for $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let $h : \pi_1(\mathscr{O}) \to \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ be the holonomy homomorphism. Suppose that $h(\pi_1(\mathscr{O}))$ is virtually reducible. Then we can choose a finite cover $\mathscr{O}_1$ so that $h(\pi_1(\mathscr{O}_1))$ is reducible since it is sufficient to prove for the finite index groups.

By Theorem 5.4.3, all generalized lens $\mathscr{R}$-ends are lens $\mathscr{R}$-ends. We may assume that $\pi_1(\mathscr{O})$ is torsion-free by taking a finite cover by Theorem 1.1.19.

We denote $\mathscr{O}_1$ by $\mathscr{O}$ for simplicity. Let S denote a proper subspace where $\pi_1(\mathscr{O})$ acts on. Suppose that S meets $\tilde{\mathscr{O}}$. Then $\pi_1(\tilde{E})$ acts on a properly convex open domain $S \cap \tilde{\mathscr{O}}$ for each p-end $\tilde{E}$. Then $S \cap \tilde{\mathscr{O}}$ for any p-end neighborhood gives a submanifold of a closed end orbifold homotopy equivalent to it. Thus, $(S \cap \tilde{\mathscr{O}})/\pi_1(\tilde{E})$ is a compact orbifold homotopy equivalent to one of the end orbifold and S must be of codimension one. However, $S \cap \tilde{\mathscr{O}}$ is $\pi_1(\tilde{E}')$-invariant and cocompact for any other p-end $\tilde{E}'$. Hence, each p-end fundamental group $\pi_1(\tilde{E})$ is virtually identical to any other p-end fundamental group. This contradicts (NA). Therefore,

$$K := S \cap \mathrm{Cl}(\tilde{\mathscr{O}}) \subset \mathrm{bd}\tilde{\mathscr{O}}, \tag{6.3.1}$$

where $g(K) = K$ for every $g \in h(\pi_1(\mathscr{O}))$.

We divide into steps:

(A) First, we show $K \neq \emptyset$.
(B) We show $K = D_j$ or $K = \{\mathrm{v}_{\tilde{E}}\} * D_j$ for some properly convex domain $D_j \subset \mathrm{bd}\tilde{\mathscr{O}} \cap \mathrm{Cl}(U)$ for a p-end neighborhood U of $\tilde{E}$.
(C) Finally $g(D_j) = D_j$ for $g \in \mathbf{\Gamma}'$ for a finite index subgroup $\mathbf{\Gamma}'$ of $\mathbf{\Gamma}$, and we use Corollary 6.3.1 to obtain a contradiction.

(A) We show that $K := \mathrm{Cl}(\tilde{\mathscr{O}}) \cap S \neq \emptyset$: Let $\tilde{E}$ be a p-end. If $\tilde{E}$ is horospherical, $\pi(\tilde{E})$ acts on a great sphere $\hat{S}$ tangent to an end vertex. Since S is $\mathbf{\Gamma}$-invariant, S has to be a subspace in $\hat{S}$ containing the end vertex by Theorem 8.1.3(iii). This implies that every horospherical p-end vertex is in S. Let p be one. Since there is no nontrivial segment in $\mathrm{bd}\tilde{\mathscr{O}}$ containing p by Theorem 8.1.3(v), p equals $S \cap \mathrm{Cl}(\tilde{\mathscr{O}})$. Hence, p is $\mathbf{\Gamma}$-invariant and $\mathbf{\Gamma} = \mathbf{\Gamma}_{\tilde{E}}$. This contradicts the condition (IE).

Suppose that $\tilde{E}$ is a generalized lens-shaped R-p-end. Then by the existence of attracting subspaces of some elements of $\mathbf{\Gamma}_{\tilde{E}}$, we have

- either S passes the end vertex $\mathrm{v}_{\tilde{E}}$ or
- there exists a subspace S' containing S and $\mathrm{v}_{\tilde{E}}$ that is $\mathbf{\Gamma}_{\tilde{E}}$-invariant.

In the first case, we have $S \cap \mathrm{Cl}(\tilde{\mathscr{O}}) \neq \emptyset$, and we are done for the step (A).

In the second case, S' corresponds to a subspace in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ and S is a hyperspace of dimension $\leq n-1$ disjoint from $\mathrm{v}_{\tilde{E}}$. Thus, $\tilde{E}$ is a virtually factorizable R-p-end. By Theorem 1.3.12, we obtain some attracting fixed points in the limit sets of $\pi_1(\tilde{E})$. If S' is a proper subspace, then $\tilde{E}$ is factorizable, and S' contains the attracting fixed set of some positive bi-semi-proximal g, $g \in \mathbf{\Gamma}_E$. The uniform middle eigenvalue condition shows that positive bi-semi-proximal g has attracting fixed sets in $\mathrm{Cl}(L)$. Since g acts on S, we obtain $S \cap \mathrm{Cl}(L) \neq \emptyset$ by the uniform middle eigenvalue condition.

If S' is not a proper subspace, then g acts on S, and S contains the attracting fixed set of g by the uniform middle eigenvalue condition. Thus, $S \cap \mathrm{Cl}(L) \neq \emptyset$.

If $\tilde{E}$ is a lens-shaped T-p-end, we can apply a similar argument using the attracting fixed sets. Therefore, $S \cap \mathrm{Cl}(\tilde{\mathscr{O}})$ is a subset K of $\mathrm{bd}\tilde{\mathscr{O}}$ of $\dim K \geq 0$ and is not empty. In fact, we showed that the closure of each p-end neighborhood meets K.

(B) By taking a dual orbifold if necessary, we assume without loss of generality that there exists a generalized lens-shaped R-p-end $\tilde{E}$ with a radial p-end vertex $\mathrm{v}_{\tilde{E}}$.

As above in (A), suppose that $\mathrm{v}_{\tilde{E}} \in K$. There exists $g \in \pi_1(\mathscr{O})$, so that

$$g(\mathrm{v}_{\tilde{E}}) \neq \mathrm{v}_{\tilde{E}}, \text{ and } g(\mathrm{v}_{\tilde{E}}) \in K \subset \mathrm{bd}\tilde{\mathscr{O}}$$

since g acts on K. The point $g(\mathrm{v}_{\tilde{E}})$ is outside the closure of the concave p-end neighborhood of $\tilde{E}$ by Corollary 6.3.1. Since K is connected, K meets $\mathrm{Cl}(L)$ for the CA-lens or generalized CA-lens L of $\tilde{E}$.

If $\mathrm{v}_{\tilde{E}} \notin K$, then again $K \cap \mathrm{Cl}(L) \neq \emptyset$ as in (A) using attracting fixed sets of some elements of $\pi_1(\tilde{E})$. Hence, we conclude $K \cap \mathrm{Cl}(L) \neq \emptyset$ for a generalized CA-lens L of $\tilde{E}$.

Let $\Sigma'_{\tilde{E}}$ denote D^o from Lemma 6.3.2. Since $K \subset \mathrm{bd}\mathscr{O}$, it follows that K cannot contain $\Sigma'_{\tilde{E}}$. Thus, $K \cap \mathrm{Cl}(\Sigma'_{\tilde{E}})$ is a proper subspace of $\mathrm{Cl}(\Sigma'_{\tilde{E}})$, and $\tilde{E}$ must be a virtually factorizable end.

By Lemma 6.3.2, there exists a totally geodesic domain $\Sigma'_{\tilde{E}}$ in the CA-lens. A p-end neighborhood of $\mathrm{v}_{\tilde{E}}$ equals $U_{\mathrm{v}_{\tilde{E}}} := (\{\mathrm{v}_{\tilde{E}}\} * \Sigma'_{\tilde{E}})^o$. Since $\pi_1(\tilde{E})$ acts reducibly,

$$\mathrm{Cl}(\Sigma'_{\tilde{E}}) = D_1 * \cdots * D_m,$$

where $K \cap \mathrm{Cl}(U_{\mathrm{v}_{\tilde{E}}})$ contains a join $D_J := *_{i \in J} D_i$ for a proper subcollection J of $\{1, \dots, m\}$. Moreover, $K \cap \mathrm{Cl}(\Sigma'_{\tilde{E}}) = D_J$.

Since $g(U_{\mathrm{v}_{\tilde{E}}})$ is a p-end neighborhood of $g(\mathrm{v}_{\tilde{E}})$, we obtain $g(U_{\mathrm{v}_{\tilde{E}}}) = U_{g(\mathrm{v}_{\tilde{E}})}$. Since $g(K) = K$ for $g \in \Gamma$, we obtain that

$$K \cap g(\mathrm{Cl}(\Sigma'_{\tilde{E}})) = g(D_J).$$

Lemma 6.3.2 implies that

$$\begin{aligned} &U_{g(\mathrm{v}_{\tilde{E}})} \cap U_{\mathrm{v}_{\tilde{E}}} = \emptyset \text{ for } g \notin \pi_1(\tilde{E}) \text{ or} \\ &U_{g(\mathrm{v}_{\tilde{E}})} = U_{\mathrm{v}_{\tilde{E}}} \text{ for } g \in \pi_1(\tilde{E}) \end{aligned} \tag{6.3.2}$$

by the similar properties of $S(g(\mathrm{v}_{\tilde{E}}))$ and $S(\mathrm{v}_{\tilde{E}})$ and the fact that $\mathrm{bd}U_{\mathrm{v}_{\tilde{E}}} \cap \tilde{\mathscr{O}}$ and $\mathrm{bd}U_{g(\mathrm{v}_{\tilde{E}})} \cap \tilde{\mathscr{O}}$ are totally geodesic domains.

Let $\lambda_J(g)$ denote the $(\dim D_J + 1)$-th root of the norm of the determinant of the submatrix of g associated with D_J for the unit norm matrix of g. There exists a sequence of virtually central diagonalizable elements $\gamma_i \in \pi_1(\tilde{E})$ by Proposition 4.4 of [21] so that

$$\{\gamma_i | D_J\} \to \mathrm{I}, \{\gamma_i | D_{J^c}\} \to \mathrm{I} \text{ satisfying } \left\{ \frac{\lambda_J(\gamma_i)}{\lambda_{J^c}(\gamma_i)} \right\} \to \infty$$

for the complement $J^c := \{1,2,\dots,m\} - J$. Since the lens-shaped ends satisfy the uniform middle eigenvalue condition by Theorem 5.4.3, we obtain

$$\{\gamma_i|D_J\} \to \mathrm{I}, \{\gamma_i|D_{J^c}\} \to \mathrm{I} \text{ for the complement } J^c := \{1,2,\dots,m\} - J,$$
$$\left\{\frac{\lambda_J(\gamma_i)}{\lambda_{\mathrm{v}_{\tilde E}}(\gamma_i)}\right\} \to \infty, \left\{\frac{\lambda_{J^c}(\gamma_i)}{\lambda_{\mathrm{v}_{\tilde E}}(\gamma_i)}\right\} \to 0, \left\{\frac{\lambda_J(\gamma_i)}{\lambda_{J^c}(\gamma_i)}\right\} \to \infty. \tag{6.3.3}$$

(See Theorem 1.4.10.)

Since $\mathrm{v}_{\tilde E}, D_J \subset K$, the uniform middle eigenvalue condition for $\mathbf{\Gamma}_{\tilde E}$ implies that one of the following holds:

$$K = D_J, K = \{\mathrm{v}_{\tilde E}\} * D_J \text{ or } K = \{\mathrm{v}_{\tilde E}\} * D_J \cup \{\mathrm{v}_{\tilde E -}\} * D_J$$

by the invariance of K under γ_i^{-1} and the fact that $K \cap \mathrm{Cl}(\Sigma'_{\tilde E}) = D_J$. Since $K \subset \mathrm{Cl}(\tilde{\mathscr{O}})$, the third case is not possible by the proper convexity of $\mathrm{Cl}(\tilde{\mathscr{O}})$. We obtain

$$K = D_J \text{ or } K = \{\mathrm{v}_{\tilde E}\} * D_J. \tag{6.3.4}$$

(C) We will explore the two cases of (6.3.4). Assume the second case. Let g be an arbitrary element of $\pi_1(\mathscr{O}) - \pi_1(\tilde E)$. Since $D_J \subset K$, we obtain $g(D_J) \subset K$. Recall that $U_{\mathrm{v}_{\tilde E}} \cup S(\mathrm{v}_{\tilde E})^o$ is a neighborhood of points of $S(\mathrm{v}_{\tilde E})^o$ in $\mathrm{Cl}(\tilde{\mathscr{O}})$. Thus, $g(U_{\mathrm{v}_{\tilde E}} \cup S(\mathrm{v}_{\tilde E})^o)$ is a neighborhood of points of $g(S(\mathrm{v}_{\tilde E})^o)$.

Recall that D_J^o is in the closure of $U_{\mathrm{v}_{\tilde E}}$. If D_J^o meets

$$g(\{\mathrm{v}_{\tilde E}\} * D_J - D_J) \subset g(U_{\mathrm{v}_{\tilde E}} \cup S(\mathrm{v}_{\tilde E})^o) \supset g(S(\mathrm{v}_{\tilde E})^o),$$

then

$$U_{\mathrm{v}_{\tilde E}} \cap g(U_{\mathrm{v}_{\tilde E}}) \neq \emptyset, \text{ and } S(\mathrm{v}_{\tilde E})^o \cap g(S(\mathrm{v}_{\tilde E})^o) \neq \emptyset$$

since these are components of $\tilde{\mathscr{O}}$ with some totally geodesic hyperspaces removed. Hence, $\mathrm{v}_{\tilde E} = g(\mathrm{v}_{\tilde E})$ by Theorems 5.4.2 and 5.4.3. Finally, we obtain $D_J = g(D_J)$ since

$$K = \{\mathrm{v}_{\tilde E}\} * D_J = g(\{\mathrm{v}_{\tilde E}\}) * g(D_J).$$

If D_J^o is disjoint from $g(\{\mathrm{v}_{\tilde E}\} * D_J - D_J)$, then $g(D_J) \subset D_J$ since $K = \{\mathrm{v}_{\tilde E}\} * D_J$ and $g(K) = K$. Since D_J and $g(D_J)$ are intersections of a hyperspace with $\mathrm{bd}\tilde{\mathscr{O}}$, we obtain $g(D_J) = D_J$.

Both cases of (6.3.4) imply that $g(D_J) = D_J$ for $g \in \pi_1(\mathscr{O})$. This implies $g(D_J) = D_J$ for $g \in \pi_1(\mathscr{O})$. Since $\mathrm{v}_{\tilde E}$ and $g(\mathrm{v}_{\tilde E})$ are not equal for $g \in \pi_1(\mathscr{O}) - \pi_1(\tilde E)$, we obtain

$$\mathrm{Cl}(U_1) \cap g(\mathrm{Cl}(U_1)) \neq \emptyset.$$

Since every virtually reducible R-ends are totally geodesic, all R-ends are lens-type ones by Theorem 5.4.3. Corollary 6.3.1 gives us a contradiction. Therefore, we deduced that the $h(\pi_1(\mathscr{O}))$-invariant subspace S does not exist.

Since parabolic subgroups of $\mathsf{PGL}(n+1,\mathbb{R})$ or $\mathsf{SL}_\pm(n+1,\mathbb{R})$ are reducible, we are done. [$\mathbb{S}^n$S]

□

6.3.1. Equivalence of lens-ends and generalized lens-ends for strict SPC-orbifolds.

COROLLARY 6.3.3. *Suppose that $\mathscr{O}$ is a strongly tame strictly SPC-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfying the conditions (IE) and (NA). Then $\mathscr{O}$ satisfies the triangle condition and every generalized lens-shaped R-ends are lens-shaped R-ends.*

PROOF. Assume first $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let $\tilde{E}$ be a generalized lens-shaped p-end neighborhood of $\tilde{\mathscr{O}}$. Let L be the generalized CA-lens so that the interior U of $\{v_{\tilde{E}}\} * L$ is a lens p-end neighborhood. Then $U - L$ is a concave p-end neighborhood. Recall the triangle condition of Definition 5.3.18. Let T be a triangle with

$$\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}, T^o \subset \tilde{\mathscr{O}} \text{ and } \partial T \cap \mathrm{Cl}(U) \neq \emptyset$$

for an R-p-end neighborhood U. By the strict convexity $\tilde{\mathscr{O}}$, each edge of T has to be inside a set of form $\mathrm{Cl}(V) \cap \mathrm{bd}\tilde{\mathscr{O}}$ for a p-end neighborhood V. Corollary 6.3.1 implies that the edges are all in $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ for a single R-p-end neighborhood U. Hence, the triangle condition is satisfied. By Theorem 5.3.21, $\tilde{E}$ is a lens-shaped p-end. [$\mathbb{S}^n$T] □

CHAPTER 7

The convex but nonproperly convex and non-complete-affine radial ends

In previous chapters, we classified properly convex or complete radial ends under suitable conditions. In this chapter, we will study radial ends that are convex but not properly convex nor complete affine. The main techniques are the theory of Fried and Goldman on affine manifolds, and a generalization of the work on Riemannian foliations by Molino, Carrière, and so on. We will show that these are quasi-joins of horospheres and totally geodesic radial ends under transverse weak middle eigenvalue conditions. These are suitable deformations of joins of horospheres and totally geodesic radial ends. Since this is the most technical chapter, we will give outlines at some places in addition to the main outline in Section 7.1.3.

7.1. Introduction

7.1.1. General setting. In this chapter, we will work with $\mathbb{S}^n$ and $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ with only a few exceptions since the purpose is to classify some objects modulo projective automorphisms. However, the corresponding results for $\mathbb{RP}^n$ can be obtained easily by results in Section 1.1.8 and then projecting back to $\mathbb{RP}^n$. Let $\tilde{E}$ be a R-p-end of a convex real projective orbifold $\mathscr{O}$ with end orbifold $\Sigma_{\tilde{E}}$ and its universal cover $\tilde{\Sigma}_{\tilde{E}}$ and the p-end vertex $\mathrm{v}_{\tilde{E}}$. We recall Proposition 1.1.4. If $\tilde{\Sigma}_{\tilde{E}}$ is convex but not properly convex and not complete affine, then we call E a *nonproperly convex and noncomplete end* (NPNC-end.) The closure $\mathrm{Cl}(\tilde{\Sigma}_{\tilde{E}})$ contains a great (i_0-1)-dimensional sphere $\mathbb{S}^{i_0-1}_{\infty}$, and the convex open domain $\tilde{\Sigma}_{\tilde{E}}$ is foliated by i_0-dimensional hemispheres with this boundary $\mathbb{S}^{i_0-1}_{\infty}$. (These follow from Section 1.4 of [36]. See also [71].)

The space of i_0-dimensional hemispheres in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ with boundary $\mathbb{S}^{i_0-1}_{\infty}$ forms a projective sphere $\mathbb{S}^{n-i_0-1}$: This follows since a complementary subspace S isomorphic to $\mathbb{S}^{n-i_0-1}$ parameterize the space by the intersection points with S. The fibration with fibers open hemispheres of dimension i_0 with boundary $\mathbb{S}^{i_0-1}_{\infty}$

$$
\begin{array}{rcl}
\hat{\Pi}_K : \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}} - \mathbb{S}^{i_0-1}_{\infty} & \longrightarrow & \mathbb{S}^{n-i_0-1} \\
\uparrow & & \uparrow \\
\tilde{\Sigma}_{\tilde{E}} & \longrightarrow & K^o
\end{array}
\tag{7.1.1}
$$

gives us an image of $\tilde{\Sigma}_{\tilde{E}}$ that is the interior K^o of a properly convex compact set K.

Let $\mathbb{S}^{i_0}_{\infty}$ be a great i_0-dimensional sphere in $\mathbb{S}^n$ containing $\mathrm{v}_{\tilde{E}}$ corresponding to the directions of $\mathbb{S}^{i_0-1}_{\infty}$ from $\mathrm{v}_{\tilde{E}}$. The space of (i_0+1)-dimensional hemispheres in $\mathbb{S}^n$ with boundary $\mathbb{S}^{i_0}_{\infty}$ again has the structure of the projective sphere $\mathbb{S}^{n-i_0-1}$, identifiable with the above one.

Each i_0-dimensional hemisphere H^{i_0} in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$ with $\mathrm{bd} H^{i_0} = \mathbb{S}^{i_0-1}_\infty$ corresponds to an (i_0+1)-dimensional hemisphere H^{i_0+1} in $\mathbb{S}^n$ with common boundary $\mathbb{S}^{i_0}_\infty$ that contains $\mathrm{v}_{\tilde E}$.

There is also a fibration with fibers open hemispheres of dimension i_0+1 and boundary $\mathbb{S}^{i_0}_\infty$:

$$
\begin{array}{ccc}
\Pi_K : \mathbb{S}^n - \mathbb{S}^{i_0}_\infty & \longrightarrow & \mathbb{S}^{n-i_0-1} \\
\uparrow & & \uparrow \\
U & \longrightarrow & K^o
\end{array}
\tag{7.1.2}
$$

since $\mathbb{S}^{i_0-1}_\infty$ corresponds to $\mathbb{S}^{i_0}_\infty$ in the projection $\mathbb{S}^n - \{\mathrm{v}_{\tilde E}, \mathrm{v}_{\tilde E-}\} \to \mathbb{S}^{n-1}$.

Let $\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbb{S}^{i_0}_\infty, \mathrm{v}_{\tilde E}}$ denote the subgroup of $\mathbf{Aut}(\mathbb{S}^n)$ acting on $\mathbb{S}^{i_0}_\infty$ and $\mathrm{v}_{\tilde E}$. The projection Π_K induces a homomorphism

$$\Pi_K^* : \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbb{S}^{i_0}_\infty, \mathrm{v}_{\tilde E}} \to \mathsf{SL}_\pm(n-i_0,\mathbb{R}).$$

Suppose that $\mathbb{S}^{i_0}_\infty$ is $h(\pi_1(\tilde E))$-invariant. We let N be the subgroup of $h(\pi_1(\tilde E))$ of elements inducing trivial actions on $\mathbb{S}^{n-i_0-1}$. The above exact sequence

$$1 \to N \to h(\pi_1(\tilde E)) \xrightarrow{\Pi_K^*} N_K \to 1 \tag{7.1.3}$$

is so that the kernel normal subgroup N acts trivially on $\mathbb{S}^{n-i_0-1}$ but acts on each hemisphere with boundary equal to $\mathbb{S}^{i_0}_\infty$ and N_K acts faithfully by the action induced from Π_K^*.

Since K is a properly convex domain, K^o admits a Hilbert metric d_K and $\mathbf{Aut}(K)$ is a subgroup of isometries of K^o. Here N_K is a subgroup of the group $\mathbf{Aut}(K)$ of the group of projective automorphisms of K, and N_K is called the *properly convex quotient* of $h(\pi_1(\tilde E))$ or $\mathbf{\Gamma}_{\tilde E}$.

We showed:

THEOREM 7.1.1. *Let $\Sigma_{\tilde E}$ be the end orbifold of an NPNC R-p-end $\tilde E$ of a strongly tame properly convex n-orbifold $\mathscr{O}$ with radial or totally geodesic ends. Let $\tilde{\mathscr{O}}$ be the universal cover in $\mathbb{S}^n$. We consider the induced action of $h(\pi_1(\tilde E))$ on $\mathbf{Aut}(\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}})$ for the corresponding end vertex $\mathrm{v}_{\tilde E}$. Then the following hold:*

- *$\tilde\Sigma_{\tilde E}$ is foliated by complete affine subspaces of dimension i_0, $i_0 > 0$. Let K be the properly convex compact domain of dimension $n-i_0-1$ whose interior is the space of complete affine subspaces of dimension i_0.*
- *$h(\pi_1(\tilde E))$ fixes the great sphere $\mathbb{S}^{i_0-1}_\infty$ of dimension i_0-1 in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde E}}$.*
- *There exists an exact sequence*

$$1 \to N \to \pi_1(\tilde E) \xrightarrow{\Pi_K^*} N_K \to 1$$

where N acts trivially on quotient great sphere $\mathbb{S}^{n-i_0-1}$ and N_K acts faithfully on a properly convex domain K^o in $\mathbb{S}^{n-i_0-1}$ isometrically with respect to the Hilbert metric d_K.

□

7.1.2. Main results. We begin with a definition of quasi-joined R-ends.

DEFINITION 7.1.2. We have the following:

- Let $\hat K$ be a compact properly convex subset of dimension
- $n-i-2$ in $\mathbb{S}^n$ for $i \geq 1$. Let $\mathbb{S}^i_\infty$ be a subspace of dimension i complementary to the subspace containing $\hat K$.

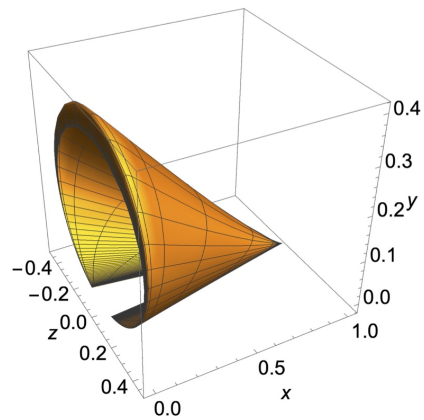

FIGURE 1. A partial development of the boundary of a quasi-joined R-p-end-neighborhood in an affine patch with Euclidean coordinates where $\langle\langle 0,0,0,1\rangle\rangle$ in $\mathbb{S}^3$ corresponds to $(0,0,0)$. In the notation of Definition 7.1.2, v is the point $(0,0,0)$, $\hat{K}$ is the singleton of $(1,0,0)$, H_x contains the upper-part of the yz-plane, and $\mathbb{S}^1_\infty$ contains the y-axis. A cusp group acts on each hyperspace containing the y-axis. H_x and $\mathbb{S}^1_\infty$ and these hyperspaces are not in the affine space of the projection. (See the mathematica file [40])

- Let v be a point in $\mathbb{S}^i_\infty$.
- A group G acts on $\hat{K}$, $\mathbb{S}^i_\infty$, and v and on an open set U containing v in the boundary.
- U/G is required to be diffeomorphic to a compact orbifold times an interval.
- There is a fibration $\Pi_K : \mathbb{S}^n - \mathbb{S}^i_\infty \to \mathbb{S}^{n-i-1}$ with fibers equal to open $(i+1)$-hemispheres with boundary $\mathbb{S}^i_\infty$.
  - The set of fibering open $(i+1)$-hemispheres H so that $H\cap U$ is a nonempty open set is projected to a convex open set in $\mathbb{S}^{n-i-1}$ onto the interior of $\{x\} * \Pi_K(\hat{K})$ in $\mathbb{S}^{n-i-1}$ for a point x not in $\Pi_K(\hat{K})$.
  - For each fibering open hemisphere H, $H\cap U$ is an open set bounded by an ellipsoid containing v unless $H\cap U$ is empty.
  - $\mathrm{Cl}(H_x)\cap \mathrm{Cl}(U) = \{\mathrm{v}\}$ for H_x the fibering open $(i+1)$-hemisphere over x.

If an R-end E of a real projective orbifold has an end neighborhood projectively diffeomorphic to U/G with the induced radial foliation corresponding to v, then E is called a *quasi-joined end* (*of a totally geodesic R-end and a horospherical end with respect to* v) and a corresponding R-p-end is said to be a *quasi-joined R-p-end* also. Also, any R-end with an end-neighborhood covered by an end-neighborhood of a quasi-joined R-end is called by the same name. In these cases, the end holonomy group is a *quasi-joined end group* (*of a totally geodesic R-end and a horospherical end with respect to* v)

We will see the example later. In this chapter, we will characterize the NPNC-ends. See Proposition 7.3.19 and Remark 7.3.20 for detailed understanding of quasi-joined ends.

Let $\tilde{E}$ be an NPNC-end. Recall from Chapter 3 that the universal cover $\tilde{\Sigma}_{\tilde{E}}$ of the end orbifold $\Sigma_{\tilde{E}}$ is foliated by complete affine i_0-dimensional totally geodesic leaves for $i_0 > 1$.

The end fundamental group $\pi_1(\tilde{E})$ acts on a properly convex domain K that is the space of i_0-dimensional totally geodesic hemispheres foliating $\tilde{\Sigma}_{\tilde{E}}$.

DEFINITION 7.1.3. A countable group G satisfies the property *NS* if every normal solvable subgroup N of a finite-index subgroup G' is virtually central in G'; that is, $N \cap G''$ is central in G'' for a finite-index subgroup G'' of G'.

By Corollary 1.4.11, the fundamental group of a closed orbifold admitting a properly convex structure has the property (NS). Clearly a virtually abelian group satisfies (NS). Obviously, the groups of Benoist are somewhat related to this condition. (see Proposition 1.4.8.)

The main result of this chapter is the following. We need the proper convexity of $\tilde{\mathscr{O}}$. The following theorem shows that given that $\tilde{\Sigma}_{\tilde{E}}$ is convex but not properly convex, the transverse weak middle eigenvalue condition implies the end is quasi-joined type. For example, this implies that the holonomy group decomposes into semi-simple part and horospherical part. (see (7.3.52)). This type is much easier to understand. Section 7.3.19 gives some detailed discussions.

THEOREM 7.1.4. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of $\mathscr{O}$ is strongly irreducible. Let $\tilde{E}$ be an NPNC R-p-end with the end orbifold $\Sigma_{\tilde{E}}$. The universal cover $\tilde{\Sigma}_{\tilde{E}}$ is foliated by i_0-dimensional totally geodesic hemispheres. The leaf space naturally identifies with the interior of a compact convex set K. Suppose that the following hold*:

- *the end fundamental group satisfies the property* (*NS*) *or* $\dim K^o = 0, 1$ *for the leaf space K^o of $\tilde{E}$.*
- *the p-end holonomy group $h(\pi_1(\tilde{E}))$ virtually satisfies the transverse weak middle-eigenvalue condition with respect to a p-end vertex $\mathrm{v}_{\tilde{E}}$.*

Then $\tilde{E}$ is a quasi-joined type R-p-end for $\mathrm{v}_{\tilde{E}}$.

See Definition 7.2.2 for the transverse weak middle-eigenvalue condition for NPNC-ends. Without this condition, we doubt that we can obtain this type of results. However, it is open to investigations. In this case, $\tilde{E}$ does not satisfy the uniform middle-eigenvalue condition as stated in Chapter 3 for properly convex ends.

We again remark that Cooper and Leitner have classified the ends when the end fundamental group is abelian. (See Leitner [**119**], [**118**] and [**120**].) Also, Ballas [**5**] and [**4**] has found some examples of quasi-joined ends when the upper-left parts are diagonal groups. Our quasi-joined ends are also classified by [**8**] when the holonomy group is nilpotent. (For virtually abelian groups, Ballas-Cooper-Leitner [**8**], [**9**] had covered much of these material but not in our generality.)

Recall the dual orbifold $\mathscr{O}^*$ given a properly convex real projective orbifold $\mathscr{O}$. (See Section 5.5.2.) The set of ends of $\mathscr{O}$ is in a one-to-one correspondence with the set of ends of $\mathscr{O}^*$. We show that a dual of a quasi-joined NPNC R-p-end is a quasi-joined NPNC R-p-end.

COROLLARY 7.1.5. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with radial or totally geodesic ends. Assume that the holonomy group of $\mathscr{O}$ is strongly irreducible. Let $\tilde{E}$ be a quasi-joined NPNC R-p-end for an end E of $\mathscr{O}$ virtually satisfying the transverse weak middle-eigenvalue condition with respect to the p-end vertex. Suppose that the end fundamental group satisfies the property* (*NS*) *or* $\dim K^o = 1$ *for the leaf space K^o of $\tilde{E}$.*

Let $\mathscr{O}^*$ denote the dual real projective orbifold of $\mathscr{O}$. Let $\tilde{E}^*$ be a p-end corresponding to a dual end of E. Then $\tilde{E}^*$ has a p-end neighborhood of a quasi-joined type R-p-end for the universal cover of $\mathscr{O}^*$ for a unique choice of a p-end vertex.

In short, we are saying that $\tilde{E}^*$ can be considered a quasi-joined type R-p-end by choosing its p-end vertex well. However, this does involve artificially introducing a radial foliation structure in an end neighborhood. We mention that the choice of the p-end vertex is uniquely determined for $\tilde{E}^*$ to be quasi-joined.

7.1.3. Outline. In Section 7.2, we discuss the R-ends that are NPNC. We introduce the transverse week middle eigenvalue condition. We will explain the main eigenvalue estimates following from the transverse weak middle eigenvalue condition for NPNC-ends. Then we will explain our plan to prove Theorem 7.1.4.

In Section 7.3, we introduce the example of the joining of horospherical and totally geodesic R-ends. We will now study a bit more general situation introducing Hypothesis 7.3.4. We will try to obtain the splitting under some hypothesis. We will outline the subsection there. By computations involving the normalization conditions, we show that the above exact sequence is virtually split under the condition (NS), and we can surprisingly show that the R-p-ends are of strictly joined or quasi-joined types. Then we show using the irreducibility of the holonomy group of $\pi_1(\mathscr{O})$ that they can only be of quasi-joined type. We divide the tasks:

- In Section 7.3.2, we introduce Hypothesis 7.3.4 under which we work. We show that K has to be a cone, and the conjugation action on $\mathscr{N}$ has to be scalar orthogonal type changes. Finally, we show the splitting of the NPNC-ends. We will outline more completely in there.
- In Section 7.3.3, we introduce Hypothesis 7.3.15, requiring more than Hypothesis 7.3.4. We will prove Proposition 7.3.19 that the ends are quasi-joins under the hypothesis.

As a final part of this section in Section 7.3.4, we discuss the case when N_K is discrete. We prove Theorem 7.1.4 for this case by showing that the above two hypotheses are satisfied.

In Section 7.4, we discuss when N_K is not discrete. There is a foliation by complete affine subspaces as above. We use some estimates on eigenvalues to show that each leaf is of polynomial growth. The leaf closures are suborbifolds V_l by the theory of Carrière [**35**] and Molino [**131**] on Riemannian foliations. They form the fibration with compact fibers. $\pi_1(V_l)$ is solvable using the work of Carrière [**35**]. One can then take the syndetic closure to obtain a bigger group that acts transitively on each leaf following Fried and Goldman [**82**]. We find a unipotent cusp group acting on each leaf transitively normalized by $\mathbf{\Gamma}_{\tilde{E}}$. Then we show that the end also splits virtually using the theory of Section 7.3. This proves Theorem 7.1.4 for this case.

In Section 7.5, we prove Corollary 7.1.5 showing that the duals of NPNC-ends are NPNS-ends, and in Section 8.1.3, we classify complete ends that are not cusps. This was needed in the proof of Theorem 8.1.2.

In Section 8.2, we will discuss some miscellaneous results. In Section 7.5.2, we discuss a counterexample to Theorem 7.3.22 when the condition (NS) is dropped.

7.1.3.1. *The plan of the proof of Theorem 7.1.4.* We will show that our NPNC-ends are quasi-joined type ones; i.e., we prove Theorem 7.1.4 by proving for discrete N_K in Section 7.3.4 in Section 7.3 and proving for nondiscrete N_K in Section 7.4.3 in Section 7.4.

- Assume Hypotheses 7.3.4.

  - We show that $\mathbf{\Gamma}_{\tilde{E}}$ acts as $\mathbb{R}_+$ times an orthogonal group on a Lie group $\mathscr{N}$ as realized as an real unipotent abelian group $\mathbb{R}^{i_0}$. See Lemmas 7.3.7. This is done by computations and coordinate change arguments and the distal group theory of Fried [80].
  - We show that K is a cone in Lemma 7.3.9.
  - We refine the matrix forms in Lemma 7.3.10 when $\mu_g = 1$. Here the matrices are in almost desired forms.
  - Proposition 7.3.14 shows the splitting of the representation of $\mathbf{\Gamma}_{\tilde{E}}$. One uses the transverse weak middle eigenvalue condition to realize the compact $(n-i_0-1)$-dimensional totally geodesic domain independent of $\mathbb{S}^{i_0}_\infty$ where $\mathbf{\Gamma}_{\tilde{E}}$ acts on.
- Now we can assume Hypothesis 7.3.15 additionally. In Section 7.3.3, we discuss joins and quasi-joins. The idea is to show that the join cannot occur by Propositions 1.4.18 and 1.4.19.
- This will settle the cases of discrete N_K in Theorem 7.3.22 in Section 7.3.4.
- In Section 7.4, we will settle for the cases of non-discrete N_K. See above for the outline of this section.

We remark that we can often take a finite index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$ during our proofs since Definition 7.3.20 is a definition given up to finite index subgroups.

REMARK 7.1.6. The main result of this chapter Theorem 7.1.4 and Corollary 7.1.5 are stated without references to $\mathbb{S}^n$ or $\mathbb{RP}^n$. We will work in the space $\mathbb{S}^n$ only. Often the result for $\mathbb{S}^n$ implies the result for $\mathbb{RP}^n$. In this case, we only prove for $\mathbb{S}^n$. In other cases, we can easily modify the $\mathbb{S}^n$-version proof to one for the $\mathbb{RP}^n$-version proof.

7.2. The transverse weak middle eigenvalue conditions for NPNC ends

We will now study the ends where the transverse real projective structures are not properly convex but not projectively diffeomorphic to a complete affine subspace. Let $\tilde{E}$ be an R-p-end of $\mathscr{O}$, and let U be the corresponding p-end-neighborhood in $\tilde{\mathscr{O}}$ with the p-end vertex $\mathrm{v}_{\tilde{E}}$. Let $\tilde{\Sigma}_{\tilde{E}}$ denote the universal cover of the p-end orbifold $\Sigma_{\tilde{E}}$ as a domain in $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$.

In Section 7.1.1, we will discuss the general setting that the NPNC-ends satisfy. In Section 7.1.3.1, we will give a plan to prove Theorem 7.1.4. We accomplished this proof in Sections 7.3 and 7.4.

We denote by $\mathscr{F}_{\tilde{E}}$ the foliation on $\tilde{\Sigma}_{\tilde{E}}$ or the corresponding one in $\Sigma_{\tilde{E}}$.

7.2.0.1. *The main eigenvalue estimations.* We denote by $\mathbf{\Gamma}_{\tilde{E}}$ the p-end holonomy group acting on U fixing $\mathrm{v}_{\tilde{E}}$. Denote the induced foliations on $\Sigma_{\tilde{E}}$ and $\tilde{\Sigma}_{\tilde{E}}$ by $\mathscr{F}_{\tilde{E}}$. We recall

$$\mathrm{length}_K(g) := \inf\{d_K(x, g(x)) | x \in K^o\}, g \in \mathbf{\Gamma}_{\tilde{E}}.$$

We recall Definition 1.3.1. Let $V^{i_0+1}_\infty$ denote the subspace of $\mathbb{R}^{n+1}$ corresponding to $\mathbb{S}^{i_0}_\infty$. By invariance of $\mathbb{S}^{i_0}_\infty$, if

$$\mathscr{R}_\mu(g) \cap V^{i_0+1}_\infty \neq \{0\}, \mu > 0,$$

then $\mathscr{R}_\mu(g) \cap V^{i_0+1}_\infty$ always contains a $\mathbb{C}$-eigenvector of g.

DEFINITION 7.2.1. Let $\Sigma_{\tilde{E}}$ be the end orbifold of a nonproperly convex R-p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathscr{O}$. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group.

- Let $\lambda^{Tr}_{\max}(g)$ denote the largest norm of the eigenvalue of $g \in \mathbf{\Gamma}_{\tilde{E}}$ affiliated with $\vec{v} \neq 0$, $(\!(\vec{v})\!) \in \mathbb{S}^n - \mathbb{S}^{i_0}_\infty$, i.e.,
$$\lambda^{Tr}_{\max}(g) := \max\{\mu \,|\, \exists \vec{v} \in \mathscr{R}_\mu(g) - V^{i_0+1}_\infty\},$$
which is the maximal norm of transverse eigenvalues.
- Also, let $\lambda^{Tr}_{\min}(g)$ denote the smallest one affiliated with a nonzero vector $\vec{v}$, $(\!(\vec{v})\!) \in \mathbb{S}^n - \mathbb{S}^{i_0}_\infty$, i.e.,
$$\lambda^{Tr}_{\min}(g) := \min\{\mu \,|\, \exists \vec{v} \in \mathscr{R}_{\mu(g)} - V^{i_0+1}_\infty\},$$
which is the minimal norm of transverse eigenvalues.
- Let $\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)$ be the largest of the norms of the eigenvalues of g with $\mathbb{C}$-eigenvectors of form $\vec{v}$, $(\!(\vec{v})\!) \in \mathbb{S}^{i_0}_\infty$ and $\lambda^{\mathbb{S}^{i_0}_\infty}_{\min}(g)$ the smallest such one.

DEFINITION 7.2.2. Let $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ denote the eigenvalue of g at $\mathrm{v}_{\tilde{E}}$. The *transverse weak middle eigenvalue condition* with respect to $\mathrm{v}_{\tilde{E}}$ or the R-p-end structure is that each element g of $h(\pi_1(\tilde{E}))$ satisfies

$$\lambda^{Tr}_{\max}(g) \geq \lambda_{\mathrm{v}_{\tilde{E}}}(g). \tag{7.2.1}$$

Theorem A.2.1 somewhat justifies our approach. We do believe that the weak middle eigenvalue conditions implies the transverse ones at least for relevant cases.

The following proposition is very important in this chapter and shows that $\lambda^{Tr}_{\max}(g)$ and $\lambda^{Tr}_{\min}(g)$ are true largest and smallest norms of the eigenvalues of g.

PROPOSITION 7.2.3. *Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathscr{O}$ with radial or totally geodesic ends. Suppose that $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ (resp. $\mathbb{RP}^n$) covers $\mathscr{O}$ as a universal cover. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group satisfying the transverse weak middle eigenvalue condition for the R-p-end structure. Let $g \in \mathbf{\Gamma}_{\tilde{E}}$. Then the following hold*:

$$\begin{gathered}\lambda^{Tr}_{\max}(g) \geq \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g) \geq \lambda_{\mathrm{v}_{\tilde{E}}}(g) \geq \lambda^{\mathbb{S}^{i_0}_\infty}_{\min}(g) \geq \lambda^{Tr}_{\min}(g), \\ \lambda_1(g) = \lambda^{Tr}_{\max}(g), \textit{ and } \lambda_{n+1} = \lambda^{Tr}_{\min}(g).\end{gathered} \tag{7.2.2}$$

PROOF. We may assume that g is of infinite order. Suppose that $\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g) > \lambda^{Tr}_{\max}(g)$. We have $\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g) > \lambda_{\mathrm{v}_{\tilde{E}}}(g)$ by the transverse weak uniform middle eigenvalue condition.

Now, $\lambda^{Tr}_{\max}(g) < \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)$ implies that
$$R_{\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)} := \bigoplus_{|\mu| = \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)} \mathscr{R}_\mu(g)$$
is a subspace of $V^{i_0+1}_\infty$ and corresponds to a great sphere $\mathbb{S}^j$, $\mathbb{S}^j \subset \mathbb{S}^{i_0}_\infty$. Hence, a great sphere $\mathbb{S}^j$, $j \geq 0$, in $\mathbb{S}^{i_0}_\infty$ is disjoint from $\{\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}\}$. Since $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^{i_0}_\infty$ is not contained in $\mathbb{S}^j$, we obtain $j+1 \leq i_0$.

A vector space V_1 corresponds $\bigoplus_{|\mu| < \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)} \mathscr{R}_\mu(g)$ where g has strictly smaller norm eigenvalues and is complementary to $R_{\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)}$. Let $C_1 = \mathbb{S}(V_1)$. The great sphere C_1 is disjoint from $\mathbb{S}^j$ but C_1 contains $\mathrm{v}_{\tilde{E}}$. Moreover, C_1 is of complementary dimension to $\mathbb{S}^j$, i.e., $\dim C_1 = n - j - 1$.

Since C_1 is complementary to $\mathbb{S}^j \subset \mathbb{S}^{i_0}_\infty$, C_1 contains a complementary subspace C_1' to $\mathbb{S}^{i_0}_\infty$ of dimension $n - i_0 - 1$ in $\mathbb{S}^n$. Considering the sphere $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ at $\mathrm{v}_{\tilde{E}}$, it follows that C_1' goes

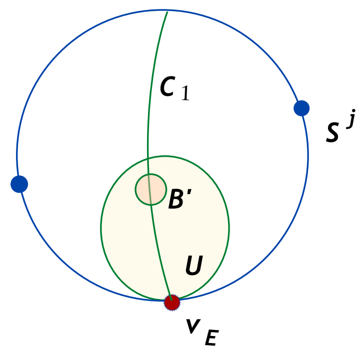

FIGURE 2. The figure for the proof of Proposition 7.2.3.

to an $n-i_0-1$-dimensional subspace C_1'' in $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$ disjoint from ∂l for any complete affine leaf l. Each complete affine leaf l of $\tilde{\Sigma}_{\tilde{E}}$ has the dimension i_0 and meets C_1'' in $\mathbb{S}^{n-1}_{v_{\tilde{E}}}$ by the dimension consideration.

Hence, a small ball B' in U meets C_1 in its interior.

$$\text{For any } ((\vec{v})) \in B', \vec{v} \in \mathbb{R}^{n+1}, \vec{v} = \vec{v}_1 + \vec{v}_2 \text{ where } ((\vec{v}_1)) \in C_1 \text{ and } ((\vec{v}_2)) \in \mathbb{S}^j.$$
$$\text{We obtain } g^k(((\vec{v}))) = \left(\left(g^k(\vec{v}_1) + g^k(\vec{v}_2)\right)\right), \tag{7.2.3}$$

where we used g to represent the linear transformation of determinant ± 1 as well (See Remark 1.1.5.) By Proposition 1.3.2, the action of g^k as $k \to \infty$ makes the former vectors very small compared to the latter ones, i.e.,

$$\left\{ \left\| g^k(\vec{v}_1) \right\| / \left\| g^k(\vec{v}_2) \right\| \right\} \to 0 \text{ as } k \to \infty.$$

Hence, $\{g^k(((\vec{v})))\}$ converges to the limit of $\{g^k(((\vec{v}_2)))\}$ if it exists.

Now choose $((\vec{w}))$ in $C_1 \cap B'$ and $\vec{v}, ((\vec{v})) \in \mathbb{S}^j$. We let $\vec{w}_1 = ((\vec{w} + \varepsilon\vec{v}))$ and $\vec{w}_2 = ((\vec{w} - \varepsilon\vec{v}))$ in B' for small $\varepsilon > 0$. Choose a subsequence $\{k_i\}$ so that $\{g^{k_i}(\vec{w}_1)\}$ converges to a point of $\mathbb{S}^n$. The above estimation shows that $\{g^{k_i}(\vec{w}_1)\}$ and $\{g^{k_i}(\vec{w}_2)\}$ converge to an antipodal pair of points in $\mathrm{Cl}(U)$ respectively. This contradicts the proper convexity of U as $g^k(B') \subset U$ and the geometric limit is in $\mathrm{Cl}(U)$.

Also the consideration of g^{-1} completes the inequality, and the second equation follows from the first one. [$\mathbb{S}^n$T] □

7.3. The general theory

7.3.1. Examples. First, we give some examples.

7.3.1.1. *The standard quadric in* $\mathbb{R}^{i_0+1}$ *and the group acting on it.* Let us consider an affine subspace A^{i_0+1} of $\mathbb{S}^{i_0+1}$ with coordinates $x_0, x_1, \dots, x_{i_0+1}$ given by $x_0 > 0$. The standard quadric in A^{i_0+1} is given by

$$x_{i_0+1} = \frac{1}{2}(x_1^2 + \cdots + x_{i_0}^2).$$

Clearly the group of the orthogonal maps $\mathsf{O}(i_0)$ acting on the planes given by $x_{i_0+1} = const$ acts on the quadric also. Also, the group of the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \frac{\|\vec{v}\|^2}{2} & \vec{v} & 1 \end{pmatrix}$$

acts on the quadric.

The group of affine transformations that acts on the quadric is exactly the Lie group generated by the above cusp group and $\mathsf{O}(i_0)$. The action is transitive and each of the stabilizer is a conjugate of $\mathsf{O}(i_0)$ by elements of the above cusp group.

The proof of this fact is simply that such a group of affine transformations is conjugate into a parabolic isometry group in the i_0+1-dimensional complete hyperbolic space H where the ideal fixed point is identified with $(\!(0,\dots,0,1)\!) \in \mathbb{S}^{i_0+1}$ and with $\mathrm{bd}H$ tangent to $\mathrm{bd}A^{i_0}$.

The group of projective automorphisms of the following forms is a unipotent cusp group.

$$\mathscr{N}'(\vec{v}) := \begin{pmatrix} 1 & \vec{0} & 0 \\ \vec{v}^T & \mathrm{I}_{i_0-1} & \vec{0}^T \\ \frac{\|\vec{v}\|^2}{2} & \vec{v} & 1 \end{pmatrix} \text{ for } \vec{v} \in \mathbb{R}^{i_0}. \tag{7.3.1}$$

(see [68] for details.) We can make each translation direction of generators of $\mathscr{N}$ in $\tilde{\Sigma}_{\tilde{E}}$ to be one of the standard vector. Therefore, we can find a coordinate system of V^{i_0+2} so that the generators are of $(i_0+2)\times(i_0+2)$-matrix forms

$$\mathscr{N}_j'(s) := \begin{pmatrix} 1 & \vec{0} & 0 \\ s\vec{e}_j^T & \mathrm{I}_{i_0} & 0 \\ \frac{1}{2} & s\vec{e}_j & 1 \end{pmatrix} \tag{7.3.2}$$

where $s \in \mathbb{R}$ and $(\vec{e}_j)_k = \delta_{jk}$ a row i-vector for $j = 1,\dots,i_0$. That is,

$$\mathscr{N}'(\vec{v}) = \mathscr{N}_1'(v_1)\cdots\mathscr{N}_{i_0}'(v_{i_0}).$$

7.3.1.2. *Examples of generalized joined ends.* We first begin with examples. In the following, we will explain the generalized joined type end.

EXAMPLE 7.3.1. Let us consider two ends E_1, a totally geodesic R-end, with the p-end-neighborhood U_1 in the universal cover of a real projective orbifold $\mathscr{O}_1$ in $\mathbb{S}^{n-i_0-1}$ of dimension $n-i_0-1$ with the p-end vertex v_1, and E_2 the p-end-neighborhood U_2, a horospherical type one, in the universal cover of a real projective orbifold $\mathscr{O}_2$ of dimension i_0+1 with the p-end vertex v_2.

- Let $\mathbf{\Gamma}_1$ denote the projective automorphism group in $\mathbf{Aut}(\mathbb{S}^{n-i_0-1})$ acting on U_1 corresponding to E_1. We assume that $\mathbf{\Gamma}_1$ acts on a great sphere $\mathbb{S}^{n-i_0-2} \subset \mathbb{S}^{n-i_0-1}$ disjoint from v_1. There exists a properly convex open domain K' in $\mathbb{S}^{n-i_0-2}$ where $\mathbf{\Gamma}_1$ acts cocompactly but not necessarily freely. We change U_1 to be the interior of the join of K' and v_1.
- Let $\mathbf{\Gamma}_2$ denote the one in $\mathbf{Aut}(\mathbb{S}^{i_0+1})$ acting on U_2 as a subgroup of a cusp group.
- We embed $\mathbb{S}^{n-i_0-1}$ and $\mathbb{S}^{i_0+1}$ in $\mathbb{S}^n$ meeting transversely at $\mathrm{v} = \mathrm{v}_1 = \mathrm{v}_2$.
- We embed U_2 in $\mathbb{S}^{i_0+1}$ and $\mathbf{\Gamma}_2$ in $\mathbf{Aut}(\mathbb{S}^n)$ fixing each point of $\mathbb{S}^{n-i_0-1}$.
- We can embed U_1 in $\mathbb{S}^{n-i_0-1}$ and $\mathbf{\Gamma}_1$ in $\mathbf{Aut}(\mathbb{S}^n)$ acting on the embedded U_1 so that $\mathbf{\Gamma}_1$ acts on $\mathbb{S}^{i_0-1}$ normalizing $\mathbf{\Gamma}_2$ and acting on U_1. One can find some

such embeddings by finding an arbitrary homomorphism $\rho : \mathbf{\Gamma}_1 \to N(\mathbf{\Gamma}_2)$ for a normalizer $N(\mathbf{\Gamma}_2)$ of $\mathbf{\Gamma}_2$ in $\mathbf{Aut}(\mathbb{S}^n)$.

We find an element $\zeta \in \mathbf{Aut}(\mathbb{S}^n)$ fixing each point of $\mathbb{S}^{n-i_0-2}$ and acting on $\mathbb{S}^{i_0+1}$ as a unipotent element normalizing $\mathbf{\Gamma}_2$ so that the corresponding matrix has only two norms of eigenvalues. Then ζ centralizes $\mathbf{\Gamma}_1|\mathbb{S}^{n-i_0-2}$ and normalizes $\mathbf{\Gamma}_2$. Let U be the strict join of U_1 and U_2, a properly convex domain. $U/\langle \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \zeta\rangle$ gives us an R-p-end of dimension n diffeomorphic to $\Sigma_{E_1} \times \Sigma_{E_2} \times \mathbb{S}^1 \times \mathbb{R}$ and the transverse real projective manifold is diffeomorphic to $\Sigma_{E_1} \times \Sigma_{E_2} \times \mathbb{S}^1$. We call the results the *a generalized joined* end and the generalized joined end-neighborhoods. Those ends with end-neighborhoods finitely covered by these are also called a *generalized joined* end. The generated group $\langle \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \zeta\rangle$ is called a *generalized joined group*.

Now we generalize this construction slightly: Suppose that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are Lie groups and they have compact stabilizers at points of U_1 and U_2 respectively, and we have a parameter of ζ^t for $t \in \mathbb{R}$ centralizing $\mathbf{\Gamma}_1|\mathbb{S}^{n-i_0-2}$ and normalizing $\mathbf{\Gamma}_2$ and restricting to a unipotent action on $\mathbb{S}^{i_0}$ acting on U_2. The other conditions remain the same. We obtain a *generalized joined homogeneous action* of the properly convex actions and cusp actions. Let U be the properly convex open subset obtained as above as a join of U_1 and U_2. Let G denote the constructed Lie group by taking the embeddings of $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ as above. G also has a compact stabilizer on U. Given a discrete cocompact subgroup of G, we obtained a p-end-neighborhood of a *generalized joined p-end* by taking the quotient of U. An end with an end-neighborhood finitely covered by such a one are also called a *generalized joined end*.

REMARK 7.3.2. We will deform this construction to a construction of quasi-joined ends (see Definition 7.1.2). This will be done by adding some translations appropriately.

We continue the above example to a more specific situation.

EXAMPLE 7.3.3. Let N be as in (7.3.10). Let N be a cusp group in conjugate to one in $\mathsf{SO}(i_0+1,1)$ acting on an i_0-dimensional ellipsoid in $\mathbb{S}^{i_0+1}$.

We find a closed properly convex real projective orbifold Σ of dimension $n-i_0-2$ and find a homomorphism from $\pi_1(\Sigma)$ to a subgroup of $\mathbf{Aut}(\mathbb{S}^{i_0+1})$ normalizing N. (We will use a trivial one to begin with.) Using this, we obtain a group Γ so that

$$1 \to N \to \Gamma \to \pi_1(\Sigma) \to 1.$$

Actually, we assume that this is "split", i.e., $\pi_1(\Sigma)$ acts trivially on N.

We now consider an example where $i_0 = 1$. Let N be 1-dimensional and be generated by N_1 as in (7.3.3).

$$N_1 := \left(\begin{array}{cc|cc} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \vec{0} & 1 & 0 & 0 \\ \hline \vec{0} & 1 & 1 & 0 \\ \vec{0} & \frac{1}{2} & 1 & 1 \end{array}\right) \tag{7.3.3}$$

where $i_0 = 1$.

We take a discrete faithful proximal semisimple representation

$$\tilde{h} : \pi_1(\Sigma) \to \mathsf{GL}(n-i_0, \mathbb{R})$$

acting on a convex cone C_Σ in $\mathbb{R}^{n-i_0}$. We define

$$h : \pi_1(\Sigma) \to \mathsf{GL}(n+1, \mathbb{R})$$

by matrices

$$h(g) := \begin{pmatrix} \tilde{h}(g) & 0 & 0 \\ \vec{d}_1(g) & a_1(g) & 0 \\ \vec{d}_2(g) & c(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g) \end{pmatrix} \tag{7.3.4}$$

where $\vec{d}_1(g)$ and $\vec{d}_2(g)$ are $n-i_0$-vectors and $g \mapsto \lambda_{\mathrm{v}_{\tilde{E}}}(g)$ is a homomorphism as defined above for the p-end vertex and $\det \tilde{h}(g) a_1(g) \lambda_{\mathrm{v}_{\tilde{E}}}(g) = 1$.

$$h(g^{-1}) := \begin{pmatrix} \tilde{h}(g)^{-1} & 0 & 0 \\ -\begin{pmatrix} \frac{\vec{d}_1(g)}{a_1(g)} \\ \frac{-c(g)\vec{d}_1(g)}{a_1(g)\lambda_{\mathrm{v}_{\tilde{E}}}(g)} + \frac{\vec{d}_2(g)}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)} \end{pmatrix} \tilde{h}(g)^{-1} & \frac{1}{a_1(g)} & 0 \\ & \frac{-c(g)}{a_1(g)\lambda_{\mathrm{v}_{\tilde{E}}}(g)} & \frac{1}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)} \end{pmatrix}. \tag{7.3.5}$$

Then the conjugation of N_1 by $h(g)$ gives us

$$\begin{pmatrix} \mathrm{I}_{n-i_0} & 0 & 0 \\ \begin{pmatrix} \vec{0} & a_1(g) \\ \vec{*} & * \end{pmatrix} \tilde{h}(g)^{-1} & \begin{matrix} 1 \\ \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g)}{a_1(g)} \end{matrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{pmatrix}. \tag{7.3.6}$$

Our condition on the form of N_1 shows that $\underbrace{(0,0,\dots,0,1)}_{n-i_0}$ has to be a common eigenvector by $\tilde{h}(\pi_1(\tilde{E}))$, and we also assume that $a_1(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)$. The last row of $\tilde{h}(g)$ equals $(\vec{0}, \lambda_{\mathrm{v}_{\tilde{E}}}(g))$. Thus, the upper left $(n-i_0)\times(n-i_0)$-part of $h(\pi_1(\tilde{E}))$ is reducible.

Some further computations show that we can take any

$$h : \pi_1(\tilde{E}) \to \mathsf{SL}(n-i_0, \mathbb{R})$$

with matrices of form

$$h(g) := \left(\begin{array}{cc|cc} S_{n-i_0-1}(g) & 0 & 0 & 0 \\ \vec{0} & \lambda_{\mathrm{v}_{\tilde{E}}}(g) & 0 & 0 \\ \hline \vec{0} & 0 & \lambda_{\mathrm{v}_{\tilde{E}}}(g) & 0 \\ \vec{0} & 0 & 0 & \lambda_{\mathrm{v}_{\tilde{E}}}(g) \end{array}\right) \tag{7.3.7}$$

for $g \in \pi_1(\tilde{E}) - N$ by a choice of coordinates by the semisimple property of the $(n-i_0)\times(n-i_0)$-upper left part of $h(g)$.

Since $\tilde{h}(\pi_1(\tilde{E}))$ has a common eigenvector, Theorem 1.1 of Benoist [21] shows that the open convex domain K that is the image of Π_K is reducible. We assume that $N_K = N'_K \times \mathbb{Z}$ for another subgroup N'_K, and the image of the homomorphism $g \in N'_K \to S_{n-i_0-1}(g)$ gives a discrete projective automorphism group acting properly discontinuously on a properly convex subset K' in $\mathbb{S}^{n-i_0-2}$ with a compact quotient.

Let $\mathscr{E}$ be the one-dimensional ellipsoid where lower-right 3×3-matrices of N acts on. From this, the end is of the join form $K'^o/N'_K \times \mathbb{S}^1 \times \mathscr{E}/\mathbb{Z}$ by taking a double cover if necessary and $\pi_1(\tilde{E})$ is isomorphic to $N'_K \times \mathbb{Z}\times\mathbb{Z}$ up to taking an index two subgroups.

We can think of this as the join of K'^o/N'_K with $\mathscr{E}/\mathbb{Z}$ as K' and $\mathscr{E}$ are on disjoint complementary projective spaces of respective dimensions $n-3$ and 2 to be denoted $\mathbb{S}(K')$ and $\mathbb{S}(\mathscr{E})$ respectively.

7.3.2. Hypotheses to derive the splitting result. These hypotheses will help us to obtain the splitting. Afterward, we will show the NPNC-ends with transverse weak middle eigenvalue conditions will satisfy these.

We will outline this subsection. In Section 7.3.2.1, we will introduce a standard coordinate system to work on, where we introduce the unipotent cusp group $\mathscr{N} \cong \mathbb{R}^{i_0}$. Our group $\mathbf{\Gamma}_{\tilde{E}}$ normalizes $\mathscr{N}$ by the hypothesis. *Similarity* Lemma 7.3.7 shows that the conjugation in $\mathscr{N}$ by an element of $\mathbf{\Gamma}_{\tilde{E}}$ acts as a similarity, a simple consequence of the normalization property. We use this similarity and the Benoist theory [21] to prove *K-is-a-cone* Lemma 7.3.9 that K decomposes into a cone $\{k\} * K''$ where $\mathscr{N}$ has a nice expression for the adopted coordinates. (If an orthogonal group acts cocompactly on an open manifold, then the manifold is zero-dimensional.) In Section 7.3.2.3, *splitting* Proposition 7.3.14 shows that the end holonomy group splits. To do that we find a sequence of elements of the virtual center expanding neighborhoods of a copy of K''. Here, we explicitly find a part corresponding to $K'' \subset \mathrm{bd}\tilde{\mathscr{O}}$ explicitly and k is realized by an (i_0+1)-dimensional hemisphere where $\mathscr{N}$ acts on.

7.3.2.1. *The matrix form of the end holonomy group.* Let $\mathbf{\Gamma}_{\tilde{E}}$ be an R-p-end holonomy group, and let $l \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ be a complete i_0-dimensional leaf in $\tilde{\Sigma}_{\tilde{E}}$. Then a great sphere $\mathbb{S}^{i_0+1}_l$ contains the great segments from $\mathrm{v}_{\tilde{E}}$ in the direction of l. Let V^{i_0+1} denote the subspace corresponding to $\mathbb{S}^{i_0}_\infty$ containing $\mathrm{v}_{\tilde{E}}$, and V^{i_0+2} the subspace corresponding to $\mathbb{S}^{i_0+1}_l$. We choose the coordinate system so that

$$\mathrm{v}_{\tilde{E}} = \underbrace{(\!(0,\cdots,0,1)\!)}_{n+1}$$

and points of V^{i_0+1} and those of V^{i_0+2} respectively correspond to

$$\left(\!\!\left(\overbrace{0,\ldots,0}^{n-i_0},*,\cdots,*\right)\!\!\right), \quad \left(\!\!\left(\overbrace{0,\ldots,0}^{n-i_0-1},*,\cdots,*\right)\!\!\right).$$

Since $\mathbb{S}^{i_0}_\infty$ and $\mathrm{v}_{\tilde{E}}$ are g-invariant, g, $g \in \mathbf{\Gamma}_{\tilde{E}}$, is of *standard* form

$$\left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & 0 \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right) \tag{7.3.8}$$

where $S(g)$ is an $(n-i_0-1)\times(n-i_0-1)$-matrix and $s_1(g)$ is an $(n-i_0-1)$-column vector, $s_2(g)$ and $c_2(g)$ are $(n-i_0-1)$-row vectors, $C_1(g)$ is an $i_0 \times (n-i_0-1)$-matrix, $a_4(g)$ is an i_0-column vectors, $A_5(g)$ is an $i_0 \times i_0$-matrix, $a_8(g)$ is an i_0-row vector, and $a_1(g), a_7(g)$, and $a_9(g)$ are scalars.

Denote

$$\hat{S}(g) = \begin{pmatrix} S(g) & s_1(g) \\ s_2(g) & a_1(g) \end{pmatrix},$$

and is called a *upper-left part* of g.

Let $\mathscr{N}$ be a unipotent group acting on $\mathbb{S}^{i_0}_\infty$ and inducing I on $\mathbb{S}^{n-i_0-1}$ also restricting to a cusp group for at least one great (i_0+1)-dimensional sphere $\mathbb{S}^{i_0+1}$ containing $\mathbb{S}^{i_0}_\infty$.

We can write each element $g \in \mathscr{N}$ as an $(n+1)\times(n+1)$-matrix

$$\begin{pmatrix} \mathrm{I}_{n-i_0-1} & 0 & 0 \\ \vec{0} & 1 & 0 \\ C_g & * & U_g \end{pmatrix} \tag{7.3.9}$$

where $C_g > 0$ is an $(i_0+1)\times(n-i_0-1)$-matrix, U_g is a unipotent $(i_0+1)\times(i_0+1)$-matrix, 0 indicates various zero row or column vectors, $\vec{0}$ denotes the zero row-vector of dimension $n-i_0-1$, and I_{n-i_0-1} is the $(n-i_0-1)\times(n-i_0-1)$-identity matrix. This follows since g acts trivially on $\mathbb{R}^{n+1}/V^{i_0+1}$ and g acts as a cusp group element on the subspace V^{i_0+2}.

For $\vec{v} \in \mathbb{R}^{i_0}$, we define

$$\mathscr{N}(\vec{v}) := \left(\begin{array}{cc|cccc} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 & \dots & 0 \\ \vec{0} & 1 & 0 & 0 & \dots & 0 \\ \hline \vec{c}_1(\vec{v}) & v_1 & 1 & 0 & \dots & 0 \\ \vec{c}_2(\vec{v}) & v_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{c}_{i_0+1}(\vec{v}) & \frac{1}{2}\|\vec{v}\|^2 & v_1 & v_2 & \dots & 1 \end{array}\right) \tag{7.3.10}$$

where $\|v\|$ is the norm of $\vec{v} = (v_1, \cdots, v_{i_0}) \in \mathbb{R}^{i_0}$. We assume that

$$\mathscr{N} := \{\mathscr{N}(\vec{v}) | \vec{v} \in \mathbb{R}^{i_0}\}$$

is a group, which must be nilpotent. The elements of our nilpotent group $\mathscr{N}$ are of this form since $\mathscr{N}(\vec{v})$ is the product $\prod_{j=1}^{i_0} \mathscr{N}(e_j)^{v_j}$. By the way we defined this, for each k, $k = 1, \dots, i_0$, $\vec{c}_k : \mathbb{R}^{i_0} \to \mathbb{R}^{n-i_0-1}$ are linear functions of $\vec{v}$ defined as

$$\vec{c}_k(\vec{v}) = \sum_{j=1}^{i_0} \vec{c}_{kj} v_j \text{ for } \vec{v} = (v_1, v_2, \dots, v_{i_0})$$

so that we form a group. (We do not need the property of $\vec{c}_{i_0+1}$ at the moment.)

From now on, we denote by $C_1(\vec{v})$ the $(n-i_0-1)\times i_0$-matrix given by the matrix with rows $\vec{c}_j(\vec{v})$ for $j = 1, \dots, i_0$ and by $c_2(\vec{v})$ the row $(n-i_0-1)$-vector $\vec{c}_{i_0+1}(\vec{v})$. The lower-right $(i_0+2)\times(i_0+2)$-matrix is form is called the *standard cusp matrix form*.

We denote by $\hat{A}$ the matrix

$$\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & A & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \tag{7.3.11}$$

for A an $i_0 \times i_0$-matrix. Denote by the group of form

$$\{\hat{O}_5 | O_5 \in \mathsf{O}(i_0)\}$$

by $\hat{\mathsf{O}}(n+1, i_0)$.

If $\mathscr{N}$ can be put the form (7.3.10) with $C_1(\vec{v}) = 0, c_2(\vec{v}) = 0$ for all $\vec{v}$, we call $\mathscr{N}$ the *standard cusp group* (*of type* $(n+1, i_0)$) in the standard form. The *standard parabolic group* (of type $(n+1, i_0)$) is a group conjugate to $\mathscr{N}\hat{\mathsf{O}}(n+1, i_0)$ where $\mathscr{N}$ is in the standard cusp group in the standard form. Ones conjugate to these are called *standard cusp group* and *standard parabolic group* respectively.

The assumptions for this subsection are as follows: We will assume that the group satisfies the condition virtually only since this will be sufficient for our purposes.

HYPOTHESIS 7.3.4.

- Let K be defined as above for an R-p-end $\tilde{E}$. Assume that N_K acts on K^o cocompactly.
- $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the transverse weak middle eigenvalue condition for the R-p-end structure.
- Elements are in the matrix form of (7.3.8) under a common coordinate system.
- A group $\mathscr{N}$ of form (7.3.10) in the same coordinate system acts on each hemisphere with boundary $\mathbb{S}^i_\infty$, and fixes $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^i_\infty$ with coordinates $(\!(0,\cdots,0,1)\!)$.
- $N \subset \mathscr{N}$ in the same coordinate system as above.
- The p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ normalizes $\mathscr{N}$.
- $\mathscr{N}$ acts on a p-end neighborhood U of $\tilde{E}$, and acts on $U \cap \mathbb{S}^{i_0+1}$ for each great sphere $\mathbb{S}^{i_0+1}$ containing $\mathbb{S}^{i_0}_\infty$ whenever $U \cap \mathbb{S}^{i_0+1} \neq \emptyset$.
- $\mathscr{N}$ freely, faithfully, and transitively acts on the space of i_0-dimensional leaves of $\tilde{\Sigma}_{\tilde{E}}$ by an induced action.

Let U be a p-end neighborhood of $\tilde{E}$. Let l' be an i_0-dimensional leaf of $\tilde{\Sigma}_{\tilde{E}}$. The consideration of the projection Π_K shows us that the leaf l' corresponds to a hemisphere $H^{i_0+1}_{l'}$ where

$$U_{l'} := (H^{i_0+1}_{l'} - \mathbb{S}^{i_0}_\infty) \cap U \neq \emptyset \tag{7.3.12}$$

holds.

LEMMA 7.3.5 (Cusp). *Assume Hypothesis 7.3.4. Let l' be an i_0-dimensional leaf of $\tilde{\Sigma}_{\tilde{E}}$. Let $H^{i_0+1}_{l'}$ denote the i_0+1-dimensional hemisphere with boundary $\mathbb{S}^{i_0}_\infty$ corresponding to l'. Then $\mathscr{N}$ acts transitively on* $\mathrm{bd} U_{l'}$ *for $U_{l'} = U \cap H_{l'}$ bounded by an ellipsoid in a component of $H^{i_0+1}_{l'} - \mathbb{S}^{i_0}_\infty$.*

PROOF. Since l' is an i_0+1-dimensional leaf of $\tilde{\Sigma}_{\tilde{E}}$, we obtain $H^{i_0}_{l'} \cap U \neq \emptyset$. Let $J_{l'} := H^{i_0+1}_{l'} \cap U \neq \emptyset$ where $\mathscr{N}$ acts on.

Now, l' corresponds to an interior point of K. We need to change coordinates of $\mathbb{S}^{n-i_0-1}$ so that l' goes to

$$\underbrace{(\!(0,\cdots,0,1)\!)}_{n-i_0} \text{ under } \Pi_K.$$

This involves the coordinate changes of the first $n-i_0$ coordinates. Now, we can restrict g to $H^{i_0+1}_{l'}$ so that the matrix form is with respect to $U_{l'}$. Now give a coordinate system on the open hemisphere $H^{i_0+1,o}_{l'}$ given by

$$\underbrace{(\!(1,\vec{x},x_{i_0+1})\!)}_{i_0+2} \text{ for } \vec{x} \in \mathbb{R}^{i_0} \text{ where } \underbrace{(\!(1,0,\ldots,0)\!)}_{i_0+2} \text{ is the origin,}$$

and $(\vec{x},x_{i_0+1})$ gives the affine coordinate system on $H^{i_0+1,o}_{l'}$.

Using (7.3.10) restricted to $\mathbb{S}^{i_0}$, the lowest row of the lower-right $(i_0+1)\times(i_0+1)$ restriction matrix has to be of form $(*,\vec{v},1)$. We obtain that each $g \in \mathscr{N}$ then has the form in $H_{l'}^{i_0+1}$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ L(\vec{v}^T) & \mathrm{I}_{i_0} & 0 \\ \kappa(v) & \vec{v} & 1 \end{pmatrix}$$

where $L : \mathbb{R}^{i_0} \to \mathbb{R}^{i_0}$ is a linear map. The linearity of L is the consequence of the group property. $\kappa : \mathbb{R}^{i_0} \to \mathbb{R}$ is some function. We consider L as an $i_0 \times i_0$-matrix.

Suppose that there exists a kernel K_1 of L. We use $t\vec{v} \in K_1 - \{O\}$. As $t \to \infty$, consider each orbit of the subgroup $\mathscr{N}(\mathbb{R}\vec{v}) \subset \mathscr{N}$ given by

$$(\!(1, \vec{x}, x_{i_0+1})\!) \to (\!(1, \vec{x}, \kappa(t\vec{v}) + t\vec{v} \cdot \vec{x} + x_{i_0+1})\!)\,.$$

This action fixes coordinates from the 2-nd to $i_0 + 1$-st ones to be $\vec{x}$. Hence, each orbit lies on an affine line from $(\!(0, 0, \dots, 1)\!)$. Since the eigenvalues of every elements of $\mathscr{N}$ all equal 1, the action is unipotent. Since the action is unipotent, either the action is trivial or the orbit is the entire complete affine line on $H_{l'}^{i_0}$. Since the action on each leaf l' is free, the action cannot be trivial. Thus, the orbit is the complete affine line, and this contradicts the proper convexity of $\tilde{\mathscr{O}}$.

Also, since $\mathscr{N}$ is abelian, the computations of

$$\mathscr{N}(\vec{v})\mathscr{N}(\vec{w}) = \mathscr{N}(\vec{w})\mathscr{N}(\vec{v})$$

shows that $\vec{v}L\vec{w}^T = \vec{w}L\vec{v}^T$ for every pair of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^{i_0}$. Thus, L is a symmetric matrix.

We may obtain new coordinates $x_{n-i_0+1}, \dots, x_n$ by taking linear combinations of these. Since L hence is nonsingular, we can find new coordinates $x_{n-i_0+1}, \dots, x_n$ so that $\mathscr{N}$ is now of standard form: We conjugate $\mathscr{N}$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for nonsingular A. We obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ AL\vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \kappa(\vec{v}) & \vec{v}A^{-1} & 1 \end{pmatrix}.$$

We thus need to solve for $A^{-1}A^{-1T} = L$, which can be done since L is nonsingular and symmetric as we showed above.

Now, we conjugate as we wished to. We can factorize each element of $\mathscr{N}$ into forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{I}_{i_0} & 0 \\ \kappa(\vec{v}) - \frac{\|\vec{v}\|^2}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \frac{\|\vec{v}\|^2}{2} & \vec{v} & 1 \end{pmatrix}.$$

Again, by the group property, $\aleph_7(\vec{v}) := \kappa(\vec{v}) - \frac{\|\vec{v}\|^2}{2}$ gives us a linear function $\aleph_7 : \mathbb{R}^{i_0} \to \mathbb{R}$. Hence $\aleph_7(\vec{v}) = \kappa_\alpha \cdot \vec{v}$ for $\kappa_\alpha \in \mathbb{R}^{i_0}$. Now, we conjugate $\mathscr{N}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{I}_{i_0} & 0 \\ 0 & -\kappa_\alpha & 1 \end{pmatrix}$$

and this will put $\mathscr{N}$ into the standard form.

Now it is clear that the orbit of $\mathscr{N}(x_0)$ for a point x_0 of $J_{l'}$ is an ellipsoid with a point removed since $\mathscr{N}$ acts so in the standard form since the standard form can be recognized as that of the parabolic group in the hyperbolic space in the Klein model in some appropriate coordinates. $\square$

Recall standard cusp group from the end of Section 7.3.2.1. For later purposes, we need:

LEMMA 7.3.6. *Let C be standard cusp group acting on a hemisphere H of dimension i_0+1 with boundary $\mathbb{S}^{i_0}$ fixing a point p in $\mathbb{S}^{i_0}$. Then the following hold:*

- *There exists an affine space $\mathbb{A}^n_C$ with $\mathbb{S}^{i_0} \subset \mathrm{bd}\mathbb{A}^n_C$ and $H^o \subset \mathbb{A}^n_C$ where orbits of points have three types*:
  - *The orbit of each point in $\mathbb{A}^n_C$ is an ellipsoid in an affine subspace of dimension i_0 parallel to the affine subspace H^o.*
  - *The orbit of each point of a great sphere $\mathbb{S}^{n-i_0} \subset \mathrm{bd}\mathbb{A}^n$ of dimension $n-i_0$ containing p transverse to $\mathbb{S}^{i_0}$ where the orbits are singletons.*
  - *The orbit of every point of $\mathrm{bd}\mathbb{A}^n_C - \mathbb{S}^{n-i_0}$ is not contained in a properly convex domain.*
- *The affine space $\mathbb{A}^n_C$ with these orbit properties is uniquely determined.*

PROOF. We choose the affine space $\mathbb{A}^n_C$ is given by $x_{n-i_0} > 0$ for the coordinate system where C is written as in (7.3.10) with c_i, $i = 1, \dots, i_0+1$, are zero. Since C is standard, There exists a sphere of dimension $\mathbb{S}^{n-i_0-1}$ complementary to $\mathbb{S}^{i_0}$ in $\mathrm{bd}\mathbb{A}^n_C$ where C acts trivially. Let $\mathbb{S}^{n-i_0}$ be the join $\{p, p_-\} * \mathbb{S}^{n-i_0-1}$. On $\mathbb{S}^{n-i_0}$ the orbits are singletons. The orbits of points of $\mathrm{bd}\mathbb{A}^n_C - \mathbb{S}^{n-i_0}$ can be understood by the matrix form. The orbits will always contain a pair of antipodal points in the closures by considering $\mathscr{N}$ with the $n-i_0$-th rows and the $n-i_0$-th columns removed. The affine translations commute with each element of C. This shows that each orbit in $\mathbb{A}^n_C$ is as claimed.

Also, since the orbit types are characterized, $\mathbb{A}^n_C$ is uniquely determined. $\square$

Let $a_5(g)$ denote $\left|\det(A^5_g)\right|^{\frac{1}{i_0}}$. Define $\mu_g := \frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_5(g)}$ for $g \in \mathbf{\Gamma}_{\tilde{E}}$ from Lemma 7.3.7.

LEMMA 7.3.7 (Similarity). *Assume Hypothesis 7.3.4. Then any element $g \in \mathbf{\Gamma}_{\tilde{E}}$ induces an $(i_0 \times i_0)$-matrix M_g given by*

$$g\mathscr{N}(\vec{v})g^{-1} = \mathscr{N}(\vec{v}M_g) \text{ where}$$

$$M_g = \frac{1}{a_1(g)}(A_5(g))^{-1} = \mu_g O_5(g)^{-1}$$

for $O_5(g)$ in a compact Lie group $G_{\tilde{E}}$, and the following hold.

- $(a_5(g))^2 = a_1(g)a_9(g)$ *or equivalently* $\frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_5(g)}$.

- *Finally, $a_1(g), a_5(g)$, and $a_9(g)$ are all positive.*

PROOF. Since the conjugation by g sends elements of $\mathcal{N}$ to itself in a one-to-one manner, the correspondence between the set of $\vec{v}$ for $\mathcal{N}$ and $\vec{v}'$ is one-to-one.

Since we have $g\mathcal{N}(\vec{v}) = \mathcal{N}(\vec{v}')g$ for vectors $\vec{v}$ and $\vec{v}'$ in $\mathbb{R}^{i_0}$ by Hypothesis 7.3.4, we consider

(7.3.13)
$$\left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & 0 \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right)\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v}) & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}) & \frac{\|\vec{v}\|^2}{2} & \vec{v} & 1 \end{array}\right)$$

where $C_1(\vec{v})$ is an $(n-i_0-1)\times i_0$-matrix where each row is a linear function of $\vec{v}$, $c_2(\vec{v})$ is a $(n-i_0-1)$-row vector, and $\vec{v}$ is an i_0-row vector. This must equal the following matrix for some $\vec{v}' \in \mathbb{R}$

(7.3.14)
$$\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v}') & \vec{v}'^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}') & \frac{\|\vec{v}'\|^2}{2} & \vec{v}' & 1 \end{array}\right)\left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & 0 \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right).$$

From (7.3.13), we compute the $(4,3)$-block of the result to be $a_8(g) + a_9(g)\vec{v}$. From (7.3.14), the $(4,3)$-block is $\vec{v}'A_5(g) + a_8(g)$. We obtain the relation $a_9(g)\vec{v} = \vec{v}'A_5(g)$ for every $\vec{v}$. Since the correspondence between $\vec{v}$ and $\vec{v}'$ is one-to-one, we obtain

$$\vec{v}' = a_9(g)\vec{v}(A_5(g))^{-1} \tag{7.3.15}$$

for the $i_0 \times i_0$-matrix $A_5(g)$ and we also infer $a_9(g) \neq 0$ and $\det(A_5(g)) \neq 0$. The $(3,2)$-block of the result of (7.3.13) equals

$$a_4(g) + A_5(g)\vec{v}^T.$$

The $(3,2)$-block of the result of (7.3.14) equals

$$C_1(\vec{v}')s_1(g) + a_1(g)\vec{v}'^T + a_4(g). \tag{7.3.16}$$

Thus,

$$A_5(g)\vec{v}^T = C_1(\vec{v}')s_1(g) + a_1(g)\vec{v}'^T. \tag{7.3.17}$$

For each g, we can choose a coordinate system so that $s_1(g) = 0$ since by the Brouwer fixed point theorem, there is a fixed point in the compact convex set $K \subset \mathbb{S}^{n-i_0-1}$. This involves the coordinate changes of the first $n-i_0$ coordinate functions only.

Let l' denote the fixed point of g. Since $\mathcal{N}$ acts on $\mathbb{S}^{i_0+1}_{l'}$ for l' as a cusp group by Lemma 7.3.5, there exists a coordinate change involving the last (i_0+1)-coordinates

$$x_{n-i_0+1}, \ldots, x_n, x_{n+1}$$

so that the matrix form of the lower-right $(i_0+2)\times(i_0+2)$-matrix of each element $\mathcal{N}$ is of the standard cusp form. This will not affect $s_1(g) = 0$ as we can check from conjugating matrices used in the proof of Lemma 7.3.5 as the change involves the above coordinates only. Denote this coordinate system by $\Phi_{g,l'}$.

We may assume that the transition to this coordinate system from the original one is uniformly bounded: First, we change for $\mathbb{S}^{n-i_0-1}$ with a bounded elliptic coordinate change since we are only picking out a single point to be a coordinate axis. This makes L and κ in the proof of Lemma 7.3.5 to be uniformly bounded functions. Hence, A and κ_α in the proof are also uniformly bounded.

Let us use $\Phi_{g,l'}$ for a while using primes for new set of coordinates functions. Now $A_5'(g)$ is conjugate to $A_5(g)$ as we can check in the proof of Lemma 7.3.5. Under this coordinate system for given g, we obtain $a_1'(g) \neq 0$ and we can recompute to show that $a_9'(g)\vec{v} = \vec{v}' A_5'(g)$ for every $\vec{v}$ as in (7.3.15). By (7.3.17) recomputed for this case, we obtain

$$\vec{v}' = \frac{1}{a_1'(g)} \vec{v} (A_5'(g))^T \tag{7.3.18}$$

as $s_1'(g) = 0$ here since we are using the coordinate system $\Phi_{g,l'}$. Since this holds for every $\vec{v} \in \mathbb{R}^{i_0}$, we obtain

$$a_9'(g)(A_5'(g))^{-1} = \frac{1}{a_1'(g)} (A_5'(g))^T.$$

Hence $\frac{1}{|\det(A_5'(g))|^{1/i_0}} A_5'(g) \in \mathsf{O}(i_0)$. Also,

$$\frac{a_9'(g)}{a_5'(g)} = \frac{a_5'(g)}{a_1'(g)}.$$

Here, $A_5'(g)$ is a conjugate of the original matrix $A_5(g)$ by linear coordinate changes as we can see from the above processes to obtain the new coordinate system.

This implies that the original matrix $A_5(g)$ is conjugate to an orthogonal matrix multiplied by a positive scalar for every g. The set of matrices $\{A_5(g) | g \in \mathbf{\Gamma}_{\tilde{E}}\}$ forms a group since every g is of a standard matrix form (see (7.3.8)). Given such a group of matrices normalized to have determinant ± 1, we obtain a compact group

$$G_{\tilde{E}} := \left\{ \frac{1}{|\det A_5(g)|^{\frac{1}{i_0}}} A_5(g) \middle| g \in \Gamma_{\tilde{E}} \right\}$$

by Lemma 7.3.8. This group has a coordinate system where every element is orthogonal by a coordinate change of coordinates $x_{n-i_0+1}, \dots, x_n$.

Also, $a_1(g), a_5(g), a_9(g)$ are conjugation invariant. Hence, we proved

$$\frac{a_9(g)}{a_5(g)} = \frac{a_5(g)}{a_1(g)} (= \mu_g) \tag{7.3.19}$$

We have $a_9(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g) > 0$. Since $a_5(g)^2 = a_1(g)a_9(g)$, we obtain $a_1(g) > 0$. Finally, $a_5(g) > 0$ by definition. □

LEMMA 7.3.8. *Suppose that G is a subgroup of a linear group $\mathsf{GL}(i_0, \mathbb{R})$ where each element is conjugate to an orthogonal element by a uniformly boounded conjugating matrices. Then G is in a compact Lie group.*

PROOF. Clearly, the norms of eigenvalues of $g \in G$ are all 1. Hence, G is virtually an orthopotent group by Theorem 1.3.7. (See [132] and [62]).) Hence, $\mathbb{R}^{i_0}$ has subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{R}^{i_0}$ where G acts as orthogonal on V_{i+1}/V_i up to a choice of coordinates. Hence, the Zariski closure $\mathscr{Z}(G)$ of G is also orthopotent.

If G is discrete, Theorem 1.3.7 shows that G is virtually unipotent. The unipotent subgroup of G is trivial since the elements must be conjugate to orthogonal elements. Thus, G is a finite group, and we finished the proof.

Suppose now that the closure $\bar{G}$ of G is a Lie group of dimension ≥ 1. Let $\mathsf{O}(\oplus_{i=1}^{m} V_i/V_{i-1})$ denote the group of linear transformations acting on each V_i/V_{i-1} orthogonally for each $i = 1, \dots, m$. By Theorem 1.3.7, there is a homomorphism $\mathscr{Z}(G) \to \mathsf{O}(\oplus_{i=1}^{m} V_i/V_{i-1})$ whose kernel U_{i_0} is the a group of unipotent matrices. Let $\hat{G}$ denote the image of $\bar{G}$ in the second group. Then $\bar{G} \cap U_{i_0}$ is trivial since every element of $\bar{G}$ is elliptic or is the identity. Thus, $\bar{G}$ is isomorphic to a compact group $\hat{G}$. $\square$

From now on, we denote by $(C_1(\vec{v}), \vec{v}^T)$ the matrix obtained from $C_1(\vec{v})$ by adding a column vector $\vec{v}^T$ and denote $O_5(g) := \frac{1}{|\det A_5(g)|^{\frac{1}{i_0}}} A_5(g)$. We also let

$$\hat{S}(g) := \begin{pmatrix} S(g), & s_1(g) \\ s_2(g), & a_1(g) \end{pmatrix}.$$

LEMMA 7.3.9 (K is a cone). *Assume Hypothesis 7.3.4. Suppose that $\mathbf{\Gamma}_{\tilde{E}}$ acts semisimply on K^o. Then the following hold*:

- *K is a cone over a totally geodesic $(n - i_0 - 2)$-dimensional domain K''.*
- *The rows of $(C_1(\vec{v}), \vec{v}^T)$ are proportional to a single vector, and we can find a coordinate system where $C_1(\vec{v}) = 0$ not changing any entries of the lower-right $(i_0+2) \times (i_0+2)$-submatrices for all $\vec{v} \in \mathbb{R}^{i_0}$.*
- *We can find a common coordinate system where*

$$\begin{aligned} &O_5(g)^{-1} = O_5(g)^T, O_5(g) \in \mathsf{O}(i_0), \\ &s_1(g) = s_2(g) = 0 \text{ for all } g \in \mathbf{\Gamma}_{\tilde{E}} \end{aligned} \tag{7.3.20}$$

  where $O_5(g) = \left|\det(A_g^5)\right|^{\frac{1}{i_0}} A_5(g)$.
- *In this coordinate system, we have*

$$s_1(g) = 0, s_2(g) = 0, a_9(g) c_2(\vec{v}) = c_2(\mu_g \vec{v} O_5(g)^{-1}) S(g) + \mu_g \vec{v} O_5(g)^{-1} C_1(g). \tag{7.3.21}$$

PROOF. The assumption implies that $M_g = \mu_g O_5(g)^{-1}$ by Lemma 7.3.7. We consider the equation

$$g \mathscr{N}(\vec{v}) g^{-1} = \mathscr{N}(\mu_g \vec{v} O_5(g)^{-1}). \tag{7.3.22}$$

We change to

$$g \mathscr{N}(\vec{v}) = \mathscr{N}(\mu_g \vec{v} O_5(g)^{-1}) g. \tag{7.3.23}$$

Considering the lower left $(n - i_0) \times (i_0 + 1)$-matrix of the left side of (7.3.23), we obtain

$$\begin{pmatrix} C_1(g), & a_4(g) \\ c_2(g), & a_7(g) \end{pmatrix} + \begin{pmatrix} a_5(g) O_5(g) C_1(\vec{v}), & a_5(g) O_5(g) \vec{v}^T \\ a_8(g) C_1(\vec{v}) + a_9 c_2(\vec{v}), & a_8(g) \cdot \vec{v}^T + a_9(g) \vec{v} \cdot \vec{v}/2 \end{pmatrix} \tag{7.3.24}$$

where the entry sizes are clear. From the right side of (7.3.23), we obtain

$$\begin{pmatrix} C_1(\mu_g \vec{v} O_5(g)^{-1}), & \mu_g O_5(g)^{-1,T} \vec{v}^T \\ c_2(\mu_g \vec{v} O_5(g)^{-1}), & \vec{v}\cdot\vec{v}/2 \end{pmatrix} \hat{S}(g) + \begin{pmatrix} C_1(g), & a_4(g) \\ \mu_g \vec{v} O_5(g)^{-1}\cdot C_1(g) + c_2(g), & a_7(g) + \mu_g \vec{v} O_5(g)^{-1}\cdot a_4(g) \end{pmatrix}. \tag{7.3.25}$$

From the top rows of (7.3.24) and (7.3.25), we obtain that

$$\left(a_5(g) O_5(g) C_1(\vec{v}), a_5(g) O_5(g) \vec{v}^T\right) = \left(\mu_g C_1\left(\vec{v} O_5(g)^{-1}\right), \mu_g O_5(g)^{-1,T} \vec{v}^T\right) \hat{S}(g). \tag{7.3.26}$$

We multiplied the both sides by $O_5(g)^{-1}$ from the right and by $\hat{S}(g)^{-1}$ from the left to obtain

$$\left(a_5(g) C_1(\vec{v}), a_5(g) \vec{v}^T\right) \hat{S}(g^{-1}) = \left(\mu_g O_5(g)^{-1} C_1(\vec{v} O_5(g)^{-1}), \mu_g O_5(g)^{-1} O_5(g)^{-1,T} \vec{v}^T\right). \tag{7.3.27}$$

Let us form the subspace V_C in the dual sphere $\mathbb{R}^{n-i_0*}$ spanned by row vectors of $(C_1(\vec{v}), \vec{v}^T)$. Let $\mathbb{S}^*_C$ denote the corresponding subspace in $\mathbb{S}^{n-i_0-1*}$. Then

$$\left\{ \frac{1}{\det \hat{S}(g)^{\frac{1}{n-i_0-1}}} \hat{S}(g) | g \in \mathbf{\Gamma}_{\tilde{E}} \right\}$$

acts on V_C as a group of bounded linear automorphisms since $O_5(g) \in G$ for a compact group G. Therefore, $\{\hat{S}(g) | g \in \mathbf{\Gamma}_{\tilde{E}}\}$ on $\mathbb{S}^*_C$ is in a compact group of projective automorphisms by (7.3.27).

We recall that the dual group N_K^* of N_K acts on the properly convex dual domain K^* of K cocompactly by Proposition 1.5.11. Notice that a finite irreducible group cannot act cocompactly on a convex open set unless the set is a singleton. Since N_K^* acts as a compact group on $\mathbb{S}^*_C$, it must be that N_K^* is reducible.

Now, we apply the theory of Vey [**151**] and Benoist [**21**]: Since N_K^* is semisimple by above premises, N_K^* acts on a complementary subspace of $\mathbb{S}^*_N$. By Proposition 1.4.13, K^* has an invariant subspace K_1^* and K_2^* so that we have strict join

$$K^* = K_1^* * K_2^* \text{ where } \dim K_1^* = \dim \mathbb{S}^*_C, \dim K_2^* = \dim \mathbb{S}^*_N$$

where

$$K_1^* = K^* \cap \mathbb{S}^*_C, K_2^* = K^* \cap \mathbb{S}^*_N.$$

Also, N_K^* is isomorphic to a cocompact subgroup of

$$N_{K,1}^* \times N_{K,2}^* \times A$$

where

- A is a diagonalizable subgroup with positive eigenvalues only isomorphic to a subgroup of $\mathbb{R}_+$,
- $N_{k,i}^*$ is the restriction image of N_K^* to K_i^* for $i = 1, 2$, and
- $N_{K,i}^*$ acts on the interior of K_i^* properly and cocompactly.

But since $N^*_{K,1}$ acts orthogonally on $\mathbb{S}^*_C$, the only possibility is that $\dim \mathbb{S}^*_C = 0$: Otherwise K_1^{*o}/N_K is not compact contradicting Proposition 1.4.13. Hence, $\dim \mathbb{S}^*_C = 0$ and K_1^* is a singleton and K_2^* is $n-i_0-2$-dimensional properly convex domain.

Rows of $(C_1(\vec{v}), \vec{v}^T)$ are elements of the 1-dimensional subspace in $\mathbb{R}^{n-i_0-1*}$ corresponding to $\mathbb{S}^*_C$. Therefore this shows that the rows of $(C_1(\vec{v}), \vec{v}^T)$ are proportional to a single row vector.

Since $(C_1(\vec{e}_j), \vec{e}_j^T)$ has 0 as the last column element except for the jth one, only the jth row of $C_1(\vec{e}_j)$ is nonzero. Let $C_1(1, \vec{e}_1)$ be the first row of $C_1(\vec{e}_1)$. Thus, each row of $(C_1(\vec{e}_j), \vec{e}_j^T)$ equals to a scalar multiple of $(C_1(1, \vec{e}_1), 1)$ for every j. Now we can choose coordinates of $\mathbb{R}^{n-i_0*}$ so that this $(n-i_0)$-row-vector now has coordinates $(0, \dots, 0, 1)$. We can also choose the coordinates so that K_2^* is in the zero set of the last coordinate. With this change, we need to do conjugation by matrices with the top left $(n-i_0-1)\times(n-i_0-1)$-submatrix being different from I and the rest of the entries staying the same. This will not affect the expressions of matrices of lower right $(i_0+2)\times(i_0+2)$-matrices involved here. Thus, $C_1(\vec{v}) = 0$ in this coordinate for all $\vec{v} \in \mathbb{R}^{i_0}$. Also, $\underbrace{(\!(0, \dots, 0, 1)\!)}_{n-i_0}$ is an eigenvector of every elements of N_K^*.

The hyperspace containing K_2^* is also N_K^*-invariant. Thus, the $(n-i_0)$-vector $(0, \dots, 0, 1)$ corresponds to an eigenvector of every element of N_K. In this coordinate system, K is a strict join of a point for an $(n-i_0)$-vector

$$k = \underbrace{(\!(0, \dots, 0, 1)\!)}_{n-i_0}$$

and a domain K'' given by setting $x_{n-i_0} = 0$ in a totally geodesic sphere of dimension $n-i_0-2$ by duality. We also obtain

$$s_1(g) = 0, s_2(g) = 0.$$

For the final item we have under our coordinate system.

$$g = \begin{pmatrix} S(g) & 0 & 0 & 0 \\ 0 & a_1(g) & 0 & 0 \\ C_1(g) & a_4(g) & a_5(g)O_5(g) & 0 \\ c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{pmatrix}, \tag{7.3.28}$$

$$\mathscr{N}(\vec{v}) = \begin{pmatrix} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vec{v}^T & \mathrm{I} & 0 \\ c_2(\vec{v}) & \frac{1}{2}\|\vec{v}\|^2 & \vec{v} & 1 \end{pmatrix}. \tag{7.3.29}$$

The normalization of $\mathscr{N}$ shows as in the proof of Lemma 7.3.7 that $O_5(g)$ is orthogonal now. (See (7.3.15) and (7.3.17).) By (7.3.22), we have

$$g\mathscr{N}(\vec{v}) = \mathscr{N}(\vec{v'})g, \vec{v}' = \mu_g \vec{v} O_5(g)^{-1}.$$

We consider the lower-right $(i_0+1)\times(n-i_0)$-submatrices of $g\mathscr{N}(\vec{v})$ and $\mathscr{N}(\vec{v}')g$. For the first one, we obtain

$$\begin{pmatrix} C_1(g), & a_4(g) \\ c_2(g), & a_7(g) \end{pmatrix} + \begin{pmatrix} a_5(g)O_5(g), & 0 \\ a_8(g), & a_9(g) \end{pmatrix}\begin{pmatrix} 0, & \vec{v}^T \\ c_2(\vec{v}), & \frac{1}{2}\|\vec{v}\|^2 \end{pmatrix}$$

For $\mathscr{N}(\vec{v}')g$, we obtain

$$\begin{pmatrix} 0, & \vec{v}'^T \\ c_2(\vec{v}'), & \frac{1}{2}\|\vec{v}'\|^2 \end{pmatrix}\begin{pmatrix} S(g), & 0 \\ 0, & a_1(g) \end{pmatrix} + \begin{pmatrix} \mathrm{I}, & 0 \\ \vec{v}', & 1 \end{pmatrix}\begin{pmatrix} C_1(g), & a_4(g) \\ c_2(g), & a_9(g) \end{pmatrix}.$$

Considering $(2,1)$-blocks, we obtain

$$\square \qquad c_2(g)+a_9(g)c_2(\vec{v}) = c_2(\vec{v}')S(g)+\vec{v}'C_1(g)+c_2(g).$$

□

From now on, we denote $O_5(g) := \left|\det(A_g^5)\right|^{\frac{1}{i_0}} A_5(g)$.

LEMMA 7.3.10. *Assume Hypothesis 7.3.4 and N_K acts semi-simply. Then we can find coordinates so that the following holds for all g :*

$$\frac{a_9(g)}{a_5(g)}O_5(g)^{-1}a_4(g) = a_8(g)^T \text{ or } \frac{a_9(g)}{a_5(g)}a_4(g)^T O_5(g) = a_8(g). \tag{7.3.30}$$

$$\text{If } \mu_g = 1, \text{ then } a_1(g) = a_9(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g) \text{ and } A_5(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)O_5(g). \tag{7.3.31}$$

PROOF. Again, we use (7.3.13) and (7.3.14). We only need to consider lower right $(i_0+2)\times(i_0+2)$-matrices.

$$\begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g) & a_5(g)O_5(g) & 0 \\ a_7(g) & a_8(g) & a_9(g) \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & \mathrm{I} & 0 \\ \frac{1}{2}\|\vec{v}\|^2 & \vec{v} & 1 \end{pmatrix} \tag{7.3.32}$$

$$= \begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g)+a_5(g)O_5(g)\vec{v}^T & a_5(g)O_5(g) & 0 \\ a_7(g)+a_8(g)\vec{v}^T+\frac{a_9(g)}{2}\|\vec{v}\|^2 & a_8(g)+a_9(g)\vec{v} & a_9(g) \end{pmatrix}. \tag{7.3.33}$$

This equals

$$\begin{pmatrix} 1 & 0 & 0 \\ \vec{v}'^T & \mathrm{I} & 0 \\ \frac{1}{2}\left\|\vec{v}'\right\|^2 & \vec{v}' & 1 \end{pmatrix}\begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g) & a_5(g)O_g^5 & 0 \\ a_7(g) & a_8(g) & a_9(g) \end{pmatrix} \tag{7.3.34}$$

$$= \begin{pmatrix} a_1(g) & 0 & 0 \\ a_1(g)\vec{v}'^T+a_4(g) & a_5(g)O_5(g) & 0 \\ \frac{a_1(g)}{2}\left\|\vec{v}'\right\|^2+\vec{v}'a_4(g)+a_7(g) & a_5(g)\vec{v}'O_5(g)+a_8(g) & a_9(g) \end{pmatrix}. \tag{7.3.35}$$

Then by comparing the $(3,2)$-blocks, we obtain

$$a_8(g)+a_9(g)\vec{v} = a_8(g)+a_5(g)\vec{v}'O_5(g).$$

Thus, $\vec{v} = \frac{a_5(g)}{a_9(g)}\vec{v}'O_5(g)$.

From the $(3,1)$-blocks, we obtain

$$a_1(g)\vec{v}'\cdot\vec{v}'/2 + \vec{v}'a_4(g) = a_8(g)\vec{v}^T + a_9(g)\vec{v}\cdot\vec{v}/2.$$

Since the quadratic forms have to equal each other, we obtain

$$\frac{a_9(g)}{a_5(g)}\vec{v}O_5(g)^{-1}\cdot a_4(g) = \vec{v}\cdot a_8(g) \text{ for all } \vec{v}\in\mathbb{R}^{i_0}.$$

Thus, $\frac{a_9(g)}{a_5(g)}(O_5(g)^T a_4(g))^T = a_8(g)^T$.

Since we have $\mu_g = 1$, we obtain

$$a_1(g) = a_9(g) = a_5(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g) \text{ and } A_5(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)O_5(g)$$

by Lemma 7.3.7. Also, $a_1(g) = a_9(g) = a_5(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)$. $\square$

Under Hypothesis 7.3.4 and assuming that N_K acts semisimply, we conclude by (7.3.30) and (7.3.19) that each $g\in\mathbf{\Gamma}_{\tilde{E}}$ has the form

$$(7.3.36)\qquad \left(\begin{array}{c|c|c|c} S(g) & 0 & 0 & 0 \\ \hline 0 & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_1(g)\vec{v}_g^T & a_5(g)O_5(g) & 0 \\ \hline c_2(g) & a_7(g) & a_5(g)\vec{v}_g O_5(g) & a_9(g) \end{array}\right)$$

defining $\vec{v}_g := \frac{a_4(g)}{a_1(g)}$.

REMARK 7.3.11. Since the matrices are of form (7.3.36), $g\mapsto\mu_g$ is a homomorphism.

COROLLARY 7.3.12. *If g of form (7.3.36) centralizes a Zariski dense subset A' of $\mathcal{N}$, then $\mu_g = 1$ and $O_5(g) = \mathrm{I}_{i_0}$.*

PROOF. $\mathcal{N}$ is isomorphic to $\mathbb{R}^{i_0}$. The subset A'' of $\mathbb{R}^{i_0}$ corresponding to A' is also Zariski dense in $\mathbb{R}^{i_0}$. $g\mathcal{N}(\vec{v}) = \mathcal{N}(\vec{v})g$ shows that $\vec{v} = \vec{v}O_5(g)$ for all $\vec{v}\in A''$. Hence $O_5(g) = \mathrm{I}$. $\square$

7.3.2.2. *Invariant $\aleph_7$.* We assume $\mu_g = 1$, $g\in\mathbf{\Gamma}_{\tilde{E}}$, identically in this subsubsection. When $\mu_g = 1$ for all $g\in\mathbf{\Gamma}_{\tilde{E}}$, by taking a finite index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$, we conclude that each $g\in\mathbf{\Gamma}_{\tilde{E}}$ has the form by Lemma 7.3.10

$$(7.3.37)\qquad M(g) := \left(\begin{array}{c|c|c|c} S(g) & 0 & 0 & 0 \\ \hline 0 & \lambda_{\mathrm{v}_{\tilde{E}}}(g) & 0 & 0 \\ \hline C_1(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g)\vec{v}_g^T & \lambda_{\mathrm{v}_{\tilde{E}}}(g)O_5(g) & 0 \\ \hline c_2(g) & a_7(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g)\vec{v}_g O_5(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g) \end{array}\right).$$

We define an invariant:

$$\aleph_7(g) := \frac{a_7(g)}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)} - \frac{\|\vec{v}_g\|^2}{2}.$$

We denote by $\hat{M}(g)$ the lower right $(i_0+1)\times(i_0+1)$-submatrix of $M(g)$. An easy computation shows that $\hat{M}(g)\hat{M}(h) = \hat{M}(gh)$ where $\vec{v}_{gh} = \vec{v} + O_5(g)\vec{v}_h$ holds. Then it is easy to show that

$$\aleph_7(g^n) = n\aleph_7(g) \text{ and } \aleph_7(gh) = \aleph_7(g) + \aleph_7(h), \text{ whenever } g,h,gh \in G. \tag{7.3.38}$$

We obtain a homomorphism to the additive group $\mathbb{R}$

$$\aleph_7 : \mathbf{\Gamma}_{\tilde{E}} \to \mathbb{R}. \tag{7.3.39}$$

(See (7.3.40).)

Here $\aleph_7(g)$ is also determined by factoring the matrix of g into commuting matrices of form

$$\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \mathrm{I}_{i_0} & 0 \\ \hline 0 & \aleph_7(g) & \vec{0} & 1 \end{array}\right) \times \tag{7.3.40}$$

$$\left(\begin{array}{c|c|c|c} S_g & 0 & 0 & 0 \\ \hline 0 & \lambda_{\mathrm{v}_{\tilde{E}}}(g) & 0 & 0 \\ \hline C_1(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g)\vec{v}_g & \lambda_{\mathrm{v}_{\tilde{E}}}(g)O_5(g) & 0 \\ \hline c_2(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g)\frac{\|\vec{v}_g\|^2}{2} & \lambda_{\mathrm{v}_{\tilde{E}}}(g)\vec{v}_g O_5(g) & \lambda_{\mathrm{v}_{\tilde{E}}}(g) \end{array}\right).$$

REMARK 7.3.13. We give a bit more explanations. Recall that the space of segments in a hemisphere H^{i_0+1} with the vertices $\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}$ forms an affine subspace $\mathbb{A}^i$ one-dimension lower, and the group $\mathbf{Aut}(H^{i_0+1})_{\mathrm{v}_{\tilde{E}}}$ of projective automorphisms of the hemisphere fixing $\mathrm{v}_{\tilde{E}}$ maps to $\mathbf{Aff}(\mathbb{A}^{i_0})$ with kernel K equal to transformations of an $(i_0+2)\times(i_0+2)$-matrix form

$$\left(\begin{array}{c|c|c} 1 & \vec{0}^T & 0 \\ \hline O & \mathrm{I}_{i_0} & \vec{0} \\ \hline b & \vec{0}^T & 1 \end{array}\right) \tag{7.3.41}$$

where $\mathrm{v}_{\tilde{E}}$ is given coordinates $⟬0,0,\dots,1⟭$, $\vec{0}$ denote the 0-vector in $\mathbb{R}^{i_0}$ and a center point of $H_l^{i_0+1}$ the coordinates $⟬1,0,\dots,0⟭$. In other words, the transformations are of form

$$\begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_{i_0} \\ x_{i_0+1} \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_{i_0} \\ x_{i_0+1}+b \end{bmatrix} \tag{7.3.42}$$

and hence b determines the kernel element. Hence $\aleph_7(g)$ indicates the translation towards $\mathrm{v}_{\tilde{E}} = ⟬0,\dots,1⟭$. (Actually the vertex corresponds to $(1,0,\dots,+\infty)$-point in this view.)

We denote by $T(n+1,n-i_0)$ the group of matrices restricting to (7.3.41) in the lower-right $(i_0+2)\times(i_0+2)$-submatrices and equal to I on upper-left $(n-i_0-1)\times(n-i_0-1)$-submatrices and zero elsewhere.

7.3.2.3. *Splitting the NPNC end.*

PROPOSITION 7.3.14 (Splitting). *Assume Hypothesis 7.3.4 for $\mathbf{\Gamma}_{\tilde{E}}$. Suppose additionally the following*:

- *Suppose that N_K acts on K in a semi-simple manner.*
- *$K=\{k\}*K''$ a strict join, and K^o/N_K is compact with k a common fixed point of N_K.*
- *Let H be a commutant of a finite index subgroup of N_K that is positive diagonalizable. Assume that $N_K\cap H$ contains a free abelian group of rank l_0 provided K'' is a strict join of compact convex subsets $K_1,\dots,K_{l_0}$ where H acts trivially on each K_j, $j=1,\dots,l_0$.*

Then the following hold:

- *there exists an exact sequence*

$$1\to\mathscr{N}\to\langle\mathbf{\Gamma}_{\tilde{E}},\mathscr{N}\rangle\xrightarrow{\Pi_K^*}N_K\to 1.$$

- *K'' embeds projectively in the closure of $\mathrm{bd}\tilde{\mathscr{O}}$ whose image is $\mathbf{\Gamma}_{\tilde{E}}$-invariant, and*
- *one can find a coordinate system so that every $\mathscr{N}(\vec{v})$ for $\vec{v}\in\mathbb{R}^{i_0}$ is in the standard form and each element g of $\mathbf{\Gamma}_{\tilde{E}}$ is written so that*
  - *$C_1(\vec{v})=0, c_2(\vec{v})=0$, and*
  - *$C_1(g)=0$ and $c_2(g)=0$.*

PROOF. (A) Let Z denote $\langle\mathbf{\Gamma}_{\tilde{E}},\mathscr{N}\rangle$. Since $N\subset\mathscr{N}$, we have homomorphism

$$Z\xrightarrow{\Pi_K^*}N_K\to 1$$

extending Π_K^* of (7.1.3). We now determine the kernel.

The function $\lambda_{\mathrm{v}_{\tilde{E}}}:\mathbf{\Gamma}_{\tilde{E}}\to\mathbb{R}_+$ extends to $\lambda_{\mathrm{v}_{\tilde{E}}}:Z\to\mathbb{R}_+$. By (7.3.36), we deduce that every element g of Z is of form:

$$\left(\begin{array}{c|c|c|c} S(g) & 0 & 0 & 0\\ \hline 0 & a_1(g) & 0 & 0\\ \hline C_1(g) & a_1(g)\vec{v}_g^T & a_5(g)O_5(g) & 0\\ \hline c_2(g) & a_7(g) & a_5(g)\vec{v}_gO_5(g) & a_9(g)\end{array}\right) \tag{7.3.43}$$

for some functions $C_1,c_2,a_i:Z\to\mathbb{R}$ for $i=1,5,9$. Notice that $\aleph_7$ is identical zero on $\mathscr{N}$. Since $N\subset\mathscr{N}$ by Hypothesis 7.3.4, $\aleph_7$ is zero on the kernel N. For $g\in\mathscr{N}$, there is an element $\vec{v}_g\in\mathbb{R}^{i_0}$ such that $\exp(\vec{v}_g)=g$. We define $C_1,c_2:\mathbb{R}^{i_0}\to\mathbb{R}$ by setting $C_1(\vec{v}_g)=C_1(g),c_2(\vec{v}_g)=c_2(g)$ for each $g\in\mathscr{N}$. Hence, $g\in N$ is of form

$$g=\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0\\ \hline 0 & 1 & 0 & 0\\ \hline C_1(\vec{v}_g) & \vec{v}_g^T & O_5(g) & 0\\ \hline c_2(\vec{v}_g) & \frac{\|\vec{v}_g\|^2}{2} & \vec{v}_gO_5(g) & 1\end{array}\right) \tag{7.3.44}$$

since $\aleph_7(g) = 0$, and $S(g) = \lambda_g \mathrm{I}_{n-i_0-1}$ and the $(n-i_0, n-i_0)$-term must be λ_g for some $\lambda_g > 0$ so that it goes to I in K.

Theorem 1.3.7 shows that N is a subgroup of $\mathscr{N}$ by taking a finite index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$. Since the kernel of $\Pi_K^*|Z$ is generated by $\mathscr{N}$ and N, we proved the first item.

(B) Lemma 7.3.9 shows that $C_1(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^{i_0}$ for a coordinate system where k has the form

$$⟦0, \dots, 0, 1⟧ \in \mathbb{S}^{n-i_0-1}.$$

Let λ_{S_g} denote the maximal norm of the eigenvalues of the upper-left part S_g of g. We define

$$\mathbf{\Gamma}_{\tilde{E},S} := \{g | \lambda_{S_g}(g) > a_1(g)\}.$$

There is always an element like this because $\mathbf{\Gamma}_{\tilde{E}}$ acts on a subspace of dimension $n - i_0 - 1$ containing a compact set projectively diffeomorphic to K. In particular, we take the inverse image of suitable diagonalizable elements of the center $H \cap N_K$ denoted in Proposition 7.4.7. We take the diagonalizable element in N_K with K'' having a largest norm eigenvalue. Let g be such an element. Since $O_5(g)$ is orthogonal, the transversal weak middle eigenvalue condition tells us

$$\max\{\lambda_S(g), a_5(g), a_1(g)\} \geq a_9(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g).$$

We have either $a_9(g) \geq a_5(g) \geq a_1(g)$ or $a_1(g) \geq a_5(g) \geq a_9(g)$ by (7.3.19) depending on $\mu_g \geq 1$ or ≤ 1. The second case can be ignored since $a_9(g)$ is the smallest eigenvalue in that case and we can consider g^{-1} to obtain that $a_1(g) = a_5(g) = a_9(g)$ by Proposition 7.2.3 reducing to the first case. Hence,for $g \in \mathbf{\Gamma}_{\tilde{E},S}$, we have $\mu_g \geq 1$ and that $\lambda_S(g)$ is the largest of norms of every eigenvalue by Proposition 7.2.3 and $a_9(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)$. $\max\{a_1(g), a_5(g)\} \leq a_9(g)$ and hence $\mu_g \geq 1$, we have

$$a_1(g) \leq a_5(g) \leq a_9(g) \leq \lambda_S(g) \text{ and } \mu_g \geq 1 \text{ for } g \in \mathbf{\Gamma}_{\tilde{E},S}. \tag{7.3.45}$$

(C) Applying Lemma 7.3.9, we modify the coordinates so that the elements of $\mathscr{N}$ are of form:

$$k = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v}_k) & \vec{v}_k^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}_k) & \frac{\|\vec{v}_k\|^2}{2} & \vec{v}_k & 1 \end{array}\right) \text{ where } C_1(\vec{v}_k) = 0. \tag{7.3.46}$$

By the group property, $\vec{v} \mapsto c_2(\vec{v})$ is a linear map.

We have coordinates so that $K'' \subset \mathbb{S}^{n-i_0-2}$. There exists a sequence of elements z_i of $N_K \cap H$ in the virtual center H so that a largest norm eigenvalue has a direction in K'' and $z_i|K'' \to \mathrm{I}_{K''}$.

Since $\mathrm{Cl}(U)$ is in properly convex $\mathrm{Cl}(\tilde{\mathscr{O}})$, it is in an affine patch where $\mathrm{v}_{\tilde{E}}$ is the origin. That is, $\mathrm{v}_{\tilde{E}} = ⟦0,0,0,1⟧$. Let $g_i \in \mathbf{\Gamma}_{\tilde{E}+}$ be the element going to z_i under Π_K^*. Then $\{g_i(x)\}$ for a point x of U converges to $⟦\lambda\vec{a}, 0, \vec{w}, 1⟧$ for $⟦a⟧ \in K''$ and some $\vec{w} \in \mathbb{R}^{i_0}$, $\lambda \in \mathbb{R}$. Here, $\lambda > 0$ since our point must project to the limit of $z_i(\Pi_K(x))$ as $i \to \infty$. Hence by (7.3.45) and

$$⟦\lambda\vec{a}, 0, \vec{w}, 1⟧ \in \mathrm{Cl}(U) \tag{7.3.47}$$

Since $z_i|K'' \to \mathrm{I}_{K''}$, we may assume that an open subset of K'' can be realized as $⟦\vec{a}⟧$ in the above.

By (7.3.46)

$$\mathscr{N}(\vec{v}_1)^k (\!(\lambda\vec{a},0,\vec{w},1)\!) = (\!(\lambda\vec{a},0,\vec{w},k\lambda\vec{a}\cdot c_2(\vec{v}_1)+k\vec{v}_1\cdot\vec{w}+1)\!) \tag{7.3.48}$$

as $C_1(\vec{v}_1) = 0$ for every $\vec{v}_1 \in \mathbb{R}^{i_0}$. Suppose that $\lambda\vec{a}\cdot c_2(\vec{v}_1)+\vec{v}_1\cdot\vec{w} \neq 0$. Then as $k \to \infty$, $\{\mathscr{N}(k\vec{v}_1)((\!(\vec{a})\!))\}$ converges to a point and as $k \to -\infty$, it converges to its antipode. The limits form an antipodal pair of points in $\mathrm{Cl}(U)$. This contradicts the proper convexity of $\tilde{\mathscr{O}}$.

Hence, $\lambda\vec{a}\cdot c_2(\vec{v}_1)+\vec{w}\cdot\vec{v}_1 = 0$ holds for every $\vec{v}_1 \in \mathbb{R}^{i_0}$. We write $\hat{c}_2$ as $(n-i_0)\times i_0$-matrix. Then $\vec{w}^T := -\lambda\hat{c}_2^T\vec{a}^T$. Let $\hat{K}$ denote the image of

$$(\!(\vec{a})\!) \mapsto (\!(\lambda\vec{a},0,\vec{w},0)\!)\,, \vec{w}^T := -\lambda\hat{c}_2^T\vec{a}^T, (\!(\vec{a})\!) \in K''.$$

Under Π_K, the compact convex set $\hat{K}$ embeds onto a compact convex set K'' of the same dimension.

Since every point of K''^{o} is a limit point of the orbit of a point of K^o under z_i, under $g_i\mathbf{\Gamma}_{\tilde{E},S}\cap\Pi_K^{*-1}(N_K\cap H)$, every point of $\hat{K}^o$ is a limit point of a point of U. Hence, we obtain $\hat{K} \subset \mathrm{bd}\tilde{\mathscr{O}}$ by convexity by (7.3.47).

Also, $\mathscr{N}$ acts on $\hat{K}$ by our discussion and hence on $\hat{K}$. We choose the coordinates so that $\hat{K}$ corresponds to $x_1 = x_2 = \cdots = x_{n-i_0-1} = 0$. Under this coordinate system,

$$C_1(\vec{v}) = 0, c_2(\vec{v}) = 0 \text{ for every } \vec{v} \in \mathbb{R}^{i_0}. \tag{7.3.49}$$

(D) Consider a sequence $\{g_i\}$ of elements $g_i \in \mathbf{\Gamma}_{\tilde{E},S}$ with $\{\Pi_K^*(g_i)(y)\}$ converging to $x \in K''$. We claim that every limit point x' of $g_i(u)$ for $u \in U$ is in $\hat{K}$: In our coordinates as above (7.3.49), we have $x' = (\!(\lambda\vec{a},0,\vec{w},1)\!)$. (7.3.48) still holds, and since $c_2(\vec{v}_1) = 0$, $\vec{v}_1\cdot\vec{w} = 0$ for every $\vec{v}_1 \in \mathbb{R}^{i_0}$. Thus, $\vec{w} = 0$, and $x' \in \hat{K}$.

Since the set of such sequences are invariant under the conjugation by $\mathbf{\Gamma}_{\tilde{E}}$, it follows that the set of accumulations points of such sequence of elements in $\hat{K}$ is $\mathbf{\Gamma}_{\tilde{E}}$-invariant. Since each point of K'' can be an accumulation point of some sequence of elements in N_K, it follows that $\hat{K}$ is $\mathbf{\Gamma}_{\tilde{E}}$-invariant. This implies that $\hat{K}$ is $\langle\mathbf{\Gamma}_{\tilde{E}},\mathscr{N}\rangle$-invariant.

We may assume in our chosen coordinates that

$$C_1(g) = c_2(g) = C_1(\vec{v}) = c_2(\vec{v}) = 0 \text{ for every } g \in \mathbf{\Gamma}_{\tilde{E}}, \vec{v} \in \mathbb{R}^{i_0}. \tag{7.3.50}$$

□

7.3.3. Strictly joined and quasi-joined ends for $\mu \equiv 1$. We will now discuss joins and their generalizations in-depth in this subsection. That is we will only consider when $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$. We will use a hypothesis and later show that the hypothesis is true in our cases to prove the main results. Again, we assume the hypothesis virtually since it will be sufficient.

HYPOTHESIS 7.3.15 ($\mu \equiv 1$). Let $\mathbf{\Gamma}_{\tilde{E}}$ be a p-end holonomy group. We continue to assume Hypothesis 7.3.4 for $\mathbf{\Gamma}_{\tilde{E}}$.

- Every $g \in \mathbf{\Gamma}_{\tilde{E}} \to M_g$ is so that M_g is in a fixed orthogonal group $\mathsf{O}(i_0)$. Thus, $\mu_g = 1$ identically.
- $\mathbf{\Gamma}_{\tilde{E}}$ acts on a subspace $\mathbb{S}^{i_0}_\infty$ containing $\mathrm{v}_{\tilde{E}}$ and the properly convex domain K''' in the subspace $\mathbb{S}^{n-i_0-2}$ forming an independent pair with $\mathbb{S}^{i_0}_\infty$ mapping homeomorphic to the factor K'' of $K = \{k\} * K''$ under Π_K.
- $\mathscr{N}$ acts on these two subspaces fixing every point of $\mathbb{S}^{n-i_0-2}$.

Let H be the closed n-hemisphere defined by $x_{n-i_0} \geq 0$. Then by the convexity of $\tilde{\Sigma}_{\tilde{E}}$, we can choose H so that $U \subset H^o$, $K''' \subset H$ and $\mathbb{S}^{i_0}_\infty \subset \mathrm{bd} H$. We identify H^o with an affine space $\mathbb{A}^n$. (See Section 1.1.6.)

By Hypothesis 7.3.15, elements of $\mathscr{N}$ have the form of (7.3.10) with

$$C_1(\vec{v}) = 0, c_2(\vec{v}) = 0 \text{ for all } \vec{v} \in \mathbb{R}^{i_0},$$

and the elements of $\mathbf{\Gamma}_{\tilde{E}}$ has the form of (7.3.37) with

$$s_1(g) = 0, s_2(g) = 0, C_1(g) = 0, \text{ and } c_2(g) = 0.$$

Again we recall the projection $\Pi_K : \mathbb{S}^n - \mathbb{S}^{i_0}_\infty \to \mathbb{S}^{n-i_0-1}$. $\mathbf{\Gamma}_{\tilde{E}}$ has an induced action on $\mathbb{S}^{n-i_0-1}$ and acts on a properly convex set K'' in $\mathbb{S}^{n-i_0-1}$ so that K equals a strict join $k * K''$ for k corresponding to $\mathbb{S}^{i_0+1}$. (Recall the projection $\mathbb{S}^n - \mathbb{S}^{i_0}_\infty$ to $\mathbb{S}^{n-i_0-1}$.)

We recall the invariants from the form of (7.3.40).

$$\aleph_7(g) := \frac{a_7(g)}{\lambda_{\mathrm{v}_{\tilde{E}}}(g)} - \frac{\left\|\vec{v}_g\right\|^2}{2}$$

for every $g \in \mathbf{\Gamma}_{\tilde{E}}$. Recall

$$\aleph_7(g^n) = n\aleph_7(g) \text{ and } \aleph_7(gh) = \aleph_7(g) + \aleph_7(h), \text{ whenever } g, h, gh \in \mathbf{\Gamma}_{\tilde{E}}. \tag{7.3.51}$$

Under Hypothesis 7.3.15, Lemma 7.3.10 shows that every $g \in \mathbf{\Gamma}_{\tilde{E}}$ is of form:

$$\left(\begin{array}{c|c|c|c} S_g & 0 & 0 & 0 \\ \hline 0 & \lambda_g & 0 & 0 \\ \hline 0 & \lambda_g \vec{v}_g^T & \lambda_g O_5(g) & 0 \\ \hline 0 & \lambda_g \left(\aleph_7(g) + \frac{\left\|\vec{v}_g\right\|^2}{2}\right) & \lambda_g \vec{v}_g O_5(g) & \lambda_g \end{array}\right), \tag{7.3.52}$$

and every element of $\mathscr{N}$ is of form

$$\mathscr{N}(\vec{v}) = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline 0 & \frac{\|\vec{v}\|^2}{2} & \vec{v} & 1 \end{array}\right). \tag{7.3.53}$$

We assumed $\mu \equiv 1$. We define

$$\lambda_k(g) := \lambda_{\mathrm{v}_{\tilde{E}}}(g) \text{ for } g \in \mathbf{\Gamma}_{\tilde{E}}. \tag{7.3.54}$$

We define $\lambda_{K''}(g)$ to be the maximal norm of the eigenvalue occurring for $S(g)$. We define $\mathbf{\Gamma}_{\tilde{E},+}$ to be a subset of $\mathbf{\Gamma}_{\tilde{E}}$ consisting of elements g so that the following hold:

- the largest norm $\lambda^{Tr}_{\max}(g)$ of the eigenvalues occurs at the vertex k, i.e., $\lambda_1(g) = \lambda_k(g)$, and
- all other norms of the eigenvalues occurring at K''' is strictly less than $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$.

Then since $\mu_g = 1$, we necessarily have $\lambda_k(g) = a_1(g) = a_5(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)$ and hence $\lambda^{Tr}_{\max}(g) = \lambda_{\mathrm{v}_{\tilde{E}}}(g)$ for $g \in \mathbf{\Gamma}_{\tilde{E},+}$.

The second largest norm $\lambda_2(g)$ equals $\lambda_{K''}(g)$. Thus, $\mathbf{\Gamma}_{\tilde{E},+}$ is a semigroup. The condition that $\aleph_7(g) \geq 0$ for $g \in \mathbf{\Gamma}_{\tilde{E},+}$ is said to be the *nonnegative translation condition*.

Again, we define

$$\mu_7(g) := \frac{\aleph_7(g)}{\log \frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g)}{\lambda_2(g)}} \text{ for } g \in \mathbf{\Gamma}_{\tilde{E},+}.$$

The condition

(7.3.55) $$\mu_7(g) > C_0, g \in \mathbf{\Gamma}_{\tilde{E},+} \text{ for a uniform constant } C_0, C_0 > 0$$

is called the *uniform positive translation condition*. (Heuristically, the condition means that we don't translate in the negative direction by too much for bounded $\frac{\lambda_{\mathrm{v}_{\tilde{E}}}(g)}{\lambda_2(g)}$.)

LEMMA 7.3.16. *The condition* $\aleph_7(g) \geq 0$ *for* $g \in \mathbf{\Gamma}_{\tilde{E},+}$ *is a necessary condition so that* $\mathbf{\Gamma}_{\tilde{E}}$ *acts on a properly convex domain.*

PROOF. Suppose that $\aleph_7(g) < 0$ for some $g \in \mathbf{\Gamma}_{\tilde{E},+}$. Let $k' \in K^o$. Now, we use (7.3.12) and see that $\{g^n(U_{k'})\}$ converges geometrically to an (i_0+1)-dimensional hemisphere since $\{\aleph_7(g^n)\} \to -\infty$ as $n \to \infty$ implies that g translates the affine subspace $H^o_{k'}$ a component to $H^o_{g^n(k')}$ toward $(\!(-1,0,\dots,0)\!)$ in the above coordinate system by (7.3.52). Thus, $\mathbf{\Gamma}_{\tilde{E}}$ cannot act on a properly convex domain. (See Remark 7.3.13 also.) □

From the matrix equation (7.3.52), we define $\vec{v}_g$ for every $g \in \langle \mathbf{\Gamma}_{\tilde{E}}, \mathscr{N} \rangle$. (We just need to do this under a single coordinate system.)

LEMMA 7.3.17. *Given* $\mathbf{\Gamma}_{\tilde{E}}$ *satisfying Hypotheses 7.3.4 and 7.3.15, let* γ_m *be any sequence of elements of* $\mathbf{\Gamma}_{\tilde{E},+}$ *so that* $\{\lambda_k(\gamma_m)/\lambda_{K''}(\gamma_m)\} \to \infty$. *Then we can replace it by another sequence* $\{g_m^{-1}\gamma_m\}$ *for* $g_m \in \mathbf{\Gamma}_{\tilde{E}}$ *so that*

$$\left\| \vec{v}_{g_m^{-1}\gamma_m} \right\| \textit{ and } \Pi^*_K(g_m) \in \mathbf{Aut}(K)$$

are uniformly bounded, and

$$\left\{ \frac{\lambda_k(g_m^{-1}\gamma_m)}{\lambda_{K''}(g_m^{-1}\gamma_m)} \right\} \to \infty.$$

PROOF. Denote $\vec{v}_m := \vec{v}_{\gamma_m}$. Suppose that N_K is discrete. Then since its action on the interior of K is properly discontinous, we have an orbifold bundle $\mathrm{bd}U/\mathbf{\Gamma}_{\tilde{E}} \to K^o/N_K$. This means that the subgroup $\mathbf{\Gamma}_{\tilde{E},l}$of $\mathbf{\Gamma}_{\tilde{E}}$ acting on a complete affine leaf l acts cocompactly on l giving us the fibers. Since the stabilizer of N_K on each point of K^o is finite, $\mathbf{\Gamma}_{\tilde{E}} \cap \ker \Pi^*_K$ acts on l cocompactly. The action of $\mathscr{N}\hat{O}(i_0)$ is proper on each leaf. Hence, $\mathbf{\Gamma}_{\tilde{E}} \cap \mathscr{N}\hat{O}(i_0)$ is a lattice in $\mathscr{N}\hat{O}(i_0)$. By cocompactness of $\mathbf{\Gamma}_{\tilde{E}} \cap \mathscr{N}\hat{O}(i_0)$ in $\mathscr{N}\hat{O}(i_0)$, we can multiply γ_m by g_m^{-1} for an element g_m of $\mathbf{\Gamma}_{\tilde{E}} \cap \mathscr{N}\hat{O}(i_0)$ nearest to $\mathscr{N}(\vec{v}_m)$. The result follows since the action on $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is given by only $\vec{v}_g$ and S_g for $g \in \mathbf{\Gamma}_{\tilde{E}}$ as we can see from (7.3.52). The last convergence follows since $S(g_m) = \mathrm{I}_{n-i_0}$ and the matrix multiplication form of $g_m^{-1}\gamma_m$ considering the top left $(n-i_0)\times(n-i_0)$-submatrix and the bottom right $(i_0+2)\times(i_0+2)$-submatrix.

We now assume that N_K is non-discrete. $\tilde{\Sigma}_{\tilde{E}}$ has a compact fundamental domain F. Thus, given $\vec{v}_m$, for $x \in F$,

$$\mathscr{N}(\vec{v}_m)(x) \in g_m(F) \text{ for some } g_m \in \mathbf{\Gamma}_{\tilde{E}}.$$

Then $g_m^{-1}\mathscr{N}(\vec{v}_m)(x) \in F$. Since

$$g_m(y) = \mathscr{N}(\vec{v}_m)(x) \in g_m(F) \text{ for } y \in F \text{ and } x \in F,$$

it follows that

$$d_K\bigg(\Pi_K(y), \Pi_K^*(g_m)(\Pi_K(y)) = \Pi_K(x)\bigg) < C_F \tag{7.3.56}$$

for a constant C_F depending on F.

(i) g_m is of form of matrix (7.3.52)
(ii) S_{g_m} is in a bounded neighborhood of I by above (7.3.56) since $\hat{S}_{g_m}$ moves a point of a compact set F to a uniformly bounded set. (This follows by considering the Hilbert metric.)

From the linear block form of $g_m^{-1}\mathscr{N}(\vec{v}_m)$ and the fact that $g_m^{-1}\mathscr{N}(\vec{v}_m)(x) \in F$, we obtain that the corresponding $\vec{v}_{g_m^{-1}\mathscr{N}(\vec{v}_m)}$ can be made uniformly bounded independent of $\vec{v}_m$.

For element γ_m above, we take its vector $\vec{v}_{\gamma_m}$ and find our g_m for $\mathscr{N}(\vec{v}_{\gamma_m})$. We obtain $\gamma_m' := g_m^{-1}\gamma_m$. Then the corresponding $\vec{v}_{g_m^{-1}\gamma_m}$ is uniformly bounded as we can see from the block multiplications and the action on $\tilde{\Sigma}_{\tilde{E}}$ in $\mathbb{S}^{n-1}_{\tilde{E}}$. The final part follows from (ii) and the fact that $\{\lambda_k(\gamma_m)/\lambda_{K''}(\gamma_m)\} \to \infty$ and the matrix multiplication form of $g_m^{-1}\gamma_m$. considering the top left $(n-i_0)\times(n-i_0)$-submatrix and the bottom right $(i_0+2)\times(i_0+2)$-submatrix. □

LEMMA 7.3.18. *Suppose that the holonomy group of $\mathscr{O}$ is strongly irreducible. Given $\mathbf{\Gamma}_{\tilde{E}}$ satisfying Hypotheses 7.3.4 and 7.3.15. let U be the properly convex p-end neighborhood of $\tilde{E}$. Let H_k be the i_0+1-hemisphere mapping to the vertex k of Hypothesis 7.3.15 under Π_K. Then the interior of* $\mathrm{Cl}(U)\cap H_k$ *is not an open domain B with* $\mathrm{bd}B \ni \mathrm{v}_{\tilde{E}}$.

PROOF. Since $\mathscr{N}$ acts on H_k, it acts on B also. The matrix form of $\mathscr{N}$ is given by the coordinates where H_k is the projectivization of the span of $e_{n-k},\dots,e_{n+1}$ as we can see from (7.3.28) in the proof of Lemma 7.3.9. Hence, B is an ellipsoid as we can see from the form of $\mathscr{N}$ in (7.3.53). First of all,

$$\aleph_7(h) = 0 \text{ for all } h \in \mathbf{\Gamma}_{\tilde{E}} \tag{7.3.57}$$

by Lemma 7.3.16 since otherwise by (7.3.52)

$$\{\gamma^i(B)\} \to H_k \text{ as } i \to \infty \text{ or } -\infty \text{ for } \gamma \text{ with } \aleph_7(\gamma) \neq 0.$$

Since $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is compact, we have a sequence $\gamma_i \in \mathbf{\Gamma}_{\tilde{E},+}$ where

$$\left\{\frac{\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma_i)}{\lambda_2(\gamma_i)}\right\} \to \infty, \aleph_7(\gamma_i) = 0, \text{ and } \{\gamma_i|K''\} \text{ are uniformly bounded.}$$

Now modify γ_i to $g_i^{-1}\gamma_i$ by Lemma 7.3.17 for g_i obtained there. Hence, rewriting γ_i as the modified one, we have

$$\left\{\frac{\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma_i)}{\lambda_2(\gamma_i)}\right\} \to \infty, \aleph_7(\gamma_i) = 0, \vec{v}_{\gamma_i} \text{ and } \gamma_i|K'' \text{ is uniformly bounded.}$$

Recall that K is a strict join $K'' * \{k\}$ for a properly convex domain $K'' \subset \mathrm{bd}\tilde{\mathscr{O}}$ of dimension $n-i_0-2$ and a vertex k. Denote by $\mathbb{S}(K'')$ and $\mathbb{S}(H)$ the subspaces spanned by K' and H_k forming a pair of complementary subspaces in $\mathbb{S}^n$.

From the form of the lower-right $(i_0+2)\times(i_0+2)$-matrix of the above matrix, h_i must act on the horosphere $H \subset \mathbb{S}(H_k)$. $\mathscr{N}$ also act transitively on H_k. Hence, for any such matrix we can find an element of $\mathscr{N}$ so that the product is in the orthogonal group acting on H_k.

Now, this is the final part of the proof: Let $H_{\max}$ denote $\mathbb{S}(H_k)\cap\mathrm{Cl}(\tilde{\mathscr{O}})$ and $K''_{\max}$ the set $\mathbb{S}(K'')\cap\mathrm{Cl}(\tilde{\mathscr{O}})$. Since $\{\vec{v}_{\gamma_m}\}$ is bounded and $\aleph_7(\gamma_m) = 0$, we have the sequence $\{\gamma_m\}$

- acting on $K''_{\max}$ is uniformly bounded and
- γ_m acting on $H_{\max}$ in a uniformly bounded manner as $m \to \infty$.

By Proposition 1.4.19 for $l = 2$ case, $\mathrm{Cl}(\tilde{\mathscr{O}})$ equals the join of $H_{\max}$ and $K''_{\max}$. This implies that $\mathbf{\Gamma}$ is virtually reducible by Proposition 1.4.18, contradicting the premise of the lemma. □

For this proposition, we do not assume N_K is discrete. The assumptions below are just Hypotheses 7.3.4 and 7.3.15. Also, we don't need the assumption of the proper convexity of $\mathscr{O}$.

PROPOSITION 7.3.19 (Quasi-joins). *Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-end $\tilde{E}$ of a strongly tame convex n-orbifold $\mathscr{O}$. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group. Let $\tilde{E}$ be an NPNC R-p-end and $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathscr{N}$ acts on a p-end-neighborhood U fixing $\mathrm{v}_{\tilde{E}}$. Let $K, K'', K''', \mathbb{S}^{i_0}_\infty$, and $\mathbb{S}^{i_0+1}$ be as in Hypotheses (7.3.4) and (7.3.15). We assume that $K^o/\mathbf{\Gamma}_{\tilde{E}}$ is compact, $K = K'' * \{k\}$ in $\mathbb{S}^{n-i_0-1}$ with a point k corresponding to $\mathbb{S}^{i_0+1}$ under the projection Π_K. Assume that*

- *$\mathbf{\Gamma}_{\tilde{E}}$ satisfies the transverse weak middle-eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$.*
- *$\mathbf{\Gamma}_{\tilde{E}}$ acts on K''' and k.*
- *$\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$.*
- *Elements of $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathscr{N}$ are of form (7.3.28) and (7.3.29). with*
$$C_1(\vec{v}) = 0, c_2(\vec{v}) = 0, C_1(g) = 0, c_2(g) = 0$$
for every $\vec{v} \in \mathbb{R}^{i_0}$ and $g \in \mathbf{\Gamma}_{\tilde{E}}$.
- *$\mathbf{\Gamma}_{\tilde{E}}$ normalizes $\mathscr{N}$, and $\mathscr{N}$ acts on U and each leaf of $\mathscr{F}_{\tilde{E}}$ of $\tilde{\Sigma}_{\tilde{E}}$.*

Then the following hold:

(i) *The uniform positive translation condition is equivalent to the existence of a properly convex p-end-neighborhood U' whose closure meets $\mathbb{S}^{i_0+1}_k$ at $\mathrm{v}_{\tilde{E}}$ only. This condition furthermore is equivalent to $\tilde{E}$ being quasi-joined p-end.*

(ii) *$\aleph_7$ is identically zero if and only if $\mathscr{C}\mathscr{H}(U)$ is the interior of the join $K''' * B$ for an open ball B in $\mathbb{S}^{i_0+1}_k$, and $\mathscr{C}\mathscr{H}(U)$ is properly convex.*

(iii) *Suppose that $\tilde{E}$ is a quasi-joined p-end. Then a properly convex p-end neighborhood U' is radially foliated by line segments from $\mathrm{v}_{\tilde{E}}$, and $\mathrm{bd}U' \cap \tilde{\mathscr{O}}$ is strictly convex with limit points only in $K''' * \{\mathrm{v}_{\tilde{E}}\}$.*

(iv) $\mathrm{Cl}(U') \cap \mathrm{bd}\mathbb{A}^n = K''' * \{\mathrm{v}_{\tilde{E}}\}$.

PROOF. (A) We will first find some coordinate system and describe the action of $\mathbf{\Gamma}_{\tilde{E}}$ on it.

Let H be the unique n-hemisphere containing segments in directions of $\tilde{\Sigma}_{\tilde{E}}$ from $\mathrm{v}_{\tilde{E}}$ where ∂H contains $\mathbb{S}^{i_0}_\infty$ and K''' in general position by Hypothesis 7.3.15. Then H^o is an affine subspace to be also denoted by $\mathbb{A}^n$ containing U. Since $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathscr{N}$ act on K''' and $\mathbb{S}^{i_0}_\infty$, $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathscr{N}$ act as an affine transformation group on $\mathbb{A}^n$.

Let H_l denote the hemisphere with boundary $\mathbb{S}^{i_0}_\infty$ and corresponding to a leaf l of the foliation on $\tilde{\Sigma}_{\tilde{E}}$. Recall that K^o denote the leaf space. Let $\mathbb{A}^n$ be the affine space whose boundary contains $\mathbb{S}^{i_0}_\infty$ and a leaf l is the i_0+1-dimensional affine subspace with transverse affine space of dimension $n-(i_0+1)$ meeting it at a point, considered as the origin. We may further require from $\mathrm{v}_{\tilde{E}}$ the space of directions to $\mathbb{A}^n$ contains those to U. Furthermore, the projection to the affine space of dimension $n-i_0-1$ with kernel vector space parallel to l, we obtain an affine space $\mathbb{A}^{n-i_0-1}$ with an origin. The image of the projection of leaves is projectively diffeomorphic to K^o. We can consider K^o as a cone from the origin in $\mathbb{A}^{n-i_0-1}$. The directions of the cone is isomorphic to K''^o. We may regard K''' as a subset of the ideal boundary of $\mathbb{A}^{n-i_0-1}$. Hence, we projectively identify

$$\bigcup_{l \in K^o} H_l^o = K^o \times \mathbb{R}^{i_0+1} \subset \mathbb{A}^n.$$

(It might be helpful to see Figure 1 where we convert the coordinates so that $(0,0,0)$, $\mathbb{S}^1_\infty$, and $\hat{K}$ are now in $\mathrm{bd}\mathbb{A}^n$.)

There exists a family of affine subspaces in $\mathbb{A}^n$ parallel to $\mathbb{A}^{n-i_0-1}$. Also, there is a transverse family of affine spaces forming a foliation $\mathscr{V}^{i_0+1}$ where each leaf is the complete affine space parallel to $\mathbb{A}^{i_0+1}$. We denote it by $\{(\vec{x}_1,1)\} \times \mathbb{R}^{i_0+1}$ for $\vec{x}_1 \in \mathbb{R}^{n-i_0-1}$. The affine coodinates are given by $(\vec{x}_1, 1, \vec{x}_2, x_{n+1})$ where 1 is at the $n-i_0$th position, $\vec{x}_1$ is an $n-i_0-1$-vector and $\vec{x}_2$ is an i_0-vector.

Now we describe $H_l^o \cap U$ for each $l \in K^o$. We use the affine coordinate system on $\mathbb{A}^n$ so that H_l^o are parallel affine i_0+1-dimensional spaces with origins in $\mathbb{A}^{n-i_0-1}$. We use the parallel affine coordinates. According to the matrix form (7.3.53), $\mathscr{N}$ acts on each $x \times \mathbb{R}^{i_0+1}$, $x \in \hat{K}^o$.

We denote each point in H_l^o by $(\tilde{x}, x_{n+1})$ where $\tilde{x}$ is a point of $\mathbb{A}^{i_0}$. Each of $E_l := H_l \cap U$ is given by

$$x_{n+1} > \|\vec{x}_2\|^2/2 + C_l, C_l \in \mathbb{R} \tag{7.3.58}$$

since $\mathscr{N}$ acts on each where C_l is a constant depending on l and U by (7.3.40). (The vertex $\mathrm{v}_{\tilde{E}}$ corresponds to the ideal point in the positive infinity in terms of the x_{n+1}-coordinate. See Section 7.3.1.1.)

There is a family of quadrics of form $Q_{l,C}$ defined by $x_{n+1} = \|\vec{x}_2\|^2/2 + C$ for each $C \in \mathbb{R}$ on each leaf l of $\mathbb{V}^{i_0+1}$ using the affine coordinate system. The family form a foliation $\mathscr{Q}_l$ for each l.

Now we describe $\mathbf{\Gamma}_{\tilde{E}}$-action. Since $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$, it follows that $\lambda_{\mathrm{v}_{\tilde{E}}} = \lambda_k$ by definition (7.3.54). Given a point $x = ((\vec{v})) \in U' \subset \mathbb{S}^n$ where $\vec{v} = \vec{v}_s + \vec{v}_h$ where $\vec{v}_s$ is in the direction of $\mathbb{S}(K''')$ and $\vec{v}_h$ is in one of $\mathbb{S}_k^{i_0+1}$. If $g \in \mathbf{\Gamma}_{\tilde{E},+}$, then we obtain

$$g((\vec{v})) = ((g\vec{v}_s + g\vec{v}_h)) \text{ where } ((g\vec{v}_s)) \in K''' \text{ and } ((g\vec{v}_h)) \in H_k. \tag{7.3.59}$$

Let $\Pi_{i_0} : U \to \mathbb{R}^{i_0+1}$ be the projection to the last i_0+1 coordinates $x_{n-i_0}, \dots, x_n$. We obtain a commutative diagram and an affine map L_g induced from g

$$\begin{array}{ccc} H_l^o & \xrightarrow{g} & g(H_l^o) \\ \Pi_{i_0} \downarrow & & \Pi_{i_0} \downarrow \\ \mathbb{R}^{i_0+1} & \xrightarrow{L_g} & \mathbb{R}^{i_0+1}. \end{array} \tag{7.3.60}$$

By (7.3.40), L_g preserves the family of quadrics $\mathscr{Q}_l$ to $\mathscr{Q}_{g(l)}$ since $\mathscr{N}$ acts on the quadrics $U \cap H_l^o$ for each l and g normalizes $\mathscr{N}$ by Hypothesis 7.3.4. Also, L_g is an affine map since L_g is a projective map sending a complete affine subspace H_l^o to a complete affine subspace $g(H_l^o)$. Finally, by (7.3.40), g sends the family of quadrics shifted in the x_{n+1}-direction by $\aleph_7(g)$ from l to $g(l)$ using the coordinates $(\vec{x}, x_{n+1})$ for $\vec{x} \in \mathbb{R}^{i_0}$. That is,

$$g : Q_{l,C} \mapsto Q_{g(l), C+\aleph_7(g)}. \tag{7.3.61}$$

(B) Now we give proofs. By assumption, $\mathbf{\Gamma}_{\tilde{E}}$ acts on $K = K'' * \{k\}$. Choose an element $\eta \in \mathbf{\Gamma}_{\tilde{E},+}$ by Proposition 1.4.10 so that $\lambda^{Tr}_{\max}(\eta) > \lambda_2(\eta)$ where $\lambda_1(\eta)$ corresponds to a vertex k and $\lambda_2(\eta)$ is associated with K''', and let F be the fundamental domain in the convex open cone K^o with respect to $\langle \eta \rangle$, which is a bounded domain in $\mathbb{A}^{n-i_0-1}$. This corresponds to a radial subset from $\mathrm{v}_{\tilde E}$ bounded away at a distance from K''' in U.

(i) This long proof will be devided as follows.

- **(i-a):** Forward part: We show U has a convex hull that is properly convex.
  - **(i-a-1):** We show that the forward images of the fundamental domain of U under $\langle \eta \rangle$ is contained in $K^o \times \sigma'^{,o}$ for some simplex σ'.
  - **(i-a-2):** Next, we will try to show that the backward images the fundamental domain under $\langle \eta \rangle$ is eventually in any neighborhood of $K''' * \{\mathrm{v}_{\tilde E}\}$.
  - **(i-a-3):** Finally, we will show that the convex hull of U is in a properly convex domain.
- **(i-b):** Converse part: We prove the uniform positive translation condition under the assumption.

(i-a) (i-a-1) Let $\lambda_{K'''}(g)$ denote the maximal eigenvalue associated with K''' for $g \in \mathbf{\Gamma}_{\tilde E}$. Choose $x_0 \in F$. Let $\mathbf{\Gamma}_{\tilde E,F} := \{g \in \mathbf{\Gamma}_{\tilde E} | g(x_0) \in F\}$. For $g \in \mathbf{\Gamma}_{\tilde E,F}$,

$$-C_F < \log \frac{\lambda_{\mathrm{v}_{\tilde E}}(g)}{\lambda_{K'''}(g)} < C_F \tag{7.3.62}$$

for a uniform $C_F > 0$ a number depending of F only: Otherwise, we can find

- a sequence g_i with $g_i(x_0) \in F$ such that $\{\lambda_k(g_i)/\lambda_{K'''}(g_i)\} \to 0$ or
- another sequence g'_i with $g'_i(x_0) \in F$ such that $\{\lambda_k(g'_i)/\lambda_{K'''}(g'_i)\} \to \infty$.

However, in the first case, let $\tilde x_0 \in U$ be a point mapping to x_0 under Π_K. Then $\{g_i(\tilde x_0)\}$ accumulates only to points of K''' by Proposition 1.3.2, which is absurd. The second case is also absurd by taking $\{g_i^{-1}(x_0)\}$ instead.

Therefore, given $g \in \mathbf{\Gamma}_{\tilde E,F}$, we can find a number $i_0 \in \mathbb{Z}$ dependent only on F and g such that $\eta^{i_0} g \in \mathbf{\Gamma}_{\tilde E,+}$ since $\log \frac{\lambda_k(\eta)}{\lambda_{K'''}(\eta)}$ is a constant bigger than 1. Now, $\aleph_7(\eta^{i_0} g)$ is bounded below by some negative number by the uniform positive eigenvalue condition (7.3.55) and the fact that $\left|\log \frac{\lambda_{\mathrm{v}_{\tilde E}}(\eta^{i_0} g)}{\lambda_{K'''}(\eta^{i_0} g)}\right|$ is also uniformly bounded. Since $\aleph_7(\eta^{i_0} g) = i_0 \aleph_7(\eta) + \aleph_7(g)$, we obtain

$$\{\aleph_7(g) | g \in \mathbf{\Gamma}_{\tilde E,F}\} > C \tag{7.3.63}$$

for a constant C by (7.3.55). F is covered by $\bigcup_{g \in \mathbf{\Gamma}_{\tilde E,F}} g(J)$ for a compact fundamental domain J of K^o by N_K.

We will be using a fixed affine coordinate system on $H^o_{k'}$ parallel under the translations preserving $\mathbb{A}^{n-i_0-1}$. In the above affine coordinates for $k' \in F$ of (7.3.60), the matrix form of (7.3.52) shows that $g \in \mathbf{\Gamma}_{\tilde E}$ send paraboloids in affine subspaces $H^o_{k'}$ for $k' \in K^o$ to paraboloids in $H^o_{g(k')}$ for $g \in \mathbf{\Gamma}_{\tilde E}$. (See Section 7.3.1.1.) Now,

$$x_n(H^o_{k'} \cap U) > C \tag{7.3.64}$$

for a uniform constant $C \in \mathbb{R}$ by (7.3.63) and the fact that $H^o_{k'}$ is the image of $H^o_{k''}$ for $k'' \in J$ by an element $g \in \mathbf{\Gamma}_{\tilde E}$. (See Remark 7.3.13.)

Since by (7.3.64),

$$\bigcup_{k' \in J} \bigcup_{g' \in \mathbf{\Gamma}_{\tilde E,F}} g(H^o_{k'} \cap U)$$

is a lower-x_n-bounded set, its convex hull D_F as a lower-x_n-bounded subset of $K^o \times \mathbb{A}^{i_0+1} \subset \mathbb{A}^n$. Each region $D_F \cap H_{l'}$ is contained an (i_0+1)-dimensional simplex σ_0 with a face in the boundary of $H_{l'}$. Since there is a lower x_n-bound, we may use one σ_0 and translations to contain every $U \cap H_{l'}$ in D_F.

Therefore, the convex hull D_F in $\mathrm{Cl}(\tilde{\mathscr{O}})$ is a properly convex set contained in a properly convex set $F \times \sigma_0$.

On K''', the sequence of norms of eigenvalues of η^i converges to 0 as $i \to \infty$ and the eigenvalue $\lambda_{\mathrm{v}_{\tilde{E}}}$ at $\mathbb{S}_k^{i_0+1}$ goes to $+\infty$. Since

$$\aleph_7(\eta^i) = i\aleph_7(\eta) \to +\infty \text{ as } i \to \infty, \tag{7.3.65}$$

we obtain that

$$\{\eta^i(D_F)\} \to \{\mathrm{v}_{\tilde{E}}\} \text{ for } i \to \infty$$

geometrically, i.e., under the Hausdorff metric $\mathbf{d}_H$ by (7.3.59). Again, for a sufficiently large integer I, because of the lower bound on the x_n-coordinates, we obtain

$$\bigcup_{i \geq I} \eta^i(D_F) \subset K^o \times \sigma_0, \tag{7.3.66}$$

which is our first main result of this proof of the forward part of (i).

(i-a-2) For each $k' = ((\vec{x}_1, 1)) \in K^o$, we can find a point in $\Pi_K^{-1}(k')$ of form $\left(\left(\vec{x}_1, 1, \vec{0}, C_{k'}\right)\right)$ in $\mathrm{bd} U \cap \mathbb{A}^n$ for $\vec{0}$ a zero vector in $\mathbb{R}^{i_0}$ and $C_{k'} \in \mathbb{R}$. Using the $\mathscr{N}$-action, we can parameterize $\mathrm{bd} U \cap \mathbb{A}^n$ starting from the point $\left(\left(\vec{x}_1, 1, \vec{0}, C_{k'}\right)\right)$. The $\mathscr{N}$-orbit of this point is given by

$$\left(\left(\vec{x}_1, 1, \vec{v}, \|\vec{v}\|^2/2 + C_{k'}\right)\right), \vec{v} \in \mathbb{R}^{i_0}. \tag{7.3.67}$$

Let

$$p_i := \left(\left(\vec{x}_1(p_i), 1, \vec{v}(p_i), \|\vec{v}(p_i)\|^2/2 + C_{k'(p_i)}\right)\right).$$

We form a sequence $\{p_i\}$ of points on ∂U for $k'(p_i) = ((\vec{x}_1(p_i), 1))$. Consider η^i in the form (7.3.52). Since $\bigcup_{i \in \mathbb{Z}} \eta^i(F)$ covers K^o, for each p_i there is an integer j_i for which $\eta^{j_i}(F)$ containing $((\vec{x}_1(p_i), 1))$. Suppose that $j_i \to \infty$ as $i \to \infty$. From considerations of η^{j_i}, we deduce that

$$\left\{C_{((\vec{x}_1(p_i),1))}\right\} \to +\infty \text{ as } \|\vec{x}_1(p_i)\| \to 0 \tag{7.3.68}$$

by (7.3.65).

Similarly, suppose that $j_i \to -\infty$ as $i \to \infty$. We deduce that

$$\left\{\frac{C_{((\vec{x}_1(p_i),1))}}{\|\vec{x}_1(p_i)\|}\right\} \to 0 \text{ as } \|\vec{x}_1(p_i)\| \to \infty \tag{7.3.69}$$

since $\aleph_7(\eta^{j_i}) = j_i\aleph_7(\eta) \to -\infty$ in a linear manner, and every sequence of $\|\vec{x}_1(p_i)\|$-coordinates of points of $\eta^{j_i}(F)$ grows uniformly exponentially as $j_i \to -\infty$.

We will now try to find all limit points of $\{p_i\}$:

- Suppose first that $\|\vec{x}_1(p_i)\| \to \infty$ and $\frac{\|\vec{x}_1(p_i)\|}{\|\vec{v}(p_i)\|^2} \to \infty$. We obtain that

  $$\left\{\left(\left(\vec{x}_1(p_i), 1, \vec{v}(p_i), \|\vec{v}(p_i)\|^2/2 + C_{k'(p_i)}\right)\right)\right\}, \vec{v}(p_i) \in \mathbb{R}^{i_0}$$

  has only limit points of form $((\vec{u}, 0, 0, 0))$ for a unit vector $\vec{u}$ in the direction of K'' by (7.3.69). Hence, the limit is in K'''.

- Suppose that we have $\|\vec{x}_1(p_i)\| \to +\infty$ with $\|\vec{v}(p_i)\|^2 \to +\infty$ with their ratios bounded between two real numbers. Then a limit point is of form $(\!(\vec{u},0,0,C)\!)$ for some $C > 0$ and a vector $\vec{u}$ in the direction a point of K''' by (7.3.69). Also, every direction of K'' occurs as a direction of $\vec{u}$ for a limit point by taking a sequence $\{p_i\}$ so that $\{\vec{x}_1(p_i)\}$ converges to $\vec{u}$ in directions. Hence, the limits are in $K''' * \{\mathrm{v}_{\tilde{E}}\}$.
- Suppose that $\|\vec{x}_1(p_i)\| \to +\infty$ with $\|\vec{v}(p_i)\|^2 \to +\infty$ and $\frac{\|\vec{x}_1(p_i)\|}{\|\vec{v}(p_i)\|^2} \to 0$. Then the only limit point is $\left(\!\left(\vec{0},0,0,1\right)\!\right)$ since the last term dominates others.
- Suppose that $1/C' \leq \|\vec{x}_1(p_i)\| \leq C'$ for a constant C'. If $\|\vec{v}(p_i)\|$ is uniformly bounded, then (7.3.64) shows that limit points in $\mathrm{bd}U$. If $\|\vec{v}(p_i)\| \to \infty$, the only limit point is $(\!(0,0,0,1)\!)$.
- Suppose that $\|\vec{x}_1(p_i)\| \to 0$. Then the limit is $(\!(0,0,0,1)\!)$ by (7.3.68).

These gives all the limit points of U in $\mathrm{bd}\mathbb{A}^n$ as we can easily deduce. Therefore,

$$\mathrm{Cl}(U) \cap \mathrm{bd}\mathbb{A}^n \subset K''' * \{\mathrm{v}_{\tilde{E}}\} \tag{7.3.70}$$

by Theorem 1.5.12.

This also shows $\eta^i(D_F)$ geometrically converges to $K''' * \{\mathrm{v}_{\tilde{E}}\}$ as $i \to -\infty$: First, the above three items show that for every $\varepsilon > 0$, there exists i_0 so that $\eta^i(D_F) \subset N_\varepsilon(K''' * \{\mathrm{v}_{\tilde{E}}\})$ for $i > i_0$. Finally, since $\Pi_K(\eta^i(D_F))$ geometrically converges to K'', and we can find a sequence as in the first item converging to any point of K''', the geometric limit is $K''' * \{\mathrm{v}_{\tilde{E}}\}$. A bit unclear

For every $\varepsilon > 0$, there exists an integer I so that

$$\bigcup_{i<I} \eta^i(D_F) \subset N_\varepsilon(K''' * \mathrm{v}_{\tilde{E}}) \cap \mathrm{Cl}(\mathbb{A}^n).$$

(i-a-3) Thus, except for finitely many i,

$$\eta^i(D_F) \subset (N_\varepsilon(K''' * \mathrm{v}_{\tilde{E}}) \cap \mathrm{Cl}(\mathbb{A}^n)) \cup K \times \sigma_0 \subset \mathrm{Cl}(\mathbb{A}^n).$$

We use the Fubini-Study metric $\mathbf{d}$ on $\mathbb{S}^n$ where the subspaces spanned and K and $\{k\} \times \mathbb{R}^{i_0}$ are all orthogonal to each other.

Assume $\varepsilon < \pi/8$. Let p denote the vertex of the simplex $\{k\} \times \sigma_0$. We may assume without loss of generality that $p = k \times O$. Then we may assume that $N_\varepsilon(K''' * \mathrm{v}_{\tilde{E}}) \cap \mathrm{Cl}(\mathbb{A}^n)$ is in $\hat{K} * \{k\}$ where $\hat{K}$ is a properly compact $(n-1)$-ball containing $K''' * \{\mathrm{v}_{\tilde{E}}\}$ and is contained its 2ε-$\mathbf{d}$-neighborhood. $\sigma_\infty := \sigma_0 \cap \mathrm{bd}\mathbb{A}^n$ is an i_0-simplex containing $\mathrm{v}_{\tilde{E}}$. The join $\sigma_\infty * K'''$ is properly convex since σ_∞ and K''' are properly convex sets in independent subspaces. The join $\sigma_\infty * \hat{K}$ is contained in a 2ε-$\mathbf{d}$-neighborhood of $\sigma_\infty * K''' = \sigma_k * K''' * \{\mathrm{v}_{\tilde{E}}\}$. Hence, for a choice of ε, $\sigma_k * \hat{K}$ is properly convex. Since $\{k\} \times \sigma_0 = \sigma_\infty * \{p\}$, we obtain that $\sigma_0 * K'''$ is also properly convex. Hence, $\sigma_0 \times \hat{K}$ is also properly convex for a sufficiently small ε. Thus, for a finite set L, the convex hull U_1 of $\bigcup_{i \in \mathbb{Z}-L} \eta^i(D_F)$ in $\mathbb{A}^n$ is properly convex.

The convex hull of U_1 and $U_L := \bigcup_{i \in L} \eta^i(D_F)$ is still properly convex: Suppose not. Then there exists an antipodal pair in

$$\mathrm{Cl}(\mathcal{CH}(U_1 \cup U_L)) = \mathcal{CH}(\mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_L)).$$

The antipodal pair must be in $\mathrm{bd}\mathbb{A}^n$ since the interior of $\mathbb{A}^n$ has no antipodal pair. However, since $\mathrm{Cl}(U_1) \cap \mathrm{bd}\mathbb{A}^n$ is a properly convex set $K''' * \{\mathrm{v}_{\tilde{E}}\}$ and $\mathrm{Cl}(U_L) \cap \mathrm{bd}\mathbb{A}^n$ is $\{\mathrm{v}_{\tilde{E}}\}$. This is a contradiction.

Let U' denote the convex hull of $U_1 \cup U_L$ in $\mathbb{A}^n$. Hence, we showed that U' is properly convex.

(i-b) Now we prove the converse part of (i). Suppose that $\mathbf{\Gamma}_{\tilde{E}}$ acts on a properly convex p-end-neighborhood U'.

By Lemma 7.3.16, we have $\aleph_7(g) \geq 0$ for $g \in \mathbf{\Gamma}_{\tilde{E},+}$. Suppose that $\aleph_7(g) = 0$ for some $g \in \mathbf{\Gamma}_{\tilde{E},+}$. Then

$$\{g^i(\mathrm{Cl}(U) \cap H_l)\} \to B \text{ as } i \to \infty \text{ under } \mathbf{d}_H$$

for a leaf l and a compact domain B at H_k bounded by an ellipsoid. This contradicts the premise of (i). Therefore,

$$\mu_7(h) > 0 \text{ for every } h \in \mathbf{\Gamma}_{\tilde{E},+}. \tag{7.3.71}$$

Suppose that $\{\mu_7(g_i)\} \to 0$ for a sequence $g_i \in \mathbf{\Gamma}_{\tilde{E},+}$. We can assume that

$$\lambda^{Tr}_{\max}(g_i)/\lambda_2(g_i) > 1 + \varepsilon \text{ for a positive constant } \varepsilon > 0$$

since we can take powers of g_i not changing μ_7.

Since $\{\mu_7(g_i)\} \to 0$, we obtain a nondecreasing sequence $\{n_i\}$, $n_i > 0$, so that

$$\{\aleph_7(g_i^{n_i}) = n_i \aleph_7(g_i)\} \to 0 \text{ and } \{\lambda^{Tr}_{\max}(g_i^{n_i})/\lambda_2(g_i^{n_i})\} \to \infty.$$

However, from such a sequence, we use (7.3.40) to shows that

$$\{g_i^{n_i}(\mathrm{Cl}(U) \cap H_l)\} \to B$$

to a ball B with a nonempty interior in H_k. Again the premise contradicts this. Hence $\mu_7(g) > C$ for all $g \in \mathbf{\Gamma}_{\tilde{E},+}$ and a uniform constant $C > 0$. This proves the converse part of (i).

(ii) Suppose that $\aleph_7$ is identically zero for $\mathbf{\Gamma}_{\tilde{E},+}$. Then by (7.3.71) in the proof of (i),

$$\{g^i(\mathrm{Cl}(U) \cap H_l)\} \to B \text{ as } i \to \infty \text{ under } \mathbf{d}_H$$

for a leaf l and a compact domain B at H_k bounded by an ellipsoid. Choose $\hat{B}$ the maximal domain of form B as arising from the situation. Then we may show that $\mathcal{CH}(U) = (K''' * \hat{B})^o$ by Proposition 1.4.19 since we can find a sequence g_i so that $g_i|K'''$ is bounded and $\{\lambda_{K'''}(g)/\lambda_k(g)\} \to \infty$ since $(K'' * k)^o/\mathbf{\Gamma}_{\tilde{E}}$ is compact. Also, $(K''' * \hat{B})^o$ is properly convex.

Conversely, we have $\aleph_7 \geq 0$ by Lemma 7.3.16. By premise $\mathcal{CH}(U) = (K''' * B)^o$ where B is a convex open ball in a hemisphere H_k in $\mathbb{S}^{i_0}_k$. Since $\mathcal{N}$ acts on $\mathcal{CH}(U)$, B bounded by an ellipsoid. If $\aleph_7(g) > 0$ for some g, Then g acts on B so that $g(B)$ is a translated image of the region B bounded by a paraboloid in the affine subspace H^o_k. We obtain $\bigcup_{k=1}^{\infty} g^{-k}(B) = H^o_k$. This contradicts the proper convexity, and hence $\aleph_7$ is identically 0.

(iii) Since $\tilde{\mathcal{O}}$ is convex, we can find a radial p-end neighborhood U of $\tilde{E}$. Let p be a point of U. Then the orbit of p has limit points only in $K''' * \{\mathrm{v}_{\tilde{E}}\}$ in (i-b). Then we take the convex hull U' of $\bigcup \mathbf{\Gamma}_{\tilde{E}}(\overline{p\mathrm{v}_{\tilde{E}}})$. Since this is a convex set, it is a radial p-end neighborhood of $\tilde{E}$. $\mathrm{bd}U'$ is the boundary of a properly convex domain $\mathrm{Cl}(U')$, and hence is the union of $(n-1)$-dimensional compact convex domains.

The boundary $\mathrm{bd}U' \cap \tilde{\mathcal{O}}$ is a union of compact simplices since it contains no straight segment ending at a point of $K''' * \{\mathrm{v}_{\tilde{E}}\}$. Hence, $\mathrm{bd}U'$ is a polyhedral hypersurface.

Since $R_p(\tilde{\mathcal{O}}) = \tilde{\Sigma}_{\tilde{E}}$ is identical with $R_p(U')$ by Lemma 3.1.5,

Again, sharply supporting hyperspaces of $\mathrm{bd}U' \cap \mathbb{A}^n$ at a fundamental domain under $\mathbf{\Gamma}_{\tilde{E}}$ is a compact set. Since $\mathrm{bd}U' \cap \mathbb{A}^n$ does not contain segment ending in $\mathrm{bd}\mathbb{A}^n$, there are no sequence of supporting hyperplanes at a point q_i where q_i forms an unbounded sequence and meeting a fixed neighborhood of a point p in $\mathrm{bd}U' \cap \mathbb{A}^n$. Hence, Lemma 4.4.6 implies that we can choose a properly convex open p-end neighborhood U' of $\tilde{E}$ so that $\mathrm{bd}U' \cap \mathbb{A}^n$ is strictly convex.

(iv) We have $\mathrm{Cl}(U') \cap \mathrm{bd}\mathbb{A}^n \subset K''' * \{\mathrm{v}_{\tilde{E}}\}$ since U' is a p-end neighborhood and we can apply the same argument as U. Since every $g \in \mathbf{\Gamma}_{\tilde{E}}$ is of form (7.3.52), the action of $\mathbf{\Gamma}_{\tilde{E}}$ on K^o is sweeping, the action of $\mathbf{\Gamma}_{\tilde{E}}$ is sweeping on $K''' * \{\mathrm{v}_{\tilde{E}}\}$ as well while the matrix forms of elements restricted to the subspaces containing K and the second one are the same.

From (A), we see that $U' \subset K \times \mathbb{A}^{n-i_0+1}$ since U' is the convex hull of U. Consider the projection $K \to K'''$ from the vertex k of K. Consider (7.1.1), and recall that U' is an orbifold-bundle over $\Sigma_{\tilde{E}}$ with fibers that are radial rays. We see that there is a projection $U' \to K'''$ that is equivariant with respect to $\mathbf{\Gamma}_{\tilde{E}}$-action since $\mathbf{\Gamma}_{\tilde{E}}$ acts on K fixing the vertex k. Since U' is open, it contains a generic point projecting to a point of K^o.

Let N_K denote the image group of the action of $\mathbf{\Gamma}_{\tilde{E}}$ on K^i. Since $\mathbf{\Gamma}_{\tilde{E}}$ has a compact fundamental domain J in $\Sigma_{\tilde{E}}$. By taking the image of J under the projections, we see that there are compact subsets J' of K^o and J''' of $K^{''',o}$ mapping onto K^o/N_K and onto $K^{''',o}/N_K$ respectively. We may assume that J''' is the image of J' under the projection from k. By taking a ray from k with an endpoint at J''', and taking a sequence of points p_i on it conveing to p in J'''. We find $g_i \in N_K$ so that $g_i(p_i) \in J'$. This means that $g_i(p) \in J'''$ for all p. Since K''' is properly convex with a Hilbert metric, $g_i|K'''$ is bounded by Proposition 1.1.14 and we can extract a convergent sequence. Hence assume that $g_i|K''' \to g_\infty$. We may assume that $g_i|K''$ is convergent. Let $g_i' \in \mathbf{\Gamma}_{\tilde{E}}$ be the one going to g_i, Here, we can see that $\lambda_{\mathrm{v}_{\tilde{E}}}(g_i') \to 0$. Therefore, a generaic point of K''' is in the image of U under the sequence of image of g_i' for a point of U. This point can be an interior point of K'''.

Since $\mathrm{Cl}(U') \cap \mathbb{A}^n$ is convex and $\mathbf{\Gamma}_{\tilde{E}}$-invariant, it contains an interior point of K'''. By Proposition 3 of [151], the convex hull of any orbit of a point in $K^{''',o}$ equals $K^{''',o}$. We conclude that $\mathrm{Cl}(U') \cap \mathbb{A}^n = K''' * \{\mathrm{v}_{\tilde{E}}\}$.

□

DEFINITION 7.3.20.

- Generalizing Example 7.3.1, an R-p-end $\tilde{E}$ satisfying the case (ii) of Proposition 7.3.19 is a *strictly joined R-p-end* (of *a totally geodesic R-end and a horospherical end*) and $\mathbf{\Gamma}_{\tilde{E}}$ now is called a *strictly joined end group*. Also, any end finitely covered by a strcitly joined R-end is called a *strictly joined R-end*.
- An R-p-end $\tilde{E}$ satisfying the case (i) of Proposition 7.3.19, is a *quasi-joined* R-p-end (of *a totally geodesic R-end and a horospherical end*) corresponding to Definition 7.1.2 and $\mathbf{\Gamma}_{\tilde{E}}$ now is a quasi-joined end holonomy group.

Also, any p-end $\tilde{E}$ with $\mathbf{\Gamma}_{\tilde{E}}$ is a finite-index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$ as above is called by the corresponding names.

7.3.3.1. *The non-existence of strictly joined cases for* $\mu \equiv 1$.

COROLLARY 7.3.21. *Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathcal{O}$ with radial or totally geodesic ends. Assume that the holonomy group of $\mathcal{O}$ is strongly irreducible. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group. Assume Hypotheses 7.3.4 only and $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$. Then $\tilde{E}$ is not a strictly joined end.*

PROOF. Suppose that $\tilde{E}$ is a strictly joined end. By premise, $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$. By Lemma 7.3.9 and Proposition 7.3.14, every $g \in \mathbf{\Gamma}_{\tilde{E}}$ is of form:

$$\left(\begin{array}{c|c|c|c} S_g & 0 & 0 & 0 \\ \hline 0 & \lambda_g & 0 & 0 \\ \hline 0 & \lambda_g \vec{v}_g^T & \lambda_g O_5(g) & 0 \\ \hline 0 & \lambda_g\left(\aleph_7(g) + \frac{\|\vec{v}_g\|^2}{2}\right) & \lambda_g \vec{v}_g O_5(g) & \lambda_g \end{array}\right) \tag{7.3.72}$$

As in the proof of

By Proposition 7.3.14, we obtain a sequence γ_m of form from the step (D) of the proof:

$$\left(\begin{array}{c|c|c|c} \delta_m S_m & 0 & 0 & 0 \\ \hline 0 & \lambda_m & 0 & 0 \\ \hline 0 & \lambda_m \vec{v}_m^T & \lambda_m O_5(\gamma_m) & 0 \\ \hline 0 & \lambda_m\left(\aleph_7(\gamma_m) + \frac{\|\vec{v}_m\|^2}{2}\right) & \lambda_m \vec{v}_m O_5(\gamma_m) & \lambda_m \end{array}\right) \tag{7.3.73}$$

as $C_{1,m} = 0$ and $c_{2,m} = 0$ where

- $\{\lambda_m\} \to \infty$,
- $\{\delta_m\} \to 0$,
- $\{S_m\}$ is in a sequence of bounded matrices in $\mathsf{SL}_{\pm}(n - i_0 - 1)$, and
- $\aleph_7(\gamma_m) = 0$ by Proposition 7.3.19 (ii).

Moreover, Hypothesis 7.3.15 now holds. By Lemma 7.3.18, we obtain a contradiction. □

7.3.4. The proof for discrete N_K. Now, we go to proving Theorem 7.1.4 when N_K is discrete. By taking a finite-index torsion-free subgroup if necessary by Theorem 1.1.19, we may assume that N_K acts freely on K^o. We have a corresponding orbifold fibration

$$\begin{array}{ccc} l/N & \rightarrow & \tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}} \\ & & \downarrow \\ & & K^o/N_K \end{array} \tag{7.3.74}$$

where the fiber and the quotients are compact orbifolds since $\Sigma_{\tilde{E}}$ is compact. Here the fiber equals l/N for generic l. The action of N_K on K is semisimple by Theorem 3 of Vey [151].

Since N acts on each leaf l of $\mathscr{F}_{\tilde{E}}$ in $\tilde{\Sigma}_{\tilde{E}}$, it also acts on a properly convex domain $\tilde{\mathscr{O}}$ and $\mathrm{v}_{\tilde{E}}$ in a subspace $\mathbb{S}_l^{i_0+1}$ in $\mathbb{S}^n$ corresponding to l. $l/N \times \mathbb{R}$ is an open real projective orbifold diffeomorphic to $(H_l^{i_0+1} \cap \tilde{\mathscr{O}})/N$ for an open hemisphere $H_l^{i_0+1}$ corresponding to l. Since elements of N restricts to I on K, we obtain

$$\lambda_{\max}^{Tr}(g) = \lambda_{\min}^{Tr}(g) \text{ for all } g \in N:$$

Otherwise, we see easily g acts not trivially on $\mathbb{S}^{n-i_0-1}$. By Proposition 7.2.3, all the norms of eigenvalues are 1. Let P_l denote the smallest subspace containing $\mathrm{v}_{\tilde{E}}$ in the direction of l in $\tilde{\Sigma}_{\tilde{E}}$.

- Since l is a complete affine subspace, Lemma 3.1.15 applied to $P_l \cap \tilde{\mathscr{O}}/N$ shows that l covers a horospherical end of $(\mathbb{S}_l^{i_0+1} \cap \tilde{\mathscr{O}})/N$.
- By Lemma 3.1.15 applied to $P_l \cap \tilde{\mathscr{O}}/N$, N is virtually unipotent, and N is virtually a cocompact subgroup of a unipotent group and $N|\mathbb{S}_l^{i_0+1}$ can be conjugated into

a maximal parabolic subgroup of $\mathsf{SO}(i_0+1,1)$ in $\mathbf{Aut}(\mathbb{S}^{i_0+1}_l)$ and acting on an ellipsoid of dimension i_0 in $H^{i_0+1}_l$.

We verify Hypothesis 7.3.4.

By the nilpotent Lie group theory of Malcev [123], the Zariski closure $\mathscr{Z}(N)$ of N is a virtually simply connected nilpotent Lie group with finitely many components and $\mathscr{Z}(N)/N$ is compact. Let $\mathscr{N}$ denote the identity component of the Zariski closure of N so that $\mathscr{N}/(\mathscr{N}\cap N)$ is compact. $\mathscr{N}\cap N$ acts on the great sphere $\mathbb{S}^{i_0+1}_l$ containing $\mathrm{v}_{\tilde{E}}$ and corresponding to l. Since $\mathscr{N}/(\mathscr{N}\cap N)$ is compact, we can modify U so that $\mathscr{N}$ acts on U by Lemma 3.1.8: i.e., we take the interior of $\bigcap_{g\in\mathscr{N}} g(U) = \bigcap_{g\in F} g(U)$ for the fundamental domain F of $\mathscr{N}$ by N.

Since $\mathbf{\Gamma}_{\tilde{E}}$ normalizes N, it also normalizes the identity component $\mathscr{N}$.

By above, $\mathscr{N}|\mathbb{S}^{i_0+1}_l$ is conjugate into a parabolic subgroup of $\mathsf{SO}(i_0+1,1)$ in $\mathbf{Aut}(\mathbb{S}^{i_0+1}_l)$, and $\mathscr{N}$ acts on $U\cap\mathbb{S}^{i_0+1}_l$, which is a horoball for each leaf l of $\tilde{\Sigma}_{\tilde{E}}$.

By taking a finite-index cover of U, we can assume that $N\subset\mathscr{N}$ since $\mathscr{Z}(N)$ is a finite extension of $\mathscr{N}$. We denote the finite index group by $\mathbf{\Gamma}_{\tilde{E}}$ again.

Since $\mathbb{S}^{i_0+1}_l$ corresponds to a coordinate i_0+2-subspace, and $\mathbb{S}^{i_0}_\infty$ and $\{\mathrm{v}_{\tilde{E}}\}$ are $\mathbf{\Gamma}_{\tilde{E}}$-invariant, we can choose coordinates so that (7.3.8) and (7.3.10) hold. Hence, Hypothesis 7.3.4 holds.

THEOREM 7.3.22. *Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathscr{O}$ with radial or totally geodesic ends. Assume that the holonomy group of $\pi_1(\mathscr{O})$ is strongly irreducible. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group satisfying the transverse weak middle-eigenvalue condition with respect to R-p-end structure. Assume also that N_K is discrete, and K^o/N_K is compact and Hausdorff. Then $\tilde{E}$ is a quasi-join of a totally geodesic R-end and a cusp R-end.*

PROOF. By Lemma 7.3.7, $h(g)\mathscr{N}(\vec{v})h(g)^{-1} = \mathscr{N}(\vec{v}M_g)$ where M_g is a scalar multiplied by an element of a copy of an orthogonal group $\mathsf{O}(i_0)$.

The group $\mathscr{N}$ is isomorphic to $\mathbb{R}^{i_0}$ as a Lie group. Since $N\subset\mathscr{N}$ is a discrete cocompact, N is virtually isomorphic to $\mathbb{Z}^{i_0}$. Without loss of generality, we assume that N is a cocompact subgroup of $\mathscr{N}$. By normality of N in $\mathbf{\Gamma}_{\tilde{E}}$, we obtain $h(g)Nh(g)^{-1} = N$ for $g\in\mathbf{\Gamma}_{\tilde{E}}$. Since N corresponds to a lattice $L\subset\mathbb{R}^{i_0}$ by the map $\mathscr{N}$, the conjugation by $h(g)$ corresponds to an isomorphism $M_g: L\to L$ by Lemma 7.3.7. When we identify L with $\mathbb{Z}^{i_0}$, $M_g: L\to L$ is represented by an element of $\mathsf{SL}_\pm(i_0,\mathbb{Z})$ since $(M_g)^{-1} = M_{g^{-1}}$. Also, by Lemma 7.3.7, $\{M_g|g\in\mathbf{\Gamma}_{\tilde{E}}\}$ is a compact group as their determinants equal ± 1. Hence, the image of the homomorphism given by $g\in h(\pi_1(\tilde{E}))\mapsto M_g\in\mathsf{SL}_\pm(i_0,\mathbb{Z}))$ is a finite order group. Moreover, $\mu_g = 1$ for every $g\in\mathbf{\Gamma}_{\tilde{E}}$ as we can see from Lemma 7.3.7. Thus, $\mathbf{\Gamma}_{\tilde{E}}$ has a finite index group $\mathbf{\Gamma}'_{\tilde{E}}$ centralizing $\mathscr{N}$.

We can now use Proposition 7.3.14 by letting G be $\mathscr{N}$ since N_K is discrete and $\bar{N}_K = N_K$. We take $\Sigma_{E'}$ to be the corresponding cover of $\Sigma_{\tilde{E}}$. By Lemma 7.3.9 and Proposition 7.3.14, Hypothesis 7.3.15 holds, and we have the result needed to apply Proposition 7.3.19. Finally, Proposition 7.3.19(i) and (ii) imply that $\mathbf{\Gamma}_{\tilde{E}}$ virtually is either a join or a quasi-joined group. Corollary 7.3.21 shows that a strictly joined end cannot occur. $\square$

7.4. The non-discrete case

This is in part a joint work with Y. Carriere. Let $\Sigma_{\tilde{E}}$ be the end orbifold of an NPNC R-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathscr{O}$ with radial or totally geodesic ends.

Let $\mathbf{\Gamma}_{\tilde{E}}$ be the p-end holonomy group. Let U be a p-end-neighborhood in $\tilde{\mathscr{O}}$ corresponding to a p-end vertex $\mathrm{v}_{\tilde{E}}$.

Recall the exact sequence

$$1 \to N \to \pi_1(\tilde{E}) \xrightarrow{\Pi_K^*} N_K \to 1$$

where we assume that $N_K \subset \mathbf{Aut}(K)$ is not discrete. Since $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is compact, K^o/N_K is compact also. However, N_K is not yet shown to be semisimple.

An element $g \in \mathbf{\Gamma}_{\tilde{E}}$ is of form:

$$g = \left(\begin{array}{c|c} K(g) & 0 \\ \hline * & U(g) \end{array} \right). \tag{7.4.1}$$

Here $K(g)$ is an $(n-i_0)\times(n-i_0)$-matrix and $U(g)$ is an $(i_0+1)\times(i_0+1)$-matrix acting on $\mathbb{S}^{i_0}_\infty$. We note $\det K(g) \det U(g) = 1$.

7.4.1. Outline of Section 7.4. In Section 7.4.2, we will take the leaf closure of the complete affine i_0-dimensional leaves. The theory of Molino [131]. shows that the space of leaf-closures will be an orbifold. In Section 7.4.2.2, we show that a leaf-closure is a compact suborbifold in $\Sigma_{\tilde{E}}$. In Section 7.4.2.3, we show that the fundamental group of each leaf-closure is virtually solvable. In Section 7.4.2.4, we find a syndetic closure S according to a theory of Fried-Goldman [82]. From this, we will find a subgroup acting on each complete affine i_0-dimensional leaf in Section 7.4.2.5 which we will show to be a cusp group.

In Section 7.4.3, we will complete the proof of Theorem 7.1.4 not covered by Theorem 7.3.22. Proposition 7.4.7 shows that N_K is semi-simple. Proposition 7.4.8 shows that $\mu_g = 1$ for every $g \in \mathbf{\Gamma}_{\tilde{E}}$. Finally, we prove Theorem 7.1.4.

7.4.2. Taking the leaf closure.

7.4.2.1. *Estimations with $KA\mathbb{U}$.* Let $\mathbb{U}$ denote a maximal nilpotent subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbb{S}^{i_0}_\infty, \mathrm{v}_{\tilde{E}}}$ given by lower triangular matrices with diagonal entries equal to 1.

The foliation on $\tilde{\Sigma}_{\tilde{E}}$ given by fibers of Π_K has leaves that are i_0-dimensional complete affine subspaces. Let us denote it by $\mathscr{F}_{\tilde{E}}$. Then K^o admits a smooth Riemannian metric μ_K invariant under N_K by Lemma 1.5.10. We consider the orthogonal frame bundle $\mathbb{F}K^o$ over K^o. A metric on each fiber of $\mathbb{F}K^o$ is induces from μ_K. Since the action of N_K is isometric on $\mathbb{F}K^o$ with trivial stabilizers, N_K acts on a smooth orbit submanifold of $\mathbb{F}K^o$ transitively with trivial stabilizers. (See Lemma 3.4.11 in [149].)

There exists a bundle $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ from pulling back $\mathbb{F}K^o$ by the projection map. Here, $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ covers $\mathbb{F}\Sigma_{\tilde{E}}$. Since $\mathbf{\Gamma}_{\tilde{E}}$ acts isometrically on $\mathbb{F}K^o$, the quotient space $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is a bundle $\mathbb{F}\Sigma_{\tilde{E}}$ over $\Sigma_{\tilde{E}}$ with compact fibers diffeomorphic to the orthogonal group of dimension $n-i_0$. Also, $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ is foliated by i_0-dimensional affine spaces pulled-back from the i_0-dimensional leaves on the foliation $\tilde{\Sigma}_{\tilde{E}}$. One can think of these leaves as being the inverse images of points of $\mathbb{F}K^o$.

LEMMA 7.4.1. *Each leaf l of $\mathbb{F}\Sigma_{\tilde{E}}$ is of polynomial growth. That is, each ball $B_R(x)$ in l of radius R for $x \in l$ has an area less than equal to $f(R)$ for a polynomial f where we are using an arbitrary Riemannian metric on $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ induced from one on $\mathbb{F}\Sigma_{\tilde{E}}$.*

PROOF. Let us choose a fundamental domain F of $\mathbb{F}\Sigma_{\tilde{E}}$. Then for each leaf l there exists an index set I_l so that l is a union of $g_i(D_i)$ $i \in I_l$ for the intersection D_i of a leaf with F and $g_i \in \mathbf{\Gamma}_{\tilde{E}}$. We have that $D_i \subset D'_i$ where D'_i is an ε-neighborhood of D_i in the leaf. Then

$$\{g_i(D'_i)|i \in I_l\}$$

cover l in a locally finite manner. The subset $G(l) := \{g_i \in \Gamma | i \in I_l\}$ is a discrete subset.

Choose an arbitrary point $d_i \in D_i$ for every $i \in I_l$. The set $\{g_i(d_i)|i \in I_l\}$ and l is quasi-isometric: a map from $G(l)$ to l is given by $f_1 : g_i \mapsto g_i(d_i)$ and the multivalued map f_2 from l to $G(l)$ given by sending each point $y \in l$ to one of finitely many g_i such that $g_i(D'_i) \ni y$. Let $\mathbf{\Gamma}_{\tilde{E}}$ be given the Cayley metric and $\tilde{\Sigma}_{\tilde{E}}$ a metric induced from $\Sigma_{\tilde{E}}$. Both maps are quasi-isometries since these maps are restrictions of quasi-isometries $\mathbf{\Gamma}_{\tilde{E}} \to \tilde{\Sigma}_{\tilde{E}}$ and $\tilde{\Sigma}_{\tilde{E}} \to \mathbf{\Gamma}_{\tilde{E}}$ defined analogously.

The action of g_i in K is bounded since it sends some points of $\Pi_K(F)$ to ones of $\Pi_K(F)$ which is a compact set in K^o. Thus, $\Pi^*_K(g_i)$ goes to a bounded subset of $\mathbf{Aut}(K)$. In the form (7.4.1),

$$K(g_i) = \det(K(g_i))^{1/(n-i_0)}\hat{K}(g_i) \text{ where } \hat{K}(g_i) \in \mathsf{SL}_{\pm}(n-i_0,\mathbb{R})$$

where $\hat{K}(g_i)$ is uniformly bounded. Let $\tilde{\lambda}_{\max}(g_i)$ and $\tilde{\lambda}_{\min}(g_i)$ denote the largest norm and the smallest norm of eigenvalues of $\hat{K}(g_i)$. Since $\Pi^*_K(g_i)$ are in a bounded set of $\mathbf{Aut}(K)$, we obtain

$$\frac{1}{C} \leq \tilde{\lambda}_{\max}(g_i), \tilde{\lambda}_{\min}(g_i) \leq C \tag{7.4.2}$$

for $C > 1$ independent of i. The largest and the smallest eigenvalues of g_i equal

$$\lambda^{Tr}_{\max}(g_i) = \det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\max}(g_i) \text{ and } \lambda^{Tr}_{\min}(g_i) = \det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\min}(g_i)$$

by Proposition 7.2.3. Denote by $a_j(g_i), j = 1,\dots,i_0+1$, the norms of eigenvalues of g_i associated with $\mathbb{S}^{i_0}_\infty$ where $a_1(g_i) \geq \cdots \geq a_{i_0+1}(g_j) > 0$ with repetitions allowed. Since $\det g_i = 1$, we have

$$\det(K(g_i))a_1(g_i)\dots a_{i_0+1}(g_i) = 1.$$

If $\{|\det(K(g_i))|\} \to 0$, then $\{a_1(g_j)\} \to \infty$ whereas by (7.4.2)

$$\{\det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\max}(g_i)\} \to 0$$

contradicting Proposition 7.2.3. If $\{|\det(K(g_i))|\} \to \infty$, then $\{a_{i_0+1}\} \to 0$ whereas by (7.4.2)

$$\{\det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\min}(g_i)\} \to \infty$$

contradicting Proposition 7.2.3. Therefore, we obtain

$$1/C < |\det(K(g_i))| < C$$

for a positive constant C. We deduce that the largest norm and the smallest norm of eigenvalues of g_i

$$\det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\max}(g_i) \text{ and } \det(K(g_i))^{1/(n-i_0)}\tilde{\lambda}_{\min}(g_i)$$

are bounded above and below by two positive numbers. Hence, $\lambda^{Tr}_{\max}(g_i)$ and $\lambda^{Tr}_{\min}(g_i)$ are all bounded above and below by a fixed set of positive numbers. $U(g_i)$ consists of $a_5(g_i)O_5(g_i)$ for an orthogonal element $O_5(g_i)$ and $a_9(g_i)$. The remaining eigenvalues of g_i have norms $a_5(g_i)$ and $a_9(g_i)$. By Proposition 7.2.3, these are bounded by the same fixed set of positive numbers.

By Corollary 1.3.4, $\{g_i\}$ is of bounded distance from $\mathbb{U}'$. Let $N_c(\mathbb{U}')$ denote a c-neighborhood of $\mathbb{U}'$. Then

$$G(l) \subset N_c(\mathbb{U}') \text{ for some } c > 0.$$

Let d denote the left-invariant metric on $\mathbf{Aut}(\mathbb{S}^n)$. By the discreteness of $\mathbf{\Gamma}_{\tilde{E}}$, the set $G(l)$ is discrete and there exists a lower bound to

$$\{d(g_i, g_j) | g_i, g_j \in G(l), i \neq j\}.$$

Also given any $g_i \in G(l)$, there exists an element $g_j \in G(l)$ so that $d(g_i, g_j) < C$ for a uniform constant C. (We need to choose g_j so that $g_j(F)$ is adjacent to $g_i(F)$.) Let $B_R(\mathrm{I})$ denote the ball in $\mathsf{SL}(n+1, \mathbb{R})$ of radius R with the center I. Then $B_R(\mathrm{I}) \cap N_c(\mathbb{U}')$ is of polynomial growth with respect to R, and so is $G(l) \cap B_R(\mathrm{I})$. Since the collection $\{g_i(D_i') | g_i \in G(l)\}$ of uniformly bounded balls cover l in a locally finite manner, l is of polynomial grow as well. $\square$

7.4.2.2. *Closures of leaves.* The foliation on $\tilde{\Sigma}_{\tilde{E}}$ given by fibers of Π_K has leaves that are i_0-dimensional complete affine subspaces. Let us denote it by $\mathscr{F}_{\tilde{E}}$. Then K^o admits a smooth Riemannian metric μ_K invariant under N_K by Lemma 1.5.10. We consider the orthogonal frame bundle $\mathbb{F}K^o$ over K^o. A metric on each fiber of $\mathbb{F}K^o$ is induced from μ_K. Since the action of N_K is isometric on $\mathbb{F}K^o$ with trivial stabilizers, we find that N_K acts on a smooth orbit submanifold of $\mathbb{F}K^o$ transitively with trivial stabilizers. (See Lemma 3.4.11 in [149].)

There exists a bundle $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ from pulling back $\mathbb{F}K^o$ by the projection map. Here, $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ covers $\mathbb{F}\Sigma_{\tilde{E}}$. Since $\mathbf{\Gamma}_{\tilde{E}}$ acts isometrically on $\mathbb{F}K^o$, the quotient space $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is a bundle $\mathbb{F}\Sigma_{\tilde{E}}$ over $\Sigma_{\tilde{E}}$ with a subbundle with compact fibers isomorphic to the orthogonal group of dimension $n - i_0$. Also, $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ is foliated by i_0-dimensional affine spaces pulled-back from the i_0-dimensional leaves on the foliation $\tilde{\Sigma}_{\tilde{E}}$. One can think of these leaves as being the inverse images of points of $\mathbb{F}K^o$.

7.4.2.3. $\pi_1(V_l)$ *is virtually solvable.* Recall the fibration

$$\Pi_K : \tilde{\Sigma}_{\tilde{E}} \to K^o \text{ which induces } \tilde{\Pi}_K : \mathbb{F}\tilde{\Sigma}_{\tilde{E}} \to \mathbb{F}K^o.$$

Since N_K acts as isometries of a Riemannian metric on K^o, we can obtain a metric on $\Sigma_{\tilde{E}}$ so that the foliation is the Riemannian foliation. Let $p_{\Sigma_{\tilde{E}}} : \mathbb{F}\tilde{\Sigma}_{\tilde{E}} \to \mathbb{F}\Sigma_{\tilde{E}}$ be the covering map induced from $\tilde{\Sigma}_{\tilde{E}} \to \Sigma_{\tilde{E}}$. The foliation on $\tilde{\Sigma}_{\tilde{E}}$ gives us a foliation of $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$.

Let A_K be the identity component of the closure of N_K the image of $\mathbf{\Gamma}_{\tilde{E}}$ in $\mathbf{Aut}(K)$, which is a Lie group of dim ≥ 1.

PROPOSITION 7.4.2. *A_K is a normal connected nilpotent subgroup of the closure of N_K.*

PROOF. Since the closure of N_K is normalized by N_K, A_K is a normal subgroup of N_K. Since l maps to a polynomial growth leaf in $\mathbb{F}\Sigma_{\tilde{E}}$ by Lemma 7.4.1, Carrière [35] shows that A_K is a connected nilpotent Lie group in the closure of N_K in $\mathbf{Aut}(K)$ acts on $\mathbb{F}K^o$ freely. $\square$

Let l be a leaf of $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$, and p be the image of l in $\mathbb{F}K^o$. Moreover, we have

$$\begin{array}{rcl} \tilde{\Pi}_K^{-1}(A_K(p)) =: \tilde{V}_l & \hookrightarrow & \mathbb{F}\tilde{\Sigma}_{\tilde{E}} \\ \downarrow & & p_{\Sigma_{\tilde{E}}} \downarrow \\ V_l & \hookrightarrow & \mathbb{F}\Sigma_{\tilde{E}} \end{array} \tag{7.4.3}$$

for $V_l := \overline{p_{\Sigma_{\tilde{E}}}(l)}$ in $\mathbb{F}\Sigma_{\tilde{E}}$. Since $\tilde{V}_l$ is closed and is a component of the inverse image of V_l which is a union of copies of $\tilde{V}_l$, the image V_l is a compact submanifold. Note V_l has a dimension independent of l since A_K acts freely.

Now, N is precisely the subgroup of $\pi_1(V_l)$ fixing a leaf l in $\mathbb{F}K^o$. For each closure V_l of a leaf l, the manifold V_l is a compact submanifold of $\mathbb{F}\Sigma_{\tilde{E}}$, and we have an exact sequence

$$1 \to N \to h(\pi_1(V_l)) \xrightarrow{\Pi_K^*} A'_K \to 1. \tag{7.4.4}$$

Since the leaf l is dense in V_l, it follows that A'_K is dense in A_K. Each leaf l' of $\tilde{\Sigma}_{\tilde{E}}$ has a realization a subset in $\tilde{\mathscr{O}}$. We have the norms of eigenvalues $\lambda_i(g) = 1$ for $g \in N$ by Proposition 7.2.3. By Theorem 1.3.7, $N = N_l$ is virtually unipotent since the norms of eigenvalues equal 1 identically and N_l is discrete.

We take a finite cover of $\Sigma_{\tilde{E}}$ so that N is nilpotent. Hence, $h(\pi_1(V_l))$ is solvable being an extension of a nilpotent group by a nilpotent group. We summarize below:

PROPOSITION 7.4.3. *Let l be a generic fiber of $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}$ and p be the corresponding point of $\mathbb{F}K^o$. Then there exists a nilpotent group A_K acting on $\mathbb{F}K^o$ so that $\tilde{\Pi}_K^{-1}(A_K(p)) = \tilde{V}_l$ covers a compact suborbifold V_l in $\mathbb{F}\Sigma_{\tilde{E}}$, a conjugate of the image of the holonomy group of V_l is a dense subgroup of A_K, and the holonomy group of V_l is solvable. Moreover, $\tilde{V}_l$ is homeomorphic to a torus times a cell or a cell.*

PROOF. We just need to prove the last statement. Since A_K is a connected nilpotent group, A_K is homeomorphic to a torus times a cell or a cell, and so is the free orbit in $\mathbb{F}K^o$. Since $\tilde{\Pi}_K$ has fibers that are i_0-dimensional open hemispheres, this last statement follows. $\square$

We remark that A_K is nilpotent but may not be unipotent.

REMARK 7.4.4. The leaf holonomy acts on $\mathbb{F}\tilde{\Sigma}_{\tilde{E}}/\mathscr{F}_{\tilde{E}}$ as a nilpotent killing field group without any fixed points. Hence, each leaf l is in $\tilde{V}_l$ with a constant dimension. Thus, $\mathscr{F}_{\tilde{E}}$ is a foliation with leaf closures of identical dimensions.

The leaf closures form another foliation $\overline{\mathscr{F}}_{\tilde{E}}$ with compact leaves by Lemma 5.2 of Molino [131]. We let $\mathbb{F}\Sigma_{\tilde{E}}/\overline{\mathscr{F}}_{\tilde{E}}$ denote the space of closures of leaves has an orbifold structure where the projection $\mathbb{F}\Sigma_{\tilde{E}} \to \mathbb{F}\Sigma_{\tilde{E}}/\overline{\mathscr{F}}_{\tilde{E}}$ is an orbifold morphism by Proposition 5.2 of [131].

7.4.2.4. *The holonomy group for a leaf closure is normalized by the end holonomy group.* Note that $\mathbf{\Gamma}_l$ is the deck transformation group of $\tilde{V}_l$ over V_l. Since $\tilde{V}_l$ is the inverse image of $A_K(x)$ for $x \in \mathbb{F}K^o$, $\mathbf{\Gamma}_l$ is the inverse image of $N_K \cap A_K$ under Π_K^*. Since $N_K \cap A_K$ is normal in N_K, $\mathbf{\Gamma}_l$ is a normal subgroup of $\mathbf{\Gamma}_{\tilde{E}}$.

Recall that $\mathbf{\Gamma}_l$ is virtually solvable, as we showed above. We let $\mathscr{Z}(\mathbf{\Gamma}_{\tilde{E}})$ and $\mathscr{Z}(\mathbf{\Gamma}_l)$ denote the Zariski closures in $\mathbf{Aut}(\mathbb{S}^n)$ of $\mathbf{\Gamma}_{\tilde{E}}$ and $\mathbf{\Gamma}_l$ respectively.

By Theorem 1.6 of Fried-Goldman [82], there exists a closed virtually solvable Lie group S_l containing $\mathbf{\Gamma}_l$ with the following four properties:

- S_l has finitely many components.
- $\mathbf{\Gamma}_l \backslash S_l$ is compact.
- The Zariski closure $\mathscr{Z}(S_l)$ is the same as $\mathscr{Z}(\mathbf{\Gamma}_l)$.
- Finally, we have solvable ranks

$$\operatorname{rank}(S_l) \leq \operatorname{rank}(\mathbf{\Gamma}_l). \tag{7.4.5}$$

We will call this the *syndetic hull* of $\mathbf{\Gamma}_l$.

We summarize:

LEMMA 7.4.5. *$h(\pi_1(V_l))$ is virtually solvable and is contained in a virtually solvable Lie group $S_l \subset \mathscr{Z}(h(\pi_1(V_l))$ with finitely many components, and $S_l/h(\pi_1(V_l))$ is compact. S_l acts on $\tilde{V}_l$. Furthermore, one can modify a p-end-neighborhood U so that S_l acts on it. Also the Zariski closure of $h(\pi_1(V_l))$ is the same as that of S_l.*

PROOF. By above, $\mathscr{Z}(S_l) = \mathscr{Z}(\mathbf{\Gamma}_l)$ acts on $\tilde{V}_l$ and normalizes $\mathbf{\Gamma}_l$. We need to prove about the p-end-neighborhood only. Let F be a compact fundamental domain of S_l under the Γ_l. Then we have

$$\bigcap_{g\in S_l} g(U) = \bigcap_{g\in F} g(U).$$

By Lemma 3.1.8, the latter set contains a S_l-invariant p-end-neighborhood. $\square$

From now on, we will let S_l to denote the only the identity component of itself for simplicity as S_l has finitely many components to begin with. We are taking a finite cover of $\mathscr{O}$ if necessary. This will be sufficient for our purposes since we only need a cusp group.

Since S_l acts on U and hence on $\tilde{\Sigma}_{\tilde{E}}$ as shown in Lemma 7.4.5, we have a homomorphism $S_l \to \mathbf{Aut}(K)$. We define by $S_{l,0}$ the kernel of this map. Then $S_{l,0}$ acts on each leaf of $\tilde{\Sigma}_{\tilde{E}}$. We have an exact sequence

$$1 \to S_{l,0} \to S_l \to A_K \to 1. \tag{7.4.6}$$

7.4.2.5. *The form of $US_{l,0}$.* Let $\mathbb{S}^{i_0+1}_l$ denote the i_0+1-dimensional great sphere containing $\mathbb{S}^{i_0}_\infty$ corresponding to each i_0-dimensional leaf l of $\mathscr{F}_{\tilde{E}}$.

PROPOSITION 7.4.6. *Let l be a generic fiber so that A_K acts with trivial stabilizers.*

- (i) *S_l acts on $\tilde{V}_l$ cocompactly, acts on ∂U properly, and acts as isometries on these spaces with respect to some Riemannian metrics.*
- (ii) *A closed subgroup $C_{l,0}$ of unipotent elements of acts transitively on each leaf l with trivial stabilizers, and $C_{l,0}$ acts on an i_0-dimensional ellipsoid $\partial U \cap \mathbb{S}^{i_0+1}_l$ passing $\mathrm{v}_{\tilde{E}}$ with an invariant Euclidean metric. Here, we may need to modify U further.*
- (iii) *$S_{l,0}$ normalizes an i_0-dimensional partial cusp group $C_{l,0}$ where $S_{l,0} \cap C_{l,0}$ are cocompact subgroups in both $S_{l,0}$ and $C_{l,0}$.*
- (iv) *$C_{l,0}$ is virtually normalized by $\mathbf{\Gamma}_{\tilde{E}}$ and also by S_l. Also, $C_{l,0}$ acts freely and properly on m for each leaf m of $\mathscr{F}_{\tilde{E}}$.*
- (v) *With setting $\mathscr{N} := C_{l,0}$, Hypothesis 7.3.4 holds virtually by $\mathbf{\Gamma}_{\tilde{E}}$ for a coordinate system.*

PROOF. (i) By Lemma 3.1.10, S_l acts properly on $\tilde{V}_l$. Since ∂U is in one-to-one correspondence with $\tilde{\Sigma}_{\tilde{E}}$, S_l acts on ∂U properly. Hence, these spaces have compact stabilizers with respect to S_l. The existence of an invariant metric follows from an argument similar to one in the proof of Lemma 1.5.10. Hence, the action is proper and the orbit is closed.

Since $\tilde{V}_l/\mathbf{\Gamma}_l$ is compact, $\tilde{V}_l/S_l$ is compact also.

(ii) We may assume that $\mathbf{\Gamma}_{\tilde{E}}$ is torsion-free by Theorem 1.1.19 taking a finite index subgroup.

Proposition 7.2.3 implies that for $g \in \mathbf{\Gamma}_l$

$$\lambda^{Tr}_{\max}(g) \geq \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g) \geq \lambda^{\mathbb{S}^{i_0}_\infty}_{\min}(g) \geq \lambda^{Tr}_{\min}(g).$$

Since $S_l = F\Gamma_l$ for a compact set F, the inequality

$$C_1\lambda^{Tr}_{\max}(g) \geq \lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g) \geq C_2\lambda^{\mathbb{S}^{i_0}_\infty}_{\min}(g) \geq C_3\lambda^{Tr}_{\min}(g), g \in S_l,$$

$$C_1\lambda^{Tr}_{\max}(g) \geq \lambda_1(g) \geq \lambda_{n+1}(g) \geq C_2\lambda^{Tr}_{\min}(g), g \in S_l, \tag{7.4.7}$$

hold for constants $C_1 > 1, 1 > C_2 > C_3 > 0$ by (7.2.3). Since $S_{l,0}$ acts trivially on K^o, we have $\lambda^{Tr}_{\max}(g) = \lambda^{Tr}_{\min}(g)$ for $g \in S_{l,0}$. Since the maximal norm $\lambda_1(g)$ of the eigenvalues of g equals $\lambda^{Tr}_{\min}(g)$ and the minimal norm of the eigenvalues of g equals $\lambda^{Tr}_{\min}(g)$, all the norms of the eigenvalues of $g \in S_{l,0}$ are bounded above. (7.4.7) implies that $|\log\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)|, |\log\lambda_1(g)|, g \in S_{l,0}$ are both uniformly bounded above. Of course we have

$$|\log\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g^n)| = |n\log\lambda^{\mathbb{S}^{i_0}_\infty}_{\max}(g)|, |\log\lambda_1(g^n)| = |n\log\lambda_1(g)|, g \in S_{l,0}.$$

We conclude that the norms of eigenvalues of $g \in S_{l,0}$ are all 1.

Theorem 1.3.7 implies that $S_{l,0}$ is a closed orthopotent group and hence a solvable Lie group. Lemma 3.1.13 gives a unipotent group $C_{l,0}$ acting on l where $C_{l,0}$ is the Zariski closure of the unipotent subgroup $S^u_{l,0}$ of $S_{l,0}$. We have $C_{l,0} \cap S_{l,0} = S^u_{l,0}$. Proposition 3.1.14 shows that $C_{l,0}$ is a cusp group. Since S_l normalizes $S_{l,0}$ and $C_{l,0} \cap S_{l,0} = S^u_{l,0}$ is cocompact in $S_{l,0}$, it follows that S_l normalizes $C_{l,0}$. This also proves (iii).

(iv) Proposition 3.1.14 shows that the action of $C_{l,0}$ on any leaf m is a free and proper action. Since $C_{m,0}$ acts on m, $B_m := H_m \cap U$ is again bounded by an ellipsoid. Since B_m has a hyperbolic metric as a Klein model, and $C_{l,0}$ is unipotent acting properly on horospheres of B_m for $\mathrm{v}_{\tilde{E}}$, $C_{l,0}$ must also be a cusp group on B_m. Hence, $C_{l,0}$ acts as a cusp group on each $H_m \cap U$.

Let $g \in \mathbf{\Gamma}_{\tilde{E}}$. By using these argument for $g(l)$ instead of l, $gC_{l,0}g^{-1}$ also acts on an ellipsoid E_m in the subspace corresponding to m from $\mathrm{v}_{\tilde{E}}$ as a unipotent Lie group freely, transitively, and faithfully. Since E_l bounds a (i_0+1)-dimensional ball with a hyperbolic metric of the Klein model, such a unipotent group is unique and hence it follows that $gC_{l,0}g^{-1}$ and $C_{l,0}$ restrict to a same group in H_m.

Let $\hat{C}$ denote the group generated by $C_{l,0}$ and its conjugates. $\hat{C}$ is obviously unipotent. Also, $\hat{C}$ acts properly on $\tilde{\Sigma}_{\tilde{E}}$ since $\mathbf{\Gamma}_{\tilde{E}}$ and $C_{l,0}$ preserve a Riemannian metric.

Let $g' \in C_{l,0}$ and $g'' \in gC_{l,0}g^{-1}$ so that $g'|H_m = g''|H_m$. Then $g'^{-1}g''$ fixes every point in m. Since and the stabilizer of the unipotent group acting properly on $\tilde{\Sigma}_{\tilde{E}}$ is trivial, $g' = g''$. Hence, the normality follows.

(v) The first two properties of Hypothesis 7.3.4 follow from Propositions 1.4.10 and 1.4.13. $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the third transverse weak middle eigenvalue condition by the premise. Since $S^u_{l,0}$ goes to I under Π^*_K, it is in the standard form where l corresponds to a great sphere $\mathbb{S}^{i_0+1}$ containing $\mathbb{S}^{i_0}_\infty$. This proves the fourth property.

Since N acts on $\tilde{V}_l$, N is a subgroup of Γ_l virtually. Hence, N is a subgroup of S_l and hence of $S_{l,0}$ virtually. Theorem 1.3.7 tells us that N is unipotent virtually and hence $N \cap \mathbf{\Gamma}'_{\tilde{E}}$ is in $S^u_{l,0}$ for a finite index subgroup $\mathbf{\Gamma}'_{\tilde{E}}$ of $\mathbf{\Gamma}_{\tilde{E}}$. (iv) showed that $\mathscr{N}$ is normalized by $\mathbf{\Gamma}_{\tilde{E}}$. (iv) also shows that $\mathscr{N}$ acts freely and properly on each complete affine leaf of $\tilde{\Sigma}_{\tilde{E}}$. □

7.4.3. The proof for non-discrete N_K. Now, we go the splitting argument for this case. We can parametrize $US_{l,0}$ by $\mathscr{N}(\vec{v})$ for $\vec{v} \in \mathbb{R}^{i_0}$ by Proposition 7.4.6. We showed that Hypothesis 7.3.4 holds virtually. For convenience, let us assume that Hypothesis 7.3.4 here.

We outline the proof strategy:

- N_K is semisimple in Proposition 7.4.7.
- $\mu_g = 1$ for every $g \in \mathbf{\Gamma}_{\tilde{E}}$.
- Hypothesis 7.3.15 holds. Now we use the results in Section 7.3.3.

Also, $N \cap US_{l,0}$ is of finite index in N since both acts on l and we took many finite index subgroups in the processes above. Again using Proposition 1.1.18, we can consider the finite covers of p-end neighborhoods. We assume $N \subset US_{l,0}$. Hypothesis 7.3.4 holds now as we showed in the above subsections. As above by Lemmas 7.3.7, we have that the matrices are of form:

$$\mathcal{N}(\vec{v}) = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v}) & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}) & \|\vec{v}\|^2/2 & \vec{v} & 1 \end{array}\right), \tag{7.4.8}$$

$$g = \left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_1(g)\vec{v}_g^T & a_5(g)O_5(g) & 0 \\ \hline c_2(g) & a_7(g) & a_5(g)\vec{v}_g^T O_5(g) & a_9(g) \end{array}\right) \tag{7.4.9}$$

where $g \in \mathbf{\Gamma}_{\tilde{E}}$. (See (7.3.36).) Recall $\mu_g = a_5(g)/a_1(g) = a_9(g)/a_5(g)$. Since S_l is in $\mathscr{Z}(\mathbf{\Gamma}_l)$ and the orthogonality of normalized $A_5(g)$ is an algebraic condition, the above form also holds for $g \in S_l$.

PROPOSITION 7.4.7 (Semisimple N_K). *Assume hypothesis 7.3.4. Suppose that $\pi_1(\tilde{E})$ satisfies (NS) or $\dim K = 0, 1$. Then the following hold*:

- *N_K or any of its finite index group acts semi-simply on K^o.*
- *There is a finite index subgroup N'_K of N_K acting on each K_i irreducibly and has the diagonalizable commutant H isomorphic to $\mathbb{R}_+^{\bar{l}-1}$ for some $\bar{l} \geq 1$.*
- *K is projectively diffeomorphic to $K_1 * \cdots * K_{\bar{l}}$, where H acts trivially on each K_j for $j = 1, \ldots, \bar{l}$.*
- *Let A_K denote the identity component of the closure $\bar{N}_K$ of N_K in $\mathbf{Aut}(K)$. Let A'_K denote the image in A_K of $\mathbf{\Gamma}_{\tilde{E}}$. Then $A'_K \cap N'_K$ is free abelian and is a diagonalizable group of matrices and is in the virtual center $N_K \cap H$.*
- *N'_K acts on each K_i strongly irreducibly, and $N'_K|K_i$ is semisimple and discrete and acts on K_i^o as a divisible action.*
- *N_K contains a free abelian group in $N_K \cap H$ of rank $\bar{l} - 1$.*

PROOF. It is sufficient to prove for N_K itself since N'_K for a finite-index subgroup N'_K of N_K acts cocompactly.

If N_K is discrete, then the conclusion follows from Proposition 1.4.10.

Suppose N_K is not discrete. For the case when $\dim K = 0, 1$, the conclusions are obvious. We will prove using induction on $\dim K$.

We now use the notation of Section 7.4.2.2. By Theorem 1.1.19, we assume that $\mathbf{\Gamma}_{\tilde{E}}$ is torsion-free. By condition (NS), since $\mathbf{\Gamma}_l$ is virtually normal in $\mathbf{\Gamma}_{\tilde{E}}$, $\mathbf{\Gamma}_l \cap G$ is central in a finite index subgroup G of $\mathbf{\Gamma}_{\tilde{E}}$ and is free-abelian.

Now, we prove by induction on $\dim K$. We recall the exact sequence

$$1 \to N \to \mathbf{\Gamma}_l \to A'_K \to 1.$$

Here, A'_K is dense in a nilpotent Lie group A_K normalized by N_K. Let N'_K denote the image of G in N_K. Let $\bar{N}'_K$ denote the closure of N'_K in $\mathbf{Aut}(K)$.

We give a bit vague outline of the rest of the proof:

- **(i):** First, we show that there is no unipotent action on K^o by length arguments.
- **(ii):** We decompose the space into invariant subspaces where N_K virtually acts on which it virtually acts discretely.
- **(iii):** Now, we prove that N_K acts semisimply.
- **(iv):** Finally, we show that N_K contains a free abelian group of certain rank.

(i) We prove some fact: We take the unipotent subgroup $\mathbf{\Gamma}_{l,u}$, $A'_{K,u}$, $A_{K,u}$ of solvable groups $\mathbf{\Gamma}_l$, A'_K, and A_K respectively. These are normalized by N_K.

Suppose that $A'_{K,u} \cap N'_K$ is nontrivial. Choose a nontrivial unipotent element g_u in $A'_{K,u} \cap N'_K$. By Lemma 1.3.10, there exists a sequence of elements $x_i \in K^o$ so that $d_K(x_i, g_u(x_i)) \to 0$. Since N'_K is still a sweeping action, let F be a compact set in K^o so that $\bigcup_{g \in N'_K} g(F) = K^o$. Now, we can choose $g_i \in G$ so that $g_i(x_i) \in F$. Then

$$\{d_\Omega(g_i g_u(x_i), g_i(x_i)) = d_\Omega(g_i g_u g_i^{-1}(g_i(x_i)), g_i(x_i))\} \to 0 \tag{7.4.10}$$

since g_i is a d_K-isometry. This implies that $\{g_i g_u g_i^{-1}\}$ converges to an element of a stabilizer of a point f, $f \in F$ in $\bar{N}_K$.

Since $\mathbf{\Gamma}_l \cap G$ is central in G, $g_i g_u g_i^{-1} = g_u$. Since g_u is unipotent, if g_u stabilizes a point of K^o, then g_u is the identity element by Lemma 1.3.8. This is a contradiction. Therefore, we conclude $A'_{K,u} \cap N'_K$ is a trivial group.

(ii) Since $\mathbf{\Gamma}_l$ is central in G, $A'_K \cap N'_K$ and its closure $A_K \cap \bar{N}'_K$ are abelian groups. Since $A_K \cap \bar{N}'_K$ is abelian, we can decompose $\mathbb{C}^{n-i_0} = V'_1 \oplus \cdots \oplus V'_{l'}$ so that each element g of $A_K \cap \bar{N}'_K$ acts irreducibly with a single eigenvalue $\lambda_i(g)$ on V'_i for each i and its conjugate $\bar{\lambda}_i(g)$ by the primary decomposition theorem. (See Theorem 12 of Section 6 of [101] and Definition 1.3.1.) The map

$$g \in A_K \cap \bar{N}'_K \mapsto (\lambda_1(g), \dots, \lambda_{l'}(g)) \in \mathbb{C}^{*l'}$$

gives us an isomorphism to the image set where we choose a representative eigenvalue $\lambda_i(g)$ for each V_i.

We define $S_1, \dots, S_{l'}$ to be the subspace in $\mathbb{S}^n$ corresponding to a real primary subspace for every g by the commutativity of elements in $A_K \cap \bar{N}'_K$.

Since N'_K commutes with A_K, N'_K also acts on or permutes the corresponding subspaces $S_1, \dots, S_{l'}$ in $\mathbb{S}^{n-i_0-1}$. We take the finite-index subgroup N''_K of N'_K acting on each S_i, and N''_K has a sweeping action on K^o. By Proposition 1.4.13, $S_i \cap K \neq \emptyset$ and $S_i \cap K^o = \emptyset$. Denote by $K_i := S_i \cap K$.

Suppose that $\lambda_i(g)$ is not real for some i and $g \in A_K \cap \bar{N}'_K$. Then there is an eigenspace for $\lambda_i(g)$ and one for $\bar{\lambda}_i(g)$ for all $g \in N_K$. Since $K \cap S$ for a corresponding subspace S' for the direct sum of two eigenspaces is properly convex, there is a global fixed point of g. The point corresponds to a positive real eigenvalue of g. This is a contradiction. The negative case violates the proper convexity. Therefore, every $\lambda_i(g) > 0$ for $g \in A_K \cap \bar{N}'_K$.

Let $\bar{N}''_K$ denote the closure of N''_K in $\bar{N}_K$. Suppose that $A_K \cap \bar{N}''_K$ acts on K_i nontrivially. $A_K \cap \bar{N}''_K | K_i$ is a unipotent action since each element of $A_K \cap \bar{N}'_K | K_i$ has a single positive eigenvalue affilated with K_i. Since $K_i^o / A'_K \cap N''_K$ is compact, we can apply the arguments in the paragraph containing (7.4.10) and the following one, and we obtain a contradiction. Hence, $A_K \cap \bar{N}''_K|K_i$ is trivial. Hence, $A_K \cap \bar{N}'_K$ is a positive diagonalizable group.

Since A_K is the identity component of $\bar{N}_K$, and A_K restricts to a trivial group for each K_i, $N_{K_i} := N''_K|K_i$ is discrete.

(iii) If $l' = 1$, then this shows $A_K \cap \bar{N}'_K$ is trivial. Then we are in the case of N_K being discrete and the result follows by Proposition 1.4.10.

Suppose now that $l' \geq 2$. Then since $\dim K_i < \dim K$, we deduce that $N_{K_i} := N''_K|K_i$ still acts cocompactly on K_i^o since otherwise this fails for $N_{K''}$ action on K^o. Hence, N_{K_i} is semi-simple and the conclusions of this proposition hold for N_{K_i} and K_i by induction.

By induction, we decompose each K_j into $K_j^{(1)} * \cdots * K_j^{l'(j)}$ with a positive diagonalizable commutant H_j for N_{K_j}. The finite index subgroup N'_{K_i} of N_{k_i} acting on each $K_j^{(i)}$ is a cocompact subgroup of $N_{K_j^{(i)}} \times \cdots \times N_{K_j^{l'(j)}} \times \Lambda_j$ for a Zariski dense subgroup Λ_j in H_j and $N_{K_j^{(i)}} := N'_{K_j}|K_j^{(i)}$ for $i = 1, \ldots, l'(j)$ by Proposition 1.4.10. Also, $N_{K_j^{(i)}}$ acts strongly irreducibly on each $K_j^{(i)}$ also by Proposition 1.4.10.

There is a commutant H_K of N'_K that just the positive diagonalizable group acting trivially on each K_i. Hence, it is isomorphic to $\mathbb{R}^{l'-1}$. Since $A_K \cap N'_K|K_i$ for each i is trivial, $H_K \cap N'_K$ contains $A_K \cap N'_K$ by the second paragraph above. This proves the third item. We define H to be the product of $H_K \times H_1 \times \cdots \times H_{l'}$.

We list out all $K_j^{(i)}$ as a single list $K_1, \ldots, K_{\bar{l}}$. Define N'''_K as a subgroup acting on each K_i for $i = 1, \ldots, \bar{l}$. Now N'_K is a subgroup of $N_{K_1} \times \cdots \times N_{K_{l'}} \times L$ for a Zariski dense L in H. Thus, N'_K is semi-simple. This means that N_K is semi-simple since N_K/N'''_K acts only as a permutation group of $K_1, \ldots, K_{\bar{l}}$.

Since N_{K_i} is discrete, and N'_K is isomorphic to a subgroup of $N_{K_1} \times \cdots \times N_{K_{l'}} \times \mathbb{R}_+^{l'-1}$, it follows that the finite extension N_K is semi-simple. This proves the first to the fourth item.

(iv) Now, we prove the last item in particular. Proposition 1.4.13 also shows the existence of a free diagonalizable subgroup Λ of $\bar{N}_K \cap H$ of rank $\bar{l} - 1$. Hence, there must be a lattice in $L \subset \Lambda$ that is Zariski dense in $\bar{N}_K \cap H$. Choose generators $\eta_1, \ldots, \eta_{\bar{l}}$ of the lattice. For each η_j, there is a sequence $\{\kappa_i^j\}$ in N_K converging to η_j in $\mathbf{Aut}(K)$. Each $\kappa_i^j|K_i$ is in a discrete group N_{K_i}. Hence, we may assume that $\kappa_i^j|K_i = \mathrm{I}_{K_i}$ for every i since $\eta_j|K_i = \mathrm{I}_{K_i}$. Hence, $\kappa_i^j \in H \cap N_K$ for every i. Since κ_i^j are sufficiently close to η_j for each $j = 1, \ldots, \bar{l}$, we can choose a set of generators $\kappa_{i'}^1, \ldots, \kappa_{i'}^{\bar{l}}$ of $H \cap N_K$. This completes the proof. $\square$

However, we have not shown Hypothesis 7.3.15 yet. We continue to have Hypothesis 7.3.4 for $\mathbf{\Gamma}_{\tilde{E}}$.

PROPOSITION 7.4.8. *Suppose that $\mathscr{O}$ is properly convex. We assume Hypothesis 7.3.4 and N_K is non-discrete. Suppose that $\pi_1(\tilde{E})$ satisfies (NS) or $\dim K = 0, 1$. Then we have $\mu_g = 1$ for every $g \in \mathbf{\Gamma}_{\tilde{E}}$.*

PROOF. We can take finite-index subgroups for $\mathbf{\Gamma}_{\tilde{E}}$ during the proof and prove for this group since μ is a homomorphism to the multiplicative group $\mathbb{R}_+$. By Proposition 7.4.7, N_K is semi-simple and Proposition 7.3.9 and Lemma 7.3.10 hold.

Propositions 7.4.7 and 1.4.13 show that $K = K_1 * \cdots * K_l$ for properly convex sets K_i and N_K is virtually isomorphic to a cocompact subgroup of $N_{K_1} \times \cdots \times N_{K_l} \times \Lambda$ where Λ is Zariski dense in a diagonalizable group $\mathbb{R}_+^{l-1}$ acting trivially on each K_i and N_{K_i} acts semisimply on each K_i for $i = 1, \dots, l$. We take a finite-index subgroup N'_K so that N'_K acts on K_i for each $i = 1, \dots, l$. We assume that N_K is this N'_K without loss of generality.

We apply Lemma 7.3.9. Then one of K_i is a vertex k. Now we can use the coordinates of (7.3.36) repeated here.

$$
\left(\begin{array}{c|c|c|c}
S(g) & 0 & 0 & 0 \\
\hline
0 & a_1(g) & 0 & 0 \\
\hline
C_1(g) & a_1(g)\vec{v}_g^T & a_5(g)O_5(g) & 0 \\
\hline
c_2(g) & a_7(g) & a_5(g)\vec{v}_g O_5(g) & a_9(g)
\end{array}\right) \tag{7.4.11}
$$

defining $\vec{v}_g := \frac{a_4(g)}{a_1(g)}$.

Also, Lemma 7.3.9 shows that $C_1(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^{i_0}$ for a coordinate system where k has the form

$$(\!(0, \dots, 0, 1)\!) \in \mathbb{S}^{n-i_0-1}.$$

By Proposition 7.3.14, we have a coordinate system where

$$
\begin{aligned}
&C_1(g) = O, c_2(g) = 0 \text{ for every } g \in \mathbf{\Gamma}_{\tilde{E}} \text{ and} \\
&C_1(\vec{v}) = O, c_2(\vec{v}) = 0 \text{ for every } \mathscr{N}(\vec{v}), \vec{v} \in \mathbb{R}^{i_0}.
\end{aligned} \tag{7.4.12}
$$

Let λ_{S_g} denote the maximal norm of the eigenvalues of the upper-left part S_g of g. We define

$$\mathbf{\Gamma}_{\tilde{E},+} := \{g | \lambda_{S_g}(g) < a_1(g)\}.$$

There is always an element like this. In particular, we take the inverse image of suitable diagonalizable elements of the center $H \cap N_K$ denoted in Proposition 7.4.7. We take the diagonalizable element in N_K with k having a largest norm eigenvalue. Let g be such an element. Then by transverse weak middle eigenvalue condition shows that $a_1(g)$ is the largest of norms of every eigenvalue by Proposition 7.2.3, and

$$a_1(g) \geq a_9(g) \text{ or } \mu_g \leq 1 \text{ for } g \in \mathbf{\Gamma}_{\tilde{E},+}.$$

By Proposition 7.4.7, $N_K \cap H$ contains a free abelian group of rank $\dim H$ which is positive diagonalizable. Hence, there exists $g_c \in \mathbf{\Gamma}_{\tilde{E},+}$ going to a center of N'_K with $\mu_{g_c} \leq 1$.

(A) We will obtain a nontrivial element of N: Let us choose $k_g \in \mathrm{Cl}(U) \cap \mathbb{A}^n$.

Since S_l acts on U, it follows that S_l acts on $\mathrm{Cl}(U) \cap \mathbb{S}^{i_0+1}_k$. By the form of the matrices (7.4.8), $\mathscr{N}$ acts on $\mathbb{S}^{i_0+1}_k$. Hence, $\mathscr{N}$ of form (7.4.8) acts on $\mathrm{Cl}(U) \cap \mathbb{S}^{i_0+1}_k \ni k_g$. Hence, we have the orbit

$$\mathscr{N}(k_g) \subset \mathrm{Cl}(U) \cap \mathbb{S}^{i_0+1}_k.$$

Since $\mathrm{Cl}(U)$ is convex, a convex domain $B = \mathrm{Cl}(U) \cap \mathbb{S}^{i_0+1}_k$ bounded by an ellipsoid is in a hemisphere H_k in $\mathbb{S}^{i_0+1}_k$ bounded by $\mathbb{S}^{i_0}_\infty$.

There exists a hyperspace $P_{k_{g_c}}$ in $\mathbb{S}^{i_0+1}_k$ tangent to ∂B at k_{g_c} where g_c acts on. We let $\hat{P}_{k_{g_c}} = P_{k_{g_c}} \cap \mathbb{S}^{i_0}_\infty$. We choose a coordinate system so that

$$
\begin{aligned}
k_{g_c} &= ((0,\dots,0,x_{n-i_0},0,\dots,0)),\ x_{n-i_0} > 0,\\
\mathbb{S}^{i_0+1}_k &= \{((0,\dots,0,x_{n-i_0},\dots,x_{n+1}))\,|\,x_i \in \mathbb{R}\},\\
P_{k_{g_c}} &= \{((0,\dots,0,x_{n-i_0},x_{n-i_0+1},\dots,x_n,0))\,|\,x_i \in \mathbb{R}\},
\end{aligned}
$$

(7.4.13)

$$
B = \{((0,\dots,0,x_{n-i_0},\dots,x_{n+1}))\,|\,x_i \in \mathbb{R}, x_{n+1} \geq \frac{1}{2}(x^2_{n-i_0+1}+\dots+x^2_n), x_{n-i_0} = 1\}.
$$

Here, k_{g_c} may be regarded as the origin of H^o_k.

Now, let g_1 be any element of $\mathbf{\Gamma}_{\tilde{E}}$. We factorize the lower-right $(i_0+2)\times(i_0+2)$-submatrix of g_1, $g_1 \in \mathbf{\Gamma}_{\tilde{E}}$,

$$
\left(\begin{array}{c|c|c}
a_1(g_1) & 0 & 0\\ \hline
a_1(g_1)\vec{v}^T_{g_1} & a_5(g_1)O_5(g_1) & 0\\ \hline
a_7(g_1) & a_5(g_1)\vec{v}_{g_1}O_5(g_1) & a_9(g_1)
\end{array}\right) = \tag{7.4.14}
$$

(7.4.15)

$$
\left(\begin{array}{c|c|c}
1 & 0 & 0\\ \hline
0 & \mathrm{I} & 0\\ \hline
\frac{a_7(g_1)}{a_1(g_1)} - \frac{\|\vec{v}_{g_1}\|^2}{2} & 0 & 1
\end{array}\right)
\left(\begin{array}{c|c|c}
1 & 0 & 0\\ \hline
\vec{v}^T_{g_1} & \mathrm{I} & 0\\ \hline
\frac{\|\vec{v}_{g_1}\|^2}{2} & \vec{v}_{g_1} & 1
\end{array}\right)
a_1(g_1)
\left(\begin{array}{c|c|c}
1 & 0 & 0\\ \hline
0 & \mu_{g_1}O_5(g_1) & 0\\ \hline
0 & 0 & \mu^2_{g_1}
\end{array}\right).
$$

Since the right two matrices act on B, we have

$$
\aleph_7(g_1) = \frac{a_7(g_1)}{a_1(g_1)} - \frac{\|\vec{v}_{g_1}\|^2}{2} = 0 \text{ for any } g_1 \in \mathbf{\Gamma}_{\tilde{E}}.
$$

Suppose that $\vec{v}_g = 0$ for every $\mathbf{\Gamma}_{\tilde{E}}$. The $\mathbf{\Gamma}_{\tilde{E}}$ fixes k_{g_c}. By Proposition 7.4.7 and Lemma 7.3.9, $K = K'' * \{k\}$ for a compact convex set K''. There is a set K''' in $\mathrm{bd}\tilde{\mathscr{O}}$ corresponding to K''. Then the interior $K''' * k_{g_c}$ in U maps to K^o under Π_K. By (7.4.12), $K''' * k_{g_c}$ is $\mathbf{\Gamma}_{\tilde{E}}$-invariant. Also, under the radial projection to $R_{\mathrm{v}_{\tilde{E}}}(\mathscr{O}) = \tilde{\Sigma}_{\tilde{E}}$, the interior of $K''' * k_{g_c}$ goes to $\mathbf{\Gamma}_{\tilde{E}}$-invariant subspace in $\tilde{\Sigma}_{\tilde{E}}$ meeting each complete affine leaf at a point. This contradicts the cocompactness of the action on $\tilde{\Sigma}_{\tilde{E}}$.

Let us take nonidentity $g \in \mathbf{\Gamma}_{\tilde{E}}$ going to N'_K. with nonzero $\vec{v}_g$. Then conjugation $g_c g g_c^{-1}$ gives us an element with $\vec{v}_{g_c g g_c^{-1}} = \vec{v}_g \mu_{g_c} O_5(g_c)^{-1}$ by Lemma 7.3.7. This is not equal to $\vec{v}_g$ since $\mu_{g_c} < 1$. Hence, a block matrix computation shows that $g_c g g_c^{-1} g^{-1}$ is not an identity element in $\mathbf{\Gamma}_{\tilde{E}}$ but maps to I in N_K. We obtain a nontrivial element n_0 of N. By Hypothesis 7.3.4, $g_c g g_c^{-1} g^{-1} \in \mathscr{N}$. Since $n_0 := g_c g g_c^{-1} g^{-1} \neq \mathrm{I}$ has the form (7.4.8), n_0 is a unipotent element. Since $N \subset \mathscr{N}$, we may let $n_0 = \mathscr{N}(\vec{v}_0)$ for some nonzero vector v_0.

(B) Now we show $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$:

Suppose that we have an element $g \in \mathbf{\Gamma}_{\tilde{E},+}$ and $\mu_g < 1$. Then we have as above

$$
\vec{v}_{g^k n_0 g^{-k}} = \vec{v}_{n_0} \mu^k_g O_5(g)^n.
$$

Also, $g^k n_0 g^{-k}$ goes to I in N_K since $\Pi^*_K(n_0) = \mathrm{I}$ in N_K. Hence, $\{g^k n_0 g^{-k}\} \to \mathrm{I}$ as $k \to \infty$ since n_0 is in the forms (7.4.8) given by (7.4.12). This contradicts the discreteness of N. Hence, $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E},+}$.

Since any element of $g \in \mathbf{\Gamma}_{\tilde{E}}$, we can take g', $g' \in \mathbf{\Gamma}_{\tilde{E},+}$ so that $gg' \in \mathbf{\Gamma}_{\tilde{E},+}$ and so $\mu_{gg'} = \mu_g \mu_{g'} = 1$. We obtain $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$. □

THE PROOF OF THEOREM 7.1.4. Suppose that $\tilde{E}$ is an NPNC R-end. When N_K is discrete, Theorem 7.3.22 gives us the result.

When N_K is non-discrete, Hypothesis 7.3.4 holds by Propositions 7.4.6. Also, N_K is semi-simple by Proposition 7.4.7.

By Proposition 7.4.8, $\mu \equiv 1$ holds. Lemmas 7.3.7 and 7.3.9, (vii) of Proposition 7.4.6 show that the premise of Proposition 7.3.14 holds. Proposition 7.3.14 shows that Hypothesis 7.3.15 holds. Proposition 7.3.19 shows that we have a strictly joined or quasi-joined end. Corollary 7.3.21 implies the result.

Note here that we may prove for finite index subgroups of $\mathbf{\Gamma}_{\tilde{E}}$ by the definition of strictly joined or quasi-joined ends. □

We give a convenient summary.

COROLLARY 7.4.9. *Let $\mathcal{O}$ be a properly convex strongly tame real projective orbifold. Assume that its holonomy group is strongly irreducible. Let $\tilde{E}$ be an NPNC p-end of the universal cover $\tilde{\mathcal{O}}$ or $\mathcal{O}$ satisfying the transverse weak middle eigenvalue condition for the R-p-end structure of $\tilde{E}$. Suppose that $\pi_1(\tilde{E})$ satisfies (NS) or $\dim K = 0, 1$. Then the holonomy group $h(\mathbf{\Gamma}_{\tilde{E}})$ is a group whose element under a coordinate system is of form:*

$$g = \left(\begin{array}{c|c|c|c} S(g) & 0 & 0 & 0 \\ \hline 0 & \lambda(g) & 0 & 0 \\ \hline 0 & \lambda(g)\vec{v}(g)^T & \lambda(g)O_5(g) & 0 \\ \hline 0 & \lambda(g)\left(\aleph_7(g) + \frac{\|\vec{v}(g)\|^2}{2}\right) & \lambda(g)\vec{v}(g)O_5(g) & \lambda(g) \end{array}\right) \tag{7.4.16}$$

where $\{S(g) | g \in \mathbf{\Gamma}_{\tilde{E}}\}$ acts cocompactly on a properly convex domain in $\mathrm{bd}\tilde{\mathcal{O}}$ of dimension $n - i_0 - 1$, $O_5 : \mathbf{\Gamma}_{\tilde{E}} \to \mathsf{O}(i_0 + 1)$ is a homomorphism, and $\aleph_7(g)$ satisfies the uniform positive translation condition given by (7.3.55).

And $\mathbf{\Gamma}_{\tilde{E}}$ virtually normalizes the group

$$\left\{ \mathcal{N}(\vec{v}) = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline 0 & \|\vec{v}\|^2/2 & \vec{v} & 1 \end{array}\right) \middle| \vec{v} \in \mathbb{R}^{i_0} \right\}. \tag{7.4.17}$$

PROOF. The proof is contained in the proof of Theorem 7.1.4. □

7.5. Applications of the NPNC-end theory

7.5.1. The proof of Corollary 7.1.5.

PROOF OF COROLLARY 7.1.5. We may always take finite index subgroups for $\mathbf{\Gamma}_{\tilde{E}}$ and consider as the end holonomy group. By Corollary 7.4.9, we obtain that the dual holonomy group $g^{-1T} \in \mathbf{\Gamma}^*_{\tilde E}$ has form under a coordinate system:

$$g^{-1T} = \left(\begin{array}{c|c|c|c} S(g)^{-1T} & 0 & 0 & 0 \\ \hline 0 & \lambda(g)^{-1} & -\lambda(g)^{-1}O_5(g)^{-1}\vec{v}_g & \lambda(g)^{-1}\left(-\aleph_7(g) + \frac{\|v(g)\|^2}{2}\right) \\ \hline 0 & 0 & \lambda(g)^{-1}O_5(g)^{-1} & -\lambda(g)^{-1}\vec{v}_g^T \\ \hline 0 & 0 & 0 & \lambda(g)^{-1} \end{array}\right). \tag{7.5.1}$$

And $\mathbf{\Gamma}^*_{\tilde E}$ virtually normalizes the group $\{\mathscr{N}(\vec{v})^{-1T} | \vec{v} \in \mathbb{R}^{i_0}\}$ where

$$\mathscr{N}(\vec{v})^{-1T} = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & -\vec{v} & \|\vec{v}\|^2/2 \\ \hline 0 & 0 & \mathrm{I}_{i_0} & -\vec{v}^T \\ \hline 0 & 0 & 0 & 1 \end{array}\right). \tag{7.5.2}$$

By using coordinate changes by reversing the order of the $n-i_0+1$-th coordinate to the $n+1$-th coordinate, we can make the lower right matrix of $\mathbf{\Gamma}_{\tilde E}$ and $\mathscr{N}$ into a lower triangular form. Hence, $\mathscr{N}^*$ is a partial i_0-dimensional cusp group.

Suppose that $\langle S(g), g \in \mathbf{\Gamma}_{\tilde E}\rangle$ acts on properly convex set $K := K'' * \{k\}$ in $\mathbb{S}^{n-i_0-1}$, a strict join, for a properly convex set $K'' \subset \mathbb{S}^{n-i_0-2} \subset \mathbb{S}^{n-i_0-1}$ and k from the proof of Proposition 7.3.19. $\mathscr{N}$ acts on $\mathbb{S}^{i_0+1}$ containing $\mathbb{S}^{i_0}_\infty$ and corresponding to a point k under the projection $\Pi_K : \mathbb{S}^n - \mathbb{S}^{i_0}_\infty \to \mathbb{S}^{n-i_0-1}$. Let K''' denote the compact convex set in $\mathbb{S}^n - \mathbb{S}^{i_0}_\infty$ mapping homeomorphic to K'' under Π_K as we showed in Proposition 7.3.14. There is also a subspace $\mathbb{S}^{n-i_0-2}_{K'''}$ that is the span of K'''. Also, $K_4 := K''' * \mathrm{v}_{\tilde E}$ is in $\mathbb{S}^{n-i_0-1\prime}$ the great sphere containing $\mathrm{v}_{\tilde E}$ and project to $\mathbb{S}^{n-i_0-2}$ under Π_K.

Recall Proposition 1.5.13 for the following: We have $\mathbb{R}^{n+1} = V \oplus W$ for subspaces V and W corresponding to $\mathbb{S}^{n-i_0-2}_{K'''}$ and $\mathbb{S}^{i_0+1}$ respectively. We may assume that $\mathscr{N}$ acts on both spaces and $\mathbf{\Gamma}_{\tilde E}$ acts on K''' and both spaces. Then $\mathbb{R}^{n+1*} = V^\dagger \oplus W^\dagger$ for subspaces $V^\dagger$ of 1-forms on V zero on W and $W^\dagger$ of 1-forms of W zero on V. Then $V^\dagger$ corresponds to the subspace $\mathbb{S}^{n-i_0-2\dagger}_{K'''}$ which equals $\mathbb{S}^{i_0+1*}$, and $W^\dagger$ corresponds to $\mathbb{S}^{i_0+1\dagger}$ which equals $\mathbb{S}^{n-i_0-2*}_{K'''}$. We let $\mathbb{S}^{n-i_0-2\dagger}_{K'''}$ and $\mathbb{S}^{i_0+1\dagger}$ denote the dual subspaces in $\mathbb{S}^{n*}$.

Hence, $\mathbf{\Gamma}^*_{\tilde E}$ and $\mathscr{N}^*$ act on both of these two spaces.

Since K'' is $\mathrm{bd}\tilde\Sigma_{\tilde E}$ as in Proposition 7.3.14, we obtain $K_4 \subset \mathrm{bd}\tilde{\mathscr{O}}$.

Let us choose a properly convex p-end neighborhood U where $\mathscr{N}$ acts on. $U \cap Q$ for any i_0+1-dimensional subspace containing $\mathbb{S}^{i_0}_\infty$ is either empty or is an ellipsoid since $\mathscr{N}$ acts on U. Any sharply supporting hyperplane P' at $\mathrm{v}_{\tilde E}$ to U or $\tilde{\mathscr{O}}$ must containing $\mathbb{S}^{i_0}_\infty$ since $P' \cap Q$ for any i_0+1-dimensional subspace Q containing $\mathbb{S}^{i_0}_\infty$ must be disjoint from $U \cap Q$ and hence $P' \cap Q \subset \mathbb{S}^{i_0}_\infty$ and hence $P' \supset \mathbb{S}^{i_0}_\infty$.

Let $P \subset \mathbb{S}^n$ be an oriented hyperspace sharply supporting $\tilde{\mathscr{O}}$ at $\mathrm{v}_{\tilde E}$ and containing $\mathbb{S}^{i_0}_\infty$ and $\mathbb{S}^{n-i_0-2}_{K'''}$. This is unique such one since the hyperspace is the join of the two. Hence, $K''' * \{\mathrm{v}_{\tilde E}\} \subset P$ where

$$\mathrm{Cl}(U) \cap P = \mathrm{Cl}(\tilde{\mathscr{O}}) \cap P = K_4 := K''' * \{\mathrm{v}_{\tilde E}\}$$

by Proposition 7.3.19.

We have the subspaces $\mathbb{S}^{n-i_0-1\prime}$ and $\mathbb{S}^{i_0+1}$ meeting at $\mathrm{v}_{\tilde E}$ and $\mathbb{S}^{i_0}$ containing $\mathrm{v}_{\tilde E}$ meeting with $\mathbb{S}^{n-i_0-1\prime}$ at the same point.

Let $P^\star$ denote the dual space of P under its intrinsic duality. Let us denote by $K^\star_{4,P} \subset P^\star$ the dual of K_4 with respect to P.

Consider a pencil $P_{t,Q}$ of hyperplanes supporting $\tilde{\mathscr{O}}$ starting from P sharing a codimension-two subspace $Q \subset P$. Then $P_{t,Q}$ exists for t in a convex interval I_Q in a projective circle. Q supports $P \cap \mathrm{Cl}(\tilde{\mathscr{O}}) = K''' * \{\mathrm{v}_{\tilde{E}}\}$. We assume $P_{0,Q} = P$ for all Q. Thus the space $S_{P,\Omega}$ of all hyperspaces in one of $P_{t,Q}$ is projectively equal to a fibration over $K^\star_4$ in $P^\star$ with fibers a singleton or a compact interval. Since $S_{P,\Omega}$ is convex, the set corresponding to the nontrivial interval fibers is the interior of $K^\star_4$.

By Proposition 1.5.13(iii), $K^\star_4$ equals $K^\dagger_4 * \mathbb{S}^{i_0-1}_P$ for a proper subspace dual $K^\dagger_4$ in $\mathbb{S}^{n-i_0-1\dagger}$ to K_4 and a great sphere $\mathbb{S}^{i_0-1}_P$ in $P^\star$.

Consider $S_{P,\Omega}$ as a subset of $\mathbb{S}^{n*}$ now. Then $P_{t,Q}$ under duality goes to a ray in $\tilde{\mathscr{O}}^*$ from P^* to a boundary point of $\tilde{\mathscr{O}}^*$. Hence, the space $R_P(S_{P,\Omega})$ of such open rays are projectively diffeomorphic to the interior of $K^\dagger_4 * \mathbb{S}^{i_0-1}_P$. Let $K'''^\dagger$ in $\mathbb{S}^{n-i_0-2\dagger}$ denote the dual of K''' in its span $\mathbb{S}^{n-i_0-2}_{K'''}$. Since $K^\dagger_4$ is projectively diffeomorphic to $K'''^\dagger * \{v\}$ for a singleton v, we have $S_{P,\Omega}$ is projectively diffeomorphic to the interior of $K'''^\dagger * H^{i_0}$ for a hemisphere H^{i_0} of dimension i_0.

By the duality argument, this space equals $R_{P^*}(\tilde{\mathscr{O}}^*)$ since such rays correspond to supporting pencils of $\mathscr{O}$ and vice versa.

Now, recall that our matrices of $\mathbf{\Gamma}_{\tilde{E}}$ in the form of (7.4.16) and matrices in $\mathscr{N}$ in the form (7.4.17). We can directly show the properness of the action on $R_p(\tilde{\mathscr{O}}^*)$:

- Let g_i be a sequence of elements of $\mathbf{\Gamma}^*_{\tilde{E}}$. Suppose that $S(g_i)^*$ is bounded for our matrices in $\mathbf{\Gamma}_{\tilde{E}}$. Then $\vec{v}_{g_i}$ blows up: Otherwise the properness of the $\mathbf{\Gamma}_{\tilde{E}}$ does not hold for $\tilde{\Sigma}_{\tilde{E}}$ and our action splits. Our action is basically that on $K''^o \times \mathbb{R}^{i_0}$, an affine form of the interior of $K * \mathbb{S}^{i_0-1}$, preserving the product structure where $K = K'' * \{k\}$.

Hence, it follows by our matrix form (7.5.1) that $\mathbf{\Gamma}^*_{\tilde{E}}$ acts properly on $R_{P^*}(\tilde{\mathscr{O}}^*)$ projectively isomorphic to the interior of $K^\dagger_4 * \mathbb{S}^{i_0-1}$. By Lemma 3.1.5, $\tilde{\mathscr{O}}^*$ can be considered a p-end neighborhood with a radial structure. Hence, we can apply our theory of the classification of NPNC-ends. The transverse weak uniform middle eigenvalue condition is satisfied by the form of the matrices. Also the uniform positive translation condition holds by the matrix forms again. Proposition 7.3.19 completes the proof.

Finally, we mention the following: Since each i_0-dimensional ellipsoid in the fiber in $\tilde{\Sigma}_{\tilde{E}^*}$ has a unique fixed point that should be common for all ellipsoid fibers, the choice of the p-end vertex is uniquely determined for $\tilde{E}^*$ so that $\tilde{E}^*$ is to be quasi-joined. □

7.5.2. A counterexample in a solvable case. We will find some nonsplit NPNC-end where the end holonomy group is solvable. Our construction is related to the construction of Carrière [34] and Epstein [77]. A related work is given by Cooper [63]. These are not a quasi-join nor a join and do not satisfy our conditions (NS) and neither the end orbifolds admit properly convex structure nor the fundamental groups are virtually abelian. Define

$$N(w,v) = \begin{pmatrix} 1 & w & w^2/2 & 0 & 0 \\ 0 & 1 & w & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & v & 1 & 0 \\ 0 & 0 & v^2/2 & v & 1 \end{pmatrix}, \text{ and} \tag{7.5.3}$$

$$g_\lambda = \begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/\lambda & 0 \\ 0 & 0 & 0 & 0 & 1/\lambda^2 \end{pmatrix}. \tag{7.5.4}$$

We compute $g_\lambda N(w,v) g_\lambda^{-1} = N(\lambda w, v/\lambda)$. We define the group

$$S := \langle N(w,v), g_\lambda : v, w \in \mathbb{R}, \lambda \in \mathbb{R}_+ \rangle.$$

Consider the affine space $\mathbb{A}^4$ given by $x_3 > 0$ with coordinates x_1, x_2, x_4, x_5 where S acts on. Then $\langle N(w,0), w \in \mathbb{R}, g_\lambda \rangle$, $\lambda \in \mathbb{R}_+$, acts on an open disk $B_{1,2}$ bounded by a quadric $x_1 > x_2^2/2$ in the plane $x_4 = 0, x_5 = 0$. (See Section 7.3.1.1.) $\langle N(0,v), v \in \mathbb{R}, g_\lambda, \lambda \in \mathbb{R}_+ \rangle$ acts on an open disk $B_{4,5}$ bounded by a quadric $x_5 > x_4^2/2$ in the plane $x_2 = 0, x_3 = 0$.

Hence, an orbit $S(⟦1,0,1,0,1⟧)$ is given by the following set as a subset of $\mathbb{A}^4$:

$$\left\{ (x_1, x_2, x_4, x_5) | x_1 = \frac{x_2^2}{2} + C^2, x_5 = \frac{x_4^2}{2} + \frac{1}{C^2}, C > 0 \right\}.$$

This is a 3-cell. Moreover,

$$\begin{aligned} & N(w,v) g_\lambda \left(\frac{x_2^2}{2} + C^2, x_2, x_4, \frac{x_4^2}{2} + \frac{1}{C^2} \right) \\ &= \left(\lambda^2 \left(\frac{x_2^2}{2} + C^2 \right) + \lambda w x_2 + \frac{w^2}{2}, \lambda x_2 + w, \frac{x_4}{\lambda} + v, \frac{x_4^2}{2\lambda^2} + \frac{1}{C^2 \lambda^2} + v \frac{x_4}{\lambda} + \frac{v^2}{2} \right) \\ &= \left(\frac{(\lambda x_2 + w)^2}{2} + \lambda^2 C^2, \lambda x_2 + w, \frac{x_4}{\lambda} + v, \frac{\left(\frac{x_4}{\lambda} + v\right)^2}{2} + \frac{1}{C^2 \lambda^2} \right) \\ &= g_\lambda N(w/\lambda, \lambda v) \left(\frac{x_2^2}{2} + C^2, x_2, x_4, \frac{x_4^2}{2} + \frac{1}{C^2} \right). \end{aligned} \tag{7.5.5}$$

Hence, there is an exact sequence

$$1 \to \{N(w,v) | w, v \in \mathbb{R}\} \to S \to \{g_\lambda | \lambda > 0\} \to 1,$$

telling us that S is a solvable Lie group (Thurston's Sol [149].).

We find a discrete solvable subgroup. We take a lattice L in $\mathbb{R}^2$ and obtain a free abelian group $N(L)$ of rank two. We can choose L so that the diagonal matrix with diagonal $(\lambda, 1/\lambda)$ acts as an automorphism. Then the group S_L generated by $\langle N(L), g_\lambda \rangle$ is a discrete cocompact subgroup of S.

We remark that such a group exists by taking a standard lattice in $\mathbb{R}^2$ and choosing an integral Anosov linear map A of determinant 1 with two eigendirections. We choose a new coordinate system so that the eigendirections are parallel to the x-axis and the y-axis. Then now L can be read from the new coordinate system, and λ is the eigenvalue of A bigger than 1.

The orbit $S(⟦1,0,1,0,1⟧)$ is a subset of $B_{1,2} \times B_{4,5}$. We may choose our end vertex v to be $⟦0,0,0,0,1⟧$ or $⟦1,0,0,0,0⟧$.

The orbit $S(⟦1,0,1,0,1⟧)$ is strictly convex: We work with the affine coordinates. We consider this point with affine coordinates $(1,0,0,1)$. The tangent hyperspace at this point is given by $x_1 + x_5 = 2$. We can show that locally the orbit meets this hyperplane only at $(1,0,0,1)$ and is otherwise in one-side of the plane.

Since Z acts transitively, the orbit is strictly convex. Also, it is easy to show that the orbit is properly embedded. Hence the orbit is a boundary of a properly convex open domain. It is now elementary to show that this is an R-p-end neighborhood for a choice of p-end vertex $⟦0,0,0,0,1⟧$ or $⟦1,0,0,0,0⟧$.

CHAPTER 8

Characterization of complete R-ends

In Chapter 8, we discuss complete affine ends. First, we explain the weak middle eigenvalue condition. We state the main result of the chapter Theorem 8.1.2, which characterizes the complete affine ends. We will prove it by the results in Chapter 7 and the results of subsequent sections: In Theorem 8.1.3, we show that pre-horospherical ends are horospherical ends. In Theorem 8.1.4, we show that a complete affine end falls into one of the two classes, one of which is a cuspidal and the other one is more complicated with two norms of eigenvalues. Theorem 8.1.2 will show that the second case is a quasi-join using results in Chapter 7. In Section 3.1.5, we concentrate on the ends with the holonomy of only unit norm eigenvalues showing that they have to be cuspidal.

The results here overlap with the results of Crampon-Marquis [68] and Cooper-Long-Tillman [67]. However, the results are of a different direction than theirs since they are interested in finite-volume Hilbert metrics, and were originally conceived before their papers appeared. We also make use of Crampon-Marquis [69].

Let $\tilde{\mathscr{O}}$ denote a convex domain in $\mathbb{S}^n$ and covering a strongly tame orbifold $\mathscr{O}$ with a holonomy homomorphism $h : \pi_1(\mathscr{O}) \to \mathrm{SL}_{\pm}(n+1,\mathbb{R})$. Let $\tilde{E}$ be an R-p-end. A *mec* for a p-end holonomy group $h(\pi_1(\tilde{E}))$ with respect to $\mathrm{v}_{\tilde{E}}$ or the R-p-end structure holds if for each $g \in h(\pi_1(\tilde{E})) - \{\mathrm{I}\}$ the largest norm $\lambda_1(g)$ of eigenvalues of g is strictly larger than the eigenvalue $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ associated with p-end vertex $\mathrm{v}_{\tilde{E}}$.

Given an element $g \in h(\pi_1(\tilde{E}))$, let $\left(\tilde{\lambda}_1(g), \dots, \tilde{\lambda}_{n+1}(g)\right)$ be the $(n+1)$-tuple of the eigenvalues listed with multiplicity given by the characteristic polynomial of g where we repeat each eigenvalue with the multiplicity given by the characteristic polynomial. The *multiplicity* of a norm of an eigenvalue of g is the number of times the norm occurs among the $(n+1)$-tuples of norms

$$\left(|\tilde{\lambda}_1(g)|, \dots, |\tilde{\lambda}_{n+1}(g)|\right).$$

DEFINITION 8.0.1. Let $\tilde{E}$ be a p-end with the holonomy group $h(\pi_1(\tilde{E}))$. A *weak middel eigenvalue condition (wmec)* for an R-p-end $\tilde{E}$ holds provided for each $g \in h(\pi_1(\tilde{E}))$ the following holds:

- whenever $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ has the largest norm of all norms of eigenvalues $h(g)$, $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ must have multiplicity ≥ 2.

We note that these definitions depend on the choice of the p-end vertices; however, they are well defined once the radial structures are assigned.

Recall the parabolic subgroup of the isometry group $\mathbf{Aut}(\mathbb{B})$ of the hyperbolic space $\mathbb{B}$ for an (i_0+1)-dimensional Klein model $\mathbb{B} \subset \mathbb{S}^{i_0+1}$ fixing a point p in the boundary of $\mathbb{B}$. Such a discrete subgroup of a parabolic subgroup group is isomorphic to extensions of a lattice in $\mathbb{R}^{i_0}$ and is Zariski closed by the Bieberbach theorem.

Let E be an i_0-dimensional ellipsoid containing the point v in a subspace P of dimension i_0+1 in $\mathbb{S}^n$. Let $\mathbf{Aut}(P)$ denote the group of projective automorphisms of P, and let

$\mathsf{SL}_\pm(n+1,\mathbb{R})_P$ the subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ acting on P. Let $r_P : \mathsf{SL}_\pm(n+1,\mathbb{R})_P \to \mathbf{Aut}(P)$ denote the restriction homomorphism $g \to g|P$. An i_0-*dimensional partial cusp subgroup* is one mapping under R_P isomorphically to a cusp subgroup of $\mathbf{Aut}(P)$ acting on $E - \{\mathrm{v}\}$, fixing v.

Suppose now that $\tilde{\mathscr{O}} \subset \mathbb{RP}^n$. Let P' denote a subspace of dimension i_0+1 containing an i_0-dimensional ellipsoid E' containing v. Let $\mathsf{PGL}(n+1,\mathbb{R})_{P'}$ denote the subgroup of $\mathsf{PGL}(n+1,\mathbb{R})$ acting on P'. Let $R_{P'} : \mathsf{PGL}(n+1,\mathbb{R})_{P'} \to \mathbf{Aut}(P')$ denote the restriction $g \mapsto g|P'$. An i_0-*dimensional partial cusp subgroup* is one mapping under $R_{P'}$ isomorphically to a cusp subgroup of $\mathbf{Aut}(P')$ acting on $E' - \{\mathrm{v}\}$, fixing v. When $i_0 = n-1$, we will drop the "partial" from the term "partial cusp group".

An i_0-*dimensional cusp group* is a finite extension of a projective conjugate of a discrete cocompact subgroup of a group of an i_0-dimensional partial cusp subgroup. If the horospherical neighborhood with the p-end vertex v has the p-end holonomy group that is a discrete $(n-1)$-dimensional cusp group, then we call the p-end to be *cusp-shaped* or *horospherical*. (See Theorem 8.1.3.)

8.1. Main results

Our main result classifies CA R-p-ends. We need some facts of NPNC-ends that will be explained in Section 7.2.

Given a hospherical R-p-end, the p-end holonomy group $\mathbf{\Gamma}_\mathrm{v}$ acts on a p-end neighborhood U and $\mathbf{\Gamma}_\mathrm{v}$ is a subgroup of an $(n-1)$-dimensional cusp group $\mathscr{H}_\mathrm{v}$. By Lemma 3.1.8,

$$V := \bigcap_{g\in\mathscr{H}_\mathrm{v}} g(U)$$

contains a $\mathscr{H}_\mathrm{v}$-invariant p-end neighborhood. Hence, V is a horospherical p-end neighborhood of $\tilde{E}$. Thus, a horospherical R-end is pre-horospherical. Conversely, a pre-horospherical R-end is a horospherical R-end by Theorem 8.1.3 under some assumption on $\mathscr{O}$ itself. (See Definition 3.1.6.)

First, we clarify by Theorem 8.1.3:

COROLLARY 8.1.1. *Let $\mathscr{O}$ be a strongly tame properly convex real projective n-orbifold. Let E be an R-end of its universal cover $\tilde{\mathscr{O}}$. Then E is a pre-horospherical R-end if and only if $\tilde{E}$ is a horospherical R-end.* $\square$

The following classifies the complete affine ends where we need some results from Chapter 7. Since these have virtually abelian holonomy groups by Theorem 8.1.3, they are classified in [8].

THEOREM 8.1.2. *Let $\mathscr{O}$ be a strongly tame properly convex real projective n-orbifold. Let $\tilde{E}$ be an R-p-end of its universal cover $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$). Let $\mathbf{\Gamma}_{\tilde{E}}$ denote the end holonomy group. Then $\tilde{E}$ is a complete affine R-p-end if and only if $\tilde{E}$ is a horospherical R-p-end or an NPNC-end with fibers of dimension $n-2$ with the virtually abelian end-fundamental group by altering the p-end vertex. Furthermore, if $\tilde{E}$ is a complete affine R-p-end and $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the weak middle eigenvalue condition, then $\tilde{E}$ is a horospherical R-p-end.*

PROOF. Again, we assume that Ω is a domain of $\mathbb{S}^n$. Theorem 8.1.4 is the forward direction. Corollary 8.1.5 implies the second case above.

Now, we prove the converse using the notation and results of Chapter 7. Since a horospherical R-p-end is pre-horospherical, Theorem 8.1.3 implies the half of the converse. Given a p-NPNC-end $\tilde{E}$ with fibers of dimension $n-2$, $\tilde{\Sigma}_{\tilde{E}}$ is projectively isomorphic to an affine half-space. Using the notation of Proposition 7.3.19, K'' is zero-dimensional and the end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$ acts on $K'' * \{\mathrm{v}\}$ for an end vertex v. There is a foliation in $\tilde{\Sigma}_{\tilde{E}}$ by complete affine spaces of dimension $n-2$ parallel to each other. The space of leaves is projectively diffeomorphic to the interior of $K'' * \mathrm{v}'$ for a point v$'$. Let U be the p-end neighborhood for $\tilde{E}$. Then bd$U$ is in one-to-one correspondence with $\tilde{\Sigma}_{\tilde{E}}$ by radial rays from v. Hence, bd$U$ has an induced foliation. Each leaf in bd$U$ lies in an open hemisphere of dimension $n-1$. (See (7.1.2) in Section 7.2.) Also, $\mathbf{\Gamma}_{\tilde{E}}$ acts on an open hemisphere $H^{n-1}_{\mathrm{v}'}$ of dimension $(n-1)$ with boundary a great sphere $\mathbb{S}^{n-2}$ containing v in the direction of v$'$.

Now we switch the p-end vertex to a singleton $\{k''\} = K''$ from v. Then $H^{n-1}_{\mathrm{v}'}$ corresponds to a complete affine space $\mathbb{A}^{n-1}_{k''}$. Each leaf projects to an ellipsoid in $\mathbb{A}^{n-1}_{k''}$ with a common ideal point v corresponding to the direction of $\overline{k''\mathrm{v}}$ oriented away from k''. The ellipsoids are tangent to a common hyperspace in $\mathbb{S}^{n-1}_{k''}$. Hence, they are parallel paraboloids in an affine subspace $\mathbb{A}^{n-1}_{k''}$. The uniform positive translation condition gives us that the union of the parallel paraboloids is $\mathbb{A}^{n-1}_{k''}$. Hence, $\tilde{E}$ is a complete R-end with k'' as the vertex. The last statement follows by Corollary 8.1.8. [$\mathbb{S}^n$S] $\square$

8.1.1. The Horosphere theorem.

THEOREM 8.1.3 (Horosphere). *Let $\mathscr{O}$ be a strongly tame convex real projective n-orbifold, $n \geq 2$. Let $\tilde{E}$ be a pre-horospherical R-end of its universal cover $\tilde{\mathscr{O}}$, $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ (resp. $\subset \mathbb{RP}^n$), and $\mathbf{\Gamma}_{\tilde{E}}$ denote the p-end holonomy group. Then the following hold*:

- (i) *The subspace $\tilde{\Sigma}_{\tilde{E}} = R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}}) \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ of directions of lines segments from the endpoint $\mathrm{v}_{\tilde{E}}$ in* bd$\tilde{\mathscr{O}}$ *into a p-end neighborhood of $\tilde{E}$ forms a complete affine subspace of dimension $n-1$.*
- (ii) *The norms of eigenvalues of $g \in \mathbf{\Gamma}_{\tilde{E}}$ are all* 1.
- (iii) *$\mathbf{\Gamma}_{\tilde{E}}$ virtually is in a conjugate of an abelian parabolic or cusp subgroup of* $\mathsf{SO}(n,1)$ (*resp.* $\mathsf{PO}(n,1)$) *of rank $n-1$ in* $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$ (*resp.* $\mathsf{PGL}(n+1,\mathbb{R})$). *And hence $\tilde{E}$ is cusp-shaped.*
- (iv) *For any compact set $K' \subset \mathscr{O}$, $\mathscr{O}$ contains a horospherical end neighborhood disjoint from K'.*

PROOF. We will prove for the case $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Let U be a pre-horoball p-end neighborhood with the p-end vertex $\mathrm{v}_{\tilde{E}}$, closed in $\tilde{\mathscr{O}}$. The space of great segments from the p-end vertex passing U forms a convex subset $\tilde{\Sigma}_{\tilde{E}}$ of a complete affine subspace $\mathbb{R}^{n-1} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ by Proposition 1.4.1. The space covers an end orbifold $\Sigma_{\tilde{E}}$ with the discrete group $\pi_1(\tilde{E})$ acting as a discrete subgroup $\Gamma'_{\tilde{E}}$ of the projective automorphisms so that $\tilde{\Sigma}_{\tilde{E}}/\Gamma'_{\tilde{E}}$ is projectively isomorphic to $\Sigma_{\tilde{E}}$.

(i) By Proposition 1.4.1, one of the following three happens:

- $\tilde{\Sigma}_{\tilde{E}}$ is properly convex.
- $\tilde{\Sigma}_{\tilde{E}}$ is foliated by complete affine subspaces of dimension i_0, $1 \leq i_0 < n-1$, with the common boundary sphere of dimension $i_0 - 1$, the space of the leaves forms a properly open convex subset K^o of $\mathbb{S}^{n-i_0-1}$, and $\mathbf{\Gamma}_{\tilde{E}}$ acts on K^o cocompactly but not necessarily discretely.
- $\tilde{\Sigma}_{\tilde{E}}$ is a complete affine subspace.

We aim to show that the first two cases do not occur.

Suppose that we are in the second case and $1 \leq i_0 \leq n-2$. This implies that $\tilde{\Sigma}_{\tilde{E}}$ is foliated by complete affine subspaces of dimension $i_0 \leq n-2$.

Since $\mathbf{\Gamma}_{\tilde{E}}$ acts on a properly convex subset K of dimension ≥ 1, an element g has a norm of an eigenvalue > 1 and a norm of eigenvalue < 1 as a projective automorphism on $\mathbb{S}^{n-i_0-1}$ by Proposition 1.1 of [18]. Hence, we obtain the largest norm of eigenvalues and the smallest one of g in $\mathbf{Aut}(\mathbb{S}^n)$ both different from 1. By Lemma 1.3.9, g is positive bi-semi-proximal. Therefore, let $\lambda_1(g) > 1$ be the greatest norm of the eigenvalues of g and $\lambda_2(g) < 1$ be the smallest norm of the eigenvalues of g as an element of $\mathsf{SL}_\pm(n+1,\mathbb{R})$. Let $\lambda_{\mathrm{v}_{\tilde{E}}}(g) > 0$ be the eigenvalue of g associated with $\mathrm{v}_{\tilde{E}}$. The possibilities for g are as follows

$$\begin{aligned} \lambda_1(g) &= \lambda_{\mathrm{v}_{\tilde{E}}}(g) &> \lambda_2(g), \\ \lambda_1(g) &> \lambda_{\mathrm{v}_{\tilde{E}}}(g) &> \lambda_2(g), \\ \lambda_1(g) &\ > \lambda_2(g) &= \lambda_{\mathrm{v}_{\tilde{E}}}(g). \end{aligned}$$

In all cases, at least one of the largest norm or the smallest norm is different from $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$. By Lemma 1.3.9, this norm is realized by a positive eigenvalue. We take $g^n(x)$ for a generic point $x \in U$. As $n \to \infty$ or $n \to -\infty$, the sequence $\{g^n(x)\}$ limits to a point x_∞ in $\mathrm{Cl}(U)$ distinct from $\mathrm{v}_{\tilde{E}}$. Also, g fixes a point x_∞, and x_∞ has a different positive eigenvalue from $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$. As $x_\infty \notin U$, it should be $x_\infty = \mathrm{v}_{\tilde{E}}$ by the definition of the pre-horospheres. This is a contradiction.

The first possibility is also shown not to occur similarly. Thus, $\tilde{\Sigma}_{\tilde{E}}$ is a complete affine subspace.

(ii) If $g \in \mathbf{\Gamma}_{\tilde{E}}$ has a norm of eigenvalue different from 1, then we can apply the second and the third paragraphs above to obtain a contradiction. We obtain $\lambda_j(g) = 1$ for each norm $\lambda_j(g)$ of eigenvalues of g for every $g \in \mathbf{\Gamma}_{\tilde{E}}$.

(iii) Since $\tilde{\Sigma}_{\tilde{E}}$ is a complete affine subspace, $\tilde{\Sigma}_{\tilde{E}}/\mathbf{\Gamma}_{\tilde{E}}$ is a complete affine orbifold with the norms of eigenvalues of holonomy matrices all equal to 1 where $\mathbf{\Gamma}'_{\tilde{E}}$ denotes the affine transformation group corresponding to $\mathbf{\Gamma}_{\tilde{E}}$. (By D. Fried [80], this implies that $\pi_1(\tilde{E})$ is virtually nilpotent.) Again by Selberg Theorem 1.1.19, we can find a torsion-free subgroup $\mathbf{\Gamma}'_{\tilde{E}}$ of finite-index. Then $\mathbf{\Gamma}'_{\tilde{E}}$ is in a cusp group by Proposition 7.21 of [68] (related to Theorem 1.6 of [68]). By the proposition, we see that $\mathbf{\Gamma}'_{\tilde{E}}$ is in a conjugate of a parabolic subgroup of $\mathsf{SO}(n,1)$ and hence acts on an $(n-1)$-dimensional ellipsoid fixing a unique point. Since a horosphere has a Euclidean metric invariant under the group action, the image group is in a Euclidean isometry group. Hence, the group is virtually abelian by the Bieberbach theorem.

Actually, there is a one-dimensional family of such ellipsoids containing the fixed point where $\mathbf{\Gamma}'_{\tilde{E}}$ acts on.

Let U denote the domain bounded by the closure of the ellipsoid. There exist finite elements $g_1,\dots,g_n$ representing cosets of $\mathbf{\Gamma}_{\tilde{E}}/\mathbf{\Gamma}'_{\tilde{E}}$. If $g_i(U)$ is a proper subset of U, the $g_i^n(U)$ is so and hence g_i^n is not in $\mathbf{\Gamma}'_{\tilde{E}}$ for any n. This is a contradiction. Hence $\mathbf{\Gamma}_{\tilde{E}}$ acts on U also. By same reasoning, $\mathbf{\Gamma}_{\tilde{E}}$ on every ellipsoid in a one-dimensional parameter space containing a unique fixed point, and an ellipsoid gives us a horosphere in the interior of a horoball. Hence, $\mathbf{\Gamma}_{\tilde{E}}$ is a cusp group.

(iv) We can choose an exiting sequence of p-end horoball neighborhoods U_i where a cusp group acts. We can consider the hyperbolic spaces to understand this.

[$\mathbb{S}^n$T] □

8.1.2. The forward direction of Theorem 8.1.2. The second case will be studied later in Corollary 8.1.5. We will show the end $\tilde{E}$ to be an NPNC-end with fiber dimension

$n-2$ when we choose another point as the new p-end vertex for $\tilde{E}$. Clearly, this case is not horospherical. (See Crampon-Marquis [68] for a similar proof.)

For the following note that Marquis classified ends for 2-manifolds [126] into cuspidal, hyperbolic, or quasi-hyperbolic ends. Since convex 2-orbifolds are convex 2-manifolds virtually, we are done for $n = 2$.

THEOREM 8.1.4 (Complete affine). *Let $\mathscr{O}$ be a strongly tame properly convex n-orbifold for $n \geq 3$. Suppose that $\tilde{E}$ is a complete-affine R-p-end of its universal cover $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$). Let $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^n$ (resp. $\in \mathbb{RP}^n$) be the p-end vertex with the p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$. Then*

- (i) *we have following two exclusive alternatives*:
  - $\mathbf{\Gamma}_{\tilde{E}}$ *is virtually unipotent where all norms of eigenvalues of elements equal* 1*, or*
  - $\mathbf{\Gamma}_{\tilde{E}}$ *is virtually abelian where*
    - *each $g \in \mathbf{\Gamma}_{\tilde{E}}$ has at most two norms of the eigenvalues,*
    - *at least one $g \in \mathbf{\Gamma}_{\tilde{E}}$ has two norms, and*
    - *if $g \in \mathbf{\Gamma}_{\tilde{E}}$ has two distinct norms of the eigenvalues, the norm of $\lambda_{\mathrm{v}_{\tilde{E}}}(g)$ has a multiplicity one.*
    - *$\mathbf{\Gamma}_{\tilde{E}}$ acts as a virtually unipotent group on the complete affine space $\tilde{\Sigma}_{\tilde{E}}$.*
- (ii) *In the first case, $\mathbf{\Gamma}_{\tilde{E}}$ is horospherical, i.e., cuspidal.*

PROOF. We first prove for the $\mathbb{S}^n$-version. Using Theorem 1.1.19, we may choose a torsion-free finite-index subgroup. We may assume without loss of generality that Γ is torsion-free since torsion elements have only 1 as the eigenvalues of norms and we only need to prove the theorem for a finite-index subgroup. Hence, Γ does not fix a point in $\tilde{\Sigma}_{\tilde{E}}$.

(i) Since $\tilde{E}$ is complete affine, $\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is identifiable with an affine subspace $\mathbb{A}^{n-1}$. $\mathbf{\Gamma}_{\tilde{E}}$ induces $\Gamma'_{\tilde{E}}$ in $\mathbf{Aff}(\mathbb{A}^{n-1})$ that are of form $x \mapsto Mx + \vec{b}$ where M is a linear map $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and $\vec{b}$ is a vector in $\mathbb{R}^{n-1}$. For each $\gamma \in \mathbf{\Gamma}_{\tilde{E}}$,

- let $\gamma_{\mathbb{R}^{n-1}}$ denote this affine transformation,
- we denote by $\hat{L}(\gamma_{\mathbb{R}^{n-1}})$ the linear part of the affine transformation $\gamma_{\mathbb{R}^{n-1}}$, and
- let $\vec{v}(\gamma_{\mathbb{R}^{n-1}})$ denote the translation vector.

A *relative eigenvalue* is an eigenvalue of $\hat{L}(\gamma_{\mathbb{R}^{n-1}})$.

At least one eigenvalue of $\hat{L}(\gamma_{\mathbb{R}^{n-1}})$ is 1 since γ acts without fixed point on $\mathbb{R}^{n-1}$. (See [113].) Now, $\hat{L}(\gamma_{\mathbb{R}^{n-1}})$ has a maximal invariant vector subspace A of $\mathbb{R}^{n-1}$ where all norms of the eigenvalues are 1.

Suppose that A is a proper γ-invariant vector subspace of $\mathbb{R}^{n-1}$. Then $\gamma_{\mathbb{R}^{n-1}}$ acts on the affine space $\mathbb{R}^{n-1}/A$ as an affine transformation with the linear parts without a norm of eigenvalue equal to 1. Hence, $\gamma_{\mathbb{R}^{n-1}}$ has a fixed point in $\mathbb{R}^{n-1}/A$, and $\gamma_{\mathbb{R}^{n-1}}$ acts on an affine subspace A' parallel to A.

A subspace H containing $\mathrm{v}_{\tilde{E}}$ corresponds to the direction of A' from $\mathrm{v}_{\tilde{E}}$. The union of segments with endpoints $\mathrm{v}_{\tilde{E}}, \mathrm{v}_{\tilde{E}-}$ in the directions in $A' \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is in an open hemisphere of dimension $< n$. Let H^+ denote this space where $\mathrm{bd} H^+ \ni \mathrm{v}_{\tilde{E}}$ holds. Since $\mathbf{\Gamma}_{\tilde{E}}$ acts on A', it follows that $\mathbf{\Gamma}_{\tilde{E}}$ acts on H^+. Then γ has at most two eigenvalues associated with H^+ one of which is $\lambda_{\mathrm{v}}(\gamma)$ and the other is to be denoted $\lambda_+(\gamma)$. Since γ fixes $\mathrm{v}_{\tilde{E}}$ and there is an eigenvector in the span of H^+ associated with $\lambda_+(\gamma)$, γ has the matrix form

$$\gamma = \begin{pmatrix} \lambda_+(\gamma)\hat{L}(\gamma_{\mathbb{R}^{n-1}}) & \lambda_+(\gamma)\vec{v}(\gamma_{\mathbb{R}^{n-1}}) & 0 \\ 0 & \lambda_+(\gamma) & 0 \\ * & * & \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma) \end{pmatrix}$$

where we have

$$\lambda_+(\gamma)^n \det(\hat{L}(\gamma_{\mathbb{R}^{n-1}}))\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma) = \pm 1.$$

(Note $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma^{-1}) = \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)^{-1}$ and $\lambda_+(\gamma^{-1}) = \lambda_+(\gamma)^{-1}$.)

We will show that $\hat{L}(\gamma_{\mathbb{R}^{n-1}})$ for every $\gamma \in \mathbf{\Gamma}_{\tilde{E}}$ is unit-norm-eigenvalued below. As before, $\lambda_1(\gamma)$ denote the largest norm of the eigevalues of γ. Note that $\lambda_1(\gamma) \geq \lambda_+(\gamma)$ since $\hat{L})(\gamma_{\mathbb{R}^{n-1}})$ has an eigenvalue equal to 1. There are following possibilities for each $\gamma \in \mathbf{\Gamma}_{\tilde{E}}$:

(a) $\lambda_1(\gamma) > \lambda_+(\gamma)$ and $\lambda_1(\gamma) > \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$.
(b) $\lambda_1(\gamma) = \lambda_+(\gamma) = \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$.
(c) $\lambda_1(\gamma) = \lambda_+(\gamma), \lambda_1(\gamma) > \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$.
(d) $\lambda_1(\gamma) > \lambda_+(\gamma), \lambda_1(\gamma) = \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$.

Suppose that γ satisfies (b). The relative eigenvalues of γ on $\mathbb{R}^{n-1}$ are all ≤ 1. Either γ is unit-norm-eigenvalued or we can take γ^{-1} and we are in case (a).

Suppose that γ satisfies (a). There exists a projective subspace S of dimension ≥ 0 where the points are associated with eigenvalues with the norm $\lambda_1(\gamma)$ where $\lambda_1(\gamma) > \lambda_+(\gamma), \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$.

Let S' be the subspace spanned by H^+ and S. Let U be a p-end neighborhood of $\tilde{E}$. Since the space of directions of U is $\mathbb{R}^{n-1}$, we have $U \cap S' \neq \emptyset$. We can choose two generic points y_1 and y_2 of $U \cap S' - H$ so that $\overline{y_1 y_2}$ meets H in its interior.

Then we can choose a subsequence $\{m_i\}$, $\{m_i\} \to \infty$, so that $\{\gamma^{m_i}(y_1)\} \to f$ and $\{\gamma^{m_i}(y_2)\} \to f_-$ as $i \to +\infty$ unto relabeling y_1 and y_2 for a pair of antipodal points $f, f_- \in S$. This implies $f, f_- \in \mathrm{Cl}(\tilde{\mathscr{O}})$, and $\tilde{\mathscr{O}}$ is not properly convex, which is a contradiction. Hence, (a) cannot be true.

We showed that if any $\gamma \in \mathbf{\Gamma}_{\tilde{E}}$ satisfies (a) or (b), then γ is unit-norm-eigenvalued.

If γ satisfies (c), then

$$\lambda_1(\gamma) = \lambda_+(\gamma) \geq \lambda_i(\gamma) \geq \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma) \tag{8.1.1}$$

for all other norms of eigenvalues $\lambda_i(\gamma)$: Otherwise, γ^{-1} satisfies (a), which cannot happen.

Similarly if γ satisfies (d), then we have

$$\lambda_1(\gamma) = \lambda_{\mathrm{v}_{\tilde{E}}}(\gamma) \geq \lambda_i(\gamma) \geq \lambda_+(\gamma) \tag{8.1.2}$$

for all other norms of eigenvalues $\lambda_i(\gamma)$. We conclude that either γ is unit-norm-eigenvalued or satisfies (8.1.1) or (8.1.2).

There is a homomorphism

$$\lambda_{\mathrm{v}_{\tilde{E}}} : \mathbf{\Gamma}_{\tilde{E}} \to \mathbb{R}_+ \text{ given by } g \mapsto \lambda_{\mathrm{v}_{\tilde{E}}}(g).$$

This gives us an exact sequence

$$1 \to N \to \mathbf{\Gamma}_{\tilde{E}} \to R \to 1 \tag{8.1.3}$$

where R is a finitely generated subgroup of $\mathbb{R}_+$, an abelian group. For an element $g \in N$, $\lambda_{\mathrm{v}_{\tilde{E}}}(g) = 1$. Since the relative eigenvalue corresponding to $\hat{L}(g_{\mathbb{R}^{n-1}})|A$ is 1, a matrix form shows that $\lambda_+(g) = 1$ for $g \in N$. (8.1.1) and (8.1.2) and the conclusion of the above paragraph show that g is unit-norm-eigenvalued. Thus, N is therefore virtually nilpotent by Theorem 1.3.7. (See Fried [80]). Taking a finite cover again, we may assume that N is nilpotent.

Since R is a finitely generated abelian group, $\mathbf{\Gamma}_{\tilde{E}}$ is solvable by (8.1.3). Since $\tilde{\Sigma}_{\tilde{E}} = \mathbb{R}^{n-1}$ is complete affine, Proposition S of Goldman and Hirsch [90] implies

$$\det(g_{\mathbb{R}^{n-1}}) = 1 \text{ for all } g \in \mathbf{\Gamma}_{\tilde{E}}.$$

If γ satisfies (c), then all norms of eigenvalues of γ except for $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$ equal $\lambda_{+}(\gamma)$ since otherwise by (8.1.1), norms of relative eigenvalues $\lambda_i(\gamma)/\lambda_+(\gamma)$ are ≤ 1, and the above determinant is less than 1. Similarly, if γ satisfies (d), then similarly all norms of eigenvalues of γ except for $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$ equals $\lambda_+(\gamma)$.

Therefore, only (b), (c), and (d) hold and $g_{\mathbb{R}^{n-1}}$ is unit-norm-eigenvalued for all $g \in \mathbf{\Gamma}_{\tilde{E}}$.

By Theorem 1.3.7, $\mathbf{\Gamma}_{\tilde{E}}|\mathbb{R}^{n-1}$ is an orthopotent group and hence is virtually unipotent by Theorem 3 of Fried [80].

Now we go back to $\mathbf{\Gamma}_{\tilde{E}}$. Suppose that every γ is orthopotent. Then we have the first case of (i). If not, then the second case of (i) holds. Immediately following Collorary 8.1.5 proves the result.

(ii) This follows by Lemma 3.1.15. [$\mathbb{S}^n$T] □

8.1.3. Complete affine ends again. We now study the second case from the conclusion of Theorem 8.1.4.

COROLLARY 8.1.5 (non-cusp complete-affine p-ends). *Let $\mathscr{O}$ be a strongly tame properly convex n-orbifold for $n \geq 3$. Let $\tilde{E}$ be a complete affine R-p-end of its universal cover $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$. Let $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^n$ be the p-end vertex with the p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$. Suppose that $\tilde{E}$ is not a cusp p-end. Then we can choose a different point as the p-end vertex for $\tilde{E}$ so that $\tilde{E}$ is a quasi-joined R-p-end with fiber homeomorphic to cells of dimension $n-2$. Also, the end fundamental group is virtually abelian.*

PROOF. We will use the terminology of the proof of Theorem 8.1.4. Theorem 8.1.4 shows that $\mathbf{\Gamma}_{\tilde{E}}$ is virtually nilpotent and with at most two norms of eigenvalues for each element. By taking a finite-index subgroup, we assume that $\mathbf{\Gamma}_{\tilde{E}}$ is nilpotent. Let Z be the Zariski closure, a nilpotent Lie group. We may assume that Z is connected by taking a finite index subgroup of $\mathbf{\Gamma}_{\tilde{E}}$. Theorem 8.1.4, says $\mathbf{\Gamma}_{\tilde{E}}$ is isomorphic to a virtually unipotent group by resticting to the affine space $\tilde{\Sigma}_{\tilde{E}}$. Hence, Z is simply connected and hence contractible. Since $\mathbf{\Gamma}_{\tilde{E}} \cap Z$ is a cocompact lattice in Z, and $\mathbf{\Gamma}_{\tilde{E}}$ has the virtual cohomological dimension $n-1$, it follows that Z is $(n-1)$-dimensional.

By Lemma 3.1.10, Z acts transitively on the complete affine subspace $\tilde{\Sigma}_{\tilde{E}}$ since $\mathbf{\Gamma}_{\tilde{E}}$ acts cocompactly on it.

The orbit map $Z \to Z(x)$ for $x \in \tilde{\Sigma}_{\tilde{E}}$ is a fiber bundle over the contractible space with fiber the stabilizer group. Since $\dim Z = n-1$, it follows that it must be discrete. Since $\tilde{\Sigma}_{\tilde{E}}$ is contractible, the stabilizer is trivial.

Since Z fixes $\mathrm{v}_{\tilde{E}}$, we have a homomorphism

$$\lambda_{\mathrm{v}_{\tilde{E}}} : Z \ni g \to \lambda_{\mathrm{v}_{\tilde{E}}}(g) \in \mathbb{R}. \tag{8.1.4}$$

Let N denote the kernel of the homomorphism. By Theorem 1.3.7, N is an orthopotent Lie group since it has only one norm of the eigenvalues equal to 1.

Since $\tilde{E}$ is a complete affine R-p-end, $R_{\mathrm{v}_{\tilde{E}}}(U)$ is a complete affine space equal to $\tilde{\Sigma}_{\tilde{E}}$. By taking a convex hull of finite number of radial rays from $\mathrm{v}_{\tilde{E}}$, we may choose a properly convex p-end neighborhood U of $\tilde{E}$. Also, we may choose so that the closure of U is in another such p-end neighborhood. Thus, $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ is in one-to-one correspondence with $\tilde{\Sigma}_{\tilde{E}}$. We modify U to

$$\bigcap_{g \in Z} g(U) = \bigcap_{g \in F} g(U).$$

This contains a nonempty properly convex open set by Lemma 3.1.8. We may assume that Z acts on a properly convex p-end neighborhood U of $\tilde{E}$. Since Z acts transitively on $\tilde{\Sigma}_{\tilde{E}}$, it acts so on an embedded convex hypersurface $\delta U := \mathrm{bd}U \cap \tilde{\mathscr{O}}$. This is the set of endpoints of maximal segments from $\mathrm{v}_{\tilde{E}}$ in the directions of the complete affine space $\tilde{\Sigma}_{\tilde{E}}$. Since this characterization is independent of $\tilde{\mathscr{O}}$, δU is an orbit of Z. Since Z is a Lie group, δU is smooth.

The smooth convex hypersurface δU is either strictly convex or has a foliation fibered by totally geodesic submanifolds. Since $\tilde{\Sigma}_{\tilde{E}}$ is complete affine, these submanifolds must be complete affine subspaces. Since $\mathrm{Cl}(U)$ contains these and $\mathrm{Cl}(U)$ is properly convex, this is a contradiction. Hence, δU is strictly convex.

Finally, since δU is in one-to-one correspondence with $\tilde{\Sigma}_{\tilde{E}}$, $\delta U/\mathbf{\Gamma}_{\tilde{E}}$ is a compact orbifold of codimension 1.

Let A be a hyperspace containing $\mathrm{v}_{\tilde{E}}$ in the direction of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}} = \mathbb{S}^{n-2} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$. Then $U_A := A \cap \mathrm{Cl}(U)$ is a properly convex compact set on which Z acts. By Lemma 8.1.7, U_A^o/Z is compact. By Lemma 8.1.6, U_A is a properly convex segment.

We will complete the proofs after Lemmas 8.1.6 and 8.1.7.

□

LEMMA 8.1.6. *Let a simply connected nilpotent Lie group S act cocompactly and effectively on a properly convex open domain J. Suppose that each element of S has at most two norms of eigenvalues and it fixes a point p in the boundary of J. Then the dimension of the domain is* 0 *or* 1.

PROOF. Suppose that $\dim J > 1$. By Lemma 3.1.10, S acts transitively on J. The action is proper since there is a Hilbert metric on J. Since S is nilpotent and is simply connected, S is contractible. Since the stabilizer of S at a point $x \in J$ is compact, it is trivial in S. Hence S is diffeomorphic to a $\dim J$-dimensional cell.

Let $\lambda_p(g)$ for $g \in S$ denote the associated eigenvalue of g at p for unit determinant matrix representatives. Let N_S denote the kernel of the homomorphism $S \to \mathbb{R}_+$ given by $g \mapsto \lambda_p(g)$. Then N_S is an orthopotent group of dimension $\dim J - 1$.

Then N_S acts on $R_p(J)$ as an orthopotent Lie group. $R_p(J)/N_S$ is compact since J/S maps onto $R_p(J)/N_S$ induced from the onto map $J \to R_p(J)$.

The stabilizer of a point of $R_p(J)$ acts on a segment s from p in J. The existence of two fixed directions of eigenvalue 1 implies that each point of s is a fixed point, and hence the stabilizer is trivial by the above. Therefore, the properness and freeness of the action of N_S on $R_p(J)$ follow.

By Proposition 3.1.14, the orbit $N_S(x)$ for $x \in J$ is an ellipsoid of dimension $\dim J - 1$. Hence, S/N_S is a 1-dimensional group. The elements of N_S are of form

$$k = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \vec{v}_k^T & \mathrm{I}_{\dim J-1} & 0 \\ \hline \frac{\|\vec{v}_k\|^2}{2} & \vec{v}_k & 1 \end{array}\right) \text{ for } \vec{v}_k \in \mathbb{R}^{\dim J-1}. \tag{8.1.5}$$

We may write for $g \in N$,

$$g = \left(\begin{array}{c|c|c} a_1(g) & 0 & 0 \\ \hline \vec{a}_4^T(g) & A_5(g) & 0 \\ \hline a_7(g) & \vec{a}_5(g) & a_9(g) \end{array}\right). \tag{8.1.6}$$

By Proposition 7.3.7 and Lemma 7.3.9, any element $g \in S$ induces an $(\dim J - 1) \times (\dim J - 1)$-matrix M_g given by $g\mathscr{N}(\vec{v})g^{-1} = \mathscr{N}(\vec{v}M_g)$ where

$$M_g = \frac{1}{a_1(g)}(A_5(g))^{-1} = \mu_g O_5(g)^{-1}$$

for $O_5(g)$ in a compact Lie group $G_{\tilde{E}}$ where $\mu_g = \frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_5(g)}$.

Reasoning as in the proof of Lemma 7.3.10, we can find coordinates so that for every $g \in N$,

$$g = \left(\begin{array}{c|c|c} a_1(g) & 0 & 0 \\ \hline a_1(g)\vec{v}_g^T & a_5(g)O_5(g) & 0 \\ \hline a_7(g) & a_5(g)\vec{v}_g O_5(g) & a_9(g) \end{array}\right), O_5(g)^{-1} = O_5(g)^T, \tag{8.1.7}$$

and the form of N_S is not changed. Also, $a_7(g) = a_1(g)(\alpha_7(g) + \|v\|^2/2)$. as we can show following the beginning of Section 7.3.3. Recall α_7 from Section 7.3.2.2. Here, $\alpha_7(g) = 0$ since otherwise J cannot be properly convex since g will translate the orbits in the affine space where p is the infinity as in Remark 7.3.13.

If there is an element g with $\mu_g \neq 1$, then the group N is solvable and not nilpotent. If $\mu_g = 1$ for all $g \in N$, then from the matrix form we see that N has only 1 as norms of eigenvalues with $a_1(g) = a_5(g) = a_9(g)$ for $g \in N$, and N acts on each ellipsoid orbit of N_S. Hence, $J/N = J/N_S$ is not compact. This is a contradiction. Therefore $\dim J = 0, 1$. □

LEMMA 8.1.7. *U_A^o/Z is compact.*

PROOF. Suppose that $\dim U_A^o = n-1$. The orbit map $Z \to Z(x)$ for $x \in U_A^o$ is a fibration over a simply connected domain. The stabilizer must be compact since U_A^o has a Hilbert metric. Since Z being a simply connected nilpotent Lie group is contractible, the stabilizer has to be trivial. Since $\dim Z = n-1$, Z acts transitively on U_A^o and U_A^o/Z is compact by Lemma 3.1.11.

Suppose now that $\dim U_A^o = j_0 < n-1$. Let L be an (j_0+1)-dimensional subspace containing U_A meeting A transversely. Let l be the j_0-dimensional affine subspace of the complete affine space $\tilde{\Sigma}_{\tilde{E}}$ corresponding to L. Since

$$g(U_A) = U_A, U_A \subset L, g(L) \text{ and } \dim L = j_0 + 1, g \in Z$$

it follows that

$$g(L) \cap L = \langle U_A \rangle \text{ or } g(L) = L, \text{ which implies } g(l) = l \text{ or } g(l) \cap l = \emptyset.$$

Recall that Z acts transitively and freely on the complete affine space $\tilde{\Sigma}_{\tilde{E}}$ from the beginning of the Section 8.1.3. Since $\dim l = j_0$, it follows that the subgroup $\hat{Z}_l := \{g \in Z | g(l) = l\}$ has the dimension j_0.

Now $\hat{Z}_l$ acts on on U_A^o. As proved above, the stabilizer of $\hat{Z}_l$ of a point of U_A^o is trivial since $U_A \cap L$ is properly convex and $\hat{Z}_l$ is nilpotent without a compact subgroup of dimension > 0. Hence, $\hat{Z}_l$ acts transitively on U_A^o as in the first paragraph by Lemma 3.1.11 since $\dim \hat{Z}_l = \dim U_A^o$, and $U_A^o/\hat{Z}_l = U_A^o/Z$ is compact. □

PROOF OF PROPOSITION 8.1.5 CONTINUED. If $\dim U_A = 0$, then U is a horospherical p-end neighborhood where $\mathbf{\Gamma}_{\tilde{E}}$ is unimodular and cuspidal by Theorem 8.1.3. Hence, $\dim U_A = 1$ by Lemma 8.1.6.

Let q denote the other endpoint of the segment U_A than $\mathrm{v}_{\tilde{E}}$. Since U is convex, $R_q(U)$ is a convex open domain. Since an element of $\mathbf{\Gamma}_{\tilde{E}}$ has two eigenvalues, Each radial segment from q maximal in $\tilde{\mathscr{O}}$ meets the smooth strictly convex hypersurface δU and transversely since a radial segment cannot be in δU. (Recall that δU is smooth and strictly convex from the first part of the proof.)

We have $R_q(U) = R_q(\delta U)$: Since $\tilde{\Sigma}_{\mathrm{v}_{\tilde{E}}}$ is complete affine, and U is properly convex, each segment from $\mathrm{v}_{\tilde{E}}$ passes δU as we lengthen it. Hence, $\mathrm{bd}U = \delta U \cup U_A$ since U_A is precisely $\mathrm{bd}U \cap A$ for the hyperspace A of $\mathbb{S}^n$ as defined earlier. (Recall that A is the hyperspace containing $\mathrm{v}_{\tilde{E}}$ in the direction of $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}} = \mathbb{S}^{n-2} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$.) Hence, for each segment in U from q must end at a point of $\mathrm{bd}U \cap \tilde{\mathscr{O}} = \delta U$ since δU is strictly convex. By the transversality, a segment from q ending at δU must have one-side in U. By strict convexity of δU, it is clear that each ray in the direction of $\tilde{\Sigma}_{\tilde{E}}$ from q meets δU transversely.

Since the $\mathbf{\Gamma}_{\tilde{E}}$-action on δU is proper, so is its action on $R_q(U)$. Hence, U can be considered a p-end neighborhood with radial lines from R_q foliating U by Lemma 3.1.5.

There is an embedding from δU to $R_q(\delta U) = R_q(U) \subset R_q(\tilde{\mathscr{O}})$. Since $\delta U/\mathbf{\Gamma}_{\tilde{E}}$ is a compact orbifold, so is $R_q(U)/\mathbf{\Gamma}_{\tilde{E}}$.

By the third item in the second item of Theorem 8.1.4, $R_q(U)$ is not complete-affine since the norm $\lambda_q(g)$ for some $g \in \mathbf{\Gamma}_{\tilde{E}}$ has a multiplicity n, $n < n+1$, and $n > 2$ by assumption.

Suppose that $R_q(U)$ is properly convex. Elements of $\mathbf{\Gamma}_{\tilde{E}}$ have at most two distinct norms of eigenvalues. Since $R_q(U)$ is homeomorphic to $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ with a compact quotient by $\mathbf{\Gamma}_{\tilde{E}}$, $R_q(U)$ has a compact quotient by $\mathbf{\Gamma}_{\tilde{E}}$, and $\dim R_q(U) = n-1 \geq 2$. By Lemma 8.1.6, this is a contradiction. Hence, U_q is not properly convex.

Thus, q is the p-end vertex of an NPNC-end for U foliated by radial segments from q. Since the associated upper-left part has only two norms of eigenvalues by Lemma 8.1.6, and the properly convex leaf space K^o is 1-dimensional and has a compact quotient, the fibers have the dimension $n-2 = n-1-1$. (Also, K^o is a properly convex segment by our definition.) Therefore, $R_q(U)$ is foliated by $n-2$-dimensional complete affine subspaces.

Suppose that $R_q(\tilde{\mathscr{O}})$ is different from $R_q(U)$. Then $R_q(\tilde{\mathscr{O}}) = K_R * \mathbb{S}^{i_0-1}$ and $R_q(U) = K_q * \mathbb{S}^{i_0-1}$ for a properly convex domain K_q and a convex domain K_R containing K_q. Then there is some point $x \in \tilde{\mathscr{O}}$ with $\overline{qx}$ not in $R_q(U)$. Then by taking elements g with maximal norm in $\hat{K} := \{\mathrm{v}_{\tilde{E}}\}$ from the forth item of Proposition 1.4.13 and using Proposition 7.3.19 and $\lambda_{\mathrm{v}_{\tilde{E}}}(g) = 1$, we obtain that $g^n(x)$ converges to the point antipodal to a point of $\hat{K}$. This contradicts the proper convexity of $\tilde{\mathscr{O}}$. Hence, $R_q(U) = R_q(\tilde{\mathscr{O}})$ and by Lemma 3.1.5 we obtain that U is a R-p-end neighborhood of a p-end vertex q.

Recall the Lie group N from (8.1.4). Since Z acts on U_q transitively and $\lambda_{\mathrm{v}_{\tilde{E}}}(g) = \lambda_q(g), g \in N$, the Lie group N acts on each complete affine leaf transitively. N is a nilpotent Lie group since it is a subgroup of Z. Also, N is orthopotent since elements of N have only one norm of the eigenvalues by Theorem 1.3.7. We can apply Proposition 3.1.14 to the hyperspace P containing the leaves with a cocompact subgroup of N acting on it. As $U \cap P$ is properly convex, $r_P(N)$ is a cusp group.

Let $x_1, \dots, x_{n+1}$ be the coordinates of $\mathbb{R}^{n+1}$. Now give coordinates so that $q = ((0,0,\dots,1))$ and $\mathrm{v}_{\tilde{E}} = ((1,0,\dots,0))$. Since these are fixed points, we obtain that elements of N can be put into forms:

$$N(\vec{v}) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vec{v}^T & \mathrm{I} & 0 \\ 0 & \frac{1}{2}\|\vec{v}\|^2 & \vec{v} & 1 \end{pmatrix} \text{ for } \vec{v} \in \mathbb{R}^{n-2}. \tag{8.1.8}$$

Now, $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the transverse weak middle eigenvalue condition with respect to q since $\mathbf{\Gamma}_{\tilde{E}}$ has just two eigenvalues and Z is generated by N and g^t for a nonunipotent element g of $\mathbf{\Gamma}_{\tilde{E}}$. N admits an invariant Euclidean structure being a cusp group. Since $\dim K = 1$, Theorem 7.1.4 shows that q is an NPNC R-p-end vertex for U covering the strongly tame convex real projective orbifold $U/\mathbf{\Gamma}_{\tilde{E}}$.

Finally, $\Gamma_{\tilde{E}}$ is virtually abelian: From (7.4.16), $S(g)$ is a 1×1-matrix or 0×0-one. From the matrix form, the Zariski closure Z is an extension of an orthopotent Lie group. Since $\tilde{\Sigma}_{\tilde{E}}$ equals $\mathbb{A}^{i_0} \times I$ for an interval or a singleton I, $i_0 = n-2, n-1$, Z acts on it. O_5 extends to a homomorphism $O_5 : Z \to \mathsf{O}(i_0)$. Let Z_K denote the kernel. Then Z_K also acts properly and cocompactly on $\tilde{\Sigma}_{\tilde{E}}$ since Z/Z_K is compact. It is easy to see Z_K is abelian from the matrix form. Also, we can put a Z-invariant Euclidean metric on the complete affine space $\tilde{\Sigma}_{\tilde{E}}$ by the product metric form. Then the Bieberbach theorem implies the result. □

If we require the weak middle eigenvalue conditions for a given vertex, the completeness of the end implies that the end is cusp.

COROLLARY 8.1.8 (cusp and complete affine). *Let $\mathcal{O}$ be a strongly tame properly convex n-orbifold. Suppose that $\tilde{E}$ is a complete affine R-p-end of its universal cover $\tilde{\mathcal{O}}$ in $\mathbb{S}^n$ (resp. in $\mathbb{RP}^n$). Let $\mathrm{v}_{\tilde{E}} \in \mathbb{S}^n$ (resp. $\in \mathbb{RP}^n$) be the p-end vertex with the p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$. Suppose that $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the weak middle eigenvalue condition with respect to $\mathrm{v}_{\tilde{E}}$. Then $\tilde{E}$ is a complete affine R-end if and only if $\tilde{E}$ is a cusp R-end.*

PROOF. It is sufficient to prove for the case $\tilde{\mathcal{O}} \subset \mathbb{S}^n$. Since a horospherical end is a complete affine end by Theorem 8.1.3, we need to show the forward direction only. In the second possibility of Theorem 8.1.4, the norm of $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$ has a multiplicity one for a nonunipotent element γ with $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma)$ or $\lambda_{\mathrm{v}_{\tilde{E}}}(\gamma^{-1})$ equal to the maximal norm. This violates the weak middle eigenvalue condition, and only the first possibility of Theorem 8.1.4 holds. [$\mathbb{S}^n$T] □

8.2. Some miscellaneous results from the above.

8.2.1. J. Porti's question. The following answers a question that we discussed with J. Porti at the UAB, Barcelona in 2013 whether there is a noncuspidal unipotent group acting as an end holonomy group of an R-end. Note that this was also proved by Theorem 5.7 in Cooper-Long-Tillman [67] using the duality theory of ends. Here we do not need to use duality.

The following generalizes the result of D.Fried [80].

COROLLARY 8.2.1. *Assume that $\mathcal{O}$ is a convex real projective strongly tame orbifold with an end E. Suppose that eigenvalues of elements of $\mathbf{\Gamma}_{\tilde{E}}$ have unit norms only. Then $\tilde{E}$ is horospherical, i.e., cuspidal.*

PROOF. First, we assume that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. By Lemma 3.1.15, we only need to show that $\tilde{E}$ is a complete affine end.

By Theorem 1.3.7, $\mathbf{\Gamma}_{\tilde{E}}$ is orthopotent. Theorem 1.4.5 shows that $\tilde{\Sigma}_{\tilde{E}}$ is complete affine. $[\mathbb{S}^n\mathrm{S}]$ ☐

8.2.2. Why $\lambda_{\mathrm{v}_{\tilde{E}}}(g) \neq 1$? We will need the following in proving Lemma 12.1.2.

COROLLARY 8.2.2. *Let $\mathscr{O}$ be a strongly tame properly convex n-orbifold. Suppose that $\tilde{E}$ is an NPNC R-p-end of its universal cover $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ or (resp. in $\mathbb{RP}^n$). Let $\mathrm{v}_{\tilde{E}}$ be the p-end vertex with the p-end holonomy group $\mathbf{\Gamma}_{\tilde{E}}$. Suppose that $\pi_1(\tilde{E})$ satisfies (NS) and* $\dim K^o = 0, 1$ *for the leaf space K^o. Then for some*

$$g \in \mathbf{\Gamma}_{\tilde{E}}, \lambda_{\mathrm{v}_{\tilde{E}}}(g) \neq 1.$$

PROOF. It is sufficient to prove for the case $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. Suppose that $\lambda_{\vec{v}_{\tilde{E}}}(g) = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$.

A $\Gamma_{\tilde{E}}$-invariant i_0-dimensional subspace $\mathbb{S}^{i_0}_\infty$ contains $\mathrm{v}_{\tilde{E}}$ as we discussed in by Section 7.1.1.

Suppose that every element of $\mathbf{\Gamma}_{\tilde{E}}$ is unit-norm-eigenvalued. By Theorem 1.3.7, $\mathbf{\Gamma}_{\tilde{E}}$ is orthopotent. By Fried [80], there exists a nonorthopotent element in $\mathbf{\Gamma}_{\tilde{E}}$ since $\tilde{\Sigma}_{\tilde{E}}$ is not complete affine. Hence, there exists an element $g \in \mathbf{\Gamma}_{\tilde{E}}$ that is not unit-norm-eigenvalued.

We show that the transverse weak middle eigenvalue condition for $\tilde{E}$ holds: Suppose not. We find an element g of $\mathbf{\Gamma}_{\tilde{E}}$ not satisfying the condition of the transverse weak middle eigenvalue condition with $\lambda_1(g) > 1$. For a real number μ equal to $\lambda_1(g)$, the subspace $\mathscr{R}_\mu(g)$ projects to $\mathbb{S}^{i_0}_\infty$ since otherwise the transverse weak middle eigenvalue condition holds. There exists a g-invariant subspace $\hat{P}_g \subset \mathbb{S}^{i_0}_\infty$ that is the projection of $\bigoplus_{\mu=\lambda_1(g)} \mathscr{R}_\mu(g)$. (See Definition 1.3.1.) This is a proper subspace since $\mathrm{v}_{\tilde{E}}$ has associated eigenvalue 1 strictly less than λ_g. Hence, $\dim \hat{P}_g \leq i_0 - 1$.

We define P_g the projection of

$$\bigoplus_{\mu<\lambda_1(g)} \mathscr{R}_\mu(g) \subset \mathbb{R}^{n+1}.$$

Then P_g is complementary to $\hat{P}_g$. Thus, $\dim P_g \geq n - i_0$, and $P_g \ni \mathrm{v}_{\tilde{E}}$.

Under the projection to $\mathbb{S}^{n-1}(\mathrm{v}_{\tilde{E}})$, P_g goes to a subspace P'_g of $\dim P_g - 1 \geq n - i_0 - 1$. As described in Section 7.2, $\tilde{S}_{\tilde{E}}$ is foliated by i_0-dimensional complete affine subspaces. Since $\dim \tilde{S}_{\tilde{E}} = n-1$, these i_0-dimensional leaves must meet P'_g of dimension $\geq n-1-i_0$.

Thus, any p-end neighborhood U of $\tilde{E}$ meets P_g. There must be an antipodal pair $\tilde{P}_g$ in $\hat{P}_g$ that is the projection of the eigenspace of g with the associated eigenvalue whose norm is $\lambda_1(g) > 1$.

$$L := U \cap (P_g * \tilde{P}_g) \subset P_g * \tilde{P}_g$$

is a nonempty open domain meeting P_g, and $L - P_g$ has two components. Let x, y be generic points in distinct components in $L - P_g$. Then $\{g^n(\{x,y\})\}$ geometrically converge to an antipodal pair of points in $\hat{P}_g$. Since this set is in $\mathrm{Cl}(\mathscr{O})$, this contradicts the proper convexity of $\mathscr{O}$. Thus, the transverse weak middle eigenvalue condition of $\tilde{E}$ holds.

The premises of Theorem 7.1.4 except for the strong irreducibility of the holonomy group of $\pi_1(\mathscr{O})$ are satisfied, and Theorem 7.1.4 classifies the ends. Suppose that $h(\pi_1(\mathscr{O}))$ is strongly irreducible. We apply Theorem 7.1.4. Let $\tilde{E}$ be a p-end corresponding to one of these, and $\pi_1(\tilde{E})$ acts on a properly convex domain K''^{oi} disjoint from $\vec{v}_{\tilde{E}}$. We showed in the proof of Proposition 7.3.14 the existence of elements where all the associated norms of eigenvalues of the subspace containing K'' are > 1 and the rest of the norms of the eigenvalues are < 1 by (7.3.43).) This is a contradiction to the assumption $\lambda_{\mathbf{v}_{\tilde{E}}}(g) = 1$. Thus, these types of ends do not occur.

Suppose that $h(\pi_1(\mathscr{O}))$ is virtually reducible. By the contrapositive of Theorem 7.3.22, (ii) of the conclusion of Proposition 7.3.19 could hold. Again $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_{\tilde{E}}$ by Proposition 7.4.8 and the corresponding part showing $\mu_g = 1$ of the proof of Theorem 7.3.22 where we do not need strong irreducibility of $h(\pi_1(\mathscr{O}))$. Now, matrices of form (7.3.43) give us the same contradiction as in the above paragraph. [$\mathbb{S}^n$T] $\square$

Part 3

The deformation space of convex real projective structures

The third part is devoted to understanding the deformation spaces of convex real projective structures on orbifolds with radial or totally geodesic ends. The end goal is to prove some versions of the Ehresmann-Thurston-Weil principle.

In Chapter 9, we give the precise definition of the deformation spaces. We show that the deformation space of real projective structures on a strongly tame orbifold with some conditions on the ends is mapped locally homeomorphically under the holonomy map to the character space of the fundamental group of the orbifold with corresponding conditions. Here, we are not concerned with convexity. Thurston's idea of deformation via John Morgan as described in Lok [121] by charts, works well here as well.

In Chapter 10, we will show that a convex real projective orbifold is strictly convex with respect to the ends if and only if the fundamental group is hyperbolic with respect to the end fundamental groups. Basic tools are from Yaman's work [157] generalizing the Bowditch's description of hyperbolic groups. That is, we look at triples of points in the boundary of the universal cover and show that the action is properly discontinuous. In addition, we show that the action of the group on the fixed points of end fundamental groups is parabolic in their sense. This generalizes the prior work of Cooper-Long-Tillman [67] and Crampon-Marquis [68] for convex real projective manifolds with cusp ends. The concept of relative hyperbolic ends depends on the types of ends here unfortunately. Our aim was to generalize to orbifolds with our class of ends.

In Chapter 11, we will show that the deformation space of convex real projective structures on a strongly tame orbifold with some conditions on the ends is identifiable with the union of components of the character space of the fundamental group of the orbifold with corresponding end conditions.

The openness part here continues that of Chapter 9. Here, the point is to prove the preservation of convexity under small perturbations. The proof consists of showing that we can patch the Hessian functions on the perturbed compact part with the Hessian functions on the end neighborhoods approximating the original Hessian metrics by finding approximating convex domains to the original covering convex domains. Cooper-Long-Tillman [67] uses the intrinsic Hessian metric instead.

The closedness part generalizes the previous work Choi-Goldman [54]. We use the end condition showing us that the sequence of covering convex domains can only degenerate to a point or a hemisphere. Then using Benzecri's work [25] putting the domains in a fixed ball and containing a fixed smaller ball, we show that the domain has to be actually properly convex.

Finally, we go to Chapter 12. We discuss our nicest cases Corollary 12.1.4 and 12.1.5 where the Ehresmann-Thurston-Weil principle holds in a simple way: the deformation space of the orbifold identifies with a union of components of character space of the orbifold fundamental group. These include the Coxeter orbifolds admitting complete hyperbolic structures.

CHAPTER 9

The openness of deformations

A real projective structure sometimes admits deformations to parameters of real projective structures. We will prove the local homeomorphism between the deformation space of real projective structures on such an orbifold with radial or totally geodesic ends with various conditions with the $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$-character space (resp. $\mathsf{PGL}(n+1,\mathbb{R})$-character space) of the fundamental group with corresponding conditions. However, the convexity issue will not be studied in this chapter. Our approach will be to work with radiant affine structures of one dimension higher by affine suspension construction and prove the results. Then the real projective versions will follow easily. In Section 9.1, we will state the main results recall some definitions such as geometric structures, boundary restrictions, and the deformation spaces. In Sedtion 9.2, we will prove the semialgebraic properties of approapriate parts of character varieties. In Section 9.3, we will introduce a way to compactify our orbifolds and related these to the local homeomorphism properties. We will also define the deformation spaces. In Section 9.4, we prove the main result of the chapter Theorem 9.4.5, showing the openness of the deformation space in the character space. We first define the end conditions for real projective structures as determined by sections. We describe how to perturb the horospherical ends to lens-shaped ones in the affine setting. Then we state the main results. In Section 9.5, we will identify the deformation spaces as defined in our earlier papers [**50**] and [**58**] as stated in Chapter 2 to the deformation spaces here.

9.1. Deformation spaces and the spaces of holonomy homomorphisms

Given a real projective orbifold $\mathscr{O}$, we add the restriction of the end to be a radial or a totally geodesic type. The end will be either assigned $\mathscr{R}$-type or $\mathscr{T}$-type.

- An $\mathscr{R}$-type end is required to be radial.
- A $\mathscr{T}$-type end is required to have totally geodesic properly convex ideal boundary components or be horospherical.

Recall that a strongly tame orbifold will always have such an assignment in this monograph, and finite-covering maps will always respect the types. Let $E_1,\dots,E_{e_1}$ be the $\mathscr{R}$-ends, and $E_{e_1+1},\dots,E_{e_1+e_2}$ be the $\mathscr{T}$-ends.

Recall that our strongly tame orbifold $\mathscr{O}$ comes with a compact orbifold $\bar{\mathscr{O}}$ with smooth boundary whose interior equals $\mathscr{O}$. Each boundary component of $\bar{\mathscr{O}}$ is said to be the ideal boundary component of $\mathscr{O}$.

DEFINITION 9.1.1. The radial foliation gives us a smooth parameterization of an end neighborhood U of a radial end or horospherical end E of $\mathscr{O}$ by $\Sigma_E\times(0,1)$ where $x\times(0,1)$ is the radial line for each $x\in\Sigma_E$ since we can choose an embedded hypersurface transverse to the radial rays. We assume that as $t\to 1$, the ray escapes to the end.

Let E be a T-end. We are given an end neighborhood U diffeomorphic to $S_E\times(0,1)$ where S_E is the ideal boundary component. We identify U with $S_E\times(0,1)$ in $S_E\times(0,1]$ by identity. Up to isotopies, $S_E \times \{1\}$ identifies with the ideal boundary of $\bar{\mathscr{O}}$ corresponding to E. This is the compatibility condition of $\bar{\mathscr{O}}$ with totally geodesic end structure.

For each R-end, we require that a vector field tangent to the leaves extends to a smooth vector field transverse to the corresponding ideal boundary component of $\bar{\mathscr{O}}$. For each T-end, the identification of U to $S_E \times (0,1)$ extends to the closure of U in $\bar{\mathscr{O}}$ and $S_E \times [0,1]$. Recall that these are the compatibility condition of R-end structures and T-end structures with the compactifiction $\bar{\mathscr{O}}$ of $\mathscr{O}$ from Section 3.1.

Also, $\bar{\mathscr{O}}$ is a very good orbifold since we can identify $\bar{\mathscr{O}}$ with $\mathscr{O} - U$ for a union U of open end neighborhoods of product forms as above. This is obtained by identifying $\mathscr{O}$ by $\mathscr{O} - \mathrm{Cl}(U)$ by an isotopy preserving radial foliations and taking the closures. (See Theorem 1.1.19.)

There is an obvious isomorphism $\pi_1(\mathscr{O}) \to \pi_1(\bar{\mathscr{O}})$ since we can perturb any $\mathscr{G}$-path in $\bar{\mathscr{O}}$ to one in $\mathscr{O}$. For the universal cover $\hat{\mathscr{O}}$ of $\bar{\mathscr{O}}$, there is an embedding $\tilde{\mathscr{O}} \to \hat{\mathscr{O}}$ as an inclusion map to a dense open subset. We will always identify $\tilde{\mathscr{O}}$ with the dense subset. (See [32].)

An *isotopy* $i : \mathscr{O} \to \mathscr{O}$ is a self-diffeomorphism so that there exists a smooth orbifold map $J : \mathscr{O} \times [0,1] \to \mathscr{O}$, so that

$$i_t : \mathscr{O} \to \mathscr{O} \text{ given by } i_t(x) = J(x,t)$$

are self-diffeomorphisms for $t \in [0,1]$ and $i = i_1, i_0 = \mathrm{I}_{\mathscr{O}}$. We require i_t to be restrictions of isotopies

$$\bar{i}_t : \bar{\mathscr{O}} \to \bar{\mathscr{O}} \text{ given by } \bar{i}_t(x) = \bar{J}(x,t)$$

are self-diffeomorphism for $t \in [0,1]$ and $\bar{J} : \bar{\mathscr{O}} \to \bar{\mathscr{O}}$ is a smooth orbifold map.

Note that the radial structures for each R-end and the totally geodesic structure for each T-end is preserved since we required the radial foliations to extend to $\bar{\mathscr{O}}$ smoothly and the ideal boundary component to be the boundary component of $\bar{\mathscr{O}}$ by the compatibility condition above in Sections 3.1.2 and 3.1.3.

We define $\mathrm{Def}_{\mathscr{E}}(\mathscr{O})$ as the deformation space of real projective structures on $\mathscr{O}$ with end structures; more precisely, this is the quotient space of the real projective structures on $\mathscr{O}$ satisfying the above conditions for ends of type $\mathscr{R}$ and $\mathscr{T}$ under the isotopy equivalence relations. We define the topology more precisely in Section 9.3.1. (See [49], [33] and [87] for more details.)

Recall that an isotopy of an orbifold $\mathscr{O}$ is a map $f : \mathscr{O} \to \mathscr{O}$ with a map $F : \mathscr{O} \times I \to \mathscr{O}$ so that

- $F_t : \mathscr{O} \to \mathscr{O}$ for $F_t(x) := F(x,t)$ every fixed t is an orbifold diffeomorphism,
- F_0 is the identity, and
- $f = F_1$.

Given an (X,G)-structure on another orbifold $\mathscr{O}'$, any orbifold diffeomorphism $f : \mathscr{O} \to \mathscr{O}'$ induces an (X,G)-structure pulled back from $\mathscr{O}'$ which is given by using the preimages in $\mathscr{O}$ of the local models of $\mathscr{O}'$.

DEFINITION 9.1.2. Let $\iota : \mathscr{O} \to \mathscr{O}$ is an isotopy. We may choose a lift $\tilde{\iota} : \tilde{\mathscr{O}} \to \tilde{\mathscr{O}}$ of ι so that for the isotopy $F : \mathscr{O} \times I \to \mathscr{O}$ with $F_0 = \mathrm{I}_{\mathscr{O}}$ and $F_t = \iota$ has a lift $\tilde{F}$ so that $\tilde{F}_0 = \mathrm{I}_{\tilde{\mathscr{O}}}$ and $\tilde{F}_1 = \tilde{\iota}$. We call such a map $\tilde{\iota}$ an *isotopy-lift*.

For now, we restrict to compact orbifolds. Suppose that $\mathscr{O}$ is compact. We define the *isotopy-equivalence space* $\widetilde{\mathrm{Def}}_{X,G}(\mathscr{O})$ as the space of development pairs $(\mathbf{dev}, h)$ quotient by the isotopy-lifts of the universal cover $\tilde{\mathscr{O}}$ of $\mathscr{O}$. The *deformation space* $\mathrm{Def}_{X,G}(\mathscr{O})$ is given by the quotient of $\widetilde{\mathrm{Def}}_{X,G}(\mathscr{O}')$ by the action of G: $g(\mathbf{dev}, h(\cdot)) = (g\circ\mathbf{dev}, gh(\cdot)g^{-1})$. (See [49] for details.) We can also interpret as follows: The deformation space $\mathrm{Def}_{X,G}(\mathscr{O})$ of the (X,G)-structures is the space of all (X,G)-structures on $\mathscr{O}$ quotient by the isotopy pullback actions.

This space can be thought of as the space of pairs $(\mathbf{dev}, h)$ with the compact open C^r-topology for $r \geq 1$ and the equivalence relation generated by the isotopy relation

- $(\mathbf{dev}, h) \sim (\mathbf{dev}', h')$ if $\mathbf{dev}' = \mathbf{dev}\circ\iota$ and $h' = h$ for an isotopy-lift ι of an isotopy and
- $(\mathbf{dev}, h) \sim (\mathbf{dev}', h')$ if $\mathbf{dev}' = k\circ\mathbf{dev}$ and $h(\cdot) = kh(\cdot)k^{-1}$ for $k \in G$.

(See [49] or Chapter 6 of [51].)

9.2. The semi-algebraic properties of $\mathrm{rep}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ and related spaces

Since $\mathscr{O}$ is strongly tame, the fundamental group $\pi_1(\mathscr{O})$ is finitely generated. Let $\{g_1,\dots,g_m\}$ be a set of generators of $\pi_1(\mathscr{O})$. As usual $\mathrm{Hom}(\pi_1(\mathscr{O}), G)$ for a Lie group G has an *algebraic topology* as a subspace of G^m. This topology is given by the notion of *algebraic convergence*

$$\{h_i\} \to h \text{ if } \{h_i(g_j)\} \to h(g_j) \in G \text{ for each } j, j = 1,\dots,m.$$

A conjugacy class of a representation is called a *character* in this monograph.

The $\mathsf{PGL}(n+1,\mathbb{R})$-*character space* (*variety*) $\mathrm{rep}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ is the quotient space of the homomorphism space

$$\mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

where $\mathsf{PGL}(n+1,\mathbb{R})$ acts by conjugation

$$h(\cdot) \mapsto gh(\cdot)g^{-1} \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R}).$$

Similarly, we define

$$\mathrm{rep}(\pi_1(\mathscr{O}), \mathrm{SL}_\pm(n+1,\mathbb{R})) := \mathrm{Hom}(\pi_1(\mathscr{O}), \mathrm{SL}_\pm(n+1,\mathbb{R}))/\mathrm{SL}_\pm(n+1,\mathbb{R})$$

as the $\mathrm{SL}_\pm(n+1,\mathbb{R})$-character space. This is not really a variety in the sense of algebraic geometry. We merely define this space as the quotient space for now, possibly nonHausdorff one.

A representation or a character is *stable* if the orbit of it or its representative is closed and the stabilizer is finite under the conjugation action in

$$\mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \text{ (resp. } \mathrm{Hom}(\pi_1(\mathscr{O}), \mathrm{SL}_\pm(n+1,\mathbb{R}))).$$

By Theorem 1.1 of [106], a representation ρ is stable if and only if it is irreducible and no proper parabolic subgroup contains the image of ρ. The stability and the irreducibility are open conditions in the Zariski topology. Also, if the image of ρ is Zariski dense, then ρ is stable. $\mathsf{PGL}(n+1,\mathbb{R})$ acts properly on the open set of stable representations in $\mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$. Similarly, $\mathrm{SL}_\pm(n+1,\mathbb{R})$ acts so on $\mathrm{Hom}(\pi_1(\mathscr{O}), \mathrm{SL}_\pm(n+1,\mathbb{R}))$. (See [106] for more details.)

A representation of a group G into $\mathsf{PGL}(n+1,\mathbb{R})$ or $\mathrm{SL}_\pm(n+1,\mathbb{R})$ is *strongly irreducible* if the image of every finite index subgroup of G is irreducible. Actually, many of the orbifolds have strongly irreducible and stable holonomy homomorphisms by Theorem 6.0.4.

An *eigen-1-form* of a linear transformation γ is a linear functional α in $\mathbb{R}^{n+1}$ so that $\alpha\circ\gamma = \lambda\alpha$ for some $\lambda \in \mathbb{R}$. We recall the lifting of Remark 1.1.5.

- $$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  to be the subspace of representations h satisfying
  The vertex condition for $\mathscr{R}$:: $h|\pi_1(\tilde{E})$ has a nonzero common eigenvector of positive eigenvalues for a lift of $h(\pi_1(\tilde{E}))$ in $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$ for each $\mathscr{R}$-type p-end fundamental group $\pi_1(\tilde{E})$, and
  The lens-condition for $\mathscr{T}$:: $h|\pi_1(\tilde{E})$ acts on a hyperspace P for each $\mathscr{T}$-type p-end fundamental group $\pi_1(\tilde{E})$ and acts discontinuously and cocompactly on a lens L, a properly convex domain with $L^o \cap P = L \cap P \neq \emptyset$ or a horoball tangent to P.
- We denote by
  $$\mathrm{Hom}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  the subspace of stable and irreducible representations, and define
  $$\mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  to be
  $$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \cap \mathrm{Hom}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$
- We define
  $$\mathrm{Hom}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  to be the subspace of representations h where
  - $h|\pi_1(\tilde{E})$ has a unique common eigenspace of dimension 1 in $\mathbb{R}^{n+1}$ with positive eigenvalues for its lift in $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$ for each p-end holonomy group $h(\pi_1(\tilde{E}))$ of $\mathscr{R}$-type
    (*) There exists a finitely many elements $g_1,\dots,g_n$ so that the intersection $\bigcap_{i=1}^n C_{\lambda_i}(h(g_i))$ is 1-dimensional where $C_{\lambda_i}(g_i)$ is the cyclic space of eigenvalue λ_i associated with the above common eigenspace. (See Definition 1.3.1.)

    and
  - $h|\pi_1(\tilde{E})$ has a common null-space P of eigen-1-forms satisfying the following:
    * $\pi_1(\tilde{E})$ acts properly and cocompactly on a lens L and $L \cap P = L^o \cap P$ with nonempty interior in P with Hausdorff quotients, or
    * $H - \{p\}$ for a horosphere H tangent to P at p

    and is unique such one for each p-end holonomy group $h(\pi_1(\tilde{E}))$ of the p-end of $\mathscr{T}$-type and dual group satisfies above (*) for the dual point of P.

  For $\mathscr{T}$-ends, the lens condition is satisfied for a hyperplane P and P is unique one satisfying the condition in other words. We define
  $$\mathrm{Hom}^s_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  to be
  $$\mathrm{Hom}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}) \cap \mathrm{Hom}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

REMARK 9.2.1. The above condition for type $\mathscr{T}$ generalizes the principal boundary condition for real projective surfaces of Goldman [88].

Since each $\pi_1(\tilde{E})$ is finitely generated and there is only finitely many conjugacy classes of $\pi_1(\tilde{E})$,
$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
is a closed semi-algebraic subset.

Define
$$\mathrm{Hom}_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
to be a subset of
$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
so that the p-end holonomy group of each R-p-end fixes finitely many points and the p-end holonomy group of each T-p-ends acts on finitely many hyperspaces. By Lemma 9.2.2, the condition that each p-end holonomy group has the isolated fixed points or isolated hyperspaces only is an open condition.

LEMMA 9.2.2. *Let V be a semi-algebraic subset of $\mathsf{PGL}(n+1,\mathbb{R})^m$ (resp. $\mathsf{SL}_{\pm}(n+1,\mathbb{R})^m$.) For each $(g_1,\dots,g_m) \in V$, suppose that there is a function*
$$E : V \mapsto \mathbb{Z}$$
where $E(g_1,\dots,g_m)$ is the maximum of
$$\{\dim W | W \text{ is a subspace of fixed points of each } g_i, i = 1,\dots,m\}$$
where we define $\dim\emptyset = -1$. Then
$$V \ni (g_1,\dots,g_m) \mapsto \dim E(g_1,\dots,g_m)$$
is an upper semi-continuous function on V.

PROOF. Suppose that we have a sequences $\{g_i^{(j)}\}$ for each $i = 1,\dots,m$ and suppose that $g_i^{(j)} \to g_i$ as $j \to \infty$ for each i. For any sequence of subspace of fixed points of $g_1^{(j)},\dots,g_m^{(j)}$, a limit subspace is contained in a subspace of fixed points of $g_1,...,g_m$. [$\mathbb{S}^n$S] $\square$

PROPOSITION 9.2.3.
$$\mathrm{Hom}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
is an open subset of a semi-algebraic subset
$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
So is
$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

PROOF. We have h in this open subset
$$\mathrm{Hom}_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

For the condition on the uniqueness, we may assume that involved $g_1,\dots,g_n$ are so that $C_{\lambda_i}(h(g_i))$ are transversal and their interseciton is 1-dimensional. Suppose that there is a sequence of holonomies $h_j : \pi_1(\tilde{E})$ converging to $h|\pi_1(\tilde{E})$ so that $h_j|\pi_1(\tilde{E})$ has more than one fixed points. Then the span of the two directions are in the sum of $C_{\lambda_{i,j}}(h_j(g_i))$ and $C_{\lambda'_{i,j}}(h_j(g_i))$ for two eigenvalues of $\lambda_{i,j}, \lambda'_{i,j}$ of $h_j(g_i)$. Clearly, $\lambda_{i,j}, \lambda'_{i,j} \to \lambda_i$. Hence, the limit of the sequence of the sum spaces must converge to the subspace of $C_{\lambda_i}(h(g_i))$ by elementary linear algebra. However, by the transversality of $C_{\lambda_i}(h(g_i)$ for $i = 1,\dots,n$, we see that the sum spaces must intersect at a 1-dimensional subspace for h_i sufficiently close to h. This means h_i has only one fixed point, a contradiction. Therefore, the uniqueness condition is an open condition.

Let $\tilde{E}$ be a T-p-end. Let

$$h \in \mathrm{Hom}_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})),$$

and let $G := h(\pi_1(\tilde{E}))$. Assume that G is not a cusp group. Let P be a hyperspace where G acts on.

Proposition 5.3.11 implies that the condition of the existence of the hyperspace P satisfying the lens-property is an open condition in $\mathrm{Hom}_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$.

Suppose that there is another hyperspace P' with a lens L' satisfying the above properties. Then $P \cap P'$ is also G-invariant. Note that $L^o \cap P$ covers a compact end-orbifold. By Proposition 1.4.10, we obtain that $\mathrm{Cl}(P \cap L)$ is a join $K * \{k\}$ for a properly convex domain K in $P \cap P'$ and a point k in $P - P'$ since there exists a codimension-one invariant subspace in P. Similarly, exchanging the role of P and P', we obtain that there is a point $k' \in P' - P$ fixed by G. G acts on the one-dimensional subspace S_G containing k and k'. There are no other fixed point on S_G since otherwise S_G is the set of fixed points and G acts on any hyperspace containing $P \cap P'$ and a point on S_G. This contradicts our assumption on the first paragraph of the proof. Hence, only k and k' are fixed points in S_G and $P \cap P'$ and $\{k, k'\}$ contain all the fixed points of G.

Now, k' is the unique fixed point outside P. The existence of lens for P tells us that k' must be a fixed point outside the closure of the lens. By Theorem 5.5.4, the existence of a lens for P tells us that every $g \in G$, the maximum norm of eigenvalues of g associated with k and $P \cap P'$ is greater than that of k'.

Now, we switch the role of P and P'. We can take a central element g' with the largest norm of eigenvalue at k' by the last item of Proposition 1.4.10 and the uniform middle eigenvalue condition from Theorem 5.5.4. This cannot happen by the above paragraph. Hence, P satisfying the lens-condition is unique.

Suppose that G is a cusp group. Then there exists a unique hyperspace P containing the fixed point of G tangent to horospheres where G acts on. Therefore,

$$\mathrm{Hom}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

is in an open subset of a union of semi-algebraic subsets of

$$\square \qquad \mathrm{Hom}_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

$\square$

We define

- $$\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  to be the set
  $$\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))/\mathsf{PGL}(n+1,\mathbb{R}).$$
- We denote by
  $$\mathrm{rep}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  the subspace of
  $$\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
  of stable and irreducible characters.

- We define
$$\mathrm{rep}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
to be
$$\mathrm{Hom}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))/\mathsf{PGL}(n+1,\mathbb{R}).$$
- We define
$$\begin{aligned} &\mathrm{rep}^s_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \\ &:= \mathrm{rep}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \cap \mathrm{rep}_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})). \end{aligned} \tag{9.2.1}$$

Let $\rho \in \mathrm{Hom}_{\mathscr{E}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$ where E is an end. Define
$$\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$
to be the subspace of representations where $\pi_1(E)$ goes into a cusp group, i.e., a parabolic subgroup in a conjugated copy of $\mathsf{PO}(n,1)$. By Lemma 9.2.4,
$$\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$
is a closed semi-algebraic set.

LEMMA 9.2.4. *Let G be a finitely presented group.* $\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(G, \mathsf{PGL}(n+1,\mathbb{R}))$ *is a closed algebraic set.*

PROOF. Let P be a maximal parabolic subgroup of a conjugated copy of $\mathsf{PO}(n+1,\mathbb{R})$ that fixes a point x. Then $\mathrm{Hom}(G,P)$ is a closed semi-algebraic set.
$$\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(G, \mathsf{PGL}(n+1,\mathbb{R}))$$
equals a union
$$\bigcup_{g \in \mathsf{PGL}(n+1,\mathbb{R})} \mathrm{Hom}(G, gPg^{-1}),$$
which is another closed semi-algebraic set. □

Let E be an end orbifold of $\mathscr{O}$. Given
$$\rho \in \mathrm{Hom}_{\mathscr{E}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R})),$$
we define the following sets:

- Let E be an end of type $\mathscr{R}$. Let
$$\mathrm{Hom}_{\mathscr{E},\mathbb{RL}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$
denote the space of representations h of $\pi_1(E)$ where $h(\pi_1(E))$ acts on a lens-cone $\{p\} * L$ for a lens L and p for $p \notin \mathrm{Cl}(L)$ of a p-end $\tilde{E}$ corresponding to E and acts properly and cocompactly on the lens L itself. Again, $\{p\} * L$ is assumed to be a bounded subset of an affine patch $\mathbb{A}^n$. Thus, it is a union of open subsets of semi-algebraic sets in $\mathrm{Hom}_{\mathscr{E}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$ by Proposition 5.3.11.
- Let E denote an end of type $\mathscr{T}$. Let
$$\mathrm{Hom}_{\mathscr{E},\mathbb{TL}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$
denote the space of totally geodesic representations h of $\pi_1(E)$ satisfying the following condition:
  - $h(\pi_1(E))$ acts on an lens L and a hyperspace P where
  - $L \cap P = L^o \cap P \neq \emptyset$ and
  - $L/h(\pi_1(E))$ is a compact orbifold with two strictly convex boundary components.

$$\mathrm{Hom}_{\mathscr{E},\mathbb{TL}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$

is again a union of open subsets of the semi-algebraic sets

$$\mathrm{Hom}_{\mathscr{E}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$

by Proposition 5.3.11.

Let

$$R_E : \mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \ni h \to h|\pi_1(E) \in \mathrm{Hom}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))$$

be the restriction map to the p-end holonomy group $h(\pi_1(E))$ corresponding to the end E of $\mathscr{O}$.

A *representative set* of p-ends of $\tilde{\mathscr{O}}$ is the subset of p-ends where each end of $\mathscr{O}$ has a corresponding p-end and a unique chosen corresponding p-end. Let $\mathscr{R}_{\mathscr{O}}$ denote the representative set of p-ends of $\tilde{\mathscr{O}}$ of type $\mathscr{R}$, and let $\mathscr{T}_{\mathscr{O}}$ denote the representative set of p-ends of $\tilde{\mathscr{O}}$ of type $\mathscr{T}$. We define a more symmetric space:

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

to be

$$\left(\bigcap_{E\in\mathscr{R}_{\mathscr{O}}} R_E^{-1}\Big(\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R})) \cup \mathrm{Hom}_{\mathscr{E},\mathbb{RL}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))\Big)\right) \cap$$
$$\left(\bigcap_{E\in\mathscr{T}_{\mathscr{O}}} R_E^{-1}\Big(\mathrm{Hom}_{\mathscr{E},\mathrm{par}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R})) \cup \mathrm{Hom}_{\mathscr{E},\mathbb{TL}}(\pi_1(E), \mathsf{PGL}(n+1,\mathbb{R}))\Big)\right).$$

The quotient space of the space under the conjugation under $\mathsf{PGL}(n+1,\mathbb{R})$ is denoted by

$$\mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

We define

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

to be

$$\mathrm{Hom}^s(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \cap \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Hence, this is a union of open subsets of semialgebraic subsets in

$$X := \mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

We don't claim that the union is open in X. These definitions allow for changes between horospherical ens to lens-shaped radial ones and totally geodesic ones.

The quotient space of this space under the conjugation under $\mathsf{PGL}(n+1,\mathbb{R})$ is denoted by

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Since

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

is the Hausdorff quotient of the above set with the conjugation $\mathsf{PGL}(n+1,\mathbb{R})$-action, this is an open subset of a semi-algebraic subset by Proposition 9.2.3 and Proposition 1.1 of [106].

We define

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

to be the subset

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{u}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \cap \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

The above shows

PROPOSITION 9.2.5.

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

is an open subset of a semi-algebraic set in

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{f}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

9.3. End structures and end compactifications for topological orbifolds

Let $\mathscr{O}$ be a strongly tame smooth orbifold with ends $E_1,\dots,E_m,E_{m+1},\dots,E_{m+l}$. For this subsection, we do not consider that $\mathscr{O}$ is the interior of a compact orbifold $\bar{\mathscr{O}}$, the associated compactification of $\mathscr{O}$, as we remind from Section 3.1.1.

An *ideal boundary structure* of an end neighborhood U of E_i is a pair (U,f) for a smooth embedding f of U into a product space $\Sigma\times(0,1]$ for a closed $n-1$-orbifold Σ where the image is $\Sigma\times(0,1)$. An ideal boundary structure (U_0,f_0) with a diffeomorphism $f_0 : U_0 \to \Sigma_0\times(0,1)$ for a closed $n-1$-orbifold Σ_0 and another one (U_1,f_1) with a diffeomorphism $f_1 : U_1 \to \Sigma_1\times(0,1)$ for a closed $n-1$-orbifold Σ_1 are *compatible* if there exists another ideal boundary structure (U_2,f_2) so that $U_2\subset U_0\cap U_1$ with a diffeomorphism $f_2 : U_2 \to \Sigma''\times(0,1)$ so that $f_i\circ f_2^{-1} : \Sigma''\times(0,1)\to\Sigma_i\times(0,1)$ extends to $\Sigma''\times(0,1]$ as an embedding restricting to a diffeomorphism $\Sigma''\times\{1\}$ to $\Sigma_i\times\{0\}$ for $i=0,1$.

Given an ideal boundary structure on $\mathscr{O}$ for an end E_i, we obtain the *completion of $\mathscr{O}$ along E_i*. We take U an end neighborhood of E_i with an embedding $f : U\to\Sigma\times(0,1]$ where the image equals $\Sigma\times(0,1)$. We paste $\mathscr{O}$ with $\Sigma\times(0,1]$ by f. The resulting orbifold $\mathscr{O}'_{(U,f)}$ is said to be the *end compactification of $\mathscr{O}$ along E_i* using (U,f).

Let U' and $f' : U'\to\Sigma'\times(0,1]$ be as above with (U',f') compatible to (U,f), and we obtain an end compactification $\mathscr{O}'_{(U',f')}$ of $\mathscr{O}$ along E_i using (U',f'). An *isotopy* ι of $\mathscr{O}$ with an ideal boundary structure for an end E_i is an isotopy of $\mathscr{O}$ extending to a diffeomorphism $\bar{\iota} : \mathscr{O}'_{(U,f)}\to\mathscr{O}'_{(U',f')}$ for at least one compatible pair $(U,f),(U',f')$.

By the following lemma, the definition of an (end-structure extendable) isotopy is independent of the choice of (U,f) and (U',f').

LEMMA 9.3.1. *Let U_1 and U_2 be end neighborhoods of an end E. Let g be an isotopy of $\mathscr{O}$ extending to an isotopy $\bar{g} : \mathscr{O}'_{(U_1,f_1)}\to\mathscr{O}'_{(f_2,U_2)}$ with diffeomorphisms $f_1 : U_1\to\Sigma_1\times(0,1)$ and $f_2 : U_2\to\Sigma_2\times(0,1)$ for closed $n-1$-orbifolds Σ_1 and Σ_2. Then for any pair $(U'_1,f'_1),(U'_2,f'_2)$ for end neighborhoods of end end E compatible to $(U_i,f_i), i=1,2$, with diffeomorphisms $f'_1 : U'_1\to\Sigma'_1\times(0,1)$ and $f'_2 : U'_2\to\Sigma'_2\times(0,1)$ for closed $n-1$-orbifolds Σ'_1 and Σ'_2, g extends to an isotopy $\bar{g}_1 : \mathscr{O}_{(U'_1,f'_1)}\to\mathscr{O}_{(U'_2,f'_2)}$.*

PROOF. This is straightforward to obtain a diffeomorphism $\bar{g}_1$ since we can take a sufficiently small product neighborhood in each of these end compactifications. To show the isotopy property of $\bar{g}_1$, we simply take $\mathscr{O}\times I$ and do the same arguments. □

A radial structure for E_i also gives us an end-compactification of $\mathscr{O}$ along E_i: Let U be an end neighborhood of E_i with a foliation by properly embedded arcs. We take a transverse hypersurface Σ_{E_i} transverse to every leaf, which is a closed orbifold. Let U' denote a component of $U-\Sigma_{E_i}$ that is an end neighborhood of E_i. We identify each leaf in U' with a leaf of $\Sigma_{E_i}\times(0,1)$ by a function $f : U'\to\Sigma_{E_i}\times(0,1)$. We call the identification orbifold $\mathscr{O}'$ of $\mathscr{O}$ with $\Sigma_{E_i}\times(0,1]$ the *end compactification of $\mathscr{O}$ along E_i*. The suborbifold of $\mathscr{O}'$ corresponding to $\Sigma_{E_i}\times\{1\}$ is call the *ideal boundary component corresponding to*

E_i. This orbifold is independent of the choices up to isotopoes of the end compactifications extending isotopies of $\mathscr{O}$ by Lemma 9.3.2.

LEMMA 9.3.2.

- *Let $\mathscr{O}$ be a strongly tame orbifold with a radial structure at an end E_i.*
- *Let $\mathscr{O}'$ be the end compactification of $\mathscr{O}$ using U and Σ_{E_i} a diffeomorphism $f : U' \to \Sigma_{E_i} \times (0,1)$.*
- *Let $\mathscr{O}_1$ be the same orbifold with an isotopic radial structure at E_i.*
- *Let $\mathscr{O}_1'$ be the end compactification of $\mathscr{O}_1$ for the second radial structure using an end neighborhood U_1 and a hypersurface Σ'_{E_i} and a diffeomorphism $f' : U_1' \to \Sigma'_{E_i} \times (0,1)$ for an end-neighborhood component U_1' of $U_1 - \Sigma'_{E_i}$.*

Suppose that an isotopy ι of $\mathscr{O}$ sends a radial structure of U for end E_i to that of U_1 for $\mathscr{O}_1$ for E_i.

Then a diffeomorphism $\iota' : \mathscr{O} \to \mathscr{O}$ extends to a diffeomorphism $\hat{\iota}' : \mathscr{O}' \to \mathscr{O}_1'$ sending the ideal boundary component corresponding to E_i in $\mathscr{O}'$ to one in $\mathscr{O}_1'$.

PROOF. First, we obtain a diffeomorphism i'. We may change ι so that $\iota(U') \subset U_1'$ by composing with an isotopy supported in U preserving leaves of the radial foliation.

We consider a diffeomorphism

$$f' \circ \iota \circ f^{-1} | \Sigma_{E_i} \times (0,1) \to \Sigma'_{E_i} \times (0,1).$$

We can find an isotopy $\iota_1 : \Sigma'_{E_i} \times (0,1) \to \Sigma'_{E_i} \times (0,1)$ preserving $\{x\} \times (0,1)$ for each $x \in \Sigma'_{E_i}$ equal to the identity map on $\Sigma'_{E_i} \times (0,\varepsilon)$ for small $\varepsilon > 0$ so that $\iota_1 \circ f' \circ \iota \circ f^{-1}$ extends to a smooth map at $\Sigma_{E_i} \times \{1\}$. This is fairly simple since every self-embedding of $\Sigma_{E_i} \times (0,1)$ preserving every fiber of form $\{x\} \times (0,1)$ for $x \in \Sigma'_{E_i}$ are isotopic. Now, we can use $\iota : \mathscr{O} - U' \to \mathscr{O} - U_1'$ and $f'^{-1} \circ \iota_1 \circ f' \circ \iota \circ f^{-1}$ on U'. Obviously, they extend each other. □

The following shows the well-definedness of an orbifold.

COROLLARY 9.3.3.

- *Let $\mathscr{O}$ be a strongly tame orbifold with a T-end structure at E_i. Then the identity map* I *of $\mathscr{O}$ is isotopic to a restriction of a diffeomorphism $\mathscr{O}'_{(U,f)} \to \mathscr{O}'_{(U',f')}$.*
- *Let $\mathscr{O}$ be a strongly tame orbifold with an R-end structure at E_i. Then the identity map* I *of $\mathscr{O}$ is isotopic to a restriction of a diffeomorphism $\mathscr{O}_1' \to \mathscr{O}_2'$ for any two end compactifications $\mathscr{O}_1'$ and $\mathscr{O}_2'$ of $\mathscr{O}$.*

PROOF. The first is a corollary of Lemma 9.3.1. The second one is a corollary of Lemma 9.3.2. □

Finally, we will say about the compactification $\bar{\mathscr{O}}$ associated with $\mathscr{O}$.

If $\bar{\mathscr{O}}$ is the compactification associated with $\mathscr{O}$, the ideal boundary structure is given by $(f, N(\Sigma_{E_i}) \cap \mathscr{O})$ where $f : N(\Sigma_{E_i}) \cap \mathscr{O} \to \Sigma_{E_i} \times (0,1]$ is an embedding for a tubular neighborhood $N(\Sigma_{E_i})$ of Σ_{E_i} in $\bar{\mathscr{O}}$.

Again, if $\bar{\mathscr{O}}$ is the associated compactification of $\mathscr{O}$, and the radial structure at E_i is compatible with $\bar{\mathscr{O}}$ as in Section 3.1.3, the radial end compactification from (f,U) can be modified to a compatible $(f', N(\Sigma_{E_i}) \cap \mathscr{O})$ for a tubular neighborhood $N(\Sigma_{E_i})$ of Σ_{E_i} to $\Sigma_{E_i} \times (0,1]$ where f' extends to a smooth diffeomorphism $N(\Sigma_{E_i}) \to \Sigma_{E_i} \times (0,1]$.

PROPOSITION 9.3.4. *We can contruct by the above end compactification process a compact orbifold $\bar{\mathscr{O}}$ of which $\mathscr{O}$ is the interior. The end compactification compatible with the given R-end and T-end structures is always diffeomorphic to $\bar{\mathscr{O}}$ by a diffeomorphism isotopic to the identity in $\mathscr{O}$.*

PROOF. Recall from Sections 3.1.2 and 3.1.3 the definitions of compatibility. Also, it is straightforward to see that the radial foliation is transverse to the added ideal boundary component corresponding to $\Sigma_E \times \{1\}$. Corollary 9.3.3 completes the proof. □

9.3.1. Definition of the deformation spaces with end structures. We will extend this notion strongly. Two real projective structures μ_0 and μ_1 on $\mathscr{O}$ with R-ends or T-ends with end structures are *isotopic* if there is an isotopy i on $\mathscr{O}$ so that $i^*(\mu_0) = \mu_1$ where $i^*(\mu_0)$ is the induced structure from μ_0 by i

- $i_*(\mu_0)$ has a radial end structure for each R-end or horospherical T-end,
- i sends the radial end foliation for μ_0 from an R-end neighborhood or horospherical T-end to the radial end foliation for real projective structure $\mu_1 = i_*(\mu_0)$ with corresponding R-end neighborhoods or a horospherical T-end, and
- i extends to a diffeomorphism of $\bar{\mathscr{O}}$ using the radial foliations and the totally geodesic ideal boundary components for μ_0 and μ_1 where we use the radial end-compactification for a horospherical T-end. (See Definition 9.1.1.)

For noncompact orbifolds with end structures, similar definitions hold except that we have to modify the notion of isotopies to preserve the end structures.

DEFINITION 9.3.5. We consider the real projective structures on orbifolds with end structures. Let $\mathscr{O}$ be one of this and $\bar{\mathscr{O}}$ be the compactification. Let $\hat{\mathscr{O}}$ denote the universal cover of $\bar{\mathscr{O}}$ containing $\tilde{\mathscr{O}}$ as a dense open set.

Let $\mathbf{dev}_\mu$ denote the developing map associated with a convex projective structure μ with R-end or T-ends. The developing map $\mathbf{dev}_\mu : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ extends to a map $\overline{\mathbf{dev}}_\mu : \hat{\mathscr{O}} \to \mathbb{RP}^n$. We will only need developing maps determined up to isotopies. For R-ends, may assume that $\overline{\mathbf{dev}}_\mu$ is smooth by Lemma 9.3.7. For T-ends, we can always isotopy $\mathbf{dev}_\mu|U$ for a p-end neighborhood U of a p-end $\tilde{E}$ so that it can extend to a smooth map by Lemma 9.3.8.

LEMMA 9.3.6. . *Let f_0 and f_1 be two immersions $\tilde{\Sigma}_{\tilde{E}} \times (0,1] \to \mathbb{RP}^n$ equivariant with respect to a holonomy representation $\rho : \pi_1(\Sigma_E) \to \mathsf{PGL}(n+1,\mathbb{R})$ fixing a point p_0. Assume the following:*

- *for each $i = 0,1$, $f_i|x \times (0,1] \to l_x$ is an embedding to a radial segment l_x with endpoint p_0 in $\mathbb{S}^n$ and $f_i(x \times t)$ converges to p_0 as $t \to 0$. Here, l_x is indepenent of $i = 0,1$.*
- *$f_0|\tilde{\Sigma} \times [\delta,1] = f_1|\tilde{\Sigma} \times [\delta,1]$ for $\delta > 0$.*

Then f_0 and f_1 are smoothly isotopic by an isotopy preserving each $x \times [0,1)$ and equals the identity on $\tilde{\Sigma} \times (\delta,1]$.

PROOF. Let $\mathscr{C}_{x,\delta,f_0}([0,1),l_x)$ denote the space of embeddings $g|(0,1]$ where $g|(\delta,1]$ is fixed to be $f_0|x \times [\delta,1] \to l_x$, and $g(t) \to p_0$ as $t \to 0$. This is a contractible space since this is a convex space if we identify l_x with a real interval. Clearly, ρ acts on this space. We can build a bundle $\tilde{B}$ over $\tilde{\Sigma}_{\tilde{E}}$ with fiber at x equal to $\mathscr{C}_{x,\delta,f_0}([0,1),l_x)$. Then $\pi_1(\tilde{E})$ acts on this space where we must act by ρ to the fibers. Hence, we can find a quotient space B fibering over $\Sigma_{\tilde{E}}$. Then consider the space $C_{\rho,x_0,\delta}(\tilde{\Sigma}_{\tilde{E}})$ to be the space of ρ-equivariant sections sending $x \in \tilde{\Sigma}_{\tilde{E}}$ to an element of $\mathscr{C}_{x,\delta,f_0}([0,1),l_x)$.

f_0 and f_1 give two such sections. This induces section $\hat{f}_0$ and $\hat{f}_1$ of B over $\Sigma_{\tilde{E}}$. By contractibility of the fibers, we can use the obstruction theory to obtain the homotopy between $\hat{f}_0$ and $\hat{f}_1$. This gives us the ρ-equivariant homotopy $f_t, t \in [0,1]$, between them using the contractibilty of fibers. We use the fact that a homotopy f'_t between two homeomorphisms f' and f'' of intervals can be realized by an isotopy $(f'_t)^{-1} \circ f'$, assuming each f'_t are homeomorphisms. Hence, these homotopies give us the desired isotopies. □

We now describe the modification of the developing map by a process that we call the *radial-end-projectivization of the developing map with respect to U and U'*. That is, we will prove Lemma 9.3.7.

Let $\tilde{U}$ and $\tilde{U}'$ denote the closed p-end neighborhoods of $\tilde{E}$ covering end-neighborhoods U and U', $\mathrm{Cl}(U) \subset U'^{o}$, respectively. We require U and U' to be compatible product neighborhoods diffeomorphic to $\Sigma_E \times (0,1]$. (Recall compatibility from Section 3.1.) Take a maximal radial ray l_x in $\tilde{U}'$ passing $x \in \mathrm{bd}\tilde{U} \cap l$. Then there exists a unique projective diffeomorphism $\Pi_x : l_x \to \mathbb{R}_+$ sending

- the endpoint of l_x in $\mathrm{bd}\tilde{U}'$ to ∞,
- the other end to 0, and
- $\mathrm{bd}\tilde{U} \cap l = \{x\}$ to 1.

We define $\Pi_{\tilde{U}',\tilde{U}} : \tilde{U}' \to \mathbb{R}_+$ by sending $z \in l_x$ to $\Pi_x(z)$. There is also a unique projective diffeomorphism $P_x : \mathbb{R}_+ \to \mathbb{RP}^n$ sending

$$0 \mapsto \mathrm{v}_{\tilde{E}}, 1 \mapsto \mathbf{dev}(l \cap \mathrm{bd}\tilde{U}) = \mathbf{dev}(x), +\infty \mapsto \mathbf{dev}(l \cap \mathrm{bd}\tilde{U}').$$

Define $\vec{v}_x$ to be the vector at $\mathrm{v}_{\tilde{E}}$ of $(\partial P_x(t)/\partial t)|_{t=0}$. This does depend on x but not on t.

Let $\Pi_{\tilde{E}} : \tilde{U}' \to \tilde{\Sigma}_E$ denote the map sending a point of a radial ray in $\tilde{U}'$ to its equivalence class in $\tilde{\Sigma}_E$. $\tilde{U}'$ has coordinate functions

$$(\Pi_{\tilde{U}',\tilde{U}}, \Pi_{\tilde{E}}) : \tilde{U}' \to \mathbb{R}_+ \times \tilde{\Sigma}_E,$$

which is a diffeomorphism. This commutes with the action of $\mathbf{\Gamma}_{\tilde{E}}$ on $\tilde{U}'$ and the action on $\mathbb{R}_+ \times \tilde{\Sigma}_E$ acting on the first factor trivial. Also, $\tilde{U}$ goes to $(0,1) \times \tilde{\Sigma}_E$ under the map.

We define a smooth map

$$\mathbf{dev}^N : \tilde{U}' \to \mathbb{RP}^n \text{ given by } \mathbf{dev}^N(y) = P_x \circ \Pi_x(y) \text{ for } y \in l_x \subset \tilde{U}'. \tag{9.3.1}$$

Then under the coordinate of $\tilde{U}'$ with affine coordinates on an affine subspace $\mathbb{A}^n_x$ with a temporary Euclidean norm $||\cdot||$ containing $\mathbf{dev}(l_x)$ and containing $\mathbf{dev}(\mathrm{v}_{\tilde{E}})$ as the origin, we can write locally

$$\mathbf{dev}^N(x,t) = f_x(t)\vec{v}_x, \|\vec{v}_x\| = 1, \text{ for } x \in \mathrm{bd}\tilde{U} \tag{9.3.2}$$

on a neighborhood of l_{x_0} for some $x_0 \in \mathrm{bd}\tilde{U}$ where $\vec{v}_x$ is a unit vector depending only on x smoothly in the direction of $\overline{\mathbf{dev}(\mathrm{v}_{\tilde{E}})\mathbf{dev}(x)}$ and

$$\begin{aligned} f_x : \mathbb{R}_+ \to \mathbb{R}_+, f_x(0) = 0, f_x(1) = \|\mathbf{dev}(x)\| \\ f_x(\infty) = \left\|\mathbf{dev}(l_x \cap \mathrm{bd}U')\right\| \text{ provided } \mathbf{dev}(l_x \cap \mathrm{bd}U') \in \mathbb{A}^n_x \end{aligned} \tag{9.3.3}$$

is a strictly increasing projective function of t. The coefficients of the 1-st order rational function f_x as a function of t depend smoothly on x since $\partial U'$ and ∂U are smooth. Here, we can actually change the affine subspaces of form $\mathbb{A}^n_x$ and the coordinates so that locally near l_x we have $f_x(\infty) < \infty$. It is easy to see that $\mathbf{dev}^N$ extends to $\tilde{\Sigma}_E \times \{0\}$ as a constant map. The expression of f_x shows that $\mathbf{dev}^N$ is a smooth extension also since it has continuous partial derivatives of all orders.

Now we change $\mathbf{dev}^N$ on $\tilde{U}' - \tilde{U}$ so that it smoothly extends to $\tilde{\bar{\mathcal{O}}} - \tilde{U}'$. Acutally on $\mathrm{bd}\tilde{U}'$ and on $\mathrm{bd}\tilde{U}$, $\mathbf{dev}^N$ and $\mathbf{dev}$ agree by our construction. On each l_x, the lines $\mathbf{dev}|l_x$ and $\mathbf{dev}^N|l_x$ have the same image to $\mathbf{dev}(l_x)$. Hence, $(\mathbf{dev}^N|l_x)^{-1} \circ \mathbf{dev}|l_x$ sends l_x to l_x as a homeomorphism. We choose U'' be the open neighborhood of U in U' whose closure is in the interior of U'. Let $\tilde{U}''$ denote the component of inverse image containing $\tilde{U}$. We choose a partition of unity function ϕ equal to 1 on $\tilde{U}' - \tilde{U}''$ and with support in a small neighborhood of $\tilde{U}' - \tilde{U}$ in $\tilde{U}'$. Of course, we need to choose ϕ equivariant with respect to $\pi_1(\tilde{E})$, which can be done as in the proof of Lemma 9.3.6. Also, the directional derivative of ϕ in the radial direction of l_x is non-negative. We define $g_x : l_x \to \mathbb{R}_+$ given by $||\mathbf{dev}(z)||$ for $z \in l_x$, which is a smooth function with positive radial directional derivatives along l_x. By isotopying for $\mathbf{dev}^N$ towards the p-end vertex equivariantly and taking smaller U' and U, we may assume without loss of generality that $g_x \geq f_x$ for all $x \in \tilde{\Sigma}_{\tilde{E}}$. Then we define new smooth function $h := (1-\phi) f_x + \phi g_x$ which still has positive directional derivatives in the radial direction of l_x. Then this agree with g_x in $\tilde{U}' - \tilde{U}''$ and f_x outside a neighborhood of $\tilde{U}' - \tilde{U}$ in $\tilde{U}'$. We define $\mathbf{dev}' : \tilde{U}' \to \mathbb{RP}^n$ by replacing Π_x with h in (9.3.1).

This map is the *radial-end-projectivization* of $\mathbf{dev}$ with respect to U' and U and agrees with $\mathbf{dev}_N$ outside a neighborhood of $\tilde{U}' - \tilde{U}$ in $\tilde{U}'$ and agrees with $\mathbf{dev}$ on $\tilde{U}' - \tilde{U}''$.

We define a new developing map $\mathbf{dev}' : \tilde{\bar{\mathcal{O}}} \to \mathbb{RP}^n$ by using $\mathbf{dev}^N$ on $\tilde{U}$ and letting it equal to $\mathbf{dev}$ on the complement $\tilde{\bar{\mathcal{O}}} - \tilde{U}''$. Lemma 9.3.6 implies that $\mathbf{dev}'$ can be obtained from $\mathbf{dev}$ by isotopies.

For other components of form $\gamma(\tilde{U})$ for $\gamma \in \pi_1(\mathcal{O})$, we do the same constructions.

LEMMA 9.3.7. *Let U and U' be a radial end-neighborhoods so that $\mathrm{Cl}_{\bar{\mathcal{O}}}(U) \subset U'$ and compatible to $\bar{\mathcal{O}}$. Let $\tilde{U}$ and $\tilde{U}'$ denote the p-end neighborhoods of $\tilde{E}$ covering U and U'. We assume that* $\mathrm{bd}U$ *and* $\mathrm{bd}U'$ *are transverse to radial rays by taking U and U' smaller if necessary. Then we can modify the developing map in $\tilde{U}$ so that the new developing map* $\mathbf{dev}'$ *agrees with* $\mathbf{dev}$ *on $\tilde{\bar{\mathcal{O}}} - p_{\mathcal{O}}(U)$ so that* $\mathbf{dev}'$ *extends smoothly on the end compactification of $\tilde{U}$ and* $\mathbf{dev}'$ *restricts to each radial line segment is a projective map. Finally,* $\mathbf{dev}' = \mathbf{dev} \circ \iota$ *for an isotopy-lift ι preserving each radial segment in $\tilde{U}'$.* □

LEMMA 9.3.8. *Let $\tilde{E}$ be a p-T-end. Let $\tilde{U}$ be a proper p-T-end neighborhood of $\tilde{E}$. Then* $\mathbf{dev}_\mu$ *can be precomposed by an isotopy so that* $\mathbf{dev}_\mu$ *extends to the ideal boundary $\tilde{\Sigma}_{\tilde{E}}$ as an immersion.*

PROOF. For any point x in $\tilde{\Sigma}_{\tilde{E}}$, we have a chart (U, ϕ) where $x \in U$. $U - \tilde{\Sigma}_{\tilde{E}}$ is inside $\tilde{\mathcal{O}}$. By definition of real projective structures as an atlas of compatible charts, we obtain $g \in \mathsf{SL}_\pm(n+1, \mathbb{R})$ so that $g \circ \phi$ agrees with $\mathbf{dev}_\mu$ on an open set in $U - \tilde{\Sigma}_{\tilde{E}}$, and hence must agree on $U - \tilde{\Sigma}_{\tilde{E}}$. Hence, $\mathbf{dev}_\mu$ extend to U as well. By continuing with all points of $\tilde{\Sigma}_{\tilde{E}}$, we obtain the result.

□

Here, $\overline{\mathbf{dev}}_\mu$ is also equivariant with respect to h if $\mathbf{dev}_\mu$ was so. We call $(\overline{\mathbf{dev}}_\mu, h)$ an extended developing pair. An *extended isotopy* is a diffeomorphism $\bar{\mathcal{O}} \to \bar{\mathcal{O}}$ extending an isotopy of $\mathcal{O}$. An *extended isotopy-lift* is an extension of an isotopy-lift $\hat{\mathcal{O}} \to \hat{\mathcal{O}}$.

Then the *isotopy-equivalence space* $\widetilde{\mathrm{Def}}_{\mathcal{E}}(\mathcal{O})$ is defined as the space of extended developing maps $\overline{\mathbf{dev}}_\mu$ of real projective structures on $\mathcal{O}$ with ends with radial structures and lens-shaped totally geodesic ends with end structures under the action of the group the extended isotopy-lfts where an extended isotopy-lift $\hat{\iota} : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$ acting by

$$(\overline{\mathbf{dev}}_\mu, h) \mapsto (\overline{\mathbf{dev}}_\mu \circ \hat{\iota}, h).$$

We explain the topology. Fix a real projective structure μ with end structure. The space $\mathscr{D}(\hat{\mathscr{O}})$ of maps of form $\overline{\mathbf{dev}}_{\mu'}: \hat{\mathscr{O}} \to \mathbb{RP}^n$ will be given the compact open C^r-topology on $\hat{\mathscr{O}}$.

For any real projective structure μ' on $\mathscr{O}$ with end structures with an isotopy $\iota_{\mu,\mu'}$ so that $\iota^*_{\mu,\mu'}(\mu') = \mu$ the end compactification $\bar{\mathscr{O}}$ has an extended isotopy-lift $\hat{\iota}_{\mu,\mu'}: \hat{\mathscr{O}} \to \hat{\mathscr{O}}$.

Now, $\mathbf{dev}_{\mu'} \circ \iota_{\mu,\mu'}$ is a developing map of $\iota^*_{\mu,\mu'}(\mu')$ sending the end structures $\tilde{\mathscr{O}}$ of μ' to radial line. Hence, $\overline{\mathbf{dev}}_{\mu'} \circ \hat{\iota}_{\mu,\mu'}$ is the unique smooth extension. Hence, we can reinterpret $\mathscr{D}(\hat{\mathscr{O}})$ as the space of extensions of developing maps of $\mathscr{O}$ with a fixed end structure for each end.

DEFINITION 9.3.9. The quotient space $\mathscr{D}(\hat{\mathscr{O}})/\mathscr{E}(\hat{\mathscr{O}})$ of $\mathscr{D}$ under the group of extended isotopy-lifts $\mathscr{E}(\hat{\mathscr{O}})$ of form $\hat{\iota}_{\mu,\mu'}: \hat{\mathscr{O}} \to \hat{\mathscr{O}}$ is in one-to-one correspondence with $\widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O})$. The topology on $\widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O})$ is given as the quotient topology of this space, which is called a *C^r-topology*.

We define $\mathrm{Def}_{\mathscr{E}}(\mathscr{O}) := \widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O})/\mathsf{PGL}(n+1,\mathbb{R})$ by the action

$$(\overline{\mathbf{dev}}, h(\cdot)) \mapsto (\phi \circ \overline{\mathbf{dev}}, \phi \circ h(\cdot) \circ \phi^{-1}), \phi \in \mathsf{PGL}(n+1,\mathbb{R})$$

as in [49] and [87]. The induced quotient topology is called a *C^r-topology* of $\mathrm{Def}_{\mathscr{E}}(\mathscr{O})$.

We can define a map

$$\mathrm{hol}': \widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O}) \to \mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$$

by sending the class of $(\mathbf{dev}, h)$ to h. This is well-defined since the isotopies do not change h. There is an induced map:

$$\mathrm{hol}: \mathrm{Def}_{\mathscr{E}}(\mathscr{O}) \to \mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R})/\mathsf{SL}_{\pm}(n+1,\mathbb{R}).$$

For these, if $\mathscr{O}$ is closed, we will simply drop the subscripts.

It is well-known:

THEOREM 9.3.10 (see Choi [49]). *Let $\mathscr{O}$ be a closed orbifold. Then* $\mathrm{hol}: \widetilde{\mathrm{Def}}(\mathscr{O}) \to \mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$ *is a local homeomorphism.*

9.4. The local homeomorphism theorems

9.4.1. The end condition for real projective structures. Now, we go over to real projective orbifolds: We are given a real projective orbifold $\mathscr{O}$ with ends $E_1, \dots, E_{e_1}$ of $\mathscr{R}$-type and $E_{e_1+1}, \dots, E_{e_1+e_2}$ of $\mathscr{T}$-type. Let us choose representative p-ends $\tilde{E}_1, \dots, \tilde{E}_{e_1}$ and $\tilde{E}_{e_1+1}, \dots, \tilde{E}_{e_1+e_2}$.

We define a subspace of $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ to be as in Section 9.2.

Let $\mathscr{V}$ be an open subset of semi-algebraic subset of

$$\mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

invariant under the conjugation action so that one can choose a continuous section $s^{(1)}_{\mathscr{V}}: \mathscr{V} \to (\mathbb{RP}^n)^{e_1}$ sending a holonomy homomorphism to a common fixed point of $h(\pi_1(\tilde{E}_i))$ for $i = 1, \dots, e_1$ and $s^{(1)}_{\mathscr{V}}$ satisfies

$$s^{(1)}_{\mathscr{V}}(gh(\cdot)g^{-1}) = g \cdot s^{(1)}_{\mathscr{V}}(h(\cdot)) \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R}).$$

There might be more than one choice of a section and the domain of definition. $s^{(1)}_{\mathscr{V}}$ is said to be a *fixed-point section*.

Again suppose that one can choose a continuous section $s^{(2)}_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^{n*})^{e_2}$ sending a holonomy homomorphism to a common dual fixed point of $\pi_1(\tilde{E}_i)$ for $i = e_1+1, \dots, e_2$, and $s^{(2)}_{\mathscr{V}}$ satisfies

$$s^{(2)}_{\mathscr{V}}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s^{(2)}_{\mathscr{V}}(h(\cdot)) \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R})).$$

There might be more than one choice of section in certain cases. $s^{(2)}_{\mathscr{V}}$ is said to be a *dual fixed-point section*.

We define $s_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$ as $s^{(1)}_{\mathscr{V}} \times s^{(2)}_{\mathscr{V}}$ and call it a *fixing section* provided the p-end holonomy group of each $\mathscr{T}$-type p-end $\tilde{E}_i$ acts on a horosphere tangent to P determined by $s^{(2)}_{\mathscr{V}}$.

Recall from Section 11.0.1. We note that the real projective structure with radial and totally geodesic ends with end structures also will determine a point of $(\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$. Conversely, if the real projective structure with radial and totally geodesic ends has the end structure *determined by* a section $s_{\mathscr{U}}$ if the following hold:

- $\tilde{E}_i$ for every $i = 1, \dots, e_1$ has a p-end neighborhood with a radial foliation with leaves developing into rays ending at the fixed point of the i-th factor of $s^{(1)}_{\mathscr{V}}$.
- $\tilde{E}_i$ for every $i = e_1+1, \dots, e_1+e_2$
  - has a p-end neighborhood with the ideal boundary component in the hyperspace determined by the i-th factor of $s^{(2)}_{\mathscr{V}}$ provided $\tilde{E}_i$ is a T-end, or
  - has a p-end neighborhood containing a $\mathbf{\Gamma}_{\tilde{E}}$-invariant horosphere tangent to the hyperspace determined by the i-th factor of $s^{(2)}_{\mathscr{V}}$ provided $\tilde{E}_i$ is a horospherical end.

EXAMPLE 9.4.1. If $\mathscr{O}$ is real projective and has some singularity of dimension one in each end-neighborhood of an $\mathscr{R}$-type end, then the universal cover of $\mathscr{O}$ has more than two lines corresponding to singular loci. The developing image of the lines must meet at a point in $\mathbb{RP}^n$, which is a common fixed point of the holonomy group of an end. If $\mathscr{O}$ has dimension 3, this is equivalent to requiring that the end orbifold has corner-reflectors or cone-points.

Hence, for an open subspace $\mathscr{V}$ of a semi-algebraic subset of

$$\mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

corresponding to the real projective structures on $\mathscr{O}$, there is a section $s^{(1)}_{\mathscr{V}}$ determined by the common fixed points.

(See Theorems 11.1.1 and 11.1.3.)

REMARK 9.4.2 (Cooper). We do caution the readers that these assumptions are not trivial and exclude some important representations. For example, these spaces exclude some incomplete hyperbolic structures arising in Thurston's Dehn surgery constructions as they have at least two fixed points for the holonomy homomorphism of the fundamental group of a toroidal end as was pointed out by Cooper. Hence, the uniqueness condition fails for this class of examples. However, if we choose a section on a subset, then we can obtain appropriate results. Or if we work with particular types of orbifolds, the uniqueness holds. See Section 12.

9.4.2. Perturbing horospherical ends. Theorems 6.1.1 and 6.1.2 study the perturbation of lens-shaped R-ends and lens-shaped T-ends.

The following concerns the deformations of $\mathbf{\Gamma}_{\tilde{E}} \to \mathsf{PGL}(n+1,\mathbb{R})$ near horospherical representations. As long as we restrict to deformed representations satisfying the lens-condition, there exist n-dimensional properly convex domains on which the groups act. (This answers a question of Tillmann near 2006. We also benefited from a discussion with J. Porti in 2011.)

Let P be an oriented hyperspace of $\mathbb{S}^n$ with a dual point $P^* \in \mathbb{S}^{n*}$ represented by a 1-form w_P defined on $\mathbb{R}^{n+1}$. Let $P^\dagger$ denote the space of oriented hyperplanes in P. Let $\mathbb{S}^{n-1*}_{P^*}$ be the space of rays from P^* corresponding to hyperspaces in P. Then the subspace $P^\dagger$ is dual to $\mathbb{S}^{n-1*}_{P^*}$: each oriented ray in $\mathbb{S}^{n*}$ from P^* define a hyperspace S' of P as the set of common zeros of the 1-forms in the ray. The orientation of S' is given by the open half-space where the 1-forms near w_P are positive. Conversely, an oriented pencil of oriented hyperspaces determined by an oriented hyperspace of P is a ray in $\mathbb{S}^{n-1*}_{P^*}$ from P^*. (We omit the obvious $\mathbb{RP}^n$-version.)

Let

$$\mathrm{Hom}_{\mathcal{E},\mathrm{lh},p}(\Gamma',\mathsf{PGL}(n+1,\mathbb{R}))\ \big(\text{resp. } \mathrm{Hom}_{\mathcal{E},\mathrm{lh},p}(\Gamma',\mathsf{SL}_\pm(n+1,\mathbb{R}))\big)$$

denote the space of representations h fixing a common fixed point p and acts properly and cocompactly on the lens of a lens-cone over vertex p or is horospherical with a horoball with vertex p.

Let

$$\mathrm{Hom}_{\mathcal{E},\mathrm{lh},P}(\Gamma',\mathsf{PGL}(n+1,\mathbb{R}))\ \big(\text{resp. } \mathrm{Hom}_{\mathcal{E},\mathrm{lh},P}(\Gamma',\mathsf{SL}_\pm(n+1,\mathbb{R}))\big)$$

denote the space of representations where $h(\Gamma')$ for each element h acts on a hyperspace P satisfying the lens-condition. (See 9.2.)

Let a convex cone $B = \partial B * \{p\}$ over a point p be diffeomorphic to $\partial B \times (0,1]$. Then B with a vertex p has a radial foliation. We complete B by identifying with $\partial B \times (0,1)$ by a diffeomorphism f sending each leaf to $x \times (0,1)$ and attaching $\partial B \times (0,1]$ by f. We denote the partial completion by $\hat{B}$ diffeomorphic to $\partial B \times [0,1]$. We call $\hat{B}$ the *p-end completion* of B. An action of a group Γ on B extends to $\hat{B}$ also. $\hat{B}/\Gamma$ is then the end-compactification of B/Γ. (See Definition 9.1.1.)

LEMMA 9.4.3 (Horospherical-end perturbation).

(A): *Let B be a horoball in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) and Γ be a group of projective automorphisms fixing p, $p \in \mathrm{bd}B$ (resp. $p \in \mathbb{S}^n$), so that B/Γ is a horospherical-end-type orbifold. Then there exists a sufficiently small neighborhood K of the inclusion homomorphism h_0 of Γ in $\mathrm{Hom}_{\mathcal{E},\mathrm{lh},p}(\Gamma,\mathsf{PGL}(n+1,\mathbb{R}))$*

$$(\text{resp. } \mathrm{Hom}_{\mathcal{E},\mathrm{lh},p}(\Gamma,\mathsf{SL}_\pm(n+1,\mathbb{R})))$$

where

- *for each $h \in K$, $h(\Gamma')$ acts on a properly convex domain B_h so that $B_h/h(\Gamma')$ is diffeomorphic to B/Γ' forming a radial end and fixes p,*
- *B_h forms the lens-shaped or horospherical p-R-end neighborhood,*
- *there is a diffeomorphism $f_h : B/\Gamma' \to B_h/h(\Gamma'), h \in K$, so that the lift $\tilde{f}_h : B \to B_h$ is a continuous family under the C^r-topology as a map into $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) where $\tilde{f}_{h_0}$ is the identity map.*

Let $\hat{B}$ and $\hat{B}_h$ denote the p-end compactifications. Then f_h extends to the end compactifications $\bar{f}_h : \hat{B}/\Gamma' \to \hat{B}_h/h(\Gamma')$ and $\bar{f}_{h_0}$ is the identity map. Furthermore, the lift of this map $\hat{f}_h : \hat{B} \to \mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is continuous in the C^r-topology where $\hat{f}_{h_0}$ is the identity map.

(B): *Let P be a hyperspace in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Let Γ' denote a projective automorphism group acting on P and a horoball B tangent to P so that B/Γ' s a horospherical-end-type orbifold. Then there exists a sufficiently small neighborhood K of the inclusion homomorphism h_0 of Γ' in $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P}(\Gamma', \mathsf{PGL}(n+1,\mathbb{R}))$ (resp. $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P}(\Gamma', \mathsf{SL}_\pm(n+1,\mathbb{R}))$) where*

- *for each $h \in K$, $h(\Gamma')$ acts on a properly convex domain B_h so that $B_h/h(\Gamma')$ is homeomorphic to B/Γ',*
- *B_h formsa lens-shaped p-T-end or horospherical p-end neighorhood, and*
- *there is a diffeomorphism $f_h : B/\Gamma' \to B_h/h(\Gamma')$, $h \in K$, so that the lift $\tilde{f}_h : B \to B_h$ is a continuous family under the C^r-topology where $\tilde{f}_{h_0}$ is the identity map.*

Let $\hat{B}$ denote the p-end compactification of B and $\hat{B}_h$ denote B_h union with the p-end ideal boundary component of B_h when h acts properly and cocompactly on a lens. Then f_h extends to the end compactifications $\bar{f}_h : \hat{B}/\Gamma' \to \hat{B}_h/h(\Gamma')$. Furthermore, the lift of this map $\hat{f}_h : \hat{B} \to \mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is a continuous family in the C^r-topology where $\hat{f}_{h_0}$ is the identity map.

PROOF. We will prove for the $\mathbb{S}^n$-version.

(A) Let us choose a larger horoball B' in B where B'/Γ' has a boundary component $S'_{\tilde{E}}$ so B'/Γ is diffeomorphic to $S'_{\tilde{E}} \times [0,1)$. $S'_{\tilde{E}}$ is strictly convex and transverse to the radial foliation. There exists a neighborhood O_1 in $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},p}(\Gamma', \mathsf{SL}_\pm(n+1,\mathbb{R}))$ corresponding to the connection on a fixed compact neighborhood N of $S'_{\tilde{E}}$ changes only by ε in the C^r-topology, $r \geq 2$, on a compact set containing a compact fundamental domain. (See the deformation theorem in [87] which generalize to the compact orbifolds with boundary.)

Let $h \in O_1$. The universal cover $\tilde{S}'_{\tilde{E}}$ is a strictly convex codimension-one manifold, and it deforms to $\tilde{S}'_{\tilde{E},h}$ that is still strictly convex for sufficiently small ε. Here, $\tilde{S}'_{\tilde{E},h}$ may not be embedded in $\mathbb{S}^n$ a priori but is a submanifold of the deformed n-manifold N_h from N by the change of connections. Every ray from p meets $\tilde{S}'_{\tilde{E},h}$ transversely also by the C^r-condition.

Let $\overrightarrow{v}_{x,h}$ be a vector in the direction of x for $x \in \tilde{S}'_{\tilde{E},h}$ which we choose equivariant with respect to the action of $h(\Gamma')$. We may choose so that $(x,h) \mapsto \overrightarrow{v}_{x,h}$ is continuous. We form a cone

$$c(\tilde{S}'_{\tilde{E},h}) := \{[t\overrightarrow{v}_p + (1-t)\overrightarrow{v}_{x,h}] | t \in [0,1], x \in \tilde{S}'_{\tilde{E},h}\}.$$

Let $\tilde{\Sigma}_{\tilde{E},h}$ denote the space of rays from p ending at $\tilde{S}'_{\tilde{E},h}$ in $c(\tilde{S}'_{\tilde{E},h})$. Here $S'_h := \tilde{S}'_{\tilde{E},h}/h(\Gamma')$ is a compact real projective orbifold of $(n-1)$-dimension.

Since Γ' is a cusp group, it is virtually abelian. Since $h \in \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\Gamma', \mathsf{SL}_\pm(n+1,\mathbb{R}))$, Lemma A.1.10 implies that $D_h : \tilde{S}'_{\tilde{E},h} \to \mathbb{S}^{n-1}_p$ is an embedding to a properly convex domain or a complete affine domain Ω_h in $\mathbb{S}^{n-1}_p$ where $h(\Gamma')$ acts properly discontinuously and cocompactly when h is not the inclusion map.

There is a one-to-one correspondence from $\tilde{S}'_{\tilde{E},h}$ to $\tilde{\Sigma}_{\tilde{E},h} =: \Omega_h$. By convexity of $\tilde{\Sigma}_{\tilde{E},h}$, the tube domain $\mathscr{T}_p(\Omega_h)$ with vertices $p, -p$ is convex. $\tilde{S}'_{\tilde{E},h}$ meets each great segments in the interior the tube domain with vertices at $p, -p$ at a unique point transversely since h is in O_1 for sufficiently small ε. The strict convexity of $\tilde{S}'_{\tilde{E},h}$ implies that B_h is convex by Lemma 1.4.3. The proper convexity of B_h follows since $\tilde{S}'_{\tilde{E},h}$ is strictly convex and meets each great segment from p in the interior of the tube domain corresponding to $\tilde{\Sigma}_{\tilde{E},h}$, and hence $\mathrm{Cl}(B_h)$ cannot contain a pair of antipodal points.

By Theorem 8.1.3, the Zariski closure of $h(\Gamma')$ is a cusp group G_h extended by a finite group and $G_h/h(\Gamma')$ is compact. Hence, Γ' is virtually abelian by the Bieberbach theorem. We take the identity component $\mathscr{N}_h$ of G_h, which is an abelian group with a uniform lattice $h(\Gamma')$. The set of orbits of $\mathscr{N}_h$ foliates B_h. Since $\mathscr{N}_h$ is a normal subgroup of G_h, $h(\Gamma')$ normalizes $\mathscr{N}_h$. Hence, the orbits give us a codimension-one foliation on $B_h/h(\Gamma')$ with compact leaves. The leaves are all diffeomorphic, and hence, we obtain a parameterization $\partial B/\Gamma' \times [0,1)$ to B_h.

Now, h induces isomorphism $\hat{h}_h : \mathscr{N}_0 \to \mathscr{N}_h$ where $\hat{h}_h \to \mathrm{I}$ as $h \to h_0 := \mathrm{I}$.

We choose a proper radial path $\alpha_h : I \to B_h$ from a point of ∂B_h and ending at p. We may assume that α_h is independent of h. We define a parameterization

$$\tilde{\phi}_h : \mathscr{N} \times [0,1) \to B_h, (m,t) \mapsto \hat{h}_h(m)(\alpha_h(t)), t \in [0,1).$$

We define $\tilde{f}_h : B \to B_h := \tilde{\phi}_h \circ \tilde{\phi}_{h_0}^{-1}$. This gives us a map f_h. (Here, we might be changing $\tilde{\Sigma}_{\tilde{E},h}$.) Since $\tilde{f}_h$ sends radial segments to radial segments, it extends to a smooth map $\hat{f}_h : \hat{B} \to \hat{B}_h$. Also, on any compact subset J of $\hat{B}$, a compact foliated set $\hat{J}$ contains it. Let $\hat{J}_h$ denote the image of $\hat{J}$ under $\hat{f}_h$. $\hat{J}_h$ is coordinatized by $\check{J} \times I$ for a fixed compact set $\check{J} \subset \mathscr{N}$. Under these coordinates of $\hat{J}$ and $\hat{J}_h$, we can write $\hat{f}_h$ as the identity map. Since $\hat{h}_h \to \hat{h}_{h_0}$, we conclude that $\hat{f}_h | \hat{J}$ uniformly converges to I as $h \to h_0$.

Now, we show that B_h is a p-R-end of horospherical or lens type. We know from above that Ω_h, $h \in O_1$, is either properly convex or complete affine. Suppose that Ω_h is properly convex. Then we have a tubular action on a tube corresponding to Ω_h. By Theorem 5.1.5, the lens condition is equivalent to the uniform middle eigenvalue condition. Hence, we have a distanced action by Proposition 5.2.5 and $\tilde{S}'_{\tilde{E},h}$ must have boundary in a distanced compact set in the boundary of the tube by Proposition 5.3.10. By taking the convex hull of $\tilde{S}'_{\tilde{E},h}$, we obtain a compact convex set distanced from p. Now, looking at from $-p$, we can obtain a smooth lens containing this. Hence, we have a p-R-end of lens type.

Suppose that Ω_h is complete affine. Then again $\tilde{S}'_{\tilde{E},h}$ is strictly convex and develops to a complete affine space in $\mathbb{S}^n_p$. By Theorem 8.1.2, we have a horospherical end since the cusp group satisfies the weak uniform middle eigenvalue condition.

(B) The second item is the dual of the first one. If $h(\Gamma')$ acts on an open horosphere B^o tangent to P with the vertex in P properly discontinuously, then the dual group $h(\Gamma')^*$ acts on a horosphere with a vertex the point P^* dual to P. By duality Proposition 5.5.5, h^* is in $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P^*}(\Gamma, \mathsf{SL}_\pm(n+1,\mathbb{R}))$. We apply the first part, and hence, there exists a properly convex domain $\Omega_{P,h}$ so that $\Omega_{P,h}/h^*(\Gamma')$ is an open orbifold for $h \in K$ for some subset K of the character space

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P^*}(\Gamma', \mathsf{SL}_\pm(n+1,\mathbb{R})).$$

By duality Proposition 5.5.5, $\Omega_{P,h}$ is foliated by radial lines from P^* and $R_{P^*}(\Omega_{P,h}) \subset \mathbb{S}^{n-1}_{P^*}$ is a properly convex domain.

Let B^o_h be the properly convex domain dual to $\Omega^o_{P,h}$ and hence is a properly convex domain, and $B^o_h/h(\Gamma')$ is a dual orbifold diffeomorphic to $\Omega_{P,h}/h(\Gamma')^*$ by Theorem 1.5.8. We have $B_{h_0} = B^o$ since the dual of the dual of a properly convex open domain is itself.

Let Γ acts properly on ∂B. Since ∂B is strictly convex, each point of ∂B has a unique hyperspace sharply supporting B. By Proposition 1.5.4, there is an open hypersurface $S, S \subset \mathrm{bd}\Omega_P$ dual to ∂B. Also, Γ'^* acts properly on S so that S/Γ'^* is a closed orbifold. (The closedness again follows since there is a torsion-free group of a finite index and hence a finite regular-covering manifold.) From the first part, there is an open surface

$S_h, S_h \subset \mathrm{bd}\Omega_{P,h}$, $h \in K$, meeting each radial ray from P^* at a unique point. Also, $S_h/h(\Gamma')^*$ is diffeomorphic to $\partial B/\Gamma'^*$.

Again, by Proposition 1.5.4, we obtain an open surface S^*_h, $S^*_h \subset \mathrm{bd}\Omega^*_{P,h}$ where Γ_h acts properly so that S^*_h/Γ_h is a closed orbifold. We define $B_h := \Omega^*_{P,h} \cup S^*_h$.

Since $\Omega_{P,h}$ is foliated by radial segments from P^* with properly convex

$$R_{P^*}(\Omega_{P,h}) \subset \mathbb{S}^{n-1}_{P^*},$$

$D_h := P \cap \mathrm{bd}\Omega^*_{P,h}$ and is a properly convex domain in P by Proposition 5.5.5.

Note that Γ' is virtually abelian, and when it is not a cusp group, then it is lens-type and hence must be virtually diagonalizable.

Define $\hat{B}_h := \Omega^*_{P,h} \cup S^*_h \cup D_h$, $h \in K$. For the second and third items, of the second part, we do as above but we choose $\alpha_h : I \to B_h$ to be a single geodesic segment starting from $x_0 \in \partial B_h$ and ending at a point of D_h where α_h converges as a parameter of functions to a geodesic $\alpha_0 : I \to \mathrm{Cl}(B) \subset \mathbb{S}^n_\infty$ ending at the vertex of the horosphere or a fixed point of h not on P whenever h is virtually diagonalizable. We assume that α_h is a C^r-family of geodesics. Now, the proof is similar to the above using an isomorphism from the identity component C_h of the Zariski closure of Γ' to that of $h(\Gamma')$, which is an abelian group since Γ' is virtually abelian. We denote by $\kappa : C_{h_0} \to C_h$ the unique homomorphism extending $h \circ h_0^{-1}$ on resricted to the abelian group of finite index of Γ'.

Here, we need the images of α_h under C_h to form a foliation. Since C_h acts on a properly convex set D_h, it acts as a diagonalizable group on P by Proposition 1.4.10. Being a free abelian group satisfying the uniform middle eigenvalue condition, C_h is a diagonalizable group acting on an n-simplex. (We dualize the situation and use Theorem 5.4.3.) We required that the extension of α_h to pass a fixed point of C_h not in $\mathrm{Cl}(D_h)$. The images of $g \circ \alpha_h$, $g \in C_h$, form a foliation of $\hat{B}_h$. Using this we define the map $\tilde{f}_h : B \to B_h$ sending leaves to leaves as given by the function

$$g(\alpha_0(t)) \mapsto \kappa(g)(\alpha_h(t)) \text{ for each } t \in [0,1], g \in C_{h_0}.$$

This map extends to ∂B to ∂B_h.

Finally notice that our constructions of f_h all are smooth from $\bar{\mathscr{O}}$. Hence, these are compatible end neighborhoods.

We have a lens by reflection about P by the fixed point not on P when Γ' is virtually diagonalizable. Hence, we have a lens-type end. Otherwisse, we have a horospherical end as in the end of the proof of the first part. [$\mathbb{S}^n$T] □

LEMMA 9.4.4 (Lens-end perturbation).

(A): *Let B be a generalized lens-cone in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) and Γ be a group of projective automorphisms fixing p, $p \in \mathrm{bd}B$ (resp. $p \in \mathbb{S}^n$), so that B/Γ is a generalized lens-end-type orbifold. Then there exists a sufficiently small neighborhood K of the inclusion homomorphism h_0 of Γ in $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},p}(\Gamma, \mathsf{PGL}(n+1,\mathbb{R}))$*

$$(\textit{resp. } \mathrm{Hom}_{\mathscr{E},\mathrm{lh},p}(\Gamma, \mathsf{SL}_\pm(n+1,\mathbb{R})))$$

where

- *for each $h \in K$, $h(\Gamma')$ acts on a properly convex domain B_h so that $B_h/h(\Gamma')$ is diffeomorphic to B/Γ' forming a radial end and fixes p,*
- *B_h forms the lens-shaped p-R-end neighborhood,*
- *there is a diffeomorphism $f_h : B/\Gamma' \to B_h/h(\Gamma'), h \in K$, so that the lift $\tilde{f}_h$: $B \to B_h$ is a continuous family under the C^r-topology as a map into $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) where $\tilde{f}_{h_0}$ is the identity map.*

Let $\hat{B}$ and $\hat{B}_h$ denote the p-end compactifications. Then f_h extends to the end compactifications $\bar{f}_h : \hat{B}/\Gamma' \to \hat{B}_h/h(\Gamma')$ and $\bar{f}_{h_0}$ is the identity map. Furthermore, the lift of this map $\hat{f}_h : \hat{B} \to \mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is continuous in the C^r-topology where $\hat{f}_{h_0}$ is the identity map.

(B): *Let P be a hyperspace in $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$). Let Γ' denote a projective automorphism group acting on P and a lens B meeting P in its interior so that B/Γ' s a lens-end-type orbifold, and a component B_1 of $B - P$ is a p-end neighborhood of an end. Then there exists a sufficiently small neighborhood K of the inclusion homomorphism h_0 of Γ' in $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P}(\Gamma', \mathsf{PGL}(n+1,\mathbb{R}))$ (resp. $\mathrm{Hom}_{\mathscr{E},\mathrm{lh},P}(\Gamma', \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$) where*

- *for each $h \in K$, $h(\Gamma')$ acts on a lens B_h so that $B_h/h(\Gamma')$ is homeomorphic to B/Γ',*
- *A component $B_h - P$ forms lens-shaped p-T-end or horospherical p-end neighborhood, and*
- *there is a diffeomorphism $f_h : B/\Gamma' \to B_h/h(\Gamma')$, $h \in K$, so that the lift $\tilde{f}_h : B \to B_h$ is a continuous family under the C^r-topology where $\tilde{f}_{h_0}$ is the identity map.*

Let $\hat{B}_1$ denote the p-end compactification of B and $\hat{B}_{1,h}$ denote $B_h - P$ union with the p-end ideal boundary component of B_h when h acts properly and cocompactly on a lens. Then $f_h|B_1$ extends to the end compactifications $\bar{f}_h : \hat{B}_1/\Gamma' \to \hat{B}_{1,h}/h(\Gamma')$. Furthermore, the lift of this map $\hat{f}_h : \hat{B}_1 \to \mathbb{RP}^n$ (resp. $\mathbb{S}^n$) is a continuous family in the C^r-topology where $\hat{f}_{h_0}$ is the identity map.

PROOF. The proofs are very similar to those of Lemma 9.4.3 using radial segments and duality. Here, we need the local homeomorphism property of for the closed orbifolds of Lok [**121**]. Hence, for nearby homeomorphisms we can choose develpong maps that are very close near the hypersurfaces bounding the p-end neighborhoods. □

We remark that we can also reinterpret the parameterization as radial projectivization in Section 9.3.1 by taking a second larger end neighborhood and some modifications of the parameters.

9.4.3. Local homeomorphism theorems. Let $\mathscr{O}$ be a noncompact strongly tame affine $(n+1)$-orbifold whose ends are assigned to be of $\mathscr{R}$-type or $\mathscr{T}$-type as is the convention in this paper.

An affine manifold affinely diffeomorphic to the affine suspension of horospherical end neighborhood is said to be the *affinely suspended horoball neighborhood*. If an end has such a neighborhood, then we call the end *affine horospherical type*. Since the projective automorphism group of a horosphere fixes a point, the fundamental group of the affine horospherical end preserves a direction. Thus, the end of an affine horospherical type is of radial type.

We define the end restricted deformation space for $\mathscr{O}$ to be the quotient space of affine structures on $\mathscr{O}$ where

- each end is radial if the end is of $\mathscr{R}$-type or
- is totally geodesic satisfying the suspended lens-condition if the end is of $\mathscr{T}$-type

under the action of group of isotopies preserving the end structures: that is, preserves the radial foliation if the end is radial or horospherical or extends to a smooth diffeomorphism if the end is totally geodesic.

Again $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{U}}}(\mathscr{O})$ is defined to be the subspace of $\mathrm{Def}_{\mathscr{E}}(\mathscr{O})$ with the stable irreducible holonomy homomorphisms in $\mathscr{U}$ and the end determined by $s_{\mathscr{U}}$, i.e.,

- each $\mathscr{R}$-type p-end has a p-end neighborhood foliated by geodesic leaves that are radial to the vector given by $s_{\mathscr{U}}$ under the developing map, or
- each $\mathscr{T}$-type p-end is totally geodesic of suspended lens-type satisfying the lens-condition or horospherical satisfying the suspended lens condition with respect to the hyperspace determined by $s_{\mathscr{U}}$. (See Section **??**.)

THEOREM 9.4.5. *Let $\mathscr{O}$ be a noncompact strongly tame real projective n-orbifold with lens-shaped radial ends or lens-shaped totally geodesic ends with types assigned. Let $\mathscr{V}$ be a conjugation-invariant open subset of the union of semialgebraic subsets of*

$$\mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Let $s_{\mathscr{V}}$ be the fixing section defined on $\mathscr{V}$ with images in $(\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$. Then the map

$$\mathrm{hol} : \mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}}}(\mathscr{O}) \to \mathrm{rep}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

sending the real projective structures with ends compatible with $s_{\mathscr{V}}$ to their conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of $\mathscr{V}'$.

COROLLARY 9.4.6. *Let $\mathscr{O}$ be a noncompact strongly tame real projective n-orbifold with lens-shaped radial ends or lens-shaped totally-geodesic ends with end structures and given types $\mathscr{R}$ or $\mathscr{T}$. Assume $\partial \mathscr{O} = \emptyset$. Then the following map is a local homeomorphism:*

$$\mathrm{hol} : \mathrm{Def}^s_{\mathscr{E},u}(\mathscr{O}) \to \mathrm{rep}^s_{\mathscr{E},u}(\pi_1(O), \mathsf{PGL}(n+1,\mathbb{R})).$$

PROOF. It is clear that the limit of a sequence of unique fixed points of representations must be a fixed point of the limit representation. Also, the limit of correpsonding cyclic spaces must be in the cyclic spaces of the limit representations. Hence, if the limit representation have elements with one-dimensional intersections of the cyclic spaces, then the corresponding elements of the sequence have one-dimensional intersections of the cyclic spaces. This proves that the map from the representation spaces to the unique fixed points is continuous. Now Theorem 9.4.5 implies the conclusion. ☐

9.4.4. The proof of Theorem 9.4.5. We wish to now prove Theorem 9.4.5 following the proof of Theorem 1 in Section 5 of [49].

Let $\mathscr{O}$ be an affine orbifold with the universal covering orbifold $\tilde{\mathscr{O}}$ with the covering map $p_{\mathscr{O}} : \tilde{\mathscr{O}} \to \mathscr{O}$ and let the fundamental group $\pi_1(\mathscr{O})$ act on it as an automorphism group.

Let $\mathscr{U}$ and $s_{\mathscr{U}}$ be as above. We will now define a map

$$\mathrm{hol} : \widetilde{\mathrm{Def}}_{\mathscr{E},s_{\mathscr{U}}}(\mathscr{O}) \to \mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$$

by sending the projective structure to the pair $(\mathbf{dev}, h)$ and to the conjugacy class of h finally.

PROOF OF THEOREM 9.4.5. We show that hol is continuous: There is a codimension-0 compact submanifold $\mathscr{O}'$ of $\mathscr{O}$ so that $\pi_1(\mathscr{O}') \to \pi_1(\mathscr{O})$ is an isomorphism. The holonomy homomorphism is determined on $\mathscr{O}'$. Since the deformation space has the C^r-topology, $r \geq 1$, induced by $\mathbf{dev} : \tilde{\mathscr{O}}' \to \mathbb{R}^{n+1}$, it follows that small changes of $\mathbf{dev}$ on compact domains in $\tilde{\mathscr{O}}'$ in the C^r-topology imply sufficiently small changes in $h(g'_i)$ for generators g'_i of $\pi_1(\mathscr{O}')$ and hence sufficiently small change of $h(g_i)$ for generators g_i of $\pi_1(\mathscr{O})$. Therefore, hol is continuous. (Actually for the continuity, we do not need any condition on ends.)

We are aiming to prove the local homeomorphism property of the map

$$\mathrm{hol}: \widetilde{\mathrm{Def}}_{\mathscr{E},s_{\mathscr{U}}}(\mathscr{O}) \to \mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$$

sending projective structures determined by the section $s_{\mathscr{U}}$ to the conjugacy classes of holonomy homomorphisms is a local homeomorphism on an open subset of $\mathscr{U}'$. The continuity of hol was proved at the beginning of this subsection.

Next, we define the local inverse map from a neighborhood in $\mathscr{U}$ of the image point. Let $\mathscr{O}'$ be a compact suborbifold of $\mathscr{O}$ so that $\mathscr{O}-\mathscr{O}'$ is a union U of end neighborhoods.

We will show how to change the proof of Theorem 1 of [49]. Let h be a representation coming from an affine orbifold $\mathscr{O}$. The task is to reassemble $\mathscr{O}$ with new holonomy homomorphisms as we vary h as in [49] following approaches of Thurston. Suppose that h' is in a neighborhood of h in $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$.

- As in Lok [121], we consider locally finite collections $\mathscr{V}$ of open domains that cover $\mathscr{O}$. We find subcollection $\mathscr{V}'$ of compact neighborhoods or end neighborhoods in the covering contractible open sets which covers $\mathscr{O}$ again. Here, each precompact element of $\mathscr{V}$ contains a compact domain in $\mathscr{V}'$ forming a cover of $\mathscr{O}'$. The end neighborhoods can meet only if they are in the same component of U.
- We will give orders to the open sets covering $\mathscr{O}$. The end neighborhoods will have orders higher than all precompact sets.
- We regard these as sets in $\mathbb{S}^{n+1}$ by charts.
- We consider the sets that are the intersection

  $$U_{i_1} \cap \cdots \cap U_{i_k}, i_1 > \cdots > i_k, \text{ where } U_{i_j} \in \mathscr{V}' \text{ for } j = 1, \dots, k,$$

  of the largest cardinality of the compact or closed domains in $\mathscr{V}'$ and find the corresponding sets in $\mathbb{S}^{n+1}$ by charts. We map it by isotopies to the corresponding intersection of deformed collections of domains in $\mathbb{S}^{n+1}$ corresponding to the $h'(\pi_1(\mathscr{O}))$-action by Lemmas 3 and 4 in [49] and using the deformations of **dev** by post-composing with maps in Lemmas 9.4.3 and 9.4.4. Here, we will follow the ordering as above when we deform as in Lok [121]. That is, we use the isotopy of U_{i_1} restricted to $U_{i_1} \cap \cdots \cap U_{i_k}$ when U_{i_1} has the largest order.
- We extend the isotopies to the sets of intersections of smaller number of sets in $\mathscr{V}'$ by Lemma 5 of [49]. By induction, we extend it to all the images of compact and closed domains in $\mathscr{V}'$.
- We patch these open sets to build an orbifold $\mathscr{O}_{h'}$ with holonomy h' referring back to $\mathscr{O}$ by isotopies.
- $\mathscr{O}_{h'}$ is diffeomorphic to $\mathscr{O}$ by the map constructed by the isotopies.

To show that the local inverse is a continuous map for the C^r-topology of $\hat{\mathscr{O}}$, we only need to consider compact suborbifolds in $\mathscr{O}$ since the holonomy representation depends only on any compact submanifold whose complement is a union of proper end neighbohoods. For this the same argument as in [49] will apply.

We now prove the local injectivity of hol. Given two structures μ_0 and μ_1 in a neighborhood of the deformation space, we show that if their holonomy homomorphisms are the same, say $h: \pi_1(\mathscr{O}) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})$, then we can isotopy one in the neighborhood to the other using vector fields as in [49].

Because of the section $s_{\mathscr{U}}$ defined on $\mathscr{U}$, given a holonomy h, we have a direction of the radial end that is unique for the holonomy homomorphism.

First assume that $\mathscr{O}$ has only $\mathscr{R}$-type ends. Recall the compact suborbifold $\mathscr{O}'$ so that $\mathscr{O}-\mathscr{O}'$ is diffeomorphic to $E_i \times (0,1)$ for each end orbifold E_i where each $x\times(0,1)$ is the image of a radial segment.

We can choose a Riemannian metric on $\bar{\mathscr{O}}$ so that an end neighborhood has a product metric of form $E_i \times (0,1]$. Let $\mathbf{dev}_j$ be the developing map of μ_j for $j=0,1$. Then the C^r-norm distance of extensions $\overline{\mathbf{dev}_0}$ and $\overline{\mathbf{dev}_1}$ to $\bar{\mathscr{O}}$ is bounded on each compact set $K\subset\bar{\mathscr{O}}$ by our assumption on the closeness of the two structures. Since we chose μ_1 and μ_2 sufficiently close, $\overline{\mathbf{dev}_0}$ and $\overline{\mathbf{dev}_1}$ can be assumed to be sufficiently close in the C^r-topology over K. The images of K under each of these maps can be assumed to lie on a neighborhood of the image of a p-end vertex, say v. Moreover, the radial lines maps to the radial lines ending at the same ideal end vertex. We may assume that they have the forms of the radial projectivizations. We can use the argument in the last part of [49] to show that $\overline{\mathbf{dev}_1}$ lifts to an immersion $\hat{\mathscr{O}}\to\hat{\mathscr{O}}$ equivariant with respect to the deck transformation group. Then we use the metrics to equivariantly isotopy it to I as in the last section of [49]. Hence, μ_0 and μ_1 represent the same point of $\widetilde{\mathrm{Def}}^{S}_{\mathbb{A},\mathscr{E},s_{\mathscr{U}}}(\mathscr{O})$.

Suppose now that $\mathscr{O}$ has some lens type $\mathscr{T}$-type ends. Suppose that μ_0 and μ_1 have a totally geodesic ideal boundary component corresponding to an end of $\mathscr{O}$. We attach the totally geodesic ideal boundary component for each end, and then we can argue as in [49] proving the local injectivity.

Suppose that μ_0 and μ_1 have horospherical end neighborhoods corresponding to an end of $\mathscr{O}$. Then these are radial ends and the same argument as the above one for $\mathscr{R}$-type ends will apply to show the local injectivity.

Finally, we cannot have the situation that μ_0 has the totally geodesic ideal boundary component corresponding to an end while μ_1 has a horoball end neighborhood for the same end. This follows since the end holonomy group acts on a properly convex domain in a totally geodesic hyperspace and as such the end holonomy group elements have some norms of eigenvalues >1. (See Proposition 1.1 of [18] for example.) $\square$

9.5. Relationship to the deformation spaces in our earlier papers

Recall $\mathfrak{D}(\hat{P})$ for a Coxeter orbifold $\hat{P}$ that is not necessarily compact in Definition 2.2.2.

Let $\mathscr{O}$ be a strongly tame orbifolds only radial ends and radial end structure $\mathscr{E}$. Recall the definition of $\mathrm{CDef}_{\mathscr{E}}(\mathscr{O})$ from Section 9.3.1.

Generalizing this, let $\mathfrak{D}_{\mathscr{E}}(\mathscr{O})$ denote the same set as $\mathrm{CDef}_{\mathscr{E}}(\mathscr{O})$. We give the topology by C^r-topology on for the set of all developing maps $\mathbf{dev}:\mathscr{O}\to\mathbb{S}^n$ and take the quotient by right actions by the isotopy lifts $\tilde{\mathscr{O}}$ and by the left action by composition with elements of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. These were the topology we used before as in [50].

By following Proposition 9.5.1, we obtain that $\mathfrak{D}(\hat{P})$ is the same as $\mathrm{CDef}_{\mathscr{E}}(P)$ as topological spaces.

We say that a diffeomorphism *preserves a radial end structure* if an end neighborhood with the radial foliation structure contains an end neighborhood is mapped into an end neighborhood sending the radial leaves to radial leaves.

Let $\tilde{\bar{\mathscr{O}}}$ denote the universal cover of $\mathscr{O}$. We can form $\bar{\mathscr{O}}$ by using the end-compactification by Definition 9.1.1. A radial end structure preserving isotopies can be extended to a homeomorphisms of end-compactifications. We call such diffeomorphism *end-structure preserving extended isotopies*.

PROPOSITION 9.5.1. *Let $\mathscr{O}$ be a stronly tame orbifold with only radial ends and radial end structure $\mathscr{E}$. Then there is a homeomorphism*

$$\mathfrak{D}_{\mathscr{E}}(\mathscr{O}) = \mathrm{CDef}_{\mathscr{E}}(\mathscr{O}).$$

PROOF. We denote by $\hat{\mathscr{O}}$ the universal cover of $\bar{\mathscr{O}}$ containing $\tilde{\mathscr{O}}$ as a dense open set. Hence, by restricting the structure on $\mathscr{O}$ only, there is a map

$$R'_r : \mathrm{CDef}_{\mathscr{E}}(\mathscr{O}) \to \mathfrak{D}(\mathscr{O}).$$

The map is one-to-one and onto since the sets are the same.

Let $D_{\tilde{\mathscr{O}}}$ denote the space of all developing maps on $\tilde{\mathscr{O}}$ with radial end structures. Let $J^r(f)$ denote the tuples of all jets of $f : \tilde{\mathscr{O}} \to \mathbb{RP}^n$ of order $\leq r$. The topology on $D_{\tilde{\mathscr{O}}}$ is given by bases of form

$$B_{K,\varepsilon}(\mathbf{dev}) := \{f \in D_{\mathscr{O}} | \mathbf{d}(J^r(\mathbf{dev})(x), J^r(f)(x)) < \varepsilon, x \in K, f \in C^r(\tilde{\mathscr{O}}, \mathbb{RP}^n)\}$$

where $K \subset \tilde{\mathscr{O}}$ is a compact set, $\varepsilon > 0$, and $\mathbf{dev} \in D_{\tilde{\mathscr{O}}}$. The topology on $D_{\hat{\mathscr{O}}}$ is given by bases of form

$$B'_{K,\varepsilon}(\mathbf{dev}) := \{f \in D_{\hat{\mathscr{O}}} | \mathbf{d}(J^r(\mathbf{dev})(x), J^r(f)(x)) < \varepsilon, x \in K, f \in C^r(\hat{\mathscr{O}}, \mathbb{RP}^n)\}$$

where $K \subset \hat{\mathscr{O}}$ is a compact set, $\varepsilon > 0$, and $\mathbf{dev} \in D_{\hat{\mathscr{O}}}$.

Consider the restriction map

$$R_r : C^r(\hat{\mathscr{O}}, \mathbb{RP}^n) \to C^r(\tilde{\mathscr{O}}, \mathbb{RP}^n)$$

inducing R'_r. Since compact subsets of $\tilde{\mathscr{O}}$ are compact subsets of $\bar{\mathscr{O}}$, the inverse image under R_r of a basis element of $C^r(\tilde{\mathscr{O}}, \mathbb{RP}^n)$ is a basis element in $C^r(\bar{\mathscr{O}}, \mathbb{RP}^n)$. Hence, the induced map

$$R'_r : \mathrm{CDef}_{\mathscr{E}}(\mathscr{O}) = D_{\bar{\mathscr{O}}}/\mathscr{G}_{\bar{\mathscr{O}}} \to \mathfrak{D}(\mathscr{O}) = D_{\mathscr{O}}/\mathscr{G}_{\mathscr{O}}$$

is continuous.

Now we find the inverse of R'_r:

Recalling Definition 9.1.2, we define:

- Let $\mathscr{G}_{\mathscr{O}}$ denote the group of isotopies of $\mathscr{O}$ preserving the radial end structures of $\mathscr{O}$, and
- let $\mathscr{G}_{\bar{\mathscr{O}}}$ denote the group of radial end structure preserving extended isotopies of $\bar{\mathscr{O}}$.
- The isotopy-lifts $\iota : \tilde{\mathscr{O}} \to \tilde{\mathscr{O}}$ form a group which we denote by $\mathscr{G}_{\tilde{\mathscr{O}}}$.
- Denote by $\mathscr{G}_{\hat{\mathscr{O}}}$ the group of the extensions of isotopy-lifts $\hat{\iota} : \hat{\mathscr{O}} \to \hat{\mathscr{O}}$ of isotopies in $\mathscr{O}$.

Fix the union U of mutually disjoint R-end neighborhoods and radial foliations on each component in $\mathscr{O}$.

Also, we choose a union U' of such neighborhoods so that $\mathrm{Cl}_{\bar{\mathscr{O}}}(U) \subset U'$. Let $\tilde{U}$ denote the inverse image of U and $\tilde{U}'$ that of U'.

- denote by $\mathscr{G}_{\mathscr{O},U',U}$ the group of isotopies of form ι of $\mathscr{O}$ acting on each component of U preserving the radial folations and U'.
- Let $\mathscr{G}_{\tilde{\mathscr{O}},U',U}$ denote the isotopy-lifts $\tilde{\iota}$ of $\mathscr{O}$ to $\tilde{\mathscr{O}}$ acting on each component of $\tilde{U}'$ and $\tilde{U}$ which are lifts of elements of $\mathscr{G}_{\mathscr{O},U',U}$.
- Let $\mathscr{G}_{\hat{\mathscr{O}},U',U}$ be the extensions of isotopy-lifts of $\tilde{\mathscr{O}}$ to $\hat{\mathscr{O}}$ acting on each component of $\tilde{U}'$ and $\tilde{U}$.

Clearly there is a natural isomorphism by extension $\mathscr{G}_{\mathscr{O},U',U} \to \mathscr{G}_{\hat{\mathscr{O}},U',U}$ with the inverse map given by the restriction to $\mathscr{O}$.

Let $D_{\mathscr{O},U,U'}$ denote the space of functions of form **dev** in a development pair $(\mathbf{dev},h)$ so that $\mathbf{dev}|\tilde{U}'$ equals $\mathbf{dev}^N|\tilde{U}'$ constructed for U and U' by the radial end-projectivization in Lemma 9.3.7. (We may have to construct on a larger union of p-end neighborhood because of the smoothing process and restrict to U and U'.) By Lemma 9.3.7, we obtain a natural map $D_{\mathscr{O},U,U'}/\mathscr{G}_{\mathscr{O},U',U} \to D_{\mathscr{O}}/\mathscr{G}$ that is a one-to-one onto map.

Also, denote by $D_{\bar{\mathscr{O}},U,U'}$ denote the space of functions of form $\overline{\mathbf{dev}}$ which are extended developing maps and $\overline{\mathbf{dev}}|\tilde{U}$ equals $\mathbf{dev}^N|\tilde{U}$ constructed for U and U' by radial end-projectivization in Lemma 9.3.7.

Recall from the elementary analysis that the C^r-topology on a compact set K is a metric topology with the metric d_K given by taking the supremum of the distances of jets up to order $\leq r$ at a compact set.

We claim that the canonical map $F: D_{\mathscr{O},U,U'}/\mathscr{G}_{\mathscr{O},U',U} \to D_{\mathscr{O}}/\mathscr{G}_{\mathscr{O}}$ is a homeomorphism: Let us take a compact neighborhood K_F of a fundamental domain F of $\tilde{\mathscr{O}} - \tilde{U}$. Then we define a metric on $D_{\mathscr{O},U,U'}$ by $d_{D_{\mathscr{O},U,U'}}$ between two developing maps f_1, f_2 is defined as $\sup_{x\in K_F} \mathbf{d}(J_r(f_1)(x), J_r(f_2)(x))$. Since the developing map on $\tilde{U}$ is determined by the developing map restricted on K_F as we can see by a radial end-projectivization with respect to U and U', we obtain a metric $d_{D_{\mathscr{O},U,U'}}$ on $D_{\mathscr{O},U,U'}$ giving us the C^r-topology on $D_{\mathscr{O},U,U'}$. The map F is continuous since it is induced by the inclusion map $D_{\mathscr{O},U',U} \to D_{\mathscr{O}}$.

There is an inverse map $G: D_{\mathscr{O}}/\mathscr{G}^u \to D_{\mathscr{O},U,U'}/\mathscr{G}^u_{U',U}$ given by taking a developing map f and modifying it by radial end-projectivization. There are choices involved, but they are well-defined up to isotopy-lifts.

Any isotopy ι induces a homeomorphism ι^* in $D_{\mathscr{O}}$. To show the continuity of G, we take a ball $B_{d_{\mathscr{O},U,U'}}(f,\varepsilon)$ for $f \in D_{\mathscr{O},U,U'}$ show that there is a neighborhood of $f\circ\iota$ in $D_{\mathscr{O}}$ in the C^r-topology going into it for $\iota \in \mathscr{G}^{\tilde{\mathscr{O}}}$. Since we can take $\iota^*(B)$ as a neighborhood for $f\circ\iota$, it is sufficient to find a neighborhood B of f in $D_{\mathscr{O}}$ in the C^r-topology. Let K_F denote a compact set in $\tilde{\mathscr{O}} - \tilde{U}$. We take a sufficiently small δ, $\delta > 0$, the neighborhood $B_{K_F,\delta}(f)$ so that each of its element g is still in $B_{d_{D_{\mathscr{O},U,U'}}}(f,\varepsilon)$ after applying the isotopy of radial end-projectivization with respect to U and U': This is because we only need to worry about the compact set $K_F \cap \mathrm{Cl}_{\tilde{\mathscr{O}}}(\tilde{U}')$ while g after radial end-projectivization with respect to U' and U is determind in this compact set. If g is sufficiently close to f on K_F in the C^r-topology already in the form of Lemma 9.3.7, then

- the leaves of the radial foliation of g intersected with K_F is C^r-close to ones for f and
- the required isotopy ι_g for radial end-projectivization of g is also sufficiently C^r-close to I on K_F in the uniform J_r-topology defined on U.

Hence, by taking sufficiently small δ, $g\circ\iota_g$ is in $B_{d_{D_{\mathscr{O},U,U'}}}(f,\varepsilon)$. Since we are estimating everything in a compact set K_F, and finding ι_g depending on $g|K_F$, these methods are possible. (By the uniform topology, we mean the topology using the norm of differences of two functions on U but not just one a compact subset of U.)

Also, $D_{\bar{\mathscr{O}},U,U'}/\mathscr{G}_{\bar{\mathscr{O}},U',U} \to D_{\bar{\mathscr{O}}}/\mathscr{G}_{\bar{\mathscr{O}}}$ is a homeomorphism similarly.

It is a triviality that the restriction map $D_{\bar{\mathscr{O}},U,U'} \to D_{\mathscr{O},U,U'}$ is a homeomorphism. Hence, the induced map $D_{\bar{\mathscr{O}},U,U'}/\mathscr{G}_{\bar{\mathscr{O}},U',U} \to D_{\mathscr{O},U,U'}/\mathscr{G}_{\mathscr{O},U',U}$ is a homeomorphism. Since there is a commutative diagram

$$
\begin{array}{ccc}
D_{\bar{\mathscr{O}},U,U'}/\mathscr{G}^{u}_{\bar{\mathscr{O}},U',U} & \to & D_{\mathscr{O},U,U'}/\mathscr{G}_{\mathscr{O},U',U} \\
\downarrow & & \downarrow \\
D_{\bar{\mathscr{O}}}/\mathscr{G}_{\bar{\mathscr{O}}} & \to & D_{\mathscr{O}}/\mathscr{G}_{\mathscr{O}}.
\end{array}
\tag{9.5.1}
$$

Since the downarrows maps are homeomorphisms by above, and the upper row map is a homeomorphism, it follows that the bottom row is continuous.

This means that there is a continous map $D_{\mathscr{O}}/\mathscr{G}_{\mathscr{O}} \to D_{\bar{\mathscr{O}}}/\mathscr{G}_{\bar{\mathscr{O}}}$ giving us the inverse of R'_r. $\square$

CHAPTER 10

Relative hyperbolicity and strict convexity

We will show the equivalence between the relative hyperbolicity of the fundamental group of the properly convex real projective orbifolds with the lens-shaped radial ends or totally geodesic ends or horospherical ends with the strict convexity of the orbifolds relative to the ends. In Section 10.1, we will show how to add some lenses to the T-ends so that the ends become boundary components and how to remove some open sets to make R-ends into boundary components. Some constructions preserve the strict convexity and so on. In Section 10.2, we describe the action of the end fundamental group on the boundary of the universal cover of a properly convex orbifold. In Section 10.3, we prove Theorem 10.3.1 that the strict convexity implies the relative hyperbolicity of the fundamental group using Yaman's work. We then present the converse of this Theorem 10.3.4. For this, we use the work of Druţu and Saphir on tree graded spaces and asymptotic cones.

10.1. Some constructions associated with ends

We will discuss some constructions to begin. It will be sufficient to prove for the case $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ in this chapter by Proposition 1.4.2. So we will not give any $\mathbb{RP}^n$ version here.

The purpose of this chapter is to prove Corollary 11.0.3 the equivalence of the strict convexity of $\mathscr{O}$ and the relative hyperbolicity of $\pi_1(\mathscr{O})$ with respect to the end fundamental groups. Cooper-Long-Tillman [67] and Crampon-Marquis [68] proved the same result when we only allow horospherical ends. Benoist told us at an IMS meeting at the National University of Singapore in 2016 that he has proof for this theorem for $n = 3$ using trees as he has done in closed 3-dimensional cases in [24] using the Morgan-Shalen's work on trees [134]. For convex cocompact actions, there are some related later work by Islam and Zimmer [104] and [103], and Weisman [155] for relatively hyperbolic groups for compact orbifold groups.

Recall that properly convex strongly tame real projective orbifolds with generalized lens-shaped or horospherical ends satisfying (NA) and (IE) have strongly irreducible holonomy groups by Theorem 6.0.4. In this chapter, we will fix the union $\mathbb{U}$ of all concave end neighborhoods for radial ends and lens end neighborhoods for T-ends and horospherical neighborhoods of ends mutually disjoint from one another. Let $\widetilde{\mathbb{U}}$ denote the inverse image in $\tilde{\mathscr{O}}$.

10.1.1. Modifying the T-ends.

For T-ends, by the lens condition, we only consider the ones that have CA-lens neighborhoods in some ambient orbifolds. First, we discuss the extension to bounded orbifolds.

THEOREM 10.1.1. *Suppose that $\mathscr{O}$ is a strongly tame properly convex real projective orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfy* (IE). *Let E be a lens-shaped T-end, and let S_E be a totally geodesic hypersurface that is the ideal boundary corresponding to E. Let L be a lens-shaped end neighborhood of S_E in an ambient real projective orbifold containing $\mathscr{O}$. Then*

- *$L \cup \mathcal{O}$ is a properly convex real projective orbifold and has a strictly convex boundary component corresponding to E.*
- *Furthermore, if $\mathcal{O}$ is strictly SPC and $\tilde{E}$ is a hyperbolic end, then so is $L \cup \mathcal{O}$ which now has one more boundary component and one less T-ends.*

PROOF. Let $\tilde{\mathcal{O}}$ be the universal cover of $\mathcal{O}$, which we can identify with a properly convex bounded domain in an affine subspace. Then S_E corresponds to a T-p-end $\tilde{E}$ and a totally geodesic hypersurface $S = \tilde{S}_{\tilde{E}}$. And L is covered by a lens $\tilde{L}$ containing S. The p-end fundamental group $\pi_1(\tilde{E})$ acts on $\tilde{\mathcal{O}}$ and $\tilde{L}_1$ and $\tilde{L}_2$ the two components of $\tilde{L} - \tilde{S}_{\tilde{E}}$ in $\tilde{\mathcal{O}}$ and outside $\tilde{\mathcal{O}}$ respectively.

□

Lemma 10.1.2 generalizes Theorem 3.7 of [88].

LEMMA 10.1.2. *Suppose that $\tilde{S}_{\tilde{E}}$ is the totally geodesic ideal boundary of a lens-shaped T-end $\tilde{E}$ of a strongly tame real projective orbifold $\mathcal{O}$.*

- *Given a $\pi_1(\tilde{E})$-invariant properly convex open domain Ω_1 with $\mathrm{bd}\Omega_1 \cap \mathbb{S}^{n-1}_\infty = \tilde{S}_{\tilde{E}}$, for each point p of $\mathrm{bd}\tilde{S}_{\tilde{E}}$, any sharply supporting hyperspace H of $\tilde{S}_{\tilde{E}}$ at p in $\mathbb{S}^{n-1}_\infty$, there exists an AS-hyperspace to $\tilde{\mathcal{O}}$ containing H.*
- *At each point of $\mathrm{bd}\tilde{S}_{\tilde{E}}$, the hyperspace sharply supporting any $\pi_1(\tilde{E})$-invariant properly convex open set Ω with $\mathrm{bd}\Omega \cap \mathbb{S}^{n-1}_\infty = \tilde{S}_{\tilde{E}}$ is unique if $\pi_1(\tilde{E})$ is hyperbolic.*
- *We are given two $\pi_1(\tilde{E})$-invariant properly convex open domains Ω_1 with $\mathrm{bd}\Omega_1 \cap \mathbb{S}^{n-1}_\infty = \tilde{S}_{\tilde{E}}$, and Ω_2 with $\mathrm{bd}\Omega_2 \cap \mathbb{S}^{n-1}_\infty = \tilde{S}_{\tilde{E}}$ from the other side. Then $\mathrm{Cl}(\Omega_1) \cup \mathrm{Cl}(\Omega_2)$ is a convex domain with*
$$\mathrm{Cl}(\Omega_1) \cap \mathrm{Cl}(\Omega_2) = \mathrm{Cl}(\tilde{S}_{\tilde{E}}) \subset \mathrm{bd}\Omega_1 \cap \mathrm{bd}\Omega_2$$
and their AS-hyperspaces at each point of $\mathrm{bd}\tilde{S}_{\tilde{E}}$ coincide.

PROOF. Let $\mathbb{A}^n$ denote the affine subspace that is the complement in $\mathbb{S}^n$ of the hyperspace containing $\tilde{S}_{\tilde{E}}$. Because $\pi_1(\tilde{E})$ acts properly and cocompactly on a lens-shaped domain, By Theorem 5.5.4, $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition.

The domain Ω_1 has an affine half-space $H(x)$ bounded by an AS-hyperspace for each $x \in \mathrm{bd}\tilde{S}_{\tilde{E}}$ containing Ω_1. Here, $H(x)$ is uniquely determined by $\pi_1(\tilde{E})$ and x and $H(x) \cap \mathbb{S}^{n-1}_\infty$ by Theorems 4.1.1 and 4.3.1. The respective AS-hyperspaces at each point of $\mathrm{Cl}(\tilde{S}_{\tilde{E}}) - \tilde{S}_{\tilde{E}}$ to Ω_1 and Ω_2 have to agree by Lemmas 4.2.12 and 4.3.8.

The second item follows by the third item and Theorem 1.1 of [22]. □

CONTINUATION OF THE PROOF OF THEOREM 10.1.1. By Lemma 10.1.2, $\tilde{L}_2 \cup \tilde{S}_{\tilde{E}} \cup \tilde{\mathcal{O}}$ is a convex domain. If $\tilde{L}_2 \cup \tilde{\mathcal{O}}$ is not properly convex, then it is a union of two cones over $\tilde{S}_{\tilde{E}}$ over of $[\pm v_x] \in \mathbb{R}^{n+1}, [v_x] = x$. This means that $\tilde{\mathcal{O}}$ has to be a cone contradicting the strong irreducibility of $h(\pi_1(\mathcal{O}))$. Hence, it follows that $\tilde{L}_2 \cup \tilde{\mathcal{O}}$ is properly convex.

Suppose that $\mathcal{O}$ is strictly SPC and $\pi_1(\tilde{E})$ is hyperbolic. Then every segment in $\mathrm{bd}\tilde{\mathcal{O}}$ or a non-C^1-point in $\mathrm{bd}\tilde{\mathcal{O}}$ is in the closure of one of the p-end neighborhood. $\mathrm{bd}\tilde{L}_2 - \mathrm{Cl}(\tilde{S}_{\tilde{E}})$ does not contain any segment in it or a non-C^1-point. $\mathrm{bd}\tilde{\mathcal{O}} - \mathrm{Cl}(\tilde{S}_{\tilde{E}})$ does not contain any segment or a non-C^1-point outside the union of the closures of p-end neighborhoods. $\mathrm{bd}(\tilde{\mathcal{O}} \cup \tilde{L}_2 \cup \tilde{S}_{\tilde{E}})$ is C^1 at each point of $\Lambda(\tilde{E}) := \mathrm{Cl}(\tilde{S}_{\tilde{E}}) - \tilde{S}_{\tilde{E}}$ by the uniqueness of the sharply supporting hyperspaces of Lemma 10.1.2.

Recall that $\tilde{S}_{\tilde{E}}$ is strictly convex since $\pi_1(\tilde{E})$ is a hyperbolic group. (See Theorem 1.1 of [22].) Thus, Λ does not contain a segment, and hence, $\mathrm{bd}(\tilde{\mathcal{O}} \cup \tilde{L}_2 \cup \tilde{S}_{\tilde{E}})$ does not contain one. Therefore, $L_2 \cup \mathscr{O}$ is strictly convex relative to the remaining ends. Now we do this for every copy $g(L_2)$ of L_2 for $g \in \pi_1(\mathscr{O})$.

Since $\tilde{L}_2 \cup \tilde{\mathscr{O}}$ has a Hilbert metric by [112], the action is properly discontinuous. $\square$

COROLLARY 10.1.3. *Suppose that $\mathscr{O}$ is a noncompact strongly tame properly convex real projective orbifold with a p-end $\tilde{E}$, and $\pi_1(\tilde{E})$ is hyperbolic.*

(i) *Let $\tilde{E}$ be a lens-shaped totally geodesic p-end. Let L be a CA-lens containing a totally geodesic properly convex hypersurface $\tilde{E}$ so that*

$$\Lambda := \mathrm{Cl}(\tilde{S}_{\tilde{E}}) - \tilde{S}_{\tilde{E}} = \mathrm{bd}L - \partial L.$$

Then each point of Λ has a unique sharply supporting hyperspace of L.

(ii) *Let $\tilde{E}$ be a lens-shaped radial p-end. Let L be a CA-lens in the p-end neighborhood. Define $\Lambda := \mathrm{bd}L - \partial L$. Then each point of Λ has a unique sharply supporting hyperspace of L.*

PROOF. (i) is already proved in Lemma 10.1.2.
(ii) is proved in Proposition 5.5.8. $\square$

10.1.2. Shaving the R-ends. We call the following construction *shaving the ends*.

THEOREM 10.1.4. *Given a strongly tame SPC-orbifold $\mathscr{O}$ and its universal cover $\tilde{\mathscr{O}}$, there exists a collection of mutually disjoint open concave p-end neighborhoods for lens-shaped p-ends. We remove a finite union of concave end-neighborhoods of some R-ends. Then*

- *we obtain a convex domain as the universal cover of a strongly tame orbifold $\mathscr{O}_1$ with additional strictly convex smooth boundary components that are closed $(n-1)$-dimensional orbifolds.*
- *Furthermore, if $\mathscr{O}$ is strictly SPC with respect to all of its ends, and we remove only some of the concave end-neighborhoods of hyperbolic R-ends, then $\mathscr{O}_1$ is strictly SPC with respect to the remaining ends.*

PROOF. If $\mathscr{O}_1$ is not convex, then there is a triangle T in $\tilde{\mathscr{O}}_1$ with three segments s_0, s_1, s_2 so that $T - s_0^o \subset \tilde{\mathscr{O}}_1$ but $s_0^o - \tilde{\mathscr{O}}_1 \neq \emptyset$. (See Theorem A.2 of [46] for details.) Since $\tilde{\mathscr{O}}_1$ is an open manifold, $s_0^o - \tilde{\mathscr{O}}_1$ is a closed subset of s_0^o. Then a boundary point $x \in s_0^o - \tilde{\mathscr{O}}_1$ is in the boundary of one of the removed concave-open neighborhoods or is in $\mathrm{bd}\tilde{\mathscr{O}}$ itself. The second possibility implies that $\mathscr{O}$ is not convex as $\tilde{\mathscr{O}}_1 \subset \tilde{\mathscr{O}}$. The first possibility implies that there exists an open segment meeting $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ at a unique point but disjoint from U. This is geometrically not possible since $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ is strictly convex towards the direction of U. These are contradictions.

Since $\tilde{\mathscr{O}}$ is properly convex, so is $\tilde{\mathscr{O}}_1$. Since $\mathrm{bd}U \cap \tilde{\mathscr{O}}$ is strictly convex, the new corresponding boundary component of $\tilde{\mathscr{O}}_1$ is strictly convex.

Now we go to the second part. We suppose that $\mathscr{O}$ is strictly SPC. Let $\mathscr{H}$ denote the set of p-ends with hyperbolic p-end fundamental groups whose concave p-end neighborhoods were removed in the equivariant manner. For each $\tilde{E} \in \mathscr{H}$, denote by $U_{\tilde{E}}$ the concave p-end neighborhood that we are removing.

Any segment in the boundary of the developing image of $\mathscr{O}$ is in the closure of a p-end neighborhood of a p-end vertex. For the p-end-vertex $\mathrm{v}_{\tilde{E}}$ of a p-end $\tilde{E}$, the domain $R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}}) \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is strictly convex by [22] if $\pi_1(\tilde{E})$ is hyperbolic. Since $\mathrm{bd}R_{\mathrm{v}_{\tilde{E}}}(\tilde{\mathscr{O}})$ contains no straight segment, only straight segments in $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ for the concave p-end neighborhood U of $\tilde{E}$ are in the segments in $\bigcup S(\mathrm{v}_{\tilde{E}})$. Thus, their interiors are disjoint from $\mathrm{bd}\tilde{\mathscr{O}}_1$, and hence $\mathrm{bd}\tilde{\mathscr{O}}_1$ contains no geodesic segment in $\bigcup_{\tilde{E} \in \mathscr{H}} \mathrm{Cl}(U_{\tilde{E}}) \cap \mathrm{bd}\tilde{\mathscr{O}}$.

Since we removed concave end neighborhoods of the lens-shaped ends with the hyperbolic end fundamental groups, any straight segment in $\mathrm{bd}\tilde{\mathscr{O}}_1$ lies in the closure of a p-end neighborhood of a remaining p-end vertex.

A non-C^1-point of $\mathrm{bd}\tilde{\mathscr{O}}_1$ is not on the boundary of the concave p-end neighborhood U for a hyperbolic p-end $\tilde{E}$ nor in $\mathrm{bd}\tilde{\mathscr{O}} - \bigcup_{\tilde{E}\in\mathscr{H}} \mathrm{Cl}(U_{\tilde{E}})$. We show that points of $\Lambda = L - \partial L$ are C^1-points of $\mathrm{bd}\tilde{\mathscr{O}}$: $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}_1$ contains the limit set $\Lambda = L - \partial L$ for the CA-lens L in a lens-neighborhood. $\tilde{\mathscr{O}}$ has the same set of sharply supporting hyperspaces as L at points of Λ since they are both $\pi_1(\tilde{E})$-invariant convex domains by Corollary 10.1.3. However, the sharply supporting hyperspaces at Λ of L are also supporting ones for $\tilde{\mathscr{O}}_1$ by Corollary 10.1.3 since $L \subset \tilde{\mathscr{O}}_1$ as we removed the outside component U of $\tilde{\mathscr{O}} - L$. Thus, $\tilde{\mathscr{O}}_1$ is C^1 at points of Λ.

Also, points of $\mathrm{bd}\tilde{\mathscr{O}} - \bigcup_{\tilde{E}\in\mathscr{H}} \mathrm{Cl}(U_{\tilde{E}})$ are C^1-points of $\mathrm{bd}\tilde{\mathscr{O}}$ since $\mathscr{O}$ is strictly SPC. Let x be a point of this set. Suppose that $x \in s^o$ for a segment s in $\mathrm{bd}\tilde{\mathscr{O}}_1$. Then $s \subset \mathrm{bd}\tilde{\mathscr{O}}$ and s is not in $\mathrm{Cl}(U_{\tilde{E}})$ for any $\mathrm{v}_{\tilde{E}}$ since we removed subsets of $\tilde{\mathscr{O}}$ to obtain $\tilde{\mathscr{O}}_1$. Hence this is not possible. Suppose that x has more than two sharply supporting hyperspaces P_1, P_2 to $\tilde{\mathscr{O}}_1$ at x. We may assume that P_1 is a sharply supporting hyperspace to $\tilde{\mathscr{O}}$. Since P_2 is not supporting $\tilde{\mathscr{O}}$, a component H_2' disjoint from $\tilde{\mathscr{O}}_1$ meets $\tilde{\mathscr{O}}$. Then $H_2' \cap \tilde{\mathscr{O}}$ is a convex domain, which we denote by Ω_2. $\Omega_2 \subset \bigcup_{\tilde{E}\in\mathscr{H}} \mathrm{Cl}(U_{\tilde{E}})$. Now, it is easy to see that $\mathrm{Cl}(U_{\tilde{E}})$ for at most one p-end $\tilde{E}$ meets H_2'. Since $x \in \tilde{\mathscr{O}}_2$, x is in the closure of $\mathrm{Cl}(U_{\tilde{E}}) \cap H_2'$. Thus, $x \in \mathrm{Cl}(U_{\tilde{E}}) \cap \mathrm{Cl}(\tilde{\mathscr{O}}_1) \subset \Lambda_{\mathrm{v}_{\tilde{E}}}$ for a limit set $\Lambda_{\mathrm{v}_{\tilde{E}}}$. However, we proved that there is a unique supporting hyperspace at x to $\tilde{\mathscr{O}}_1$ in the above paragraph. Hence, $\mathscr{O}_1$ is strictly SPC. □

10.2. The strict SPC-structures and relative hyperbolicity

10.2.1. The Hilbert metric on $\mathscr{O}$. Recall Hilbert metrics from Section 1.1.3. A Hilbert metric on an orbifold with an SPC-structure is defined as a distance metric given by cross ratios. (We do not assume strictness here.)

Given an SPC-structure on $\mathscr{O}$, there is a Hilbert metric which we denote by $d_{\tilde{\mathscr{O}}}$ on $\tilde{\mathscr{O}}$ and hence on $\tilde{\mathscr{O}}$. Actually, we will make $\mathscr{O}$ slightly small by inward perturbations of $\partial\mathscr{O}$ preserving the strict convexity of $\partial\mathscr{O}$ by Lemma 1.4.6. The Hilbert metric will be defined on original $\tilde{\mathscr{O}}$. (We call this metric the *perturbed Hilbert metric*.) This induces a metric on $\mathscr{O}$, including the boundary now. We will denote the metric by $d_{\mathscr{O}}$.

Given an open properly convex domain Ω, we note that given any two points x, y in Ω, there is a geodesic arc $\overline{xy}$ with endpoints x, y so that its interior is in Ω.

PROPOSITION 10.2.1. *Let Ω be a properly convex open domain. Let P be a subspace meeting Ω, and let x be a point of $\Omega - P$:*

- **(i):** *There exists a shortest path m from x to $P \cap \Omega$ that is a line segment.*
- **(ii):** *The set of shortest paths to P from a point x of $\Omega - P$ have endpoints in a compact convex subset K of $P \cap \Omega$.*
- **(iii):** *For any line m' containing m and $y \in m'$, the segment in m' from y to the point of $P \cap \Omega$ is one of the shortest segments.*
- **(iv):** *When P is a complete geodesic in Ω with $x \in \Omega - P$, outside the compact set K, $K \subset P$, of endpoints of shortest segments from x to P, the distance function from $P - K$ to x is strictly increasing or strictly decreasing.*

PROOF. (i) The distance function $f : P \cap \Omega \to \mathbb{R}$ defined by $f(y) = d_\Omega(x, y)$ is a proper function where $f(x) \to \infty$ as $x \to z$ for any boundary point z of $P \cap \Omega$ in P. Hence, there exists a shortest segment with an endpoint x_0 in $P \cap \Omega$. (iv) is also proved.

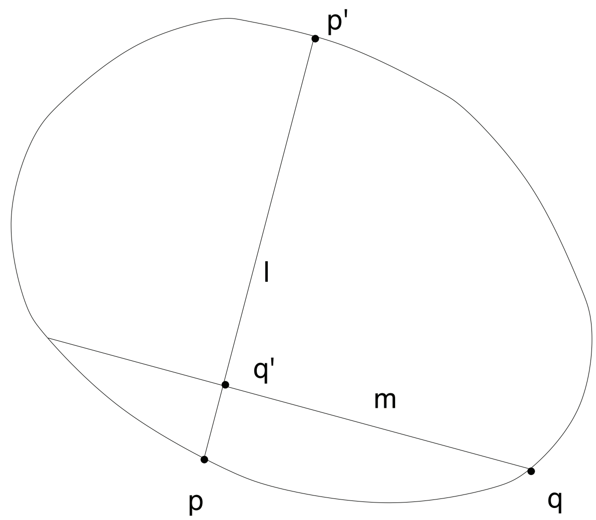

FIGURE 1. The shortest geodesic m to a geodesic l.

(ii) Let γ be any geodesic in $P \cap \Omega$ passing x_0. We need to consider the 2-dimensional subspace Q containing γ and x. The set of endpoints of shortest segments of Ω in Q is a connected compact subset containing x_0 by Proposition 1.4. of [45]. Hence, by considering all geodesics in $P \cap \Omega$ passing x_0, we obtain that the endpoints of the shortest path to P from x is a connected compact set. We take two points z_1, z_2 on it. Then the segment connecting z_1 and z_2 is also in the set of endpoints by Proposition 1.4 of [45]. Hence, the set is convex.

(iii) Suppose that there exists $y \in m'$, so that the shortest geodesic m'' to $P \cap \Omega$ is not in m'. Consider the 2-dimensional subspace Q containing m' and m''. Then this is a contradiction by Corollary 1.5 of [45].

(iv) Again follows by considering a 2-dimensional subspace containing P and m. (See Proposition 1.4 of [45] for details.) □

An endpoint in P of a shortest segment is called a *foot of the perpendicular* from x to γ.

10.2.2. Strict SPC-structures and the group actions. By Corollary 6.3.3, strict SPC-orbifolds with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends have only lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends.

An *elliptic element* of g is a nonidentity element of $\pi_1(\mathscr{O})$ fixing an interior point of $\tilde{\mathscr{O}}$. Since $\pi_1(\mathscr{O})$ acts discretely on the space $\tilde{\mathscr{O}}$ with a metric, an elliptic element has to be of finite order.

LEMMA 10.2.2. *Let $\mathscr{O}$ be a strongly tame strict SPC-orbifold. Let $\tilde{E}$ be a p-end of $\tilde{\mathscr{O}}$.*

(i) *Suppose that $\tilde{E}$ is a horospherical p-end. Let B be a horoball p-end neighborhood with a p-end vertex p corresponding to $\tilde{E}$. There exists a homeomorphism $\Phi_{\tilde{E}}$: $\mathrm{bd}B - \{p\} \to \mathrm{bd}\tilde{\mathscr{O}} - \{p\}$ given by sending a point x to the endpoint of maximal convex segment containing x and p in $\mathrm{Cl}(\tilde{\mathscr{O}})$.*

(ii) *Suppose that $\tilde{E}$ is a lens-shaped radial p-end. Let U be a lens-shaped radial p-end neighborhood with the p-end vertex p corresponding to $\tilde{E}$. There exists a* *homeomorphism $\Phi_{\tilde E} : \mathrm{bd}U \cap \tilde{\mathscr{O}} \to \mathrm{bd}\tilde{\mathscr{O}} - \mathrm{Cl}(U)$ given by sending a point x to the other endpoint of the maximal convex segment containing x and p in $\mathrm{Cl}(\tilde{\mathscr{O}})$.*

Moreover, each of the maps denoted by $\Phi_{\tilde E}$ commutes with elements of $h(\pi_1(\tilde E))$.

PROOF. (i) By Theorem 8.1.3(i) $\Phi_{\tilde E}$ is well-defined. The same proposition implies that $\mathrm{bd}B$ is smooth at p and $\mathrm{bd}\tilde{\mathscr{O}}$ has a unique sharply supporting hyperspace. Therefore the map is onto.

(ii) The second item follows from Theorems 5.4.2 and 5.4.3 since they imply that the segments in $S(p)$ are maximal ones in $\mathrm{bd}\tilde{\mathscr{O}}$ from p. □

We now study the fixed points in $\mathrm{Cl}(\tilde{\mathscr{O}})$ of elements of $\pi_1(\mathscr{O})$. Recall that a great segment is a geodesic arc in $\mathbb{S}^n$ with antipodal p-end vertices. It is not properly convex.

Note that we can replace a generalized lens to a lens for a strongly tame strictly SPC-orbifold by Corollary 6.3.3.

LEMMA 10.2.3. *Let $\mathscr{O}$ be a strongly tame strict SPC-orbifold with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. Let g be an infinite order element of a p-end fundamental group $\pi_1(\tilde E)$. Then every fixed point x of g in $\mathrm{Cl}(\tilde{\mathscr{O}})$ satisfies one of the following*:

- *x is in the closure of a p-end-neighborhood that is a concave end-neighborhood of an R-p-end,*
- *x is in the closure of a p-ideal boundary component of a T-p-end or*
- *x is the fixed point of a horospherical R-p-end.*

PROOF. Suppose that the p-end $\tilde E$ is a lens-shaped R-end. The direction of each segment in the interior of the lens cone with an endpoint $\mathrm{v}_{\tilde E}$ is fixed by only the finite-order element of $\pi_1(\tilde E)$ since $\pi_1(\tilde E)$ acts properly discontinuously on $\tilde S_{\tilde E}$. Thus, the fixed points are on the rays in the direction of the boundary of $\tilde E$. They are in one of $S(\mathrm{v}_{\tilde E})$ for the p-end vertex $\mathrm{v}_{\tilde E}$ corresponding to $\tilde E$ by Theorems 5.4.2 and 5.4.3. Hence, the fixed points of the holonomy homomorphism of $\pi_1(\tilde E)$ is in the closure of the lens-cone with end vertex $\mathrm{v}_{\tilde E}$ and nowhere else in $\mathrm{Cl}(\tilde{\mathscr{O}})$.

If $\tilde E$ is horospherical, then the p-end vertex $\mathrm{v}_{\tilde E}$ is not contained in any segment s in $\mathrm{bd}\tilde{\mathscr{O}}$ by Theorem 8.1.3. Hence $\mathrm{v}_{\tilde E}$ is the only point $S \cap \mathrm{bd}\tilde{\mathscr{O}}$ of any invariant subset S of $\pi_1(\tilde E)$ by Lemma 10.2.2. Thus, the only fixed point of $\pi_1(E)$ in $\mathrm{bd}\tilde{\mathscr{O}}$ is $\mathrm{v}_{\tilde E}$.

Suppose that E is a lens-shaped T-p-end. Since $\tilde E$ is a properly convex real projective orbifold that is closed, we obtain an attracting fixed point a and a repelling fixed point r of $g|\mathrm{Cl}(\tilde S_{\tilde E})$ by [17]. Then a and r are attracting and repelling fixed points of $g|\mathrm{Cl}(\tilde{\mathscr{O}})$ by the existence of the CA-lens neighborhood of $\tilde S_{\tilde E}$ since Theorem 5.5.4 implies the uniform middle eigenvalue condition.

Suppose that we have a fixed point $s \in \mathrm{bd}\tilde{\mathscr{O}}$ distinct from a and r. We claim that $\overline{as}$ and $\overline{rs}$ are in $\mathrm{bd}\tilde{\mathscr{O}}$. The norm of the eigenvalue associated with s is strictly between those of r and s by the uniform middle eigenvalue condition. Let P denote the two-dimensional subspace containing r, s, a. Suppose that one of the segment meets $\tilde{\mathscr{O}}$ at a point x. We take a convex open-ball-neighborhood B of x in $P \cap \tilde{\mathscr{O}}$. Suppose that $x \in \overline{rs}^o$. Then using the sequence $\{g^n(B)\}$, we obtain a great segment in $\mathrm{Cl}(\tilde{\mathscr{O}})$ by choosing $n \to \infty$. This is a contradiction. If $x \in \overline{as}^o$, we can use $\{g^{-n}(B)\}$ as $n \to \infty$, again giving us a contradiction. Hence, $\overline{as}, \overline{rs} \subset \mathrm{bd}\tilde{\mathscr{O}}$.

Since $\tilde E$ has a one-sided neighborhood U in a CA-lens neighborhood of $\tilde S_{\tilde E}$ by choosing a smaller such neighborhood U if necessary, we may assume that $\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathscr{O}}$ is in $\mathrm{Cl}(\tilde S_{\tilde E})$. By the strict convexity of $\tilde{\mathscr{O}}$, we see that the nontrivial segments $\overline{as}$ and $\overline{rs}$ have to be in $\mathrm{Cl}(\tilde S_{\tilde E})$. (See Definition 6.0.3.) □

See Crampon and Marquis [68] and Cooper-Long-Tillmann [67] for similar work to the following. We remind the reader that generalized lens-shaped R-ends are lens-shaped R-ends in the following assumption by Corollary 6.3.3,

PROPOSITION 10.2.4. *Suppose that $\mathscr{O}$ is a strongly tame strict SPC-orbifold with lens-shaped ends or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends satisfying* (*IE*) *and* (*NA*). *Then each nonidentity and infinite-order element g of $\pi_1(\mathscr{O})$ has two exclusive possibilities*:

- *$g|\mathrm{Cl}(\tilde{\mathscr{O}})$ has exactly two fixed points in $\mathrm{bd}\tilde{\mathscr{O}}$ none of which is in the closures of the p-end neighborhoods for distinct ends, and g is positive proximal.*
- *g is in a p-end fundamental group $\pi_1(\tilde{E})$, and $g|\mathrm{Cl}(\tilde{\mathscr{O}})$*
  - *has all fixed points in $\mathrm{bd}\tilde{O}$ in the closure of a concave p-end neighborhood of a lens-shaped radial p-end $\tilde{E}$.*
  - *has all fixed points in $\mathrm{bd}\tilde{O}$ in $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$ for the ideal boundary component $\tilde{S}_{\tilde{E}}$ of a lens-shaped totally geodesic p-end $\tilde{E}$, or*
  - *has a unique fixed point in $\mathrm{bd}\tilde{O}$ at the horospherical p-end vertex.*

PROOF. Suppose that g has a fixed point at a horospherical p-end vertex v for a p-end $\tilde{E}$. We can choose the horoball U at v that maps into an end-neighborhood of $\mathscr{O}$. A horoball p-end neighborhood is either sent to a disjoint one or sent to the identical one. Since $g(U)\cap U \neq \emptyset$ by the geometry of a horoball having a smooth boundary at v, g must act on the horoball, and hence g is in the p-end fundamental group. The p-end vertex is the unique fixed point of g in $\mathrm{bd}\tilde{\mathscr{O}}$ by Lemma 10.2.3.

Similarly, suppose that $g \in \pi_1(\mathscr{O})$ fixes a point of the closure U of a concave p-end neighborhood of a p-end vertex v of a lens-shaped end. $g(\mathrm{Cl}(U))$ and $\mathrm{Cl}(U)$ meet at a point. By Corollary 6.3.1, $g(\mathrm{Cl}(U))$ and $\mathrm{Cl}(U)$ share the p-end vertex and hence $g(U) = U$ as g is a deck transformation. Therefore, g is in the p-end fundamental group of the p-end of v. Lemma 10.2.3 implies the result.

Suppose that $g \in \pi_1(\mathscr{O})$ fixes a point of $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$ for a totally geodesic ideal boundary $\tilde{S}_{\tilde{E}}$ corresponding to a p-end $\tilde{E}$. Again Corollary 6.3.1 and Lemma 10.2.3 imply the result for this case.

Suppose that an element g of $\pi_1(\mathscr{O})$ is not in any p-end fundamental subgroup. Then by above, g does not fix any of the above types of points. We show that g has exactly two fixed points in $\mathrm{bd}\tilde{\mathscr{O}}$:

Suppose that $g \in \pi_1(\mathscr{O})$ fixes a unique point x in the closure of $\mathrm{bd}\tilde{\mathscr{O}}$ and x is not in the closure of p-end neighborhoods by the first part of the proof. Then x is a C^1-point by the strict convexity. (See Definition 6.2.3.) Suppose that we have two eigenvalues with the largest norm > 1 and the smallest norm < 1 respectively. If the largest norm eigenvalue is not positive real, $\mathrm{Cl}(\tilde{\mathscr{O}})$ contains a nonproperly convex subset as we can see by an action of g^n on a generic point of $\tilde{\mathscr{O}}$. Hence, the largest norm eigenvalue is positive and so is the smallest norm eigenvalue. We obtain attracting and repelling subspaces easily with these, and there are at least two fixed points. This is a contradiction. Therefore, g has only eigenvalues of unit norms. However, Lemma 1.3.10 shows that there is a sequence of simple closed curves c_i whose the sequence of Hilbert lengths is going to zero. Hence, g must be freely homotopic to an end neighborhood. This is absurd.

We conclude that $g \in \pi_1(\mathscr{O})$ not in a p-end fundamental groups fixes at least two points a and r in $\mathrm{bd}\tilde{\mathscr{O}}$. We choose the two fixed points to have the positive real eigenvalues that are largest and smallest absolute values of the eigenvalues of g. (As above, the largest and smallest norm eigenvalues must be positive for $\tilde{\mathscr{O}}$ to be properly convex.)

No fixed point of g in $\mathrm{bd}\tilde{\mathcal{O}}$ is in the closures of p-end neighborhoods by the first part of the proof. By strict convexity, the interior of $\tilde{\mathcal{O}}$ contains an open line segment l connecting a and r.

Suppose that there is a third fixed point t in $\mathrm{Cl}(\tilde{\mathcal{O}})$, which must be a boundary point. Let S denote the subspace spanned by a, r, t. The point t is not in the closures of p-end neighborhoods as we assumed that g is not in the p-end fundamental group. Then the line segment connecting t to the a or r must be in $\mathrm{bd}\tilde{\mathcal{O}}$: Assume without loss of generality $\overline{ta}^o \subset \tilde{\mathcal{O}}$ by taking g^{-1} and switching notation of a and r if necessary. Since g acts properly, the norms of eigenvalues of g at t or a are distinct. We can form a segment s in $\tilde{\mathcal{O}} \cap S$ transverse to the segment. Then $\{g^k(s)\}$ geometrically converges to a segment in $\mathrm{bd}\tilde{\mathcal{O}}$ containing t with endpoints r and r_- as $k \to -\infty$. Thus, the existence of t contradicts the proper convexity of $\mathrm{Cl}(\tilde{\mathcal{O}})$.

Hence, there are exactly two fixed points of g in $\mathrm{bd}\tilde{\mathcal{O}}$ of the positive real eigenvalues that are largest and smallest absolute values of the eigenvalues of g. $\square$

PROPOSITION 10.2.5. *Suppose that $\mathcal{O}$ is a noncompact strongly tame strict SPC-orbifold with lens-shaped ends or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends. Let $\tilde{E}$ be an end. Then for a p-end $\tilde{E}$, $(\mathrm{bd}\tilde{\mathcal{O}} - K)/\pi_1(\tilde{E})$ is a compact orbifold where $K = \bigcup S(\tilde{E})$ for a lens-shaped radial p-end $\tilde{E}$, $K = \mathrm{Cl}(\tilde{S}_{\tilde{E}})$ for totally geodesic p-end $\tilde{E}$, or $K = \{\mathrm{v}_{\tilde{E}}\}$ for horospherical p-end $\tilde{E}$.*

PROOF. Suppose that $\tilde{E}$ is a lens-shaped R-p-end or horospherical type. By Lemma 10.2.2, the homeomorphism $\Phi_{\tilde{E}} : \tilde{S}_{\tilde{E}} \to \mathrm{bd}\tilde{\mathcal{O}} - K$ gives us the result.

Suppose that $\tilde{E}$ is a lens-shaped T-p-end. Let $\tilde{\mathcal{O}}^*$ denote the dual domain. Then there exists a dual radial p-end $\tilde{E}^*$ corresponding to $\tilde{E}$. Hence, $(\mathrm{bd}\tilde{\mathcal{O}}^* - K')/\pi_1(\tilde{E}^*)$ is compact for K' equal to the closure of p-end neighborhoods of $\tilde{E}^*$ in the radial case or the vertex in the horospherical case.

Recall Section 1.5. Let $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}}$ be the augmented boundary with the fibration Π_{Ag}, and let $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}}^*$ be the augmented boundary with the fibration map Π^*_{Ag}. Let $K'' := \Pi^{-1}_{\mathrm{Ag}}(K)$ and $K''' := \Pi^{*-1}_{\mathrm{Ag}}(K')$. The discussion on in the proof of Corollary 5.5.1 shows that there is a duality homeomorphism

$$\mathcal{D}_{\tilde{\mathcal{O}}} : \mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}} - K'' \to \mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}}^* - K'''.$$

Now $(\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}}^* - K''')/\pi_1(\tilde{E}^*)$ is compact since $\mathrm{bd}\tilde{\mathcal{O}}^* - K'$ has a compact fundamental domain, and the space is the inverse image in $\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}}^*$ of $\mathrm{bd}\tilde{\mathcal{O}}^* - K'$. By (iv) of Proposition 1.5.4, $(\mathrm{bd}^{\mathrm{Ag}}\tilde{\mathcal{O}} - K'')/\pi_1(\tilde{E})$ is compact also. Since the image of this set under the map induced by a proper map Π_{Ag} is $(\mathrm{bd}\tilde{\mathcal{O}} - K)/\pi_1(\tilde{E})$. Hence, it is is compact. $\square$

10.3. Bowditch's method

10.3.1. The strict convexity implies the relative hyperbolicity. There are results proved by Cooper, Long, and Tillmann [67] and Crampon and Marquis [68] similar to below. However, the ends have to be horospherical in their work. By Lemma 6.3.3, for strict SPC-orbifold, generalized lens-shaped ends are lens-shaped. We will use Bowditch's result [30] to show

THEOREM 10.3.1. *Let $\mathcal{O}$ be a noncompact strongly tame strict SPC-orbifold with lens-shaped ends or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends $E_1, \dots, E_k$ and satisfies (IE) and (NA). Assume $\partial\mathcal{O}$ is smooth and strictly convex. Let $\tilde{\mathbb{U}}_i$ be the inverse image $\mathbb{U}_i$ in $\tilde{\mathcal{O}}$ for a mutually disjoint collection of neighborhoods $\mathbb{U}_i$ of the ends E_i for each $i = 1, \dots, k$. Then*

- *$\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to the end fundamental groups*
$$\pi_1(E_1), \dots, \pi_1(E_k).$$
Hence $\mathscr{O}$ is relatively hyperbolic with respect to $\mathbb{U} := \mathbb{U}_1 \cup \dots \cup \mathbb{U}_k$ as a metric space.
- *If $\pi_1(E_{l+1}), \dots, \pi_1(E_k)$ are hyperbolic for some $1 \leq l \leq k$ (possibly some of the hyperbolic ones), then $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to the end fundamental group $\pi_1(E_1), \dots, \pi_1(E_l)$.*

PROOF. We show that $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to the end fundamental groups $\pi_1(E_1), \dots, \pi_1(E_k)$.

- We now collapse each set of form $\mathrm{Cl}(\mathbb{U}_i) \cap \mathrm{bd}\tilde{\mathscr{O}} = \tilde{S}_{\tilde{E}}$ for a concave p-end neighborhood $\mathbb{U}_i$ to a point and
- collapse $\mathrm{Cl}(\tilde{S}_{\tilde{E}})$ for each lens-shaped totally geodesic end $\tilde{E}$ to a point.

By Corollary 6.3.1, these sets are mutually disjoint balls. Let $\mathscr{C}_B$ denote the collection, and let $C_B := \bigcup \mathscr{C}_B$.

We claim that for each closed set J in $\mathrm{bd}\tilde{\mathscr{O}}$, the union of C_J of elements of $\mathscr{C}_B$ meeting J is also closed: Let us choose a sequence $\{x_i\}$ for $x_i \in C_i$, $C_i \cap J \neq \emptyset$, $C_i \in \mathscr{C}_B$. Suppose that $\{x_i\} \to x$. Let $y_i \in C_i \cap J$. Let v_i be the p-end vertex of C_i if it is from a R-p-end. Then define $s_i := \overline{x_i \mathrm{v}_i} \cup \overline{\mathrm{v}_i, y_i} \subset C_i$ if C_i is radial or else $s_i := \overline{x_i y_i} \subset C_i$. Choose a subsequence so that $\{s_i\}$ geometrically converges to a limit containing x. The limit s_∞ is a singleton, a segment or a union of two segments. By the strict convexity of $\tilde{\mathscr{O}}$, we obtain that s_∞ is a subset of an element of $\mathscr{C}_B$ and s_∞ meets J. Thus, $x \in s_\infty \subset C_j$ for $C_j \cap J \neq \emptyset$. We conclude that C_J is closed.

We denote this quotient space $\mathrm{bd}\tilde{\mathscr{O}}/\sim$ by B. By Proposition 10.3.8, B is a metrizable space.

We show that $\pi_1(\mathscr{O})$ acts on the metrizable space B as a geometrically finite convergence group. By Theorem 0.1 of Yaman [157] following Bowditch [30], this shows that $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to $\pi_1(E_1), \dots, \pi_1(E_k)$. The definition of conical limit points and so on are from the article.

(I) We first show that the group acts properly discontinuously on the metric space of ordered mutually distinct triples in $B = \partial\tilde{\mathscr{O}}/\sim$. Suppose not. Then there exists a sequence of nondegenerate triples $\{(p_i, q_i, r_i)\}$ of points in $\mathrm{bd}\tilde{\mathscr{O}}$ converging to a mutually distinct triple $\{(p, q, r)\}$ so that
$$p_i = \gamma_i(p_0), q_i = \gamma_i(q_0), \text{ and } r_i = \gamma_i(r_0)$$
where $\{\gamma_i\}$ is a sequence of mutually distinct elements of $\pi_1(\mathscr{O})$ and the equivalence classes $[p_0], [q_0], [r_0]$ are mutually distinct and so are $[p], [q], [r]$. By multiplying by some uniformly bounded element R_i in $\mathsf{PGL}(n+1, \mathbb{R})$ but not necessarily in $h(\pi_1(\mathscr{O}))$, we obtain that $R_i \circ \gamma_i$ for each i fixes p_0, q_0, r_0 and restricts to a diagonal matrix with entries $\lambda_i, \delta_i, \mu_i$ on the plane with coordinates so that $p_0 = e_1, q_0 = e_2, r_0 = e_3$.

Then we can assume that
$$\lambda_i \delta_i \mu_i = 1, \lambda_i \geq \delta_i \geq \mu_i > 0$$
by restricting to the plane and up to choosing subsequences and renaming. Thus $\{\lambda_i\} \to \infty$ and $\{\mu_i\} \to 0$: otherwise, both of these two sequences are bounded. Let P_i denote the 2-dimension subspace spanned by p_i, q_i, r_i. Then $\gamma_i | P_i$ is a sequence of uniformly bounded automorphisms. Let $D_i = \mathrm{Cl}(\tilde{\mathscr{O}} \cap P_i)$. Then the sequence of maximal $\mathbf{d}$-distances from points of D_i to $\mathrm{bd}\tilde{\mathscr{O}}$ is uniformly bounded below by a positive number: If not, the geometric

limit of D_i is a nontrivial disk in $\mathrm{bd}\tilde{\mathscr{O}}$ containing p,q,r. Since p,q,r are in mutually distinct equivalence classes, this contradicts the strict convexity. Then for a compact subset K of $\tilde{\mathscr{O}}$, $K \cap D_i \subset \tilde{\mathscr{O}}$ is not empty for sufficiently large i. Choose $p_i \in K \cap D_i$. Then $\gamma_i(p_i)$ is in a $d_{\tilde{\mathscr{O}}}$-bounded neighborhood of K independent of i since otherwise γ_i is not uniformly bounded on $\tilde{\mathscr{O}} \cap P_i$ as indicated by the Hilbert metric. Hence, $\gamma_i(K)$ is in a $d_{\tilde{\mathscr{O}}}$-bounded neighborhood of K. This contradicts the proper discontinuity of the action by $\mathbf{\Gamma}$.

Let P_0 denote the 2-dimensional subspace containing p_0, q_0, and r_0. By strictness of convexity, as we collapsed each of the p-end balls, the interiors of the segments $\overline{p_0q_0}$, $\overline{q_0r_0}$, and $\overline{r_0p_0}$ are in the interior of $\tilde{\mathscr{O}}$.

We claim that one of the sequence $\{\lambda_i/\delta_i\}$ or the sequence $\{\delta_i/\mu_i\}$ are bounded: Suppose not. Then $\{\lambda_i/\delta_i\} \to \infty$ and $\{\delta_i/\mu_i\} \to \infty$. We choose generic segments s_0 and t_0 in $\tilde{\mathscr{O}}$ with a common endpoint q_0 and the respective other endpoint $\hat{s}_0$ and $\hat{t}_0$ in different components of $P \cap \mathscr{O} - \overline{p_0q_0}$ so that

$$\mathbf{d}(\hat{s}_0, q_0), \mathbf{d}(\hat{t}_0, q_0) \geq \delta \text{ for a uniform } \delta > 0.$$

We choose s_0 and t_0 so that their directions from q_0 differ from those of $\overline{p_0q_0}$ and $\overline{q_0r_0}$ at least by a small constant $\delta' > 0$. Then the sequence $\{R_i \circ \gamma_i(s_0 \cup t_0)\}$ geometrically converges to the segment with endpoint p_0 passing q_0. The segment is a great segment. Since R_i is bounded, this implies that there exists such a segment in $\mathrm{Cl}(\tilde{\mathscr{O}})$. This is a contradiction to the proper convexity of $\tilde{\mathscr{O}}$.

Suppose now that the sequence λ_i/δ_i is bounded: Now the sequence of segments $\{\overline{p_iq_i}\}$ converges to $\overline{pq}$ whose interior is in $\tilde{\mathscr{O}}$. Then we see that $\overline{pq}$ must be in the boundary of $\tilde{\mathscr{O}}$: Each point in $\overline{pq}$ must be the limit points of a sequence $\{y_i\}$ for $y_i \in \gamma_i(s)$ for some compact subsegment $s \subset \overline{p_1q_1}^o$ by the boundedness of the above ratio and the proper-discontinuity of the action. This contradicts the strict convexity as we assumed that p, q, and r represent distinct points in B. If we assume that δ_i/μ_i is bounded, then we obtain a contradiction similarly.

This proves the proper discontinuity of the action on the space of distinct triples.

(II) By Propositions 10.2.4 and 10.2.5, each group of form Γ_x for a point x of $B = \tilde{\mathscr{O}}/\sim$ is a bounded parabolic subgroup in the sense of Bowditch [157].

Now we take lens-cone end neighborhood for each radial end instead. We still choose ones mutually disjoint from themselves and nonradial ones in $\mathbb{U}$. We denote by $\mathbb{U}'$ the union of the modified end-neighborhoods. Let $\mathbb{U}'_1, \dots, \mathbb{U}'_k$ denote its components. Let $\tilde{\mathbb{U}}'_k$ denote the inverse image of $\mathbb{U}'_k$ in $\tilde{\mathscr{O}}$ for each k.

(III) Let $p \in \mathrm{bd}\tilde{\mathscr{O}}$ be a point that is not in a horospherical endpoint or an equivalence class corresponding to a lens-shaped p-end of radial or totally geodesic type of B. Hence, $[p] = \{p\}$ in $\mathrm{bd}\tilde{\mathscr{O}}/\sim$. That is, there is no segment containing p in one of the collapsed sets. We show that $[p]$ is a conical limit point. This will complete our proof by Theorem 0.1 of [157].

To show that $[p]$ is a conical limit point, we will find a sequence of holonomy transformations γ_i and distinct points $a, b \in \partial B$ so that $\{\gamma_i([p])\} \to a$ and $\{\gamma_i(q)\} \to b$ locally uniformly for $q \in \partial B - \{p\}$: To do this, we draw a line l in $\tilde{\mathscr{O}}$ from a point of the fundamental domain to p where as $t \to \infty$, $l(t) \to p$ in $\mathrm{Cl}(\tilde{\mathscr{O}})$. We may assume that the other endpoint p' of l is in distinct equivalence class from $[p]$. Since $l(t)$ is not eventually in a p-end neighborhood, there is a sequence $\{t_i\}$ going to ∞ so that $l(t_i)$ is not in any of the p-end neighborhoods in $\tilde{\mathbb{U}}'_1 \cup \cdots \cup \tilde{\mathbb{U}}'_k$. Let p' be the other endpoint of the complete extension of $l(t)$ in $\tilde{\mathscr{O}}$. We can assume without generality that p' is not in the closure of any p-end neighborhood by choosing the line $l(t)$ differently if necessary.

Since $(\tilde{\mathscr{O}} - \tilde{\mathbb{U}}'_1 - \cdots - \tilde{\mathbb{U}}'_k)/\Gamma$ is compact, we have a compact fundamental domain F of $\tilde{\mathscr{O}} - \tilde{\mathbb{U}}'_1 - \cdots - \tilde{\mathbb{U}}'_k$ with respect to Γ. Note that for the minimum distance, we have $\mathbf{d}(F, \mathrm{bd}\tilde{\mathscr{O}}) > C_0$ for some constant $C_0 > 0$.

We note: Given any line m passing F, the two endpoints must be in distinct equivalence classes because of the convexity of each component of $\tilde{\mathbb{U}}'$.

We find a sequence of points $z_i \in F$ so that $\gamma_i(l(t_i)) = z_i$ for a deck transformation γ_i. Then $\{\gamma_i\}$ is an unbounded sequence.

Using Definition 1.3.15, we may choose a set-convergent subsequence of $\{(\!(\gamma_i)\!)\}$ that is convergent in $\mathbb{S}(M_{n+1}(\mathbb{R}))$ to $(\!(\gamma_\infty)\!)$ for $\gamma_\infty \in M_{n+1}(\mathbb{R})$. Hence, $A_*(\{\gamma_i\}) = \mathbb{S}(\mathrm{Im}\,\gamma_\infty) \cap \mathrm{Cl}(\tilde{\mathscr{O}})$. Also, on $\mathrm{Cl}(\tilde{\mathscr{O}}) - N_*(\{\gamma_i\}) \cap \mathrm{Cl}(\tilde{\mathscr{O}})$, $\{\gamma_i\}$ is convergent to a subset of $A_*(\{\gamma_i\})$ locally uniformly as we can easily deduce by linear algebra and some estimation.

Since $N_*(\{\gamma_i\})$ is a convex subset of $\mathrm{bd}\tilde{\mathscr{O}}$ and $A_*(\{\gamma_i\})$ is a convex subset of $\mathrm{bd}\tilde{\mathscr{O}}$ by Theorem 1.3.21, they are in collapsed sets of $\mathrm{bd}\tilde{\mathscr{O}}$ by the strictness of the convexity.

If $p \notin N_*(\{\gamma_i\})$, then $\gamma_i(l(t_i))$ is also bounded away from $N_*(\{\gamma_i\})$, and hence $\gamma_i(l(t_i))$ accumulates only to $A_*(\{\gamma_i\})$. This is a contradiction. Thus, $p \in N_*(\{\gamma_i\})$. Since $N_*(\{\gamma_i\})$ is a convex compact subset of $\mathrm{bd}\tilde{\mathscr{O}}$, we must have

$$N_*(\{\gamma_i\}) \subset [p].$$

Thus, for all $q \in \mathrm{bd}\tilde{\mathscr{O}} - [p]$, we obtain a local uniform convergence under γ_i to $A_*(\{\gamma_i\})$. This shows that p is a conical limit point. We let b be the collapsed set containing $A_*(\{\gamma_i\})$.

Our line l equals the interior of $\overline{pp'}$. We choose a subsequence of γ_i so that the corresponding subsequence $\{\gamma_i(\overline{pp'})\}$ geometrically converges to a line passing F. Since p and p' are in district equivalence classes, $[\gamma_i(p')]$ converges to b, and $\gamma_i(\overline{pp'})$ passes F, it follows that $\gamma_i(p)$ converges to a point of the equivalence class a distinct from b by our note above.

Finally, we remove concave end-neighborhoods for $E_{l+1}, \dots, E_k$ or add lens end neighborhoods by Theorems 10.1.4 and 10.1.1. The resulting orbifold is a strict SPC-orbifold again and we can apply the result (i) to this case and obtain (ii). □

10.3.2. The theorem of Druţu. The author obtained a proof of the following theorem from Druţu. See [75] for more details.

THEOREM 10.3.2 (Druţu). *Let $\mathscr{O}$ be a strongly tame properly orbifold with generalized lens-shaped ends and horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Let $\pi_1(E_1), \dots, \pi_1(E_m)$ be end fundamental groups where $\pi_1(E_{l+1}), \dots, \pi_1(E_m)$ for $l \leq m$ are hyperbolic groups. Then $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to $\pi_1(E_1), \dots, \pi_1(E_m)$ if and only if $\pi_1(\mathscr{O})$ is one with respect to $\pi_1(E_1), \dots, \pi_1(E_l)$.*

PROOF. With the terminology in the paper [75], $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end fundamental groups $\pi_1(E_1), \dots, \pi_1(E_m)$ if and only if $\pi_1(\mathscr{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathscr{O})$ and $i = 1, \dots, m$.

We claimed that $\pi_1(\mathscr{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathscr{O})$ and $i = 1, \dots, m$ if and only if $\pi_1(\mathscr{O})$ with a word metric is asymptotically tree graded with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathscr{O})$ and $i = 1, \dots, l$.

Conditions (α_1) and (α_2) of Theorem 4.9 in [75] are satisfied still when we drop end fundamental groups $\pi_1(E_{n+1}), \dots, \pi_1(E_m)$ or add them. (See also Theorem 4.22 in [75].)

For the condition (α_3) of Theorem 4.9 of [75], it is sufficient to consider only hexagons. According to Proposition 4.24 of [75] one can take the fatness constants as large as one wants, in particular θ (measuring how fat the hexagon is) much larger than χ prescribing how close the fat hexagon is from a left coset.

If θ is very large, left cosets containing such hexagons in their neighborhoods can never be cosets of hyperbolic subgroups since hyperbolic groups do not contain fat hexagons. So the condition (α_3) is satisfied too whether one adds $\pi_1(E_{n+1}), \dots, \pi_1(E_m)$ or drop them. $\square$

10.3.3. Converse. We will prove the converse to Theorem 10.3.1. We will use the theory of tree-graded spaces and asymptotic cones [76] and its appendix.

- We shave off every generalized lens-shaped R-ends of $\mathscr{O}$ by Theorem 10.1.4 to obtain $\mathscr{O}^{(1)}$.
- We expand $\mathscr{O}$ to $\mathscr{O}^{(2)}$ by adding lens neighborhoods to totally geodesic ideal boundary components by Theorem 10.1.1.
- We then take out the interior of outside parts of the lens of $\mathscr{O}^{(2)}$ for every T-ends to obtain $\mathscr{O}^{(3)}$.
- Next we remove a collection of the mutually disjoint horospherical end neighborhoods. Let the resulting compact orbifold be denoted $\mathscr{O}^{(4)}$.

Now $\tilde{S}_{\tilde{E}}$ for every totally geodesic p-end $\tilde{E}$ is in $\tilde{\mathscr{O}}^{(i)}$ for $i = 2, 3, 4$.

PROPOSITION 10.3.3. *Let $\mathscr{O}$ be a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends. Let $\mathscr{O}^{(4)}$ have the restricted metric $d_{\tilde{\mathscr{O}}^{(2)}}$. Then $\tilde{\mathscr{O}}^{(4)}$ is quasi-isometric with $\pi_1(\mathscr{O})$.*

PROOF. Let $\pi_1(\mathscr{O})$ have the set of generators $g_1, \dots, g_q$. Since we removed all the end-neighborhoods of $\mathscr{O}^e$, our orbifold $\mathscr{O}^M$ is compact. Hence, $\tilde{\mathscr{O}}^{(4)}$ has a compact fundamental domain F. We find a function $\tilde{\mathscr{O}}^{(4)} \to \pi_1(\mathscr{O})$ by defining gF^o to go to g and defining arbitrarily the faces of F to go to gg_i. and hence there is a function from it to $\pi_1(\mathscr{O})$ decreasing distances up to a positive constant.

Conversely, there is a function from $\pi_1(\mathscr{O})$ to $\tilde{\mathscr{O}}^M$ by sending g to $g(x_0)$ for a fixed $x_0 \in F^o$. This is also distance decreasing up to a positive constant. Hence, this proves the result. $\square$

THEOREM 10.3.4. *Let $\mathscr{O}$ be a strongly tame properly convex real projective orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial\mathscr{O}$ is smooth and strictly convex. Suppose that $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \dots, \pi_1(E_k)$ where E_i are horospherical for $i = 1, \dots, m$ and generalized lens-shaped for $i = m+1, \dots, k$ for $0 \leq m \leq k$. Then $\mathscr{O}$ is strictly SPC with respect to the ends $E_1, \dots, E_k$ with lens-type R-ends and T-ends or horospherical ends.*

PROOF. Since an ε-mc-p-end-neighborhood is always proper by Corollary 6.2.12 for sufficiently small ε, we choose the end neighborhood of any generalized lens-shaped R-p-end $\tilde{E}_i$ to be the image of an ε-mc-p-end-neighborhood for some $\varepsilon > 0$. Assume that all shaved-off parts are inside the union of these. We can choose all such neighborhoods and horospherical end neighborhoods and lens-shaped end neighborhoods for T-ends to be mutually disjoint by Corollaries 6.2.9 and 6.2.12. Let $\tilde{\mathbb{U}}$ denote the union of the inverse images of the end neighborhoods.

Suppose that $\mathscr{O}$ is not strictly convex. We divide into two cases: First, we assume that there exists a segment in $\mathrm{bd}\tilde{\mathscr{O}}$ not contained in the closure of a p-end neighborhood. Second, we assume that there exists a non-C^1-point in $\mathrm{bd}\tilde{\mathscr{O}}$ not contained in the closure of a p-end neighborhood.

(I) We assume the first case now. We will obtain a triangle with boundary in $\mathrm{bd}\tilde{\mathscr{O}}^{(3)}$: Let l be a nontrivial maximal segment in $\mathrm{bd}\tilde{\mathscr{O}}^{(3)}$ not contained in the closure of a p-end neighborhood intersected with $\mathrm{bd}\tilde{\mathscr{O}}^{(3)}$. First, l does not meet the closure of a horospherical p-end neighborhood by Theorem 8.1.3. By Theorems 5.4.2 and 5.4.3 if l^o meets the closure of a lens-shaped R-p-end neighborhood, then l^o is in the closure. Also, suppose that l^o meets $\tilde{S}_{\tilde{E}}$ for a totally geodesic p-end $\tilde{E}$. Then $l^o \cap \partial \mathrm{Cl}(\tilde{S}_{\tilde{E}}) \neq \emptyset$. l is in the hyperspace P containing $\tilde{S}_{\tilde{E}}$ since otherwise we have some points of $\tilde{S}_{\tilde{E}}$ in the interior of $\mathrm{Cl}(\tilde{\mathscr{O}})$. We take a convex hull of $l \cup \tilde{S}_{\tilde{E}}$ which is a domain D containing $\tilde{S}_{\tilde{E}}$ where $\pi_1(\tilde{E})$ acts on. Then D is still properly convex since so is $\mathrm{Cl}(\tilde{\mathscr{O}})$. Since D^o has a Hilbert metric, $\pi_1(\tilde{E})$ acts properly on D^o. By taking a torsion-free subgroup by Theorem 1.1.19, we obtain that $\tilde{S}_{\tilde{E}}/\pi_1(\tilde{E}) \to D^o/\pi_1(\tilde{E})$ has to be surjective. Hence, $D^o = \tilde{S}_{\tilde{E}}$. Therefore, $l^o \subset \mathrm{Cl}(\tilde{S}_{\tilde{E}})$, a contradiction. (See Theorem 4.1 of [**61**] and [**20**].) Therefore, l meets the closures of p-end neighborhoods possibly only at its endpoints.

Let P be a 2-dimensional subspace containing l and meeting the interior of $\tilde{\mathscr{O}}^{(3)}$ outside $\tilde{\mathbb{U}}$. By above, l^o is in the boundary of $P \cap \tilde{\mathscr{O}}^{(3)}$. Draw two segments s_1 and s_2 in $P \cap \tilde{\mathscr{O}}^{(3)}$ from the endpoint of l meeting at a vertex p in the interior of $\tilde{\mathscr{O}}^{(3)}$.

Let I denote the index set of components of $\tilde{\mathbb{U}}$. Let $\mathbb{U}_i$ be a component of $\tilde{\mathbb{U}}$.

Define A_i to be the set of points x of l^o with an open $\mathbf{d}$-metric ball-neighborhood in $\mathrm{Cl}(\mathscr{O}) \cap P$ in the closure of a single component $\mathbb{U}_i$. By definition, A_i is open in l^o. Also, l^o is not a subset of single A_i since otherwise l is in the closure of U_i, a contradiction. Since l^o is connected, $l^o - \bigcup_{i \in I} A_i$ is not empty. Choose a point x in it. For any open $\mathbf{d}$-metric ball-neighborhood B of x in $\mathrm{Cl}(\tilde{\mathscr{O}}) \cap P$, we cannot have $B \cap \tilde{\mathscr{O}} \subset \tilde{\mathbb{U}}$ since otherwise B is in a single $\mathrm{Cl}(\mathbb{U}_i)$. For each open ball-neighborhood B of x, $B - \tilde{\mathbb{U}}$ is not empty. We conclude that $(\tilde{\mathscr{O}} - \tilde{\mathbb{U}}) \cap P$ has a sequence of points $\{x_i\}$ converging to a point x of l^o.

Then we claim

$$d_{\mathscr{O}^{(2)}}(x_i, s_1 \cup s_2) \to \infty :$$

Consider any sequence of any maximal straight segment t_i from x_i passing a point y_i of s_1 or s_2. Let us orient it in the direction of y_i from x_i. Then let $\delta_+ t_i$ be the forward endpoint of t_i and $\delta_- t_i$ the backward one. Then the $\mathbf{d}$-distance from y_i to $\delta_+ t_i$ goes to zero by the maximality of l, which implies the Hilbert metric result by the cross-ratio consideration.

Recall that there is a compact fundamental domain F of $\tilde{\mathscr{O}} - \tilde{U}$ under the action of $\pi_1(E)$. Now, we can take x_i to the fundamental domain F by g_i. We choose g_i to be a sequence of mutually distinct elements of $\pi_1(\mathscr{O})$. We choose a subsequence so that we assume without loss of generality that $\{g_i(T)\}$ geometrically converges to a convex set, which could be a point or a segment or a nondegenerate triangle. Since $g_i(T) \cap F \neq \emptyset$, and the sequence $\partial g_i(T)$ exits any compact subsets of $\tilde{\mathscr{O}}$ always while

$$\{d_{\mathscr{O}^{(2)}}(g_i(x_i), \partial g_i(T))\} \to \infty$$

and $g_i(T)$ passes F, we see that a subsequence of $\{g_i(T)\}$ converges to a nondegenerate triangle, say T_∞.

By following Lemma 10.3.5, T_∞ is so that ∂T_∞ is in $\bigcup S(\mathrm{v}_{\tilde{E}})$ for a generalized lens-shaped R-p-end $\tilde{E}$.

Now, T_∞ is so that $\partial T_\infty \subset \mathrm{Cl}(U_1)$ for a p-end neighborhood U_1 of a generalized lens-shaped end $\tilde{E}$. Then for sufficiently small $\varepsilon > 0$, the ε-$d_{\mathscr{O}}$-neighborhood of $T_\infty \cap \tilde{\mathscr{O}}$ is a subset of U_1 as U_1 was chosen to be an ε-mc-p-end-neighborhood (see Lemma 6.2.11). However as $\{g_i(T)\} \to T_\infty$ geometrically, for any compact subset K of $\tilde{\mathscr{O}}$, $g_i(T) \cap K$ is a subset of U_1 for sufficiently large i. But $g_i(T) \cap F \neq \emptyset$ for all i and the compact fundamental domain F of $\tilde{\mathscr{O}} - \tilde{U}$, disjoint from U_1. This is a contradiction.

Also, since the triangle condition is satisfied, the generalized lens R-end must also be a lens R-end by Lemma 5.3.20.

(II) Now we suppose that $\mathrm{bd}\tilde{\mathscr{O}}$ has a non-C^1-point x outside the closures of p-end neighborhoods. Then we go to the dual $\tilde{\mathscr{O}}^*$ and the dual group Γ^* where $\tilde{\mathscr{O}}^*/\Gamma^*$ is a strongly tame properly convex orbifold with horospherical ends, lens-shaped T-ends or generalized lens-shaped R-ends by Corollary 5.5.7 and Theorem 1.5.8. Here the type $\mathscr{T}$ and $\mathscr{R}$ are switched for the correspondence between the ends of $\mathscr{O}$ and $\mathscr{O}^*$ by Corollary 5.5.7.

Then we have a one-to-one correspondence of the set of p-ends of $\tilde{\mathscr{O}}$ to the set of p-ends of $\tilde{\mathscr{O}}^*$, and we obtain that x corresponds to a convex subset of dim ≥ 1 in $\mathrm{bd}\tilde{\mathscr{O}}$ containing a segment l not contained in the closure of p-end neighborhoods using the map $\mathscr{D}$ in Proposition 5.5.5. Thus, the proof reduces to the case (I).

By Theorem 6.0.4, we obtain that our orbifold is strictly SPC. $\square$

Recall that the interior of a triangle has a Hilbert metric called the hex metric by de la Harpe [72]. The metric space is isometric with a Euclidean space with norms given by regular hexagons. The unit norm of the metric is a regular hexagon ball for this metric. A *regular hexagon of side length l* is a hexagon in the interior of a triangle T with geodesic edges parallel to the sides of the unit norms and with all edge lengths equal to l. The regular hexagon is the boundary of a ball of radius l. The *center* of a hexagon is the center of the ball.

LEMMA 10.3.5. *Assume the premise of Theorem 10.3.4. Let T be a triangle in $\tilde{\mathscr{O}}^{(3)}$ with $T^o \cap \tilde{\mathscr{O}}^{(3)} \neq \emptyset$ and $\partial T \subset \mathrm{bd}\tilde{\mathscr{O}}^{(3)}$. Then $\partial T \subset \bigcup S(\tilde{E})$ for an R-p-end $\tilde{E}$.*

PROOF. Let F be the fundamental domain of $\tilde{\mathscr{O}}^{(3)}$.

Again, we assume that $\pi_1(\mathscr{O})$ is torsion-free by Theorem 1.1.19 since it is sufficient to prove the result for the finite cover of $\mathscr{O}$. Hence, $\pi_1(\mathscr{O})$ acts freely on $\tilde{\mathscr{O}}$.

Let T' be a triangle with $T'^o \cap \tilde{\mathscr{O}}^{(3)} \neq \emptyset$ and $\partial T' \subset \mathrm{bd}\tilde{\mathscr{O}}^{(3)}$. Suppose that T' meets infinitely many horoball p-end neighborhoods in $\tilde{\mathbb{U}}$ of horospherical p-ends, and the $d_{\mathscr{O}^{(2)}}$-diameters of T' intersected with these are not bounded. We consider a sequence of such sets A_i with $d_{\mathscr{O}^M}$-diameter A_i going to $+\infty$, and we choose a deck transformation g_i so that $g_i(\mathrm{Cl}(A_i))$ intersects the fundamental domain F of $\tilde{\mathscr{O}}^M$. We choose a subsequence so that $\{g_i(T')\}$ and $\{g_i(A_i)\}$ geometrically converge to a triangle T'' and a compact set A_∞ respectively. Here, T'' intersects F and the interior of T'' is in $\tilde{\mathscr{O}}$. $g_i(A_i) = g_i(T'') \cap H_i$ for a horoball H_i whose closure meets F. Since only finitely many closures of the horoball p-end neighborhoods in $\tilde{\mathscr{O}}$ meet F, there are only finitely many such H_i, say $H_{i_1}, \dots, H_{i_m}$. Now, T'' meets one such H_{i_j} so that its vertex is in the boundary of T'' since the $d_{\mathscr{O}^M}$-diameter of $g_i(T'') \cap H_i = g_i(A_i)$ goes to $+\infty$. This contradicts Theorem 8.1.3.

Thus, the $d_{\mathscr{O}^{(2)}}$-diameters of horospherical p-end neighborhoods intersected with L are bounded above uniformly. Therefore, by choosing a horospherical end neighborhood sufficiently far inside each horospherical end neighborhood by Corollary 6.2.9, we may assume that L does not meet any horospherical p-end neighborhoods. That is we choose a horoball V' inside a one V so that

$$d_{\mathscr{O}^M}(V', \partial V) > \frac{1}{2} \sup\{d_{\mathscr{O}^{(2)}}\text{-diam}\{V \cap T'\} | V \in \mathscr{V}, T' \in \mathscr{T}\}$$

where $\mathscr{V}$ is the collection of horoball p-end neighborhoods that we were given in the beginning and $\mathscr{T}$ is the collection of all triangles T' meeting with $\tilde{\mathscr{O}}^{(3)}$ and with boundary in $\mathrm{bd}\tilde{\mathscr{O}}^{(3)}$ –(*).

For i in the index set I of p-ends, we define $L_{1,i}$ to be the following subsets of $\tilde{\mathscr{O}}^{(3)}$:

- $\mathrm{Cl}(U(\mathrm{v}_{\tilde{E}})) \cap \tilde{\mathscr{O}}^{(3)}$ where $U(\mathrm{v}_{\tilde{E}})$ is the open shaved-off concave p-end neighborhood of $\tilde{E}$ when $\tilde{E}$ is a generalized lens-shaped R-p-end,
- $\mathrm{Cl}(\tilde{S}_{\tilde{E}}) \cap \tilde{\mathscr{O}}^{(3)}$ if $\tilde{E}$ is a lens-shaped totally geodesic end, or
- $\mathrm{Cl}(U_{\tilde{E}}) \cap \tilde{\mathscr{O}}^{(3)}$ for a horoball $U_{\tilde{E}}$ for a horospherical end $\tilde{E}$.

By Theorem 1.5 of [75], $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to

$$\pi_1(E_1), \dots, \pi_1(E_k)$$

if and only if every asymptotic cone $\pi_1(\mathscr{O})$ is asymptotically tree graded with respect to the collection of left cosets of

$$\mathscr{L} = \{g\pi_1(E_i) | g \in \pi_1(\mathscr{O})/\pi_1(E_i), i = 1, \dots, k\}.$$

By Theorem 5.1 of [76], $\tilde{\mathscr{O}}^{(4)}$ with the metric $d_{\tilde{\mathscr{O}}^{(2)}}$ is asymptotically tree-graded with respect to $L_{1,i}, i = 1, 2, \dots,$ since $\pi_1(\mathscr{O})$ is quasi-isometric with $\tilde{\mathscr{O}}^{(4)}$ with the cosets of $\pi_1(E_i)$ mapping quasi-isometric into $L_{1,j}$ by Proposition 10.3.3.

Now, we consider $\tilde{\mathscr{O}}^{(3)}$ and $\mathrm{bd}\tilde{\mathscr{O}}^{(3)}$.

For any $\theta > 0, \nu \geq 8$, a regular hexagon in T'^o with side length $l > \nu\theta$ is (θ, ν)-fat according to Definition 5.1 of Druţu [75]. By Theorem 4.22 of [75], there is $\chi > 0$ so that a regular hexagon H_l with side length $l > \nu\theta$ is in χ-neighborhood V_1 of $L_{1,i}$ with respect to $d_{\mathscr{O}^{(2)}}$ that is either contained in a concave p-end neighborhood of an R-p-end, a CA-lens of a T-p-end or a horoball p-end neighborhood.

Choose a family of regular hexagons

$$\{H_l | H_l \subset T'^o, l > \nu\theta\}$$

with a common center in T'^o. Hence, $\bigcup_{l > \nu\theta} H_l = T'^o - K_1$ for a bounded set $K_1, K_1 \subset T'^o$. By the above paragraph, $T'^o - K_1 \subset V_1$. Now, $\partial T' \subset \mathrm{bd}\tilde{\mathscr{O}}^{(3)}$ must be in the closure of $L_{1,i}$ by Lemma 10.3.6.

If $L_{i,1}$ is from a horospherical p-end, $\partial T'$ must be a point. This is absurd. In the case of a T-p-end, the hyperspace containing T' must coincide with one containing $\tilde{S}_{\tilde{E}}$. This is absurd since T'^o is a subset of $\tilde{\mathscr{O}}^o$. In the case of an R-p-end $\tilde{E}$, $\partial T'^o$ must lie on a subset that has segments extending those segments in $S(\mathrm{v}_{\tilde{E}})$. Theorems 5.4.2 and 5.4.3, this means $\partial T \subset \bigcup S(\mathrm{v}_{\tilde{E}})$. □

LEMMA 10.3.6. *Let V be a χ-neighborhood of $L_{1,i}$ in $\tilde{\mathscr{O}}^{(3)}$ under the metric $d_{\mathscr{O}^{(3)}}$. Then* $\mathrm{Cl}(V) \cap \mathrm{bd}\tilde{\mathscr{O}}^{(3)} = \mathrm{Cl}(L_{1,i}) \cap \mathrm{bd}\tilde{\mathscr{O}}^{(3)}$.

PROOF. We recall the metric. We first extend $\tilde{\mathscr{O}}$ and shave off to $\tilde{\mathscr{O}}^e$. Then we remove the parts of the lenses outside the ideal end orbifold for T-p-ends and remove horoballs of ends to obtain $\tilde{\mathscr{O}}^{(3)}$.

Clearly, $\mathrm{Cl}(V) \cap \mathrm{bd}\tilde{\mathscr{O}}^{(3)} \supset \mathrm{Cl}(L_{1,i}) \cap \mathrm{bd}\tilde{\mathscr{O}}^{(3)}$. Suppose that $L_{1,i}$ is from a horospherical p-end. Then the equality is clear since V is contained in a horospherical p-end neighborhood.

Suppose that $L_{1,i}$ is from a T-p-end of lens type. Then there is a CA-lens L containing $L_{1,i}$. The closure of V in $\tilde{\mathscr{O}}^{(3)}$ has a compact fundamental domain F_V. Theorem 5.5.4 and Lemma 4.4.2 applied to any sequence of images of F_V imply the equality.

Suppose that $L_{1,i}$ is from an R-p-end of lens type. Then the closure of $L_{1,i}$ in $\tilde{\mathscr{O}}^{(3)}$ has a compact fundamental domain F_V. Theorem 5.3.21 and Lemma 5.3.9 and Proposition 5.3.10 again show the equality since the limit sets are independent of the choice of neighborhoods. □

We recapitulate the results:

COROLLARY 10.3.7. *Assume that $\mathscr{O}$ is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*). *Let*

$$E_1, \dots, E_m, E_{m+1}, \dots, E_k$$

be the ends of $\mathscr{O}$ where $E_{m+1}, \dots, E_k$ are some or all of the hyperbolic ends. Assume $\partial \mathscr{O} = \emptyset$. Then $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \dots, \pi_1(E_m)$ if and only if $\mathscr{O}_1$ as obtained by Theorem 10.1.4 is strictly SPC with respect to ends $E_1, \dots, E_m$.

PROOF. If $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \dots, \pi_1(E_m)$, then $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \dots, \pi_1(E_k)$ by Theorem 10.3.2. By Theorem 10.3.4, it follows that $\mathscr{O}$ is strictly SPC with respect to the ends $E_1, \dots, E_k$. Theorem 10.1.4 shows that $\mathscr{O}_1$ is strictly SPC with respect to $E_1, \dots, E_m$.

For converse, if $\mathscr{O}_1$ is strictly SPC with respect to $E_1, \dots, E_m$, then $\mathscr{O}$ is strictly SPC with respect $E_1, \dots, E_k$. By Theorem 10.3.1, $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), \dots, \pi_1(E_k)$. The conclusion follows by Theorem 10.3.2. □

10.3.4. A topological result.

PROPOSITION 10.3.8. *Let X be a compact metrizable space. Let $\mathscr{C}_X$ be a countable collection of mutually disjoint compact connected sets. The collection has the property that $C_K := \bigcup_{C \in \mathscr{C}_X, C \cap K \neq \emptyset} C$ is closed for any closed set K. We define the quotient space $X/\sim$ with the equivalence relation $x \sim y$ iff $x, y \in C$ for an element $C \in \mathscr{C}_X$. Then $X/\sim$ is metrizable.*

PROOF. We show that $X/\sim$ is Hausdorff, 2-nd countable, and regular and use the Urysohn metrization theorem. We define a countable collection $\mathscr{B}$ of open sets of X as follows: We take an open subset L of X that is an ε-neighborhood for $\varepsilon \in \mathbb{Q}, \varepsilon > 0$ of an element of $\mathscr{C}_X$ or a point of a dense countable set Y in $X - \bigcup \mathscr{C}_X$. We form

$$L - \bigcup_{C \cap \mathrm{bd} L \neq \emptyset, C \in \mathscr{C}_X} C$$

for all such L containing an element of $\mathscr{C}_X$ or a point of Y. This is an open set by the premise since $\mathrm{bd}L$ is closed. The elements of $\mathscr{B}$ are neighborhoods of elements of $\mathscr{C}_X$ and Y. Also, each element of $\mathscr{C}_X$ or a point of Y is contained in an element of $\mathscr{B}$. Furthermore, each element of $\mathscr{B}$ is a saturated open set under the quotient map. Hence, $X/\sim$ is Hausdorff and 2-nd countable.

Now, the proof is reduced to showing that $X/\sim$ is regular. For any saturated compact set K in X and a disjoint element Y of $\mathscr{C}_X$ or a point of X not in any of $\mathscr{C}_X$, let U_K and U_Y denote the disjoint neighborhoods of X of K and Y respectively. We form

$$U := U_K - \bigcup_{C \cap \mathrm{bd} U_K \neq \emptyset, C \in \mathscr{C}_X} C, \text{ and } V := V_K - \bigcup_{C \cap \mathrm{bd} V_K \neq \emptyset, C \in \mathscr{C}_X} C.$$

Then these are disjoint open neighborhoods. □

CHAPTER 11

Openness and closedness

Lastly, we will prove the openness and closedness of the properly (resp. strictly) convex real projective structures on the deformation spaces of a class of orbifolds with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends. We need the theory of Crampon and Marquis and Cooper, Long, and Tillmann on the Margulis lemma for convex real projective manifolds. The theory here partly generalizes that of Benoist on closed real projective orbifolds. In Section 11.1, we give some definitions and state the main results of the monograph: various Ehresmann-Thurston-Weil principles holding in certain circumstances. In Section 11.2, we state the openness results that we will prove in this chapter. Mainly, we will use fixing-sections to prove the results here. We will show that the small deformations preserve the convexity. The idea is to use the Hessian functions in the compact part and use the approximation of the original domain by the covering domains of the end neighborhoods. In Section 11.3, we will show the closedness of the convexity under the deformations, first assuming the irreducibility of the holonomy representations. In Section 11.3.3, we show that we actually do not need to assume the irreducibility a priori. Any sequence of properly convex real projective structures will converge to the one whenever the corresponding sequence of representations converges algebraically. In Section 11.4, we prove Theorem 11.1.4, the most general result of this monograph. Here, we show the natural existence of the fixing section.

11.0.0.1. *Main theorems.* We now state our main results:

- We define $\mathrm{Def}^{s}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ to be the subspace of $\mathrm{Def}_{\mathcal{E}}(\mathcal{O})$ consisting of real projective structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and stable irreducible holonomy homomorphisms.
- We define $\mathrm{CDef}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ to be the subspace of $\mathrm{Def}_{\mathcal{E}}(\mathcal{O})$ consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends.
- We define $\mathrm{CDef}_{\mathcal{E},\mathrm{u},\mathrm{lh}}(\mathcal{O})$ to be the subspace of $\mathrm{Def}_{\mathcal{E},\mathrm{u}}(\mathcal{O})$ consisting of SPC-structures with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends.
- We define $\mathrm{SDef}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ to be the subspace of $\mathrm{Def}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends.
- We define $\mathrm{SDef}_{\mathcal{E},\mathrm{u},\mathrm{lh}}(\mathcal{O})$ to be the subspace of $\mathrm{Def}_{\mathcal{E},\mathrm{u},\mathrm{lh}}(\mathcal{O})$ consisting of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends.

By defintion, these spaces all have stable holonomies only. (By Theorem 6.0.4, some of these has to have stable holonomies by a topological conditions.) We remark that these spaces are dual to the same type of the spaces but we switch the $\mathcal{R}$-end with $\mathcal{T}$-ends and vice versa by Proposition 5.5.5. Also by Corollary 6.3.3, for strict SPC-orbifolds with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends have only lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends.

The following theorems are to be regarded as examples of the so-called *Ehresmann-Thurston-Weil principle*.

THEOREM 11.0.1. *Let $\mathscr{O}$ be a noncompact strongly tame n-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. Assume $\partial\mathscr{O} = \emptyset$. Suppose that $\mathscr{O}$ satisfies (IE) and (NA). Then the subspace*

$$\mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) \subset \mathrm{Def}^{s}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$$

is open.

Suppose further that every finite-index subgroup of $\pi_1(\mathscr{O})$ contains no nontrivial infinite nilpotent normal subgroup. Then hol *maps* $\mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ *homeomorphically to a union of components of*

$$\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

THEOREM 11.0.2. *Let $\mathscr{O}$ be a strict SPC noncompact strongly tame n-dimensional orbifold, $n \geq 2$, with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Assume $\partial\mathscr{O} = \emptyset$. Then*

- *$\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to its end fundamental groups.*
- *The subspace $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) \subset \mathrm{Def}^{s}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ of strict SPC-structures with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends is open.*

Suppose further that every finite-index subgroup of $\pi_1(\mathscr{O})$ contains no nontrivial infinite nilpotent normal subgroup. Then hol *maps the deformation space $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ of strict SPC-structures on $\mathscr{O}$ with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends homeomorphically to a union of components of*

$$\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Theorems 11.0.1 and 11.0.2 are proved by dividing into the openness result in Section 11.0.1 and the closedness result in Section 11.0.2.

11.0.1. Openness. For openness of $\mathrm{SDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$, we will make use of:

COROLLARY 11.0.3 (Corollary 10.3.7). *Assume that $\mathscr{O}$ is a noncompact strongly tame SPC n-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Let*

$$E_1, \dots, E_k$$

be the ends of $\mathscr{O}$. Assume $\partial\mathscr{O} = \emptyset$. Then $\pi_1(\mathscr{O})$ is a relatively hyperbolic group with respect to the end groups $\pi_1(E_1), ..., \pi_1(E_k)$ if and only if $\mathscr{O}$ is strictly SPC with respect to ends $E_1, \dots, E_k$.

THEOREM 11.0.4. *Let $\mathscr{O}$ be a noncompact strongly tame real projective n-orbifold, $n \geq 2$, and satisfies (IE) and (NA). Assume $\partial\mathscr{O} = \emptyset$. In $\mathrm{Def}^{s}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$, the subspace $\mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ of SPC-structures with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends is open, and so is $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$.*

PROOF. $\mathrm{Hom}^{s}_{\mathscr{E},u}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})$ is an open subset of $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ by Proposition 9.2.3. On $\mathrm{Hom}^{s}_{\mathscr{E},u}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ has a uniqueness section defined by Lemma 11.0.5. Now, Theorem 11.0.6 proves the result. □

We are given a properly real projective orbifold $\mathscr{O}$ with ends $E_1, \dots, E_{e_1}$ of $\mathscr{R}$-type and $E_{e_1+1}, \dots, E_{e_1+e_2}$ of $\mathscr{T}$-type. Let us choose representative p-ends $\tilde{E}_1, \dots, \tilde{E}_{e_1}$ and $\tilde{E}_{e_1+1}, \dots, \tilde{E}_{e_1+e_2}$. Again, e_1 is the number of $\mathscr{R}$-type ends, and e_2 the number of $\mathscr{T}$-type ends of $\mathscr{O}$.

We define a subspace of $\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ to be as in Section 9.2.

Let $\mathscr{V}$ be an open subset of a semi-algebraic subset of

$$\mathrm{Hom}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

invariant under the conjugation action of $\mathsf{PGL}(n+1,\mathbb{R})$ so that the following hold:

- one can choose a continuous section $s^{(1)}_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^n)^{e_1}$ sending a holonomy homomorphism to a common fixed point of $\mathbf{\Gamma}_{\tilde{E}_i}$ for $i = 1, \dots, e_1$ and
- $s^{(1)}_{\mathscr{V}}$ satisfies

$$s^{(1)}_{\mathscr{V}}(gh(\cdot)g^{-1}) = g \cdot s^{(1)}_{\mathscr{V}}(h(\cdot)) \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R}).$$

$s^{(1)}_{\mathscr{V}}$ is said to be a *fixed-point section*.

If $\tilde{E}_i$ for every $i = 1, \dots, e_1$ has a p-end neighborhood with a radial foliation with leaves developing into rays ending at the point of the i-th factor of $s^{(1)}_{\mathscr{V}}$, we say that radial end structures are *determined by* $s^{(1)}_{\mathscr{V}}$.

Again we assume that $\mathscr{V}$ is a open subset of a semi-algebraic subset of

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

invariant under the conjugation action by $\mathsf{PGL}(n+1,\mathbb{R})$, and the following hold:

- one can choose a continuous section $s^{(2)}_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^{n*})^{e_2}$ sending a holonomy homomorphism to a common dual fixed point of $\mathbf{\Gamma}_{\tilde{E}_i}$ for $i = e_1+1, \dots, e_1+e_2$,
- $s^{(2)}_{\mathscr{V}}$ satisfies $s^{(2)}_{\mathscr{V}}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s^{(2)}_{\mathscr{V}}(h(\cdot))$ for $g \in \mathsf{PGL}(n+1,\mathbb{R})$, and
- letting $P_{\mathscr{V}}(\tilde{E}_i)$ denote the null space of the i-th value of $s^{(2)}_{\mathscr{V}}$ for $i = e_1+1, \dots, e_1+e_2$, $\mathbf{\Gamma}_{\tilde{E}_i}$ acts on the hyperspace $P_{\mathscr{V}}(\tilde{E}_i)$ satisfying the lens-condition for $\tilde{E}_i$.

$s^{(2)}_{\mathscr{V}}$ is said to be a *dual fixed-point* section.

If each $\tilde{E}_i$ for every $i = e_1+1, \dots, e_1+e_2$

- has a p-end neighborhood with the ideal boundary component in the hyperspace determined by the i-th factor of $s^{(2)}_{\mathscr{V}}$ provided $\tilde{E}_i$ is a T-end, or
- has a p-end neighborhood containing a $\mathbf{\Gamma}_{\tilde{E}}$-invariant horosphere tangent to the hyperspace determined by the i-th factor of $s^{(2)}_{\mathscr{V}}$ provided $\tilde{E}_i$ is a horospherical end,

we say that end structures for the totally geodesic end are *determined by* $s^{(2)}_{\mathscr{V}}$.

We define $s_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$ as $s^{(1)}_{\mathscr{V}} \times s^{(2)}_{\mathscr{V}}$ and call it a *fixing section*.

LEMMA 11.0.5. *We can define section*

$$s_u : \mathrm{Hom}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$$

by choosing for each holonomy and each p-end the unique fixed point and the unique hyperspace as the images.

PROOF. s_u is a continuous function since a sequence of fixed points or dual fixed points of end holonomy group is a fixed point or a dual fixed point of the limit end holonomy group. □

We call s_u the *uniqueness section*.

Let $\mathscr{V}$ and $s_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$ be as above.

- We define $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$ to be the subspace of $\mathrm{Def}_{\mathscr{E},s_{\mathscr{V}}}(\mathscr{O})$ of real projective structures with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-end structures determined by $s_{\mathscr{V}}$, and stable irreducible holonomy homomorphisms in $\mathscr{V}$.
- We define $\mathrm{CDef}_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$ to be the subspace consisting of SPC-structures with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-end structures determined by $s_{\mathscr{V}}$ and holonomy homomorphisms in $\mathscr{V}$ in $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$.
- We define $\mathrm{SDef}_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$ to be the subspace of consisting of strict SPC-structures with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-end structures determined by $s_{\mathscr{V}}$ and holonomy homomorphisms in $\mathscr{V}$ in $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$.

THEOREM 11.0.6. *Let $\mathscr{O}$ be a noncompact strongly tame real projective n-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial \mathscr{O} = \emptyset$. Choose an open $\mathsf{PGL}(n+1,\mathbb{R})$-conjugation invariant subset of a union of semialgebraic subsets of*

$$\mathscr{V} \subset \mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})),$$

and a fixing section $s_{\mathscr{V}} : \mathscr{V} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$.

Then $\mathrm{CDef}_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$ is open in $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$, and so is $\mathrm{SDef}_{\mathscr{E},s_{\mathscr{V}},\mathrm{lh}}(\mathscr{O})$.

This is proved in Theorem 11.2.1.

By Theorems 11.0.4 and 11.0.6, we obtain:

COROLLARY 11.0.7. *Let $\mathscr{O}$ be a noncompact strongly tame real projective n-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial \mathscr{O} = \emptyset$. Then*

$$\mathrm{hol} : \mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) \to \mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

is a local homeomorphism.

Furthermore, if $\mathscr{O}$ has a strict SPC-structure with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends, then so is

$$\mathrm{hol} : \mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) \to \mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

11.0.2. The closedness of convex real projective structures. The results here will be proved in Chapter 11 in Part 3.

We recall

$$\mathrm{rep}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

the subspace of stable irreducible characters of

$$\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

which is shown to be the open subset of a semi-algebraic subset in Section 9.2, and denote by $\mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ the subspace of stable irreducible characters of $\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$, an a union of open subsets of semialgebraic sets.

THEOREM 11.0.8. *Let $\mathscr{O}$ be a noncompact strongly tame SPC n-orbifold, $n \geq 2$, with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial \mathscr{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathscr{O})$ are trivial. Then the following hold:*

- *The deformation space $\mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ of SPC-structures on $\mathscr{O}$ with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends maps under hol homeomorphically to a union of components of $\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$.*

- *The deformation space* $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ *of strict SPC-structures on* $\mathscr{O}$ *with lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends maps under* hol *homeomorphically to the union of components of* $\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$.

PROOF. $\mathrm{Hom}^s_{\mathscr{E},u}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})$ is an open subset of $\mathrm{Hom}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})$ by Proposition 9.2.3. Corollary 11.3.5 proves this by the existence of the uniqueness section of Lemma 11.0.5. □

The following is probably the most general result.

THEOREM 11.0.9 (Theorem 11.1.4). *Let* $\mathscr{O}$ *be a noncompact strongly tame SPC* n*-orbifold,* $n \geq 2$*, with generalized lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends and satisfies* (*IE*) *and* (*NA*)*. Assume* $\partial\mathscr{O} = \emptyset$*. Then*

- *Suppose that every finite-index subgroup of* $\pi_1(\mathscr{O})$ *contains no nontrivial infinite nilpotent normal subgroup and* $\partial\mathscr{O} = \emptyset$*. Then* hol *maps the deformation space* $\mathrm{CDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ *of SPC-structures on* $\mathscr{O}$ *with generalized lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$
- *Suppose that every finite-index subgroup of* $\pi_1(\mathscr{O})$ *contains no nontrivial infinite nilpotent normal subgroup and* $\partial\mathscr{O} = \emptyset$*. Then* hol *maps the deformation space* $\mathrm{SDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ *of strict SPC-structures on* $\mathscr{O}$ *with lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

For example, these apply to the projective deformations of hyperbolic manifolds with torus boundary as in [6].

11.1. Introduction

We will allow for these structures that a radial lens-cone end could change to a horospherical type and vice versa, and a totally geodesic lens end could change to a horospherical one and vice versa. However, we will not allow a radial lens-cone end to change to a totally geodesic lens end.

For a strongly tame orbifold $\mathscr{O}$, we recall conditions in Definition 6.0.1.

(IE) $\mathscr{O}$ or $\pi_1(\mathscr{O})$ satisfies the infinite-index end fundamental group condition (IE) if $[\pi_1(E) : \pi_1(\mathscr{O})] = \infty$ for the end fundamental group $\pi_1(E)$ of each end E.

(NA) $\mathscr{O}$ or $\pi_1(\mathscr{O})$ satisfies the nonannular property if
$$\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$$
is finite for two distinct p-ends $\tilde{E}_1$ and $\tilde{E}_2$ of $\mathscr{O}$.

The following theorems are to be regarded as examples of the so-called Ehresmann-Thurston-Weil principle.

THEOREM 11.1.1. *Let* $\mathscr{O}$ *be a strongly tame* n*-orbifold with generalized lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends. Assume* $\partial\mathscr{O} = \emptyset$*. Suppose that* $\mathscr{O}$ *satisfies* (*IE*) *and* (*NA*)*. Then*

- *the subspace of SPC-structures*
$$\mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) \subset \mathrm{Def}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$$
is open.

- *Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial\mathcal{O} = \emptyset$. Then* hol *maps* $\mathrm{CDef}_{\mathcal{E},\mathrm{u,lh}}(\mathcal{O})$ *homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathcal{E},\mathrm{u,lh}}(\pi_1(\mathcal{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

The proof of Theorem 11.1.1 and that of following Theorem 11.1.3 are as follows: The openness follows from Theorem 11.2.1 by the uniqueness section obtained by Lemma 11.0.5. Corollary 11.3.5 proves the closedness. For the first item of Theorem 11.1.3, we give:

REMARK 11.1.2. A strongly tame SPC-orbifold $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends satisfying (IE) and (NA) is strictly SPC with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends if and only if $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups. Corollary 6.3.3 shows this by Theorems 10.3.1 and 10.3.4.

THEOREM 11.1.3. *Let $\mathcal{O}$ be a strongly tame strictly SPC n-dimensional orbifold with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial\mathcal{O} = \emptyset$. Then*

- *$\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its end fundamental groups.*
- *The subspace* $\mathrm{SDef}_{\mathcal{E},\mathrm{u,lh}}(\mathcal{O}) \subset \mathrm{Def}^s_{\mathcal{E},\mathrm{u,lh}}(\mathcal{O})$*, of strict SPC-structures with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends is open.*
- *Suppose further that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial\mathcal{O} = \emptyset$. Then* hol *maps the deformation space* $\mathrm{SDef}_{\mathcal{E},\mathrm{u,lh}}(\mathcal{O})$ *of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathcal{E},\mathrm{u,lh}}(\pi_1(\mathcal{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Finally, we use the eigenvector-sections to prove in Section 11.4:

THEOREM 11.1.4 (Main result of the monograph). *Let $\mathcal{O}$ be a strongly tame n-dimensional SPC-orbifold with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends and satisfies* (*IE*) *and* (*NA*)*. Assume $\partial\mathcal{O} = \emptyset$. Then*

- *Suppose that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial\mathcal{O} = \emptyset$. Then* hol *maps the deformation space* $\mathrm{CDef}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ *of SPC-structures on $\mathcal{O}$ with generalized lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathcal{E},\mathrm{lh}}(\pi_1(\mathcal{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$
- *Suppose that every finite-index subgroup of $\pi_1(\mathcal{O})$ contains no nontrivial infinite nilpotent normal subgroup and $\partial\mathcal{O} = \emptyset$. Then* hol *maps the deformation space* $\mathrm{SDef}_{\mathcal{E},\mathrm{lh}}(\mathcal{O})$ *of strict SPC-structures on $\mathcal{O}$ with lens-shaped or horospherical $\mathcal{R}$- or $\mathcal{T}$-ends homeomorphically to a union of components of*
$$\mathrm{rep}_{\mathcal{E},\mathrm{lh}}(\pi_1(\mathcal{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

For example, these apply to hyperbolic manifolds with torus boundary as in [6].

11.2. The openness of the convex structures

In this section also, we will only need $\mathbb{RP}^n$ versions. Given a strongly tame real projective orbifold $\mathcal{O}$ with e_1 $\mathcal{R}$-ends and e_2 $\mathcal{T}$-ends, each end E_i, $i = 1,\dots,e_1,e_1+1,\dots,e_1+e_2$, has an orbifold structure of dimension $n-1$ and inherits a real projective structure.

Let $\mathcal{U}$ and $s_{\mathcal{U}} : \mathcal{U} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$ be as in Section 9.4.1.

- We define $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ to be the subspace of $\mathrm{Def}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ of real projective structures with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends determined by $s_{\mathscr{U}}$, and stable irreducible holonomy homomorphisms in $\mathscr{U}$.
- We define $\mathrm{CDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ to be the subspace consisting of SPC-structures with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends in $\mathrm{Def}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$.
- We define $\mathrm{SDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ to be the subspace of consisting of strict SPC-structures with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends in $\mathrm{Def}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$.

THEOREM 11.2.1. *Let $\mathscr{O}$ be a strongly tame real projective n-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Assume that $\partial\mathscr{O} = \emptyset$. For a $\mathsf{PGL}(n+1,\mathbb{R})$-conjugation invariant open subset $\mathscr{U}$ of a union of semi-algebraic subsets of*

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})),$$

and a $\mathsf{PGL}(n+1,\mathbb{R})$-equivariant fixing section $s_{\mathscr{U}} : \mathscr{U} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$, the following are open subspaces

$$\mathrm{CDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}) \subset \mathrm{Def}^s_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}),$$
$$\mathrm{SDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}) \subset \mathrm{Def}^s_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}).$$

For orbifolds such as these, the deformation space of convex structures may only be a proper subset of space of the characters.

By Theorem 11.2.1 and Theorem 9.4.5, we obtain:

COROLLARY 11.2.2. *Let $\mathscr{O}$ be a strongly tame real projective n-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Assume that $\partial\mathscr{O} = \emptyset$. Let $\mathscr{U}$ and $s_{\mathscr{U}}$ be as in Theorem 11.2.1. Suppose that $\mathscr{U}$ has its image $\mathscr{U}'$ in*

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Then

$$\mathrm{hol} : \mathrm{CDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}) \to \mathscr{U}'$$

is a local homeomorphism, and so is

$$\mathrm{hol} : \mathrm{SDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O}) \to \mathscr{U}'.$$

PROOF. Theorem 9.4.5 shows that the map

$$\mathrm{hol} : \mathrm{Def}^s_{\mathscr{E},s_{\mathscr{V}}}(\mathscr{O}) \to \mathrm{rep}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

is an open one when we don't require "ce" condition. Proposition 11.2.6 tells us the openness of the images here. Theorem 11.2.1 completes the proof. □

Here, in fact, one needs to prove for every possible continuous section.

Koszul [114] proved these facts for closed affine manifolds and expanded by Goldman [88] for the closed real projective manifolds. See [49], [61] and also Benoist [23].

11.2.1. The proof of the openness. The major part of showing the preservation of convexity under deformation is Proposition 11.2.4 on Hessian function perturbations. (These parts are already explored in [66]; however, we are studying R-ends and T-ends, and we also have conceived these ideas independently.)

We mention that our approach for openness is slightly different from that of Cooper-Long-Tillman [67] since they are using their canonical invariant hessian metrics for end-neighborhoods. Our hessian metrics for end-neighborhoods are not canonical ones as theirs are.

Recall that a convex open cone V is a convex cone of $\mathbb{R}^{n+1}$ containing the origin O in the boundary. Recall that a properly convex open cone is a convex cone so that its closure does not contain a pair of $\vec{v}, -\vec{v}$ for a nonzero vector in $\mathbb{R}^{n+1}$. Equivalently, it does not contain a complete affine line in its interior.

A dual convex cone V^* to a convex open cone is a subset of $\mathbb{R}^{n+1*}$ given by the condition $\phi \in V^*$ if and only if $\phi(\vec{v}) > 0$ for all $\vec{v} \in \mathrm{Cl}(V) - \{O\}$.

Recall that V is a properly convex open cone if and only if so is V^* and $(V^*)^* = V$ under the identification $(\mathbb{R}^{n+1*})^* = \mathbb{R}^{n+1}$. Also, if $V \subset W$ for a properly convex open cone, then $V^* \supset W^*$.

For properly convex open subset Ω of $\mathbb{S}^n$, its dual Ω^* in $\mathbb{S}^{n*}$ is given by taking a cone V in $\mathbb{R}^{n+1}$ corresponding to Ω and taking the dual V^* and projecting it to $\mathbb{S}^{n*}$. The dual Ω^* is a properly convex open domain if so was Ω.

Recall the Koszul-Vinberg function for a properly convex cone V and the dual properly convex cone V^*

$$f_{V^*}: V \to \mathbb{R}_+ \text{ defined by } x \in V \mapsto f_{V^*}(x) = \int_{V^*} e^{-\phi(x)} d\phi \tag{11.2.1}$$

where the integral is over the euclidean measure in $\mathbb{R}^{n+1*}$. This function is strictly convex if V is properly convex. f_{V^*} is homogeneous of degree $-(n+1)$. Writing D as the affine connection, we will write the Hessian $Dd\log(f)$. The hessian is positive definite and norms of unit vectors are strictly bounded below in a compact subset K of $V - \{O\}$. (See Chapter 4 of [86] and [152].) The metric $Dd\log(f)$ is invariant under the group $\mathbf{Aff}(V)$ of affine transformation acting on V. (See Theorem 6.4 of [86].) In particular, it is invariant under scalar dilatation maps. (For extensive survey, see Shima [143].)

A *Hessian metric* on an open subset V of an affine space is a metric of form $\partial^2 f/\partial x_i \partial x_j$ for affine coordinates x_i and a function $f: V \to \mathbb{R}$ with a positive definite Hessian defined on V. A Riemannian metric on an affine manifold is a *Hessian metric* if the manifold is affinely covered by a cone and the metric lifts to a Hessian metric of the cone.

Let $\mathscr{O}$ have an SPC-structure μ with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. Clearly $\tilde{\mathscr{O}}$ is a properly convex open domain. Then an affine suspension of $\mathscr{O}$ has an affine Hessian metric defined by $Dd\phi$ for a function ϕ defined on the cone in $\mathbb{R}^{n+1}$ corresponding to $\tilde{\mathscr{O}}$ by above.

A *parameter of real projective structures* $\mu_t, t \in [0,1]$ on a strongly tame orbifold $\mathscr{O}$ is a collection so that the restriction $\mu_t|K$ to each compact suborbifold K is a continuous parameter; In other words, the associated developing map $\mathbf{dev}_t|\hat{K}: \hat{K} \to \mathbb{S}^n$ (resp. $\mathbb{RP}^n$) for every compact subset $\hat{K}$ of $\tilde{\mathscr{O}}$ is a family in the C^r-topology continuous for the variable t. (See Definition 9.3.9, Choi [49] and Canary-Epstein-Green [33].)

DEFINITION 11.2.3. Let $\mathscr{O}$ be a strongly tame orbifold with ends. Let U be a union of mutually disjoint end neighborhoods, and let μ_0 and μ_1 be two real projective structures on $\mathscr{O}$. Let $\overline{\mathbf{dev}}_0, \overline{\mathbf{dev}}_1: \hat{\mathscr{O}} \to \mathbb{S}^n$ be extended developing maps. We say that μ_0 and μ_1 on $\mathscr{O}$ are *δ-close* in the C^r-topology, $r \geq 2$, on the compactification $\bar{\mathscr{O}}$ if for a compact path-connected domain K in $\hat{\mathscr{O}}$ mapping onto $\bar{\mathscr{O}}$, the associated developing maps $g_0 \circ \overline{\mathbf{dev}}_0|K$ and $g_1 \circ \overline{\mathbf{dev}}_1|K$ are δ-close in C^r-topology for some g_0 and g_1 in $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$.

Recall the Vinberg metric from Sections 4.4 and 12.3 of Goldman [89], which is a Hessian metric.

PROPOSITION 11.2.4. *Let $\mathscr{O}$ be a strongly tame orbifold with ends and satisfies* (*IE*) *and* (*NA*). *Suppose that $\mathscr{O}$ has an SPC structures μ_0 with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and the affine suspension of $\mathscr{O}$ with μ_0 has a Hessian metric.*

(See Section 1.2.1.) The ends of $\mathscr{O}$ are given $\mathscr{R}$-type or $\mathscr{T}$-types. Suppose that one of the following holds:

- *μ_0 is SPC, and a C^r-continuous parameter, $r \geq 2$, of real projective structure μ_t, $t \in [0,1]$, radial or totally geodesic ends with end holonomy groups of generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends where the $\mathscr{R}$-types or $\mathscr{T}$-types of ends are preserved.*
- *We may also let μ_{t_i} to be a sequence for $t_i \in [0,1]$ with $\{\mu_{t_i}\} \to \mu_0$ as $t_i \to 0$ in the C^r-topology for $r \geq 0$.*

Then for sufficiently small t, the affine suspension $C(\tilde{\mathscr{O}})$ for $\tilde{\mathscr{O}}$ with μ_t also has a Hessian metric invariant under the group of dilatations.

PROOF. We will prove for $\mathbb{S}^n$. It will be sufficient since we aim to obtain the Hessian metric on $C(\tilde{\mathscr{O}})$. We will keep $\tilde{\mathscr{O}}$ and the action of the deck transformation group fixed and only change the structures on it. Note that the subsets here remain fixed and the only changes are on the real projective structures, i.e., the atlas of charts to $\mathbb{S}^n$.

Let $\tilde{\mathscr{O}}$ in $\mathbb{S}^n$ denote the universal covering domain corresponding to μ_0. Again $\mathbf{dev}_0$ being an embedding identifies the first with subsets of $\mathbb{S}^n$ but $\mathbf{dev}_t$ is not known to be so. We shall prove this below.

We will prove this by steps:

(A) The first step is to understand the deformations of the end-neighborhoods.

(B) We change the Hessian function on the cone associated with the universal covers. We need to obtain one for the deformed end neighborhoods by Hessian functions from Koszul-Vinberg integrals and another one the outside of the union of end neighborhoods by isotopies and patch the two together.

(A) Let $\tilde{E}'$ be a p-end of $\tilde{\mathscr{O}}$, and it corresponds to a p-end of $\tilde{\mathscr{O}}'$ as well. Let E be the end of $\mathscr{O}$ corresponding to $\tilde{E}$. There exists a C^r-parameter of real projective structures μ_t with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. We can also find a parameter of developing maps $\mathbf{dev}_t$ associated with μ_t where $\mathbf{dev}_t|K$ is a continuous with respect to t for each compact $K \subset \hat{\mathscr{O}}$. To begin with, we assume that $\tilde{E}'$ keeps being a lens-shaped or horospherical p-end.

Let h_t denote the holonomy homomorphism associated with $\mathbf{dev}_t$ for each t. Recall that Theorems 6.1.1 and 6.1.2 study the perturbation of lens-shaped R-ends and T-ends. Lemma 9.4.3 studies the perturbations of a horospherical $\mathscr{R}$- or $\mathscr{T}$-end to either a R-end or to a T-end.

In this monograph, we do not allow $\mathscr{R}$-type ends to change to $\mathscr{T}$-type ends and vice versa as this will make us to violate the local injectivity property from the deformation space to a space of characters. (See Theorem 9.4.5.) Let $\tilde{E}$ be a p-end. Thus, we need to consider only four cases to prove openness:

(I): $h_t(\tilde{E})$ changes from the holonomy group of a radial p-end to that of a radial p-end in the cases:
- **(a):** $h_t(\tilde{E})$ changes from the holonomy group of a radial p-end of generalized lens-shaped becoming that of a generalized lens-shaped radial p-end.
- **(b):** $h_t(\tilde{E})$ changes from the holonomy group of a horospherical p-end to that of a generalized lens-shaped radial p-end or a horospherical p-end.

(II): $h_t(\tilde{E})$ changes from the holonomy group of totally geodesic ends of lens type or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends changes to that of themselves here.
- **(a):** $h_t(\tilde{E})$ changes from the holonomy group of a lens-shaped totally geodesic p-end to that of a lens-shaped totally geodesic p-end.

(b): $h_t(\tilde{E})$ changes from the holonomy group of a horospherical p-end to that of a horospherical p-end or to a lens-shaped totally geodesic p-end.

These hold for the corresponding holonomy homomorphisms of the fundamental groups of ends by the premise. (The above happens in actuality as well. See [4],[5], and [6].)

We will now work on one end at a time: Let us fix a p-end $\tilde{E}$ of R-type of $\tilde{\mathcal{O}}$. Let v be the p-end vertex of $\tilde{E}$ for μ_0 and v′ that for μ_1. We denote by $\mathrm{v} = \mathrm{v}_0$ and $\mathrm{v}' = \mathrm{v}_1$. Assume that v_t is the p-end vertex of $\tilde{E}$ for μ_t. Let $\mathbf{dev}_t$ and h_t denote the developing map and the holonomy homomorphism of μ_t. Assume first that the corresponding p-end for μ is of radial or horospherical type. By post-composing the developing map by a transformation near the identity, we assume that the perturbed vertex v_t of the corresponding p-end $\tilde{E}$ is mapped to v_0, i.e., $\mathrm{v} = \mathbf{dev}_t(\mathrm{v}_t)$.

(I) Suppose that $\tilde{E}$ is a generalized lens-shaped radial p-end or a horospherical p-end for μ_0. Then the holonomy group of $\tilde{E}$ is that of a generalized lens-shaped radial p-end or a horospherical p-end for μ_t under (I).

Let Λ_0 denote the limit set in the tube of the radial p-end $\tilde{E}$ for $\tilde{\mathcal{O}}$ if $\tilde{E}$ is lens-shaped radial p-end, or $\{\mathrm{v}_{\tilde{E}} = \mathrm{v}\}$ if $\tilde{E}$ is a horospherical type for μ_0. (See Definition 6.2.1.)

- Recall that $R_{\mathrm{v}}(\mathbf{dev}_t(\tilde{\mathcal{O}}))$ denotes the space of directions of segments from v in $\mathbf{dev}_t(\tilde{\mathcal{O}})$,
- $R_{\mathrm{v}}(\mathbf{dev}_t(A_t))$ denotes the space of directions of segments from v of $\mathbf{dev}_t(\tilde{\mathcal{O}})$ in $R_{\mathrm{v}}(\tilde{\mathcal{O}})$ passing through the set $\mathbf{dev}(A_t) \subset \mathbf{dev}_t(\tilde{\mathcal{O}})$.

(i) We first find domains Ω_{s_0} with smooth boundary approximating $\tilde{\mathcal{O}}$ on which $h(\pi_1(\tilde{E}))$ acts. Here s_0 will be a parameter that we will use to vary Ω_{s_0} for fixed holonomy representations. Here, Ω_{s_0} is a lens-cone for $\tilde{\mathcal{O}}$ with μ_0 so that $\partial\Omega_{s_0} = \mathrm{bd}_{\tilde{\mathcal{O}}}\Omega_{s_0}$ which is the top boundary component of the lens.

- Suppose that $\tilde{E}$ is a lens-shaped lens-shaped R-p-end for μ_0. Then we obtain a hypersurface $\partial\Omega_{s_0}$ as the top boundary component of a CA-lens as obtained by Theorem 5.1.4. The property of the strict lenses of Theorems 5.4.2 and 5.4.3 imply
$$\mathrm{Cl}(\partial\Omega_{s_0}) - \partial\Omega_{s_0} \subset \Lambda_{s_0}$$
for the limit set Λ_{s_0} of $\tilde{E}$ since a generalized lens-shaped end also satisfies the uniform middle eigenvalue condition by Theorem 5.3.21. Also, each radial geodesic is transverse to $\partial\Omega_{s_0}$. $\partial\Omega_{s_0} \cup \bigcup S(\mathrm{v})$ bounds a properly convex domain Ω_{s_0}.
- Suppose that $\tilde{E}$ is a horospherical R-p-end for μ_0. Then we obtain a $\partial\Omega_{s_0}$ as a boundary of a convex domain invariant under $h_0(\pi_1(\tilde{E}))$. Again $\Omega_{s_0} \cup \{\mathrm{v}\}$ bounds a properly convex domain Ω_{s_0}.

Choose some small $\varepsilon > 0$. By Lemma 6.2.8, if $\tilde{E}$ is a lens-shaped R-p-end or a horospherical R-p-end, we may choose Ω_{s_0} that
$$\mathbf{d}_H(\Omega_{s_0}, \tilde{\mathcal{O}}) < \varepsilon.$$

Suppose that $\tilde{E}$ is merely a generalized lens-shaped R-p-end. Then we can still form a lens-cone neighborhood V of in the tube domain corresponding to the directions of the p-end domain $\tilde{\Sigma}_{\tilde{E}}$ for $\tilde{E}$. We can choose a smooth top boundary component approximates $\mathrm{bd}\tilde{\mathcal{O}} - \bigcup S(\vec{v}_{\tilde{E}})$ from outside by Proposition 5.3.14. Hence, any $\varepsilon > 0$, sufficiently large 0 satisfies
$$\mathbf{d}_H(\Omega_{s_0}, \tilde{\mathcal{O}}) < \varepsilon$$

still holds. Here, $\partial\Omega_{s_0}$ may not be a subset of $\tilde{\mathscr{O}}$ unless $\tilde{E}$ is lens-shaped and not just generalized lens-shaped. However, this is irrelevant for our purposes.

For each ε, we will choose the parameter s_0 so that the above is satisfied. So, s_0 is considered to vary for our purposes.

(ii) Now our purpose is to find a convex domain $\Omega_{s_0,t}$ where $h_t(\pi_1(E)$ acts on and approximating Ω_0, and show that it contains an embedded image of a p-end neighborhood. We denote by $\tilde{\Sigma}_{\tilde{E},t}$ the universal cover of the end orbifold associated with for a p-end $\tilde{E}$ of $\tilde{\mathscr{O}}$ with μ_t. By our end holonomy group condition in the premise, Corollary A.1.12 shows that $\tilde{\Sigma}_{\tilde{E},t}$ is again complete affine or properly convex.

For the $\mathscr{R}$-type end E, $\mathscr{O}$ has a concave end-neighborhood or a horospherical end-neighborhood for E bounded by a smooth compact end orbifold S'_E transverse to the radial rays. Here, S'_E is diffeomorphic to Σ_E clearly.

For a sufficiently small t in μ_t, we obtain a domain $U_t \subset \mathscr{O}$ with $U_0 \subset \Omega_{s_0}/h_0(\pi_1(\tilde{E}))$ bounded by an inverse image of a compact orbifold $S'_{E,t}$ diffeomorphic to S'_E still transverse to radial rays by Propositions 5.3.11 and 5.3.12. $S'_{E,t}$ is either strictly concave if S_E was strictly convex if Σ_E was horospherical. (The strict convexity and the transversality follow since the change of affine connections are small as the argument of Koszul [114].)

Since the change was sufficiently small, we may assume that $S'_{E,t}$ still bounds an end-neighborhood U_t of product form by Lemma 11.2.5.

□

LEMMA 11.2.5. *Suppose that $\tilde{U}$ is a R-p-end neighborhood of $\tilde{\mathscr{O}}$ covering an end-neighborhood U of an end E in $\mathscr{O}$. Then for sufficiently small change of real projective structures in C^r-sense, $r \geq 2$, in the compact open topology, hypersurface S_t sufficiently close to S in the C^0-sense in terms of a parameterizing map still bounds an end neighborhood of E. Letting $\tilde{S}_t$ be a component of the inverse image of S on which $\pi_1(\tilde{E})$ acts, we still have that $\tilde{S}_t$ bounds a p-end neighborhood of $\tilde{E}$.*

PROOF. Straightforward. □

PROOF OF PROPOSITION 11.2.4 CONTINUED. Let Λ_t denotes the limit set in $\bigcup S(\mathrm{v})_t$ for generalized radial p-end cases and $\Lambda_t = \{\mathrm{v}\}$ for the horospherical case. Let $S(\mathrm{v})_t$ denote the set of maximal segments in the closure of U_t from v corresponding to $\mathrm{bd}R_{\mathrm{v}}(\mathbf{dev}_t(\tilde{\mathscr{O}}))$ of μ_t.

Suppose that $\tilde{E}$ is a lens-shaped R-p-end for μ_0. We showed above that the C^r-change $r \geq 2$ of μ_t from μ_0 be sufficiently small so that we obtain a region $\Omega_{s_0,t}$ in B^o_t with $\partial\Omega_{s_0,t}$ strictly convex and transverse to radial rays under $\mathbf{dev}_t$. Here, $\Omega_{s_0,0} = \Omega_{s_0}$.

Choose a compact domain F in $\partial\Omega_{s_0}$. Let F_t denote the corresponding deformed set in $\partial\Omega_{s_0,t}$. By Theorem 8.1.2, $\pi_1(\tilde{E})$ is virtually abelian. For sufficiently small t, $0 < t < 1$, $\mathbf{dev}_t(F_t)$ is a subset of the tube B_t determined by $\mathbf{dev}_t(U_t)$ since B_t and a paramterization of $\mathbf{dev}_t(F_t)$ depends continuously on t by Corollary A.1.13.

- By transversality to the segments mapping to ones from v under $\mathbf{dev}_t$, it follows that $\mathbf{dev}_t|\partial\Omega_{s_0,t}$ gives us a smooth immersion to a convex domain $\tilde{\Sigma}_{\tilde{E},t}$ that equals the space of maximal segments in B_t with vertices v and v_-.
- By Corollary A.1.12, the immersion $\mathbf{dev}_t|\partial\Omega_{s_0,t}$ to a properly convex domain $\tilde{\Sigma}_{\tilde{E},t}$ is a diffeomorphism if $\tilde{\Sigma}_{\tilde{E},t}$ is properly convex. It is also so if $h_t(\pi_1(\tilde{E}))$ is horospherical since this follows from the classical Bieberbach theory using the Euclidean metric on $\tilde{\Sigma}_{\tilde{E}}$ where it develops.

Since $\pi_1(\tilde{E})$ is the fundamental group of a generalized lens-shaped or horospherical p-end, it follows that

$$\mathbf{dev}_t(\mathrm{Cl}(\partial\Omega_{s_0,t}) - \partial\Omega_{s_0,t}) \subset \mathbf{dev}_t(\Lambda_t)$$

by Theorems 8.1.3, 5.4.2, and 5.4.3.

Suppose that $\tilde{E}$ is a generalized lens-shaped or lens-shaped R-p-end for μ_t, $t > 0$. Since $\partial\Omega_{s_0,t}$ is convex, each point of $\partial\Omega_{s_0,t} \cup \bigcup S(\mathrm{v})_t$ has a neighborhood that maps under the completion $\widehat{\mathbf{dev}_t}$ to a convex open ball. Thus, $\partial\Omega_{s_0,t} \cup \bigcup S(\mathrm{v})_t$ bounds a compact ball $\Omega_{s_0,t} \cup \bigcup S(\mathrm{v})_t$ by Lemma 1.4.3 since the local convexity implies the global convexity and they are in B_t.

Suppose that $\tilde{E}$ is a horospherical R-p-end. For $t > 0$, $\partial\Omega_{s_0,t} \cup \{\mathrm{v}\}$ bounds a convex domain $\Omega_{s_0,t}$ by the local convexity of the boundary set $\partial\Omega_{s_0,t} \cup \{\mathrm{v}\}$ and Lemma 1.4.3. □

PROPOSITION 11.2.6. *Assume as in Proposition 11.2.4, and (I) in the proof. Then for μ_t for sufficiently small t, the end corresponding to E is always generalized lens-shaped R-end or a horospherical R-end. Also, if μ' is sufficiently C^r-close to μ, then the end of $\mathscr{O}$ with μ' is a generalized lens-shaped R-end or a horospherical R-end.*

PROOF. The above arguments prove this since we are studying arbitrary deformations. □

PROOF OF PROPOSITION 11.2.4 CONTINUED. (iii) We will show how these regions deform approximating Ω_{s_0} in the Hausdorff metric sense. We define $\mathbb{r}_{\mathrm{v}}(K)$ the union of great segments with an endpoint v in directions of K, $K \subset R_{\mathrm{v}}$.

- Let K be a compact convex subset of $\Omega_{s_0,0}$ with smooth boundary, and K_t the perturbed one in $\Omega_{s_0,t}$ and $\tilde{E}$ be the corresponding p-end. We can form a compact set inside $\Omega_{s_0,t}$ consisting of segments from the p-end vertex to K in the set of radial segments. For μ_t from μ_0 changed by a sufficiently small manner, a compact subset $\mathbb{r}_{\mathrm{v}}(K) \subset \mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}})$ is changed to a compact convex domain $\mathbb{r}_{\mathrm{v}}(K_t) \subset \mathbb{r}_{\mathrm{v}}(\tilde{\Sigma}_{\tilde{E},t})$.

We choose s_0 large enough so that $K \subset \Omega^o_{s_0}$. For sufficiently small t, $\Omega_{s_0,t} \cap \mathbb{r}_{\mathrm{v}}(K_t)$ is a convex domain since $\partial\Omega_{s_0,t}$ is strictly convex and transverse to great segments from v and hence embeds to a convex domain under $\mathbf{dev}_t$. We may assume that $\Omega_{s_0,t} \cap \mathbb{r}_{\mathrm{v}}(K_t)$ is sufficiently close to $\Omega_{s_0} \cap \mathbb{r}_{\mathrm{v}}(K)$ as we changed the real projective structures sufficiently small in the C^r-sense. See Definition 11.2.3.

An ε-*thin* space is a space which is an ε-neighborhood of its boundary for small $\varepsilon > 0$. By Lemma 3.1.5 and Corollary A.1.13, we may assume that $\mathrm{Cl}(\mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}}))$ and $\mathrm{Cl}(\mathbb{r}_{\mathrm{v},t}(\tilde{\mathscr{O}}))$ are ε-$\mathbf{d}$-close convex domains in the Hausdorff sense for sufficiently small t. Thus, given an $\varepsilon > 0$, we can choose K and K'_t and a sufficiently small deformation of the real projective structures so that $\Omega_{s_0} \cap (\mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}}) - \mathbb{r}_{\mathrm{v}}(K))$ is an ε-thin space, and so is $\Omega_{s_0,t} \cap (\mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}}) - \mathbb{r}_{\mathrm{v}}(K_t))$ for sufficiently small changes of t. Moreover,

$$\mathrm{Cl}(\Omega_{s_0}) \cap (\mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}}) - \mathbb{r}_{\mathrm{v}}(K)) \subset N_\varepsilon(\mathrm{Cl}(\Omega_{s_0} \cap \mathbb{r}_{\mathrm{v}}(K))) \text{ and}$$

$$\mathrm{Cl}(\Omega_{s_0,t}) \cap (\mathbb{r}_{\mathrm{v}}(\tilde{\mathscr{O}}) - \mathbb{r}_{\mathrm{v}}(K_t)) \subset N_\varepsilon(\mathrm{Cl}(\Omega_{s_0,t} \cap \mathbb{r}_{\mathrm{v}}(K_t))); \tag{11.2.2}$$

The reason is that the sharply supporting hyperspaces of $\mathrm{Cl}(\Omega_{s_0})$ at points of $\partial\mathbb{r}_{\mathrm{v}}(K) \cap \mathrm{Cl}(\Omega_{s_0})$ are in arbitrarily small acute angles from geodesics from v and similarly for those of $\mathrm{Cl}(\Omega_{s_0,t})$ for sufficiently small t by Corollary 5.5.8.

Therefore we conclude for (I) that for any $\varepsilon > 0$, there exists $\delta, \delta > 0$

$$\mathbf{d}_H(\mathrm{Cl}(\Omega_{s_0}), \mathrm{Cl}(\Omega_{s_0,t})) < \varepsilon \tag{11.2.3}$$

provided $|t| < \delta$; that is, we choose $\mu_{0,t}$ sufficiently close to μ_0: First, we choose K and deformation K_t so that it satisfies (11.2.2) for $t < \delta$ for some $\delta > 0$. Then we choose t sufficiently small so that $\Omega_{s_0,t} \cap R(K_t)$ is sufficiently close to $\Omega_{s_0} \cap R(K)$.

Also, $\Omega_{s_0,t}$ contains a concave p-end neighborhood of $\tilde{E}$ for μ_t for sufficiently small $t > 0$. This can be assured by taking K sufficiently large containing a fundamental domain of $R_x(\tilde{\mathscr{O}})$ and sufficiently large K_t containing a fundamental domain F_t of $R_t(\mathbf{dev}_t(\mathscr{O}))$ for $h_t(\pi_1(\tilde{E}))$ deformed from F by Proposition 11.2.6.

(II) Now suppose that $\tilde{E}$ is a lens-shaped T-p-end or horospherical p-end of type $\mathscr{T}$, and we suppose that $h_t(\tilde{E})$ is a lens-shaped T-p-end for μ_t for $t > 0$. Other cases are similar to (I).

We take Ω_{s_0} to be the convex domain obtained as in Lemma 6.2.8 with strictly convex boundary component $\partial_1\Omega_{s_0}$ and totally geodesic one $\tilde{S}_{\tilde{E},0}$ in $\mathrm{bd}\Omega_{s_0}$. Now, $\Omega_{s_0}/h(\pi_1(\tilde{E}))$ has strictly convex boundary component $\partial\Omega_{s_0}/h(\pi_1(\tilde{E}))$ and totally geodesic boundary $\tilde{S}_{\tilde{E},0}/h(\pi_1(\tilde{E}))$ when $\tilde{E}$ is a T-p-end. If $\tilde{E}$ is a horospherical end, $\partial\Omega_{s_0}/h(\pi_1(\tilde{E}))$ still is a strictly convex compact $(n-1)$-orbifold.

Here, we note $\mathrm{bd}\Omega_{s_0} = \partial\Omega_{s_0} \cup \mathrm{Cl}(\tilde{S}_{\tilde{E}})$.

Suppose that μ_t is sufficiently close to μ_0. Then by Theorem 6.1.2, we deform the lens-shaped T-end for $\tilde{E}$:

- we obtain a properly convex domain $\Omega_{s_0,t}$ for sufficiently small t with a strictly convex boundary $\partial_1\Omega_{s_0,t}$ and $\tilde{S}_t$, and
- $\partial\Omega_{s_0,t}$ is also cocompact under the $\pi_1(\tilde{E})$-action associated with μ_1 and strictly convex.

See Proposition 5.3.11. We choose Ω_{s_0} to be $L_0 \cap \tilde{\mathscr{O}}$ for a lens in an ambient orbifold containing L_0. We also have $\mathrm{bd}\Omega_{s_0,t} = \partial\Omega_{s_0,t} \cup \mathrm{Cl}(\tilde{S}_{\tilde{E},t})$ where $\tilde{S}_{\tilde{E},t}$ is the ideal boundary component for $\tilde{E}$ for μ_t. By Proposition 5.3.11, L_0 deforms to a properly convex domain L_t so that $L_t \cap \tilde{\mathscr{O}}_t$ is $\Omega_{s_0,t}$ where $\tilde{\mathscr{O}}_t$ is $\tilde{\mathscr{O}}$ with a real projective structure μ_t. We have

$$\mathrm{Cl}(\partial\Omega_{s_0,t}) - \partial\Omega_{s_0,t} = \partial\mathrm{Cl}(\tilde{S}_{\tilde{E},t})$$

for a totally geodesic ideal boundary component $\tilde{S}_{\tilde{E},t}$ by Theorem 4.4.1. Therefore the union of $\partial\Omega_{s_0,t}$ and a totally geodesic ideal boundary component $\mathrm{Cl}(\tilde{S}_{\tilde{E},t})$ bounds a properly convex compact n-ball in $\mathbb{S}^n$. We can find a properly convex lens L_0 for $\tilde{E}$ at μ_0 with $L_0 \cap \tilde{\mathscr{O}}$ in a p-end neighborhood Ω_{s_0}. Since the change of μ_t is sufficiently small, $L_t \cap \tilde{\mathscr{O}}_t$ is still in a p-end neighborhood of $\tilde{E}$ by Lemma 11.2.5. We obtain a lens-shaped p-end neighborhood for $\tilde{E}$ and $\mathscr{O}$ with μ_t by Theorem 4.4.1.

We may assume without loss of generality that the hyperspace $V_{\tilde{E}}$ containing $\tilde{S}_{\tilde{E},t}$ is fixed. Moreover, we may assume by choosing sufficiently large Ω_{s_0} without loss of generality that $\mathbf{d}_H(\Omega_{s_0}, \tilde{\mathscr{O}}) < \varepsilon$ for any $\varepsilon > 0$.

By Lemma 5.3.9, a sharply supporting hyperspace at a point of $\mathrm{bd}\tilde{S}_{\tilde{E}}$ is uniformly bounded away from $V_{\tilde{E}}$. A sequence of sharply supporting hyperspaces can converge to a sharply supporting one. Let us choose a sufficiently small $\varepsilon > 0$. Let B be a compact ε-neighborhood of $\partial\Omega_{s_0}$ so that

$$\mathbf{d}^H(\partial\Omega_{s_0} - B, \partial\mathrm{Cl}(\tilde{S}_{\tilde{E}})) < \varepsilon.$$

Given a sharply supporting hyperspace W_x of a point x of $\partial\Omega_{s_0}$ of Ω_{s_0}, there exists a sharply supporting closed hemisphere H_x bounded by W_x.

We define the *shadow* S of ∂B as the set

$$\bigcap_{x\in\partial B} H_x \cap V_{\tilde E}.$$

Then we can choose sufficiently small ε so that $\mathbf{d}^H(S, \mathbf{dev}_0(\mathrm{Cl}(\tilde S_{\tilde E}))) \leq \varepsilon$. We can also assure that W_x meets $V_{\tilde E}$ in angles in $(\delta, \pi-\delta)$ for some $\delta > 0$ by compactness of ∂B and the continuity of map $x \mapsto W_x$.

Suppose that we change the structure from μ_0 to μ_t with a small C^2-distance. Then B will change to B'_t with W_x change by small amount. The new shadow S'_t will have the property $\mathbf{d}^H(S'_t, \mathrm{Cl}(\tilde S_{\tilde E,t})) \leq \varepsilon$ for a sufficiently small C^r-change, $r \geq 2$, of μ_t from μ_0. Hence, we obtain that for each $\varepsilon > 0$, there exists $\delta, \delta > 0$ so that

$$\mathbf{d}_H(\partial\Omega_{s_0,t} - B'_t, \partial\mathrm{Cl}(\tilde S_{\tilde E,t})) < \varepsilon \tag{11.2.4}$$

provided $|t| < \delta$. Therefore by Corollary A.1.13 for each $\varepsilon > 0$, there exists $\delta, \delta > 0$ so that

$$\mathbf{d}_H(\mathrm{Cl}(\tilde S_{\tilde E}), \mathrm{Cl}(\tilde S_{\tilde E,t})) < \varepsilon.$$

Recall that $\mathrm{bd}\Omega_{s_0} = \partial\Omega_{s_0} \cup \mathrm{Cl}(\tilde S_{\tilde E})$ and $\mathrm{bd}\Omega_{s_0,t} = \partial\Omega_{s_0,t} \cup \mathrm{Cl}(\tilde S_{\tilde E,t})$. Combining with (11.2.4) and the sufficiently small change of B to B_t, we obtain that each $\varepsilon > 0$, there exists $\delta, \delta > 0$ so that

$$\mathbf{d}_H(\mathrm{Cl}(\Omega_{s_0}), \mathrm{Cl}(\Omega_{s_0,t})) < \varepsilon \tag{11.2.5}$$

provided $|t| < \delta$.

Suppose that $\tilde E$ was a horospherical p-end of type $\mathscr{T}$. Again, the argument is similar. We start with Ω_{s_0} which is horospherical with strictly convex boundary, and tangent to a hyperspace P_0. The deformation gives us strictly convex hypersurface $\partial\Omega_{s_0,t}$ and a hyperspace P_t where $h_t(\pi_1(\tilde E))$ acts on.

We assume without loss of generality that $P_t = P$ for small $t > 0$. For sufficiently small t, we obtain a domain bounded by $\partial\Omega_{s_0,t}$ and the closure $\mathrm{Cl}(\tilde S_{\tilde E,t})$ of a totally geodesic ideal boundary component. By taking duals by Corollary 5.5.7, we have lens-shaped or horospherical R-p-end $\tilde E$. Proposition 11.2.6 shows that we obtain a domain $\Omega^*_{s_0,t}$ approximating $\Omega^*_{s_0}$ where $\partial\Omega^*_{s_0,t}$ is dual to $\partial\Omega_{s_0,t}$. Then $\Omega^{**}_{s_0,t}$ approximates Ω_{s_0} as much as one wishes to by Lemma 1.5.14.

We show that we have an embedded image of a p-end neighborhood in $\Omega_{s_0,t}$. The hypersurface $\partial\Omega_{s_0}$ is embedded in $\tilde{\mathscr{O}}$ with μ_0. Let F be a compact fundamental domain of $\partial\Omega_{s_0}$ by $h(\pi_1(\tilde E))$. Now, $\mathbf{dev}_t^{-1}(\partial\Omega_{s_0,t})$ contains a compact fundamental domain F_t perturbed from F. Let $\tilde S_t$ denote a component of the inverse image of $\partial\Omega_{s_0,t}$ under $\mathbf{dev}_t$ containing perturbed fundamental domain F_t deformed from F. We deduce that $\pi_1(\tilde E)$ acts on $\tilde S_t$ since $h_t(\pi_1(\tilde E))$ acts on $\partial\Omega_{s_0,t}$.

By above, $h_t(\pi_1(\tilde E))$ acts on $\partial\Omega_{s_0,t}$ properly and cocompactly giving us a closed orbifold as a quotient. Since $\mathbf{dev}_t|F_t$ is an embedding for sufficiently small t, the equivariance tells us that $\mathbf{dev}_t : \tilde S_t \to \partial\Omega_{s_0,t}$ is a diffeomorphism.

Since the change is sufficiently small $\tilde S_t$ still bounds a p-end neighborhood of $\mathscr{O}$ by Lemma 11.2.5.

Before going on with the part (B) of the proof we briefly do a slight generalization.

□

COROLLARY 11.2.7. *We consider all cases (I), (II). For each $\varepsilon > 0$, there exists $\delta, \delta > 0$ depending only on $\mathrm{Cl}(\Omega_{s_0})$ so that we can choose a convex domain $\Omega_{s_0,1}$ where $h_1(\pi_1(\tilde E))$ acts on for the holonomy homomorphism h_1 so that*

- $\Omega_{s_0,1}$ *contains a domain that is the embedded image of a p-end neighborhood of* $\tilde{E}$ *by a developing map* $\mathbf{dev}_1$ *for* μ_1 *associated with* h_1 *and*
-

$$\mathbf{d}_H(\mathrm{Cl}(\Omega_{s_0}), \mathrm{Cl}(\Omega_{s_0,1})) < \varepsilon \tag{11.2.6}$$

provided μ_0 *and* μ_1 *are* δ*-close in* C^r*-topology,* $r \geq 2$*, on the compact set* $\mathscr{O} - U$ *for a union of* U *of end neighborhoods of* $\mathscr{O}$*.*

PROOF. Suppose that this is false. There exists a sequence of real projective structures μ_{t_i} with $\mu_{t_i} \to \mu_0$ in the C^2-topology on $\mathscr{O} - U$. Then letting the associated holonomy of μ_{t_i} be denoted by h_{t_i}, $\{h_{t_i}\}$ converges to h_0 for μ_0. We can apply the argument of cases (I), (II), to show that (11.2.6) holds for every $\varepsilon > 0$. □

PROOF OF PROPOSITION 11.2.4 CONTINUED. (B) With $\tilde{\mathscr{O}}$ with μ_t, we obtain a special affine suspension on $\mathscr{O} \times \mathbb{S}^1$ with the affine structure $\hat{\mu}_t$. Let $C(\tilde{\mathscr{O}})$ be the cone over $\tilde{\mathscr{O}}$. Then this covers the special affine suspension. Let $\tilde{\mu}_t$ denote the affine structure on $C(\tilde{\mathscr{O}})$ corresponding to $\hat{\mu}_t$. For each μ_t, it has an affine structure $\tilde{\mu}_t$, different from the induced one from $\mathbb{R}^{n+1}$ as for $t = 0$. We recall the scalar multiplication

$$s \cdot \mathrm{v} = s\mathrm{v}, \mathrm{v} \in C(\tilde{\mathscr{O}}), s \in \mathbb{R}$$

for any affine structure $\tilde{\mu}_t$. Also, given a subset K of $\tilde{\mathscr{O}}$, we denote by $C(K)$ the corresponding set in $C(\tilde{\mathscr{O}})$. This set is independent of $\tilde{\mu}_t$ but will have different affine structures nearby.

For μ_0, $\tilde{\mathscr{O}}$ is a domain in $\mathbb{S}^n$. Recall the Koszul-Vinberg function $f : C(\tilde{\mathscr{O}}) \to \mathbb{R}_+$ homogeneous of degree $-n-1$ as given by (11.2.1). (See Lemma 11.2.9.) By our choice above, the Hausdorff distance between $\mathrm{Cl}(\Omega_{s_0})$ and $\tilde{\mathscr{O}}$ can be made as small as desired for some choice of 0.

By the proof of Theorem 9.4.5 in Page 252 constructing the local inverse maps applied to strongly tame orbifolds with boundary, there exists a diffeomorphism

$$F_t : \Omega_{s_0}/h_0(\pi_1(E)) \to \Omega_{s_0,t}/h_t(\pi_1(E)) \text{ with a lift } \tilde{F}_t : \Omega_{s_0} \to \Omega_{s_0,t}$$

so that $\tilde{F}_t \to \mathrm{I}$ on every compact subset of $C(\tilde{\mathscr{O}})$ in the C^r-topology as $t \to 0$. That is, on every compact subset K of $C(\tilde{\mathscr{O}})$,

$$\{\left\|D^j\tilde{F}_t - D^j\mathrm{I}|K\right\|\} \to 0 \text{ for every multi-index } j, 0 \leq |j| \leq r.$$

We may assume that $\tilde{F}_t$ commutes with the radial flow $\hat{\Psi}_s : \tilde{\mathscr{O}} \to \tilde{\mathscr{O}}$ for $s \in \mathbb{R}$ by restricting F_t to a cross-section of $C(\tilde{\mathscr{O}})$ of the radial flow and extending radially. (See the paragraph after Definition 1.2.1.)

By the third item of Lemma 1.5.14 and Lemma 11.2.9, the Hessian functions $f'_t \circ \tilde{F}_t$ defined by (11.2.1) on the inverse image $\tilde{F}_t^{-1}(C(\Omega_{s_0,t})^o)$ is as close to the original Hessian function f in any compact subset of $C(\tilde{\mathscr{O}})$ in the C^r-topology, $r \geq 2$, as we wish provided $|t - t_0|$ is sufficiently small. By construction, f'_t is homogeneous of degree $-n-1$.

The holonomy groups $h(\pi_1(\mathscr{O}))$ and $h_t(\pi_1(\tilde{E}))$ being in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ preserve f and f'_t under deck transformations respectively.

Now do this for all p-ends. Let $\mathbb{U}_t$ be the $\pi_1(\mathscr{O})$-invariant mutually disjoint union of p-end neighborhoods of p-ends of $\tilde{\mathscr{O}}$. We construct a function f'_t on $C(\mathbb{U}_t)$ for μ_t and sufficiently small $|t|$.

Let $\mathbb{U}$ be the corresponding $\pi_1(\mathscr{O})$-invariant union of proper p-end neighborhoods of $\tilde{\mathscr{O}}$ for μ_0. For each component $\mathbb{U}_i$ of $\mathbb{U}$, we construct $f'_{i,t} \circ \tilde{F}_i$ on $C(\mathbb{U}_i)$ using Ω_{s_0} so that

$f'_{i,t}$ satisfies the above properties and $\tilde{F}_i$ is constructed as above for $\mathbb{U}_i$. We call f'_t the union of these functions.

Let $\mathbb{V}$ be a $\pi_1(\mathscr{O})$-invariant compact neighborhood of the complement of $\mathbb{U}$ in $\tilde{\mathscr{O}}$.

- Let $\partial_s\mathbb{V}$ be the image of $\partial\mathbb{V}\times\{s\}$ inside the regular neighborhood of $\partial\mathbb{V}$ in $\mathbb{U}$ parameterized as $\partial\mathbb{V}\times[-1,1]$ for $s\in[-1,1]$.
- We assign $\partial\mathbb{V}=\partial_0\mathbb{V}$.
- Let $\partial_{[s_1,s_2]}\mathbb{V}$ denote the image of $\partial\mathbb{V}_t\times\{[s_1,s_2]\}$ inside the regular neighborhood of $\partial\mathbb{V}$ in $\mathbb{V}\cap\mathbb{U}'$ for a neighborhood $\mathbb{U}'$ of $\mathrm{Cl}(\mathbb{U})\cap\tilde{\mathscr{O}}$.

We find a C^∞ map $\phi_t : C(\mathbb{U}')\cap C(\mathbb{V})\to\mathbb{R}_+$ so that $\phi_t(s\overrightarrow{v})=\phi_t(\overrightarrow{v})$ for every $s>0$ and $f'_t(\overrightarrow{v})=\phi_t(\overrightarrow{v})f(\overrightarrow{v})$ and ϕ_t is very close to the constant value 1 function. By making f'_t/f near 1 and the derivatives of f'_t/f up to two near 0 as possible, we obtain ϕ_t that has derivatives up to order two as close to 0 in a compact subset as we wish: This is accomplished by taking a partition of unity functions p_1,p_2 invariant under the radial flow so that

- $p_1=1$ on $C(W)$ for
$$W:=\partial_{[0,s_1]}\mathbb{V}\cup(\mathbb{U}'-\mathbb{V})\text{ for }s_1<1,$$
- $p_1=0$ on $C(\tilde{\mathscr{O}}-N)$ for a neighborhood N of W in $\partial_{(-1,1)}\mathbb{V}\cup(\mathbb{U}'-\mathbb{V})$, and
- $p_1+p_2=1$ identically.

We assume that
$$1-\varepsilon<f'_t/f<1+\varepsilon\text{ in }C(\mathbb{U}'\cap\mathbb{V}),$$
and f'_t/f has derivatives up to order two sufficiently close to 0 by taking f'_t and f sufficiently close in $C(\mathbb{U}')\cap C(\mathbb{V})$ by taking sufficiently small t. We define
$$\phi_t=(f'_t/f-(1-\varepsilon))p_1+\varepsilon p_2+(1-\varepsilon),0<t<1$$
as f'_t and f are homogeneous of degree $-n-1$. Then $1-\varepsilon<\phi_t<1+\varepsilon$ and derivatives of ϕ_t up to order two are sufficiently close to 0 by taking sufficiently small ε as we can see easily from computations. Thus, using ϕ_t we obtain a function f obtained from f'_t and $\phi_t f$ on $C(W)$ and extending them smoothly for sufficiently small $|t|$.

We can check the welded function from f'_t and $\phi_t f$ has the desired Hessian properties for μ_t for sufficiently small t since the derivatives of ϕ_t up to order two can be made sufficiently close to zero. Now we do this for every p-end of $\tilde{\mathscr{O}}$.

The $-(n+1)$-homogeneity gives us the invariance of the Hessian metric under the scalar dilatations and the affine lifts of the holonomy groups. (See Chapter 4 of [86].) This completes the proof for Proposition 11.2.4. [$\mathbb{S}^n$S] □

This is a strengthened version of Proposition 11.2.4.

COROLLARY 11.2.8. *Let $\mathscr{O}$ be a strongly tame orbifold with ends and satisfies (IE) and (NA). Suppose that $\mathscr{O}$ has an SPC-structures μ_0 with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and the suspension of $\mathscr{O}$ with μ_0 has a Hessian metric. Let U be a union of mutually disjoint end neighborhoods of $\mathscr{O}$. Suppose the following hold*:

- *Let μ_1 be an SPC-structure with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends so that μ_0 and μ_1 are ε-close in C^r-topology on $\mathscr{O}-U$ for $r\geq 2$, and*
- *the R-end holonomy group of μ_1 are either lens-type or horospherical type and the T-end holonomy group of μ_1 are totally geodesic satisfying the lens conditions.*

Then for sufficiently small ε, the affine suspension $C(\tilde{\mathscr{O}})$ for $\tilde{\mathscr{O}}$ with μ_1 also has a Hessian metric invariant under dilations and the affine suspensions of the holonomy homomorphism for μ_1.

PROOF. We use Corollary 11.2.7 so that (11.2.6) holds with sufficiently small ε. Now we use the step (B) of the proof of Proposition 11.2.4. $[\mathbb{S}^n\mathrm{S}]$ $\square$

LEMMA 11.2.9. *Let V be a properly convex cone, and let V^* be a dual cone. Suppose that a Koszul-Vinberg function $f_{V^*}(x)$ is defined on a compact neighborhood B of x contained in a convex cone V. Let V_1 be another properly convex cone containing the same neighborhood. Let $\Omega := \mathbb{S}(V^*)$ and $\Omega_1 := \mathbb{S}(V_1^*)$ for the dual V_1^* of V_1. For given any integer $s \geq 1$ and $\varepsilon > 0$, there exists $\delta > 0$ so that if the Hausdorff distance between Ω and Ω_1 is δ-close, then $f_{V^*}(x)$ and $f_{V_1^*}(x)$ are ε-close in B in the C^r-topology.*

PROOF. We prove for $\mathbb{S}^n$. By Lemma 1.5.14, we have

$$\begin{aligned} &\Omega^* \subset N_\delta(\Omega_1^*), \Omega_1^* \subset N_\delta(\Omega^*), \\ &(\Omega - N_\delta(\partial\Omega))^* \subset \Omega_1^*, \text{ and} \\ &(\Omega_1 - N_\delta(\partial\Omega_1))^* \subset \Omega^* \end{aligned} \tag{11.2.7}$$

provided δ is sufficiently small. We choose sufficiently small $\delta > 0$ so that

$$B \subset \Omega - N_\delta(\partial\Omega), \Omega_1 - N_\delta(\partial\Omega_1).$$

Recall the Koszul-Vinberg integral (11.2.1). For fixed $\phi \in V^*$ or $\in V_1^*$, the functions $e^{-\phi(x)}$ and the derivatives of $e^{-\phi(x)}$ with respect to x in the domains are uniformly bounded on B since ϕ and its derivatives are bounded function on B. The integral is computable from an affine hyperspace meeting V^* and V_1^* in bounded precompact convex sets. Also, the integration is with respect to ϕ. The result follows by (11.2.7). (See Section 4.1.2 of [86].) $[\mathbb{S}^n\mathrm{S}]$ $\square$

11.2.2. The proof of Theorem 11.2.1.

THE PROOF OF THEOREM 11.2.1. Suppose that $\mathscr{O}$ has an SPC-structure μ with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. Let $\mathbb{U}$ be the union of end neighborhoods of product form with mutually disjoint closures. By premises, the end structures are given.

We assume that μ_0 and μ_s correspond to elements of $\mathscr{U}$ in $\mathrm{Def}^s_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$. We show that a structure μ_s that has generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends sufficiently close μ in $\mathscr{O} - \mathbb{U}$ is also SPC.

Let $h : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$ be the lift of the holonomy homomorphism corresponding to μ_0 where $\tilde{\mathscr{O}} \subset \mathbb{S}^n$ is a properly convex domain covering $\mathscr{O}$. Let $h_s : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$ be the lift of the holonomy homomorphism corresponding to μ_s sufficiently close to h in

$$\mathscr{U} \subset \mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{SL}_\pm(n+1,\mathbb{R})).$$

By Theorem 9.4.5, it corresponds to a real projective structure μ_s on $\mathscr{O}$. Since $[\mu_s] \in \mathrm{Def}^s_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$, it is sufficient to show that μ_s is a properly convex real projective structure.

Let $\mathscr{O} := C(\tilde{\mathscr{O}})/h(\pi_1(\mathscr{O}))$ with $C(\tilde{\mathscr{O}})$ as the universal cover. Let $\tilde{\mathscr{O}}_s$ denote $\tilde{\mathscr{O}}$ with μ_s. One applies special affine suspension to obtain an affine orbifold $\mathscr{O} \times \mathbb{S}^1$. (See Section 1.2.1.) The universal cover is still $C(\tilde{\mathscr{O}})$ and has a corresponding affine structure $\tilde{\mu}_s$. We denote $C(\tilde{\mathscr{O}})$ with the lifted affine structure of $\tilde{\mu}_s$ by $C(\tilde{\mathscr{O}})_s$. Recall the Kuiper completion $\hat{C}(\tilde{\mathscr{O}})_s$ of $C(\tilde{\mathscr{O}})_s$. This is a completion of $C(\tilde{\mathscr{O}})_s$ the path metric induced from the pull-back of the standard Riemannian metric on $\mathbb{S}^{n+1}$ by the developing map $\mathbf{dev}_s$ of $\tilde{\mu}_s$. (Here the image is in $\mathbb{R}^{n+1}$ as an affine subspace of $\mathbb{S}^{n+1}$.) The developing maps always extend to ones on $\hat{C}(\tilde{\mathscr{O}})_s$ which we denote by $\mathbf{dev}_s$ again. (See [43] and [46] for details.)

By Corollary 11.2.8, an affine suspension $\tilde{\mu}_s$ of μ_s also have a Hessian function ϕ since μ_s is in a sufficiently small C^2-neighborhood of μ in $\mathscr{O}-\mathbb{U}$. The Hessian metric $Dd\phi$ is invariant under affine automorphism groups of $C(\tilde{\mathscr{O}})$ by construction. We prove that $\tilde{\mu}_s$ is properly convex, which will show μ_s is properly convex:

Suppose that $\tilde{\mu}_s$ is not convex. Then there exists a triangle T embedded in $\hat{C}(\tilde{\mathscr{O}})_s$ where a point in the interior of an edge of T is in the ideal set

$$\delta_\infty\tilde{\mathscr{O}}_s := \hat{C}(\tilde{\mathscr{O}})_s - C(\tilde{\mathscr{O}})_s$$

while T^o and the union l' of two other edges are in $C(\tilde{\mathscr{O}})_s$. We can move the triangle T so that the interior of an edge l has a point x_∞ in $\delta_\infty\tilde{\mathscr{O}}_s$, and $\mathbf{dev}_s(l)$ does not pass the origin. We form a parameter of geodesics $l_t, t\in[0,\varepsilon]$ in T so that

$$l_0 = l \text{ and } l_t \subset C(\tilde{\mathscr{O}})_s \text{ with } \partial l_t \subset l'$$

is close to l in the triangle. (See Theorem A.2 of [46] for details.)

Let p,q be the endpoints of l. Then the Hessian metric is $D^s d\phi$ for a function ϕ defined on $C(\tilde{\mathscr{O}})_s$. And $d\phi|p$ and $d\phi|q$ are bounded, where D^s is the affine connection of μ_s. This should be true for p_t and q_t for sufficiently small t uniformly. Let u, $u\in[0,1]$, be the affine parameter of l_t, i.e., $l_t(s)$ is a constant speed line in $\mathbb{R}^{n+1}$ when developed. We assume that $u\in(\varepsilon_t, 1-\varepsilon_t)$ parameterize l_t for sufficiently small t where $\varepsilon_t\to 0$ as $t\to 0$ and $dl_t/ds=\vec{v}$ for a parallel vector $\vec{v}$. The function $D^s_v d_v\phi(l_t(u))$ is uniformly bounded since its integral $d_{\vec{v}}\phi(l_t(u))$ is strictly increasing by the strict convexity and converges to certain values as $u\to\varepsilon_t, 1-\varepsilon_t$.

Since

$$\int_{\varepsilon_t}^{1-\varepsilon_t} D^s_v d_v\phi(l_t(u))du = d\phi(p_t)(\vec{v}) - d\phi(q_t)(\vec{v}),$$

the function $\sqrt{D^s_{\vec{v}} d_{\vec{v}}\phi(l_t(u))}$ is also integrable by Jensen's inequality, and the length of l_t

$$\int_{\varepsilon_t}^{1-\varepsilon_t}\sqrt{D^s_{\vec{v}} d_{\vec{v}}\phi(l_t(u))}du$$

under the Hessian metric $Dd\phi$ have an upper bound $\sqrt{d\phi(p_t)(\vec{v})-d\phi(q_t)(\vec{v})}$ by the same inequality. Since

$$\sqrt{d\phi(p_t)(\vec{v})-d\phi(q_t)(\vec{v})} \to \sqrt{d\phi(p_0)(\vec{v})-d\phi(q_0)(\vec{v})} \text{ as } t\to 0,$$

the length of l_t is uniformly bounded.

$\mathbb{U}$ corresponds to an inverse image $\widetilde{\mathbb{U}}$ in $\tilde{\mathscr{O}}$ and to $C(\tilde{\mathbb{U}})_s$ the inverse image in $C(\tilde{\mathscr{O}})_s$. The minimum distance between components of $\mathbb{U}$ is bounded below since the metric is invariant under scalar dilatations in $C(\tilde{\mathscr{O}})_s$. Since $(C(\tilde{\mathscr{O}}_s)-C(\tilde{\mathbb{U}})_s)/\langle\mathbb{R}_+\mathrm{I}\rangle$ is compact, if l meets infinitely many components of $C(\widetilde{\mathbb{U}})_s$, then the length is infinite.

As $t\to 0$, the number is thus bounded, l can be divided into finite subsections, each of which meets at most one component of $C(\widetilde{\mathbb{U}})_s$.

Let $\hat{l}$ be the subsegment of l in $C(\tilde{\mathscr{O}})_s$ containing x_∞ in the ideal set of the Kuiper completion of $C(\tilde{\mathscr{O}}_s)$ with respect to $\mathbf{dev}_s$ and meeting only one component $C(\tilde{\mathbb{U}}_1)_s$ of $C(\widetilde{\mathbb{U}})_s$ with $\mathrm{bd}s\in\mathrm{bd}C(\mathbb{U}_1)_s$. Let $\hat{l}_t$ be the subsegment of l_t so that the parameter of the endpoints of segements of form $\hat{l}_t$ converges to those of $\hat{l}$ as $t\to 0$. Let p' and q' be the endpoint of $\hat{l}$.

Suppose that $C(\tilde{\mathbb{U}}_1)_s \subset C(\tilde{\mathscr{O}})_s$ corresponds to a lens-shaped or horospherical R-p-end neighborhood $\tilde{\mathbb{U}}'_1$ in $\tilde{\mathscr{O}}_s$. and x_∞ is on a line corresponding to the p-end vertex of $\tilde{\mathbb{U}}'_1$. We project to $\mathbb{S}^n$ from by the projection $\Pi' : \mathbb{R}^{n+1} - \{O\} \to \mathbb{S}^n$. Then by the proper-convexity of $\tilde{\Sigma}_{\tilde{E}}$ contradicts this when $\tilde{E}$ is an R-p-end of generalized lens-type. When $\tilde{E}$ is a horospherical p-end, the whole segment must be in $\mathrm{bd}\tilde{\mathscr{O}}$ by convexity. Theorem 8.1.3 contradicts this.

Now suppose that $\Pi'(x_\infty)$ is in the middle of the radial line from the p-end vertex. Then the interior of the triangle is transverse to the radial lines. Since our p-end orbifold $\tilde{\Sigma}_{\tilde{E}}$ is convex, there cannot be such a line with a single interior point in the ideal set.

If $C(\tilde{\mathbb{U}}_1)_s$ is the inverse image in $C(\tilde{\mathscr{O}})_s$ of a generalized lens-shaped T-p-end neighborhood $\tilde{\mathbb{U}}_1$ in $\tilde{\mathscr{O}}_s$, then clearly there is no such a segment l containing an ideal x_∞ in its interior similarly.

Now suppose that a subsegment l_1 of l contains an ideal pojnt in its interior but is disjoint from $\tilde{\mathbb{U}}$. There is connected arc in $l_1 \cap C(\tilde{\mathscr{O}} - \tilde{\mathbb{U}})_s$ ending at an ideal point x_∞. This is arc is never in a compact subset of $C(\tilde{\mathscr{O}})_s$. However, we showed above that the Hessian length of l_t is bounded. Since for a subarc $l_{1,t}$ of l_t, the parameter $\{l_{1,t}\}$ converges to l_1 as $t \to 0$. Thus, the Hessian length of l_1 is also finite. Since $C(\tilde{\mathscr{O}} - \tilde{\mathbb{U}})$ covers a compact orbifold that is the affine suspended over $\mathscr{O} - \mathbb{U}$, the Hessian metric is compatible with any Riemannian metric. Since l_1 is in a compact orbifold, it cannot have a finite Riemannian length.

This is again a contradiction. Therefore, $\tilde{\mathscr{O}}_s$ is convex.

Finally, for sufficiently small deformations, the convex real projective structures are properly convex. Suppose not. Then there is a sequence $\{\mu_{s_i}\}$ of sufficiently small deformed convex real projective structures which are not properly convex. By Proposition 1.1.4, there exists a unique great sphere $\mathbb{S}^{i_0}$ in the boundary of the nonproperly convex set. By uniqueness, the holonomy h_{s_i} acts on $\mathbb{S}^{i_0}$.

The sequence of structures converges to the beginning μ in $\tilde{\mathscr{O}} - U$. By taking limits, the original holonomy has to be reducible.

Suppose now that $\mathscr{O}$ with μ is strictly SPC with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. The relative hyperbolicity of $\tilde{\mathscr{O}}$ with respect to the p-ends is stable under small deformations since it is a metric property invariant under quasi-isometries by Theorem 10.3.1.

The irreducibility and the stability follow since these are open conditions in

$$\mathrm{Hom}(\pi_1(\mathscr{O}), \mathsf{SL}_\pm(n+1,\mathbb{R})).$$

Also, the ends are lens-shaped or horospherical.

By Theorem 6.0.4, the holonomy is not in a parabolic group. This completes the proof of Theorem 11.2.1. $[\mathbb{S}^n\mathrm{S}]$ $\square$

11.3. The closedness of convex real projective structures

We recall $\mathrm{rep}^s_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ the subspace of stable irreducible characters of $\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ which is shown to be an open subset of a semialgebraic set in Section 9.2, and denote by $\mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$ the subspace of stable irreducible characters of $\mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$, an open subset of a semialgebraic set.

11.3.1. Preliminary of the section. Recall the definition of compatible end-compactification from Sections 3.1.2 and 3.1.3.

LEMMA 11.3.1. *Let* $h_i, h \in \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O})), \mathrm{SL}_\pm(n+1,\mathbb{R}))$
(*resp.* $\in \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O})), \mathsf{PGL}(n+1,\mathbb{R})))$.

Suppose that the following hold:

- *Let $\mathscr{O}$ be a strongly tame real projective orbifold with ends assigned types $\mathscr{R}$ and $\mathscr{T}$ satisfying* (*IE*) *and* (*NA*) *with a compatible end compactification.*
- *Let Ω_i be a properly convex open domain in $\mathbb{S}^n$* (*resp.* $\mathbb{RP}^n$).
- *Suppose that $\Omega_i/h_i(\pi_1(\mathscr{O}))$ is an n-dimensional noncompact strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends.*
- *Assume that each p-end holonomy group $h_i(\pi_1(E_j))$ of $h_i(\pi_1(\mathscr{O}))$ of type $\mathscr{R}$ has a p-end vertex v_i^j corresponding to the p-end structure where $\{\mathrm{v}_i^j\}$ forms a convergent sequence as $i \to \infty$. We assume that $h_i \mapsto \mathrm{v}_i^j$ extends to an analytic function near h.*
- *Assume that each p-end holonomy group $h_i(\pi_1(E_j))$ of $h_i(\pi_1(\mathscr{O}))$ of type $\mathscr{T}$ has a hyperplane P_i^j containing the p-ideal boundary component where $\{P_i^j\}$ forms a convergent sequence as $i \to \infty$. We assume that $h_i \mapsto P_i^j$ extends to an analytic function near h.*
- *Suppose that $\{h_i\} \to h$ algebraically where h is discrete and faithful.*
- $\mathrm{Cl}(\Omega_i) \to K$ *for a compact properly convex domain $K \subset \mathbb{S}^n$, $K^o \neq \emptyset$.*

Then the following holds:

- *$\mathscr{O}_h := K^o/h(\pi_1(\mathscr{O}))$ is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends to be denoted $\mathscr{O}_h$ diffeomorphic to $\mathscr{O}$.*
- *For each p-end $\tilde{E}$ of the universal cover $\tilde{\mathscr{O}}_h$ of $\mathscr{O}_h$, K^o has a subgroup $h(\pi_1(\tilde{E}))$ acting on a $h(\pi_1(\tilde{E}))$-invariant open set $U_{\tilde{E}}$ where $U_{\tilde{E}}/h(\pi_1(\tilde{E}))$ is an end neighborhood that is one of the following*:
  - *a horospherical or lens-shaped totally geodesic end neighborhood provided $\tilde{E}$ is a $\mathscr{T}$-p-end,*
  - *a horospherical or concave end neighborhood provided $\tilde{E}$ is a $\mathscr{R}$-p-end.*
- *Finally, suppose that there is a fixed strongly tame properly orbifold $\mathscr{O}'$ with an ideal boundary structure and diffeomorphism $f_i : \mathscr{O}' \to \Omega_i/h_i(\pi_1(\mathscr{O}))$ for sufficiently large i extending to a diffeomorphism of an end-compactification $\bar{\mathscr{O}}'$ to the end compactification of $\Omega_i/h_i(\pi_1(\mathscr{O}))$ compatible with R-end and T-end structures given by v_i^j and P_i^j. Then $K^o/h(\pi_1(\mathscr{O}))$ is an orbifold with a diffeomorphism f from $\mathscr{O}'$ extending to a diffeomorphism from $\bar{\mathscr{O}}'$ to an end compactification of $K^o/h(\pi_1(\mathscr{O}))$ with the above R-end and T-end structures.*

PROOF. Again, we prove for $\mathbb{S}^n$. The holonomy group $h(\pi_1(\mathscr{O}))$ acts on K^o with a Hilbert metric. Hence, $K^o/h(\pi_1(\mathscr{O}))$ is an orbifold to be denoted $\mathscr{O}_h$. (See Lemma 1 of [54].)

Since $h \in \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathrm{SL}_\pm(n+1,\mathbb{R}))$ holds, each p-end holonomy group $h(\pi_1(\tilde{E}))$ acts on a horoball $H \subset \mathbb{S}^n$, a generalized lens-cone, or a totally geodesic hypersurface $\tilde{S}_{\tilde{E}}$ with a CA-lens L. In the first case, we can choose a sufficiently small horoball U inside K^o and in H since the sharply supporting hyperspaces at the vertex of H must coincide by the invariance under $h(\pi_1(\tilde{E}))$ by a limiting argument. By Lemma 5.5.2, U component of $K^o - \mathrm{bd}U$ is a p-end neighborhood.

Now, we consider the second case. Let v be a limit of the sequence $\{\mathrm{v}_{\tilde{E},i}\}$ of the fixed p-end vertices of $h_i(\pi_1(\tilde{E}))$. We obtain $\mathrm{v} \in K$. Also, $\mathrm{v} \notin K^o$ since otherwise the elements fixing it has to be of finite order by the proper discontinuity of the Hilbert isometric action of $h(\pi_1(\tilde{E}))$. For each i, $h_i(\pi_1(\tilde{E}))$ acts on a lens-cone $L_i * \{\mathrm{v}_{\tilde{E},i}\}$. We may assume without loss of generality that $\mathrm{v}_{\tilde{E},i}$ is constant by changing $\mathbf{dev}_i$ by a convergent sequence $\{g_i\}$ of elements of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. We may assume that h_i in a continuous parameter converging to h since there are finitely many components in the above real algebraic set. By Corollary A.1.13 and the condition "ce", we may assume that

$$\{\mathrm{Cl}(R_{\mathrm{v}_{\tilde{E},i}}(\Omega_i))\} \to K_{\mathrm{v}}$$

for a properly convex domain K_{v} on which $h(\pi_1(\tilde{E}))$ acts. Since Ω_i is a subset of a tube domain for $R_{\mathrm{v}_{\tilde{E},i}}(\Omega_i)$, we deduce that Ω is a subset of a tube domain for K_{v}^o. Since $K_{\mathrm{v}}^o/h(\pi_1(\tilde{E}))$ is a closed properly convex orbifold, and $R_{\mathrm{v}_{\tilde{E}}}(\Omega) \subset K_{\mathrm{v}}^o$, it follows that they are equal by Lemma 1.4.16. Hence, $R_{\mathrm{v}_{\tilde{E}}}(\Omega)$ is properly convex. By the Hilbert metric on this domain, $h(\pi_1(\tilde{E}))$ acts properly discontinuously on it. Since $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition by premise, Theorem 5.3.21 shows that the action is distanced in a tubular domain corresponding to $R_{\mathrm{v}_{\tilde{E}}}(K^o)$. Hence, $h(\pi_1(\tilde{E}))$ acts properly and cocompactly on a generalized lens L in K. The group $h(\pi_1(\tilde{E}))$ acts on an open set $U_L := L * \{\mathrm{v}_{\tilde{E}}\} - L$. We may choose one with sufficiently large L so that the lower boundary component $\partial_- L$ is a subset in K^o since we can make $U_L \cap \mathscr{T}_{\mathrm{v}_{\tilde{E}}}(F)$ be as small as we wish for any compact fundamental domain F for $R_{\mathrm{v}_{\tilde{E}}}(K^o)$ and $h(\pi_1(\tilde{E}))$. By Lemma 5.5.2, U_L is a p-end neighborhood of $\tilde{E}$.

In the third case, we can find a CA-lens neighborhood of a totally geodesic domain $\tilde{S}_{\tilde{E},i} \subset \mathrm{Cl}(\Omega_i) \cap P_i$ in a hypersurface P_i on which $h_i(\pi_1(\tilde{E}))$ acts. We may assume without loss of generality that P_i is constant by changing $\mathbf{dev}_i$ by a convergent sequence g_i in $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$. We may assume by taking subsequences that $\{\mathrm{Cl}(\tilde{S}_{\tilde{E},i})\} \to D$ for a properly convex domain D by Corollary A.1.13.

By Corollary 1.4.17, $D^o/h(\pi_1(\tilde{E}))$ is a closed orbifold homotopy equivalent to $S_{E,i}$ up to finite manifold covers. By Theorem 5.5.4, $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition with respect to the hyperspace containing D. By Theorem 4.4.1, the group $h(\pi_1(\tilde{E}))$ acts on a component L_1 of $L - P$ is in K^o for a lens L. Then L_1^o is a p-end neighborhood of $\tilde{E}$ by Lemma 5.5.2. Hence, we constructed R-end and T-end structures for each end for $K^o/\pi_1(\mathscr{O})$.

We apply Theorem 6.0.4 to show that the end structure is SPC.

The end compactification $\bar{\mathscr{O}}_h$ of $K^o/h(\pi_1(\mathscr{O}))$ is given by attaching the cover of ideal boundary component for each p-T-end and attaching $\Sigma_E \times [1,0)$ to an end-neighborhood of an R-end E by a diffeomorphism restricting to a proper map as in Section 9.3.

For the final part, recall from Section 9.2 that $\mathrm{Hom}^s_{\mathscr{E},f}(\pi_1(\mathscr{O}), \mathsf{SL}_{\pm}(n+1,\mathbb{R}))$ is Zariski dense. The homomorphism h is in it by Proposition 5.4.4. We choose p-end vertices the hyperspace containing the p-ideal boudndary components be obtained by respective limits of the corresponding sequence of corresponding objects of h_i. By premise on the analytic extension, we can build a fixed section $s_{\mathscr{U}}$ on a Zariski open subset $\mathscr{U}$ containing h by Proposition 5.4.4. By Theorem 9.4.5, we obtain a subspace of the parameter of real projective structures on a strongly tame orbifold $\mathscr{O}'$ with end structures determined by $s_{\mathscr{U}}$ for each point of $\mathscr{U}$.

By premise, the p-end vertices of Ω_i and the hypersurfaces containing the p-ideal boundary components are determined also by $s_{\mathscr{U}}$ for sufficiently large i. Now, $\mathscr{O}' := K^o/h(\pi_1(\mathscr{O}))$ is realized as a convex real projective orbifold with ends determined by $s_{\mathscr{U}}$ also. By Theorem 9.4.5, there is a neighborhood $\mathscr{U}' \subset \mathscr{U}$ where every holonomy is realized by a convex real projective structure with end structures determined by $s_{\mathscr{U}}$. Since h_i may be assumed to be in $\mathscr{U}'$ except for finitely many is, a structure μ_i on $\mathscr{O}'$

has has holonomy h_i. By Theorem 11.3.2, $\Omega_i/h_i(\pi_1(\mathscr{O}))$ is projectively diffeomorphic to $\mathscr{O}'$ with a convex real projective structure μ_i with identical R-end and T-end structures for sufficiently large i.

By Theorem 11.3.2, for μ_i with holonomy in $\mathscr{U}'$, $\Omega_i/h_i(\pi_1(\mathscr{O}))$ has an end compactification $\bar{\mathscr{O}}'_i$. Since this end compactification is compatible with the R-end and T-end structures also, f_i extends to a diffeomorphism $\bar{\mathscr{O}}' \to \bar{\mathscr{O}}'_i$. Since $h_i \in \mathscr{U}'$ deformed from h, $\bar{\mathscr{O}}'_i$ is isotopic to $\bar{\mathscr{O}}_h$.

By premise, the diffeomorphism $f_i : \mathscr{O}' \to \Omega_i/h_i(\pi_1(\mathscr{O}))$ extends smoothly as a diffeomorphism from $\bar{\mathscr{O}}'$ to $\bar{\mathscr{O}}'_i$. Hence, $\mathscr{O}$ and $\mathscr{O}'$ are diffeomorphic with end structures preserved. By Corollary 9.3.3, the respective end compactifications of $\mathscr{O}$ and $\mathscr{O}'$ are diffeomorphic.

Again the $\mathbb{RP}^n$-version follows by Proposition 1.4.2. [$\mathbb{S}^n$P] $\square$

THEOREM 11.3.2 (Uniqueness of domains). *Let Γ be a discrete projective automorphism group of a properly convex open domain $\Omega \subset \mathbb{S}^n$. Suppose that Ω/Γ is a strongly tame SPC n-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Assume $\partial\mathscr{O} = \emptyset$. Suppose that for each $\mathrm{v}_{\tilde{E}} \in \mathrm{bd}\Omega$ for each R-p-end $\tilde{E}$ is specified up to $\mathscr{A}$ in $\mathbb{S}^n$ and so is each hyperplane for each T-p-end $\tilde{E}$ meeting $\mathrm{bd}\Omega$. Then Ω is a unique domain with these properties up to the antipodal map $\mathscr{A}$.*

PROOF. Suppose that Ω_1 and Ω_2 are distinct open domains in $\mathbb{S}^n$ satisfying the above properties. For this, we assume that Γ is torsion-free by taking a finite-index subgroup by Theorem 1.1.19. We claim that Ω_1 and Ω_2 are disjoint:

Suppose that $\Omega' := \Omega_1 \cap \Omega_2$ is a nonempty open set. Since Ω_1, Ω_2, and Ω' are all n-cells, the set of p-ends of Ω_1, the set of those of Ω_2, in one-to-one correspondences by considering their p-end fundamental groups. The types are also preserved by the premise.

Suppose that $\tilde{E}_1$ and $\tilde{E}_2$ are corresponding R-p-ends of generalized lens-type. The p-end vertex $\mathrm{v}_{\tilde{E}_j}$ of a generalized lens-shaped R-p-end $\tilde{E}_j$ of Ω_j, $j = 1,2$, is determined up to $\mathscr{A}$ by the premise.

Suppose that $\mathrm{v}_{\tilde{E}_1}$ and $\mathrm{v}_{\tilde{E}_2}$ are antipodal. The interior of $\Omega' * \{\mathrm{v}_{\tilde{E}_j}\}$, $j = 1,2$, are in Ω_j by the convexity of $\mathrm{Cl}(\Omega_j)$. Also, $\Omega_1 \cup \Omega_2$ is in a convex tube $\mathscr{T}_{\mathrm{v}_{\tilde{E}_1}}(\tilde{\Sigma}_{\tilde{E}_1})$ with the vertices $\mathrm{v}_{\tilde{E}_1}$ and its antipode $\mathrm{v}_{\tilde{E}_2}$ in the direction of $\tilde{\Sigma}_{\tilde{E}_1}$. Also, the convex hull of $\mathrm{Cl}(\Omega_1) \cup \mathrm{Cl}(\Omega_2)$ equals $\mathscr{T}_{\mathrm{v}_{\tilde{E}_1}}(\tilde{\Sigma}_{\tilde{E}_1})$. $h(\pi_1(\mathscr{O}))$ acts on the unique pair of antipodal points $\{\mathrm{v}_{\tilde{E}_1}, \mathrm{v}_{\tilde{E}_2}\}$. Hence, $h(\pi_1(\mathscr{O}))$ is reducible contradicting the premise.

Suppose that $\mathrm{v}_{\tilde{E}_1} = \mathrm{v}_{\tilde{E}_2}$. Then $R_{\mathrm{v}_{\tilde{E}_1}}(\Omega_1)$ and $R_{\mathrm{v}_{\tilde{E}_1}}(\Omega_2)$ are not disjoint since otherwise Ω_1 and Ω_2 are disjoint. Lemma 1.4.16 shows that they are equal. The generalized lens-cone p-end neighborhood U_1 in Ω_1 and one U_2 in Ω_2 must intersect. Hence, by Lemma 5.5.2, the intersection of a generalized lens-cone p-end neighborhood of Ω_1 and that of Ω_2 is one for Ω':

Suppose that $\tilde{E}_j$ is a horospherical p-end of Ω_j, $j = 1,2$. Then p-end vertices $\mathrm{v}_{\tilde{E}_j}$, $j = 1,2$, are either equal or antipodal since there is a unique antipodal pair of fixed points for the cusp group $\Gamma_{\tilde{E}_j}$, $j = 1,2$. Since the fixed point in $\mathrm{v}_{\tilde{E}_j}$ is the unique limit point of $\{\gamma^n(p)\}$ as $n \to \infty$ for any $p \in \Omega_j$, it follows that $\mathrm{v}_{\tilde{E}_1} = \mathrm{v}_{\tilde{E}_2}$. We can verify that Ω_1, Ω_2, and Ω' share a horospherical p-end neighborhood from this by Lemma 5.5.2.

Similarly, consider the ideal boundary component $\tilde{S}_{\tilde{E}_1}$ for a T-p-end $\tilde{E}_1$ of Ω_1 and the corresponding $\tilde{S}_{\tilde{E}_2}$ for a T-p-end $\tilde{E}_2$ of Ω_2. Since $\mathbf{\Gamma}_{\tilde{E}_1}$ acts on a properly convex domain Ω', Theorem 5.5.4 and Lemma 4.4.2 show that $\mathrm{Cl}(\Omega') \cap P$ is a nonempty properly convex set in $\mathrm{Cl}(\tilde{S}_{\tilde{E}_1})$.

We claim that a point $\mathrm{Cl}(\tilde{S}_{\tilde{E}_2})$ for Ω_2 cannot be antipodal to any point of $\mathrm{Cl}(\tilde{S}_{\tilde{E}_1})$: Suppose not. Then $\mathrm{Cl}(\tilde{S}_{\tilde{E}_2}) = R_2(\mathrm{Cl}(\tilde{S}_{\tilde{E}_1}))$ for a projective automorphism R_2 acting as I or $\mathscr{A}$ on a collection of independent subspaces of P by Lemma 1.4.16. There exists a pair of extremal points $p_1 \in \mathrm{Cl}(\tilde{S}_{\tilde{E}_1})$ and $p_2 \in \mathrm{Cl}(\tilde{S}_{\tilde{E}_2})$, antipodal to each other. Here, there exists a point $c \in \tilde{S}_{\tilde{E}_j}$ and a sequence $g_i^{(j)}$ such that $\{g_i^{(j)}(c)\} \to p_j$ by Lemma 5 of [151]. By Lemma 4.4.2, $\{g_i^{(j)}(d)\} \to p_j$ for a point $d \in \Omega'$. This implies that Ω' is not properly convex, a contradiction.

Thus, we obtain $\tilde{S}_{\tilde{E}} = \tilde{S}'_{\tilde{E}}$ again by Lemma 1.4.16. Since Ω' is a $h(\pi_1(\tilde{E}))$-invariant open set in one side of P, it follows that Ω' contains a one-sided lens neighborhood L_1 by Lemma 4.3.1. By Lemma 5.5.2, L_1 is a p-end neighborhood of Ω'.

We have concave p-end neighborhoods for radial p-ends, lens p-end neighborhoods for totally geodesic p-ends, and horoball p-end neighborhoods of p-ends for each of Ω_1, Ω_2, and Ω'. We verify from above discussions that a p-end neighborhood of Ω_1 exists if and only if a p-end neighborhood of Ω_2 exists and their intersection is a p-end neighborhood of Ω'. Ω'/Γ is a closed submanifold in Ω_1/Γ and in Ω_2/Γ. Thus, $\Omega_1/\Gamma, \Omega_2/\Gamma$, and Ω'/Γ are all homotopy equivalent relative to the union of disjoint end-neighborhoods. The map has to be onto in order for the map to be a homotopy equivalence as we can show using relative homology theories, and hence, $\Omega' = \Omega_1 = \Omega_2$.

Suppose that Ω_1 and $\mathscr{A}(\Omega_2)$ meet. Then similarly, $\Omega_1 = \mathscr{A}(\Omega_2)$.

Suppose now that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cap \mathscr{A}(\Omega_2) = \emptyset$. Suppose that Ω_1 has a p-end $\tilde{E}$ of type $\mathscr{R}$. The corresponding pair of the p-end neighborhoods share the p-end vertex or have antipodal p-end vertices by the premise. Since Ω_1 and Ω_2 are disjoint, it follows that $\mathrm{Cl}(\Omega_1) \cap \mathrm{Cl}(\Omega_2)$ or $\mathrm{Cl}(\Omega_1) \cap \mathscr{A}(\mathrm{Cl}(\Omega_2)$ is a compact properly convex subset of dimension $< n$ and is not empty since the end vertex of the p-ends are in it. The minimal hyperspace containing it is a proper subspace and is invariant under Γ. This contradicts the strong irreducibility of $h(\pi_1(\mathscr{O}))$ as can be obtained from Theorem 6.0.4. This also applies to the case when $\tilde{E}$ is a horospherical end of type $\mathscr{T}$.

Suppose that Ω_1 has a p-end $\tilde{E}_1$ of type $\mathscr{T}$. Then Ω_2 has a p-end $\tilde{E}_2$ of type $\mathscr{T}$. Now, $\Gamma_{\tilde{E}_1} = \Gamma_{\tilde{E}_2}$ acts on a hyperspace P containing the $\tilde{S}_{\tilde{E}_i}$ in the boundary of $\mathrm{bd}\Omega_i$ for $i = 1, 2$. Here, $\tilde{S}_{\tilde{E}_1}/\Gamma_{\tilde{E}_1}$ and $\tilde{S}_{\tilde{E}_2}/\Gamma_{\tilde{E}_2}$ are closed $n-1$-orbifolds. Lemma 1.4.16 shows that their closures always meet or they are antipodal. Hence, up to $\mathscr{A}$, their closures always meet. Again

$$\mathrm{Cl}(\Omega_1) \cap \mathrm{Cl}(\Omega_2) \neq \emptyset \text{ or } \mathrm{Cl}(\Omega_1) \cap \mathscr{A}(\mathrm{Cl}(\Omega_2)) \neq \emptyset$$

while we have $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cap \mathscr{A}(\Omega_2) = \emptyset$. We obtain a lower-dimensional convex subspace fixed by Γ. This is a contradiction. $\square$

11.3.2. The main result for the section. This generalizes Theorem 4.1 of [61] for closed orbifolds which is really due to Benoist [23].

THEOREM 11.3.3. *Let $\mathscr{O}$ be a strongly tame SPC n-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies (IE) and (NA). Assume $\partial \mathscr{O} = \emptyset$. We have an open $\mathsf{PGL}(n+1, \mathbb{R})$-conjugation invariant set $\mathscr{U}$ in a semi-algebraic subset of*

$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1, \mathbb{R})),$$

and a $\mathsf{PGL}(n+1, \mathbb{R})$-equivariant fixing section $s_{\mathscr{U}} : \mathscr{U} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$. Let $\mathscr{U}'$ denote the quotient set under $\mathsf{PGL}(n+1, \mathbb{R})$. Assume that every finite index subgroup of $\pi_1(\mathscr{O})$ has no nontrivial nilpotent normal subgroup. Then the following hold:

- *The deformation space* $\mathrm{CDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ *of SPC-structures on* $\mathscr{O}$ *with generalized lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends maps under* hol *homeomorphically to a union of components of*
$$\mathscr{U}' \subset \mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$
- *The deformation space* $\mathrm{SDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ *of strict SPC-structures on* $\mathscr{O}$ *with lens-shaped or horospherical* $\mathscr{R}$*- or* $\mathscr{T}$*-ends maps under* hol *homeomorphically to the union of components of*
$$\mathscr{U}' \subset \mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

PROOF. Define $\widetilde{\mathrm{CDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ to be the inverse image of $\mathrm{CDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ in the isotopy-equivalence space $\widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O})$ in Definition 9.3.5. Let $\widetilde{\mathscr{U}}$ denote the points of the inverse image of $\mathscr{U}$ in $\widetilde{\mathrm{CDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ where the vertices of R-p-ends and hyperspaces of T-p-ends are determined by $s_{\mathscr{U}}$. Then $\widetilde{\mathscr{U}}$ is an open subset by Theorem 9.4.5 and Theorem 11.1.4.

We show that
$$\mathrm{hol}: \widetilde{\mathscr{U}} \to \mathscr{U} \subset \mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
is a homeomorphism onto a union of components. This will imply the results. Theorem 11.3.2 shows that hol is injective.

Now, hol is an open map by Theorems 11.2.1 and 9.4.5. To show that the image is of hol is closed, we show that the subset of $\mathscr{U}$ corresponding to elements in $\widetilde{\mathscr{U}}$ is closed. Let $(\mathbf{dev}_i, h_i)$ be a sequence of development pairs so that we have $\{h_i\} \to h$ algebraically. Let $\Omega_i = \mathbf{dev}_i(\tilde{\mathscr{O}})$ denote the corresponding properly convex domains for each i. The limit h is a discrete representation by Lemma 1.1 of Goldman-Millson [92]. Let $\hat{\Omega}_i$ denote the lift of Ω_i in $\mathbb{S}^n$ and let $\hat{h}_i: \pi_1(\mathscr{O}) \to \mathrm{SL}_{\pm}(n+1,\mathbb{R})$ be the corresponding lift of h_i by Theorem 1.1.20. The sequence $\{\mathrm{Cl}(\Omega_i)\}$ also geometrically converges to a compact convex set $\hat{\Omega}$ up to choosing a subsequence by Proposition 1.1.9 where $\hat{h}(\pi_1(\mathscr{O}))$ acts on as in Lemma 1 of [54]. If $\hat{\Omega}$ have the empty interior, h is reducible, and $h \notin \mathscr{U}$, contradicting the premise. If $\hat{\Omega}^o$ is not empty and is not properly convex, then the lift of Ω to $\mathbb{S}^n$ contains a maximal great sphere $\mathbb{S}^i$, $i \geq 1$, or a unique pair of antipodal points $\{p, p_-\}$ by Proposition 1.1.4. In the both cases, h is reducible. Thus, $\hat{\Omega}^o$ is not empty and is properly convex. Let Ω denote the image of $\hat{\Omega}$ under the double covering map. As in [54], since Ω^o has a Hilbert metric, $h(\pi_1(\mathscr{O}))$ acts on Ω^o properly discontinuously.

By Lemma 11.3.1, the condition of the generalized lens or horospherical condition for $\mathscr{R}$-ends or lens or horospherical condition for $\mathscr{T}$-ends of the holonomy representation is a closed condition in the
$$\mathrm{Hom}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$
as we defined above. (It is of course redundant to say that it is not a closed condition when we drop the notation "ce" from the above character space.)

Define $\mathscr{O}' := \Omega^o/h(\pi_1(\mathscr{O}))$. We can deform $\mathscr{O}'$ with holonomy in an open subset of $\widetilde{\mathscr{U}}$ using the openness of hol by Theorem 11.2.1. We can find a deformed orbifold $\mathscr{O}''_i$ that has a holonomy h_i for some large i. Now, $\Omega_i/h_i(\mathscr{O})$ is diffeomorphic to $\mathscr{O}$ being in the deformation space. $\mathscr{O}''_i$ is diffeomorphic to $\mathscr{O}$ with the corresponding end-compactifications since they share the same open domain as the universal cover by the uniqueness for each holonomy group by Theorem 11.3.2. By the openness of the map hol for $\mathscr{O}'$, $\mathscr{O}''$ is diffeomorphic to $\mathscr{O}'$. Hence, $\mathscr{O}'$ is diffeomorphic to $\mathscr{O}$.

Therefore, we conclude that $\widetilde{\mathscr{U}}$ goes to a closed subset of $\mathscr{U}$. The proof up to here imply the first item.

Now, we go to the second item. Define $\widetilde{\mathrm{SDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ to be the inverse image of $\mathrm{SDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ in the isotopy-equivalence space $\widetilde{\mathrm{Def}}_{\mathscr{E}}(\mathscr{O})$. Let $\widetilde{\mathscr{U}}$ denote the inverse image of $\mathscr{U}$ in $\widetilde{\mathrm{SDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$. We show that

$$\mathrm{hol} : \widetilde{\mathscr{U}} \to \mathscr{U}$$

is a homeomorphism onto a union of components of $\mathscr{U}$. Theorem 11.2.1 shows that hol is a local homeomorphism to an open set. The injectivity of hol follows the same way as in the above item.

We now show the closedness. By Theorem 10.3.1, $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to the end fundamental groups. Let h be the limit of a sequence of holonomy representations $\{h_i : \pi_1(\mathscr{O}) \to \mathsf{PGL}(n+1,\mathbb{R})\}$. As above, we obtain Ω as the limit of $\mathrm{Cl}(\Omega_i)$ where Ω_i is the image of the developing map associated with h_i. Ω is properly convex and Ω^o is not empty. Since h is irreducible and acts on Ω^o properly discontinuously, it follows that $\Omega^o/h(\pi_1(\mathscr{O}))$ is a strongly tame properly convex orbifold $\mathscr{O}'$ with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends by the above part of the proof. By Theorem 10.3.4 and Corollary 6.3.3, $\mathscr{O}'$ is a strict SPC-orbifold with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends. The rest is the same as above. □

REMARK 11.3.4 (Thurston's example). We remark that without the end controls we have, there might be counter-examples as we can see from the examples of geometric limits differing from algebraic limits for sequences of hyperbolic 3-manifolds. (See Anderson-Canary [2].)

11.3.3. Dropping of the superscript s. We can drop the superscript s from the above space. Hence, the components consist of stable irreducible characters. This is a stronger result.

COROLLARY 11.3.5. *Let $\mathscr{O}$ be a noncompact strongly tame SPC n-dimensional orbifold with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends and satisfies* (IE) *and* (NA)*. Assume $\partial\mathscr{O} = \emptyset$. Assume that no finite-index subgroups $\pi_1(\mathscr{O})$ has a nontrivial nilpotent normal subgroup. We have a $\mathsf{PGL}(n+1,\mathbb{R})$-conjugation invariant set $\mathscr{U}$ open in a union of semi-algebraic subsets of*

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})),$$

and a $\mathsf{PGL}(n+1,\mathbb{R})$-equivariant fixing section $s_{\mathscr{U}} : \mathscr{U} \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$. Let $\mathscr{U}'$ denote the quotient set under $\mathsf{PGL}(n+1,\mathbb{R})$. Then the following hold:

- *The deformation space $\mathrm{CDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ of SPC-structures on $\mathscr{O}$ with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends maps under* hol *homeomorphically to a union of components of*

$$\mathscr{U}' \subset \mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

- *The deformation space $\mathrm{SDef}_{\mathscr{E},s_{\mathscr{U}},\mathrm{lh}}(\mathscr{O})$ of SPC-structures on $\mathscr{O}$ with lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends maps under* hol *homeomorphically to the union of components of*

$$\mathscr{U}' \subset \mathrm{rep}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

Furthermore, $\mathscr{U}'$ has to be in $\mathrm{rep}^s_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$.

PROOF. We define $\widetilde{\mathrm{CDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ and $\widetilde{\mathrm{SDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$ as above. Let $\widetilde{\mathscr{U}}$ be the inverse image of $\mathscr{U}$. We will show that the image of $\widetilde{\mathscr{U}}$ under hol in $\mathscr{U}$ is closed and consists of stable irreducible characters. Now Theorem 11.3.3 implies the result.

We will prove by lifting to $\mathbb{S}^n$. Using Theorem 1.1.20, let

$$\{h_i : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})\}$$

be a sequence of holonomy homomorphisms of real projective structures corresponding to liftings of elements of $\widetilde{\mathscr{U}}$. These are stable and strongly irreducible representations by Theorem 6.0.4. Let Ω_i be the sequence of associated properly convex domains in $\mathbb{S}^n$, and $\Omega_i/h_i(\pi_1(\mathscr{O}))$ is diffeomorphic to $\mathscr{O}$ and has the structure that lifts an element of $\mathrm{CDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$. We assume that $\{h_i\} \to h$ algebraically, i.e., for a fixed set of generators $g_1,\dots,g_m$ of $\pi_1(\mathscr{O})$, $\{h_i(g_j)\} \to h(g_j) \in \mathrm{SL}_\pm(n+1,\mathbb{R})$ as $i \to \infty$. The limit h is a discrete representation by Lemma 1.1 of Goldman-Millson [92]. We will show that h is a lifted holonomy homomorphism of an element of $\mathrm{CDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$, and hence h is stable and strongly irreducible.

Here we are using the definition of convexity for $\mathbb{S}^n$ as given in Definition 1.1.1. Since the Hausdorff metric space $\mathbf{d}_H$ of compact subsets of $\mathbb{S}^n$ is compact, we may assume that $\{\mathrm{Cl}(\Omega_i)\} \to K$ for a compact convex set K by taking a subsequence if necessary as in [54]. We take a dual domain $\Omega_i^* \subset \mathbb{S}^{n*}$. Then the sequence $\{\mathrm{Cl}(\Omega_i^*)\}$ also geometrically converges to a convex compact set K^* by Proposition 1.5.15. (See Section 1.5.4.)

Recall the classification of compact convex sets in Proposition 1.1.4. For any 1-form α positive on the cone C_K, any sufficiently close 1-form is still positive on C_K. If K has an empty interior and properly convex, then we can easily show that K^* has a nonempty interior. Also, if K^* has an empty interior and properly convex, K has a nonempty interior.

(I) The first step is to show that at least one of K and K^* has nonempty interior. We divide into four cases (i)-(iv) where the types change for R-ends and T-ends.

(i) To begin, suppose that there exists a radial p-end $\tilde{E}$ for Ω_i and h_i and the type does not become horospherical. We may assume that $\mathrm{v}_{\tilde{E},h_i} = \mathrm{v}_{\tilde{E},h}$ by conjugating h_i by a bounded sequence of projective automorphisms. Then the mc-p-end neighborhood must be in K since this is true for all structures in Ω_i and holonomy homomorphisms in

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathrm{SL}_\pm(n+1,\mathbb{R})).$$

For each i, h_i acts on a lens-cone $\mathrm{v}_{\tilde{E},h_i} * L_i$ in Ω_i for each p-end $\tilde{E}$. By Theorems 5.4.2 and 5.4.3, L_i can be chosen to be the convex hull of the closure of the union of attracting fixed sets of elements of $h_i(\pi_1(\tilde{E}))$. Hence, h is also in it, and Theorem 5.1.4 shows that there exists a distanced compact convex set L distanced away from a point x, and the lens-cone $\{\mathrm{v}_{\tilde{E},h}\} * L - \{\mathrm{v}_{\tilde{E},h}\}$ has a nonempty interior.

Choose an element $g_1 \in \pi_1(\tilde{E})$ so that $h(g_1)$ is positive bi-semi-proximal by Theorem 1.3.12. Then $h_i(g_1)$ is also positive bi-semi-proximal for sufficiently large i since $\{h_i(g_1)\} \to h(g_1)$ as a sequence. We may assume that

$$\{A_{h_i(g_1)}\} \to A'_{h(g_1)} \subset A_{h(g_1)} \subset \mathrm{bd} L$$

for attracting-fixed-point sets $A_{h_i(g_1)}$ and $A_{h(g_1)}$ and a compact subset $A'_{h(g_1)}$ since the limits of sequences of eigenvectors are eigenvectors. Since $A'_{h(g_1)}$ is a geometric limit of a sequence compact convex sets, it is compact and convex. (See Section 1.3.2.)

We claim that the convex hull

$$\mathscr{C}\mathscr{H}\left(\bigcup_{q\in h(\pi_1(\tilde{E}))} h(q)A'(h(g_1))\cup\{\mathrm{v}_{\tilde{E}}\}\right)$$

has a nonempty interior: Suppose not. Then it is in a proper subspace where $\pi_1(\tilde{E})$ acts on. This means that $\pi_1(\tilde{E})$ is virtually-factorizable, and $\tilde{E}$ is a totally geodesic R-end by Theorem 5.4.3. We choose another g_2 with $A(h(g_2))$ is not contained in this subspace by Proposition 1.4.10. Again, we find $A'(h(g_2))\subset A(h(g_2))$. Now,

$$\mathscr{C}\mathscr{H}\left(\bigcup_{q\in h(\pi_1(\tilde{E})),g=g_1,g_2} h(q)A'(h(g))\cup\{\mathrm{v}_{\tilde{E}}\}\right)$$

is in a strictly larger subspace where $h(\pi_1(\tilde{E}))$ acts on. By induction, we stop at certain point, and we obtain

$$\mathscr{C}\mathscr{H}\left(\bigcup_{q\in h(\pi_1(\tilde{E})),g=g_1,\dots,g_m} h(q)A'(h(g))\cup\{\mathrm{v}_{\tilde{E}}\}\right)$$

that is not contained in a proper subspace.

It is easy to show $h(q)A'(h(g))=A'(h(qgq^{-1}))$. There exists a finitely many elements $g_1,\dots,g_m$ in $\pi_1(\tilde{E})$ so that the attracting fixed set

$$\mathscr{C}\mathscr{H}\left(\{a_1,\dots,a_m,\mathrm{v}_{\tilde{E},h}\}\right), a_j\in A'(h(g_j))$$

has a nonempty interior for some choice of a_j.

We have $A(h_j(g_i))\subset \mathrm{Cl}(L_j)$ for each j and i. The sequence $\{A(h_j(g_i))\}$ accumulates only to points of $A'(h(g_j))$ since $\{h_j(g_i)\}\to h(g_i)$. There is a sequence $\{a_{i,j}\}$, $a_{i,j}\in A(h_j(g_i))$, converging to $a_j\in A'(h(g_j))$ as $i\to\infty$. Lemma 1.1.23 implies that the sequence

$$\{\mathscr{C}\mathscr{H}(\{a_{1,j},\dots,a_{m,j},\mathrm{v}_{\tilde{E},h_i}\})\}\text{ in }\Omega_j$$

converges to $\mathscr{C}\mathscr{H}(\{a_1,\dots,a_m,\mathrm{v}_{\tilde{E}}\})$ geometrically. Since $\mathrm{Cl}(\Omega_j)\to K$ as $j\to\infty$,

$$\mathscr{C}\mathscr{H}(\{a_1,\dots,a_m,\mathrm{v}_{\tilde{E},h}\})\subset K,$$

by Proposition 1.1.7; thus, K has nonempty interior in the case. (i) is accomplished.

(ii) Suppose that there is a lens-shaped totally geodesic p-end $\tilde{E}$ for $\tilde{\mathscr{O}}$ and the holonomy group $h(\pi_1(\tilde{E}))$, and the type for Ω_i and $h_i(\tilde{E})$ do not become horospherical. Then the dual Ω_i^* and K^* have a nonempty interior by above arguments since Ω_i^* has lens-shaped R-p-ends by Corollary 5.5.7. Now, we do the argument for (i) and use the duality at the end by Proposition 1.5.15.

(iii) Suppose that there is a lens-shaped R-p-end $\tilde{E}$ for $\tilde{\mathscr{O}}$ and the holonomy group $h(\pi_1(\tilde{E}))$, and the type for Ω_i and $h_i(\tilde{E})$ becomes horospherical.

This will be sufficient for (I) since when lens-type T-end changes to a horospherical $\mathscr{R}$- or $\mathscr{T}$-end, we can use the duality as in (ii).

We have $\tilde{\Sigma}_{\tilde{E},h}\subset\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ is a complete affine space as $h|\pi_1(\tilde{E})$ is parabolic. We have properly convex domain or a complete affine space $\tilde{\Sigma}_{\tilde{E},h_i}\subset\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}_i}}$. By multiplying the developing maps by a convergent sequence of elements of $\mathsf{SL}_\pm(n+1,\mathbb{R})$, we may assume $\mathrm{v}_{\tilde{E}}=\mathrm{v}_{\tilde{E}_i}$.

Recall that the map from the deformation space of real projective structures on a closed orbifold to the space of representations is a local homeomorphism by Theorem 9.3.10. (See [49].) Corollary A.1.13 shows that $\mathrm{Cl}(\tilde{\Sigma}_{\tilde{E},h_i})\to\tilde{\Sigma}_{\tilde{E},h}$ up to a choice of subsequences.

Now $h(\pi_1(\tilde{E}))$ is the algebraic limit $h_i(\pi_1(\tilde{E}))$. Then $\mathscr{P} \cap h(\pi_1(\tilde{E}))$ is a lattice in a cusp group $\mathscr{P}$. We conjugate $\mathscr{P}$ so that it is a standard unipotent cusp group in $\mathsf{SO}(n,1) \subset \mathsf{SL}_{\pm}(n+1,\mathbb{R})$.

We choose any great segment s_j with vertex v in the direction of $\tilde{\Sigma}_{\tilde{E},h}$. We may assume that these are all inside $\tilde{\Sigma}_{\tilde{E},h_i}$ since this is true for sufficiently large i. Choose finitely many l_j, $j = 1,\dots,m$, so that the directions in the convex domain $\tilde{\Sigma}_{\tilde{E},h_i}$ has a convex hull with a nonempty interior.

Suppose that

$$\{\mathbf{d}\text{-length}(l_j)\} > \varepsilon \text{ for } l_{i,j} := \Omega_i \cap s_j, j = 1,\dots,m, \tag{11.3.1}$$

for a uniform $\varepsilon > 0$. Let l be the geometric limit of a subsequence $\{l_{j_i}\}$ of $\{l_i\}$ with a nonzero $\mathbf{d}$-length. Then $\{h_{j_i}(g)(l_{j_i})\} \to h(g)(l)$ for $g \in \pi_1(\tilde{E})$. For any finite set $F \subset \pi_1(\tilde{E})$, the set

$$\{h_{j_i}(g)(l_{j_i}) | g \in F\} \to \{h(g)(l) | g \in F\}$$

geometrically. By Lemma 1.1.23, we have the geometric convergence of the sequence of convex hulls

$$\{\mathscr{CH}(\bigcup_{g\in F} h_{j_i}(g)(l_{j_i}))\} \to \mathscr{CH}(\bigcup_{g\in F} h(g)(l)).$$

Since the later set has a nonempty interior by our assumption on $\mathbf{d}$-lengths of l_i and $h(g), g \in F$, is in a group conjugate to a cusp group, the convex hull

$$\mathscr{CH}(h_{j_i}(\pi_1(\tilde{E}_{j_i}))(s_{j_i})) \subset \Omega_{j_i}$$

contains a fixed open ball B for sufficiently large i. This means $B \subset K$ showing (I).

Now for the final case, we suppose

$$\{\mathbf{d}\text{-length}(l_{j,i})\} \to 0 \text{ for } l_{j,i} := \Omega_i \cap s_j, j = 1,\dots,m \text{ as } i \to \infty. \tag{11.3.2}$$

LEMMA 11.3.6. *Let* v, $\mathrm{v} = ⟦1,0,\dots,0⟧$ *be a fixed point of the standard unipotent cusp group* $\mathscr{P}$ *and let* L *be a lattice in* $\mathscr{P}$. *Let* H *be a* $\mathscr{P}$*-invariant hemisphere with* v *in the boundary, and let* l *be the maximal line with endpoints* v *and* v_- *perpendicular to* ∂H *with respect to* $\mathbf{d}$. *Then there exists a finite subset* F *of* L *so that the following holds*:

- *for any point* $x \in l$ *and a* $\mathbf{d}$*-perpendicular hyperspace at* x *bounding a closed hemisphere* H_x,
$$I_x := \bigcap_{g\in F} g(H_x)$$
is a properly convex domain, and
- *as* $x \to \mathrm{v}$, *the parameter* $\{I_x\}$ *geometrically converges to* $\{\mathrm{v}\}$.

PROOF. If F is large enough, then $\{g(H_x) | g \in F\}$ is in a general position. We choose the affine coordinate system of H^o as in Section 7.3.1.1 where we let $i_0 = n$.

The set of outer normal vectors of $\{g(\partial H_x)\}$ in the affine subspace H^o are independent of the choice of x. Hence, as $x \to \mathrm{v}$, the corresponding intersection set must be contained in any arbitrarily small ball. □

PROOF OF PROPOSITION 11.3.5 CONTINUED. We may assume $\mathrm{v}_{\tilde{E},h_i} = \mathrm{v}_{\tilde{E},h} = \mathrm{v}$ without loss of generality by changing the developing map by a sequence of bounded automorphisms g_i. Let H denote the $\mathscr{P}$-invariant hemisphere containing K. We assume that $\Omega_i \subset H$ and recall that radial p-end vertices are fixed to be v. We assume without loss of generality that the direction of a segment l of the $\mathbf{d}$-length π is in $\mathbf{F}_i$ always.

Let ε_i be the maximum $\mathbf{d}$-length of a maximal segment s_i' in Ω_i from v in direction of $\mathbf{F}_i$ for $i \geq I$. Let $\mathbf{F}_i'$ denote the set of endpoints of the maximal segments in Ω_i in a direction of $\mathbf{F}_i$. Then $\{\varepsilon_i\} \to 0$ by (11.3.2). A hyperspace perpendicular to l at $x_i \in l$ bounds a closed hemisphere H_i' containing $\mathbf{F}_i'$.

$$\text{For } \delta_i := \mathbf{d}(\mathrm{v}, x_i), \text{ we have } \{\delta_i\} \to 0$$

since otherwise (11.3.1) does not hold. By Lemma 11.3.6, there is a finite set $F \subset \pi_1(\tilde{E})$ so that

$$\hat{K}_i := \bigcap_{g \in F} h_j(g)(H_i') \cap H$$

is properly convex for sufficiently large j since $\{h_j(g)\} \to h(g), g \in F$. This set $\hat{K}_i$ contains $\mathrm{Cl}(\Omega_i)$ since $H_i' \supset \mathrm{Cl}(\Omega_i)$.

As $\{x_i\} \to \mathrm{v}$ and $\{g(x_i')\} \to \mathrm{v}, g \in F$, it follows that $\{\hat{K}_i\} \to \{\mathrm{v}\}$: We just need to show that $\{h_i(g)(\partial H_i')\}, g \in F$, are uniformly bounded away from that of ∂H and $\partial H_i'$ under the Hausdorff metric $\mathbf{d}_H$. Since $\{h_i(g)(\partial H_i')\}$ for each $g \in F$ geometrically converges to to $h(g)(\partial H_i')$, we are done by Lemma 11.3.6.

Therefore, we conclude that K is a singleton.

By Proposition 1.5.15, $\{\Omega_i^*\}$ geometrically converges to K^* dual to K as $i \to \infty$. (See 1.5.4.) In case, K is a singleton, K^* must be a hemisphere by Proposition 1.5.13. We now conclude that K or its dual K^* has a nonempty interior.

Thus, by choosing h_i^* and h^* if necessary, we may assume without loss of generality that K has a nonempty interior. We will show that K is a properly convex domain and this implies that so is K^*.

(II) The second step is to show K is properly convex.

Assume that $h(\pi_1(\mathscr{O}))$ acts on a convex open domain K^o. We may assume that $K^o \subset \mathbb{A}$ for an affine subspace $\mathbb{A}$ and $\Omega_i \subset \mathbb{A}$ as well by acting by an orthogonal $\kappa_i \in \mathsf{O}(n+1,\mathbb{R})$, where $\{\kappa_i\}$ is converging to I. We can accomplish this by moving Ω_i into $\mathbb{A}$. Since $\{\kappa_i\} \to$ I, we still have $\{\mathrm{Cl}(\Omega_i)\} \to K$ by Lemma 1.1.8. Take a ball B_{2C} of $\mathbf{d}$-radius $2C$, $C > 0$, in K. By Lemma 1.1.10, $\mathbb{A}$ contains a $\mathbf{d}$-radius C ball $B_C \subset \Omega_i$ for sufficiently large i. Without loss of generality assume $B_C \subset \Omega_i$ for all i. Choose the $\mathbf{d}$-center x_0 of B_C as the origin in the affine coordinates.

Let $g_1, \dots, g_m$ denote the set of generator of $\pi_1(\mathscr{O})$. Then by extracting subsequences, we may assume without loss of generality that $\{h_i(g_j)\}$ converges to $h(g_j)$ for each $j = 1, \dots, m$.

LEMMA 11.3.7. *For each g_j, $j = 1, \dots, m$,*

$$\mathbf{d}(h_i(g_j)(x_0), \mathrm{bd}\Omega_i) \geq C_0 \textit{ for a uniform constant } C_0. \tag{11.3.3}$$

PROOF. Suppose not. Then there is a sequence of a $\mathbf{d}$-length constant C segment s_i, $s_i \subset \Omega_i$, with an origin x_0 is sent to the segment $h_i(g_j)(s_i)$ in Ω_i with endpoint $h_i(g_j)(x_0)$ and lying on the shortest $\mathbf{d}$-length segment from $h_i(g_j)(x_0)$ to $\mathrm{bd}\Omega_i$. Thus, the sequence of the $\mathbf{d}$-length of $h_i(g_j)(s_i)$ is going to zero. This implies that $h_i(g_j)$ is not in a compact subset of $\mathsf{SL}_{\pm}(n+1,\mathbb{R})$, a contradiction. □

PROOF OF PROPOSITION 11.3.5 CONTINUED. By estimation from (11.3.3), and the cross-ratio expression of the Hilbert metric, a uniform constant C satisfies

$$d_{\tilde{\mathscr{O}}_i}(x_0, h_i(g_j)(x_0)) < C. \tag{11.3.4}$$

By Benzécri [25] (see Proposition 4.3.8 of Goldman [86]), there exists a constant $R_B > 1$ and $\tau_i \in \mathsf{SL}_\pm(n+1,\mathbb{R})$ so that

$$B_1 \subset \tau_i(\Omega_i) \subset B_{R_B}.$$

Now, $\tau_i h_i(\pi_1(\mathscr{O}))\tau_i^{-1}$ acts on $\tau_i(\Omega_i)$. By Theorem 7.1 of Cooper-Long-Tillmann [67], we obtain that $\tau_i h_i(g_j)\tau_i^{-1}$ for $j = 1,\dots,n$ are in a compact subset of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ independent of i.

Therefore, up to choosing subsequences, we have $\{\tau_i(\Omega_i)\}$ geometrically converges to a properly convex domain $\hat{K}$ in B_R containing B_1 and

$$\{\tau_i h(\cdot)\tau_i^{-1} : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})\}$$

algebraically converges to a holonomy homomorphism

$$h' : \pi_1(\mathscr{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R}).$$

And the image of h' acts on the interior of the properly convex domain $\hat{K}$.

Suppose that the sequence $\{\tau_i\}$ is not bounded. Then $\tau_i = k_i d_i k_i'$ where d_i is diagonal with respect to a standard basis of $\mathbb{R}^{n+1}$ and $k_i, k_i' \in O(n+1,\mathbb{R})$ by the KTK-decomposition of $\mathrm{SL}_\pm(n+1,\mathbb{R})$. Then the sequence of the maximum modulus of the eigenvalues of d_i are not bounded above. We assume without loss of generality

$$\{k_i\} \to k, \{k_i'\} \to k' \text{ in } O(n+1,\mathbb{R}).$$

Thus, $\{k_i' h_i(g_j)k_i^{-1}\}$ converges to $k'h(g_j)k^{-1}$ for $k' \in O(n+1,\mathbb{R})$. Since

$$\{k_i d_i k_i' h_i(g_j) k'^{-1} d_i^{-1} k_i^{-1}\}$$

is convergent to $h'(g_j)$, we obtain

$$\{d_i k_i' h_i(g_j) k_i'^{-1} d_i^{-1}\} \to k^{-1}h'(g_j)k \text{ for each } j.$$

Thus, $\{d_i k_i' h_i(\pi_1(\mathscr{O}))k_i'^{-1}d_i^{-1}\}$ converges algebraically to a group $kh'(\pi_1(\mathscr{O}))k^{-1}$ acting on $k^{-1}(\hat{K})$.

Since the sequence of the norms of d_i is divergent, $kh'(\pi_1(\mathscr{O}))k^{-1}$ is reducible: We may assume up to a choice of subsequence and a change of coordinates that the diagonal entries of d_i satisfy

$$d_{i,1} \geq d_{i,2} \geq \cdots \geq d_{i,n+1}.$$

Up to a choice of subsequences, there is j so that $d_{i,k}/d_{i,j} \geq 1$ for $k \leq j$ and $\{d_{i,k}/d_{i,j}\} \to 0$ for $k > j$. Then $\{d_i k_i' h(\pi_1(\mathscr{O})k_i'^{-1}d_i^{-1}\}$ being a bounded sequence converges to a matrix with entries at $(k+1,\dots,n+1)\times(1,\dots,k)$ are identically zero. (Compare to the proof of Lemma 1 of [49].)

By Lemma 11.3.1, $k^{-1}(\hat{K})^o/kh'(\pi_1(\mathscr{O}))k^{-1}$ is a strongly tame SPC-orbifold with horospherical or generalized lens-shaped ends.

By Theorem 6.0.4, the algebraic limit of

$$\{d_i k_i' h(\pi_1(\mathscr{O})k_i'^{-1}d_i^{-1}\}$$

cannot be reducible. Therefore the sequence of the norms of d_i is uniformly bounded. This is a contradiction to the unboundedness of τ_i.

By Lemma 11.3.1, we obtain that $\mathscr{O}_h := \hat{K}^o/h(\pi_1(\mathscr{O}))$ is a strongly tame SPC-orbifold with generalized lens-shaped or horospherical $\mathscr{R}$- or $\mathscr{T}$-ends diffeomorphic to $\mathscr{O}$. This completes the proof for $\widetilde{\mathscr{U}} \subset \widetilde{\mathrm{CDef}}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$.

To prove for $\mathrm{SDef}_{\mathscr{E},\mathrm{lh}}(\mathscr{O})$, we need additionally Theorems 10.3.1 and 10.3.4 as in the last paragraph of the proof of Theorem 11.3.3. This completes the proof of Corollary 11.3.5. $\square$

11.4. General cases without the uniqueness condition: The proof of Theorem 11.1.4.

We will construct a section by the following. Let

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R})) \text{ (resp. } \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{SL}_\pm(n+1,\mathbb{R})))$$

denote the space of representations h fixing a common fixed point p and acting on a lens L of a lens-cone of form $\{p\} * L$ with $p \notin \mathrm{Cl}(L)$ or is horospherical with a fixed point p.

Let

$$\mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R})) \text{ (resp. } \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{SL}_\pm(n+1,\mathbb{R})))$$

denote the space of representations where $h(\pi_1(\tilde{E}))$ for each element h acts on P satisfying the lens-condition or acts on a horosphere tangent to P. (See Section 9.2.)

We define the sections

$$s_R : \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R})) \to \mathbb{RP}^n, \qquad s_T : \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R})) \to \mathbb{RP}^{n*}$$

$$\big(\text{resp. } s_R : \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{SL}_\pm(n+1,\mathbb{R})) \to \mathbb{S}^n, \qquad s_T : \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\tilde{E}), \mathsf{SL}_\pm(n+1,\mathbb{R})) \to \mathbb{S}^{n*}\big)$$

by Propositions 5.4.4 and 5.4.5.

PROPOSITION 11.4.1. *The maps s_R and s_T for $\mathbb{RP}^n$ and $\mathbb{RP}^{n*}$ (resp. $\mathbb{S}^n$ and $\mathbb{S}^{n*}$) are both continuous.*

PROOF. We will only prove for $\mathbb{RP}^n$. The version for $\mathbb{S}^n$ is obvious. Let h be an element of $\mathrm{Hom}_{\mathscr{E},\mathrm{ce},p}(\pi_1(\tilde{E}), \mathsf{PGL}(n+1,\mathbb{R}))$. The vertex of a lens-cone is a common fixed point of all $h(\pi_1(\tilde{E}))$. Let F be the set of generators of $\pi_1(\tilde{E})$ so that $\{v\} = \{w | g(w) = w, g \in F\}$. Otherwise, we will have a line of fixed points for Γ_E and we obtain a contradiction as in the proof of Proposition 5.4.4. Hence, the holonomies of elements of F determine the vertex. The continuity follows by a sequence argument.

For s_T, we take the dual by by Proposition 5.5.5 and prove the continuity. □

LEMMA 11.4.2. *We can construct the uniqueness section of lens-type*

$$s : \mathrm{Hom}_{\mathscr{E},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \to (\mathbb{RP}^n)^{e_1} \times (\mathbb{RP}^{n*})^{e_2}$$

PROOF. We can always choose a vertex and the hyperspace by Propositions 5.4.4 and 5.4.5. s is continuous by Proposition 11.4.1. □

PROOF OF THEOREM 11.1.4. Using the uniqueness section of lens-type, we apply Corollary 11.3.5. □

CHAPTER 12

Nicest cases

This is a chapter now.

We will now present the cases when the theory presented in this monograph works best.

In Section 12.1, we will discuss the results where the results of this monograph applies. In Section 12.2, we will end with two examples where the results applies.

12.1. Main results

Let us start with an example:

EXAMPLE 12.1.1. Let M be a complete hyperbolic 3-orbifold and each end orbifold has a sphere or a disk as the base space. The end fundamental group is generated by a finite order elements. By Lemma 12.1.2, a properly convex real projective structure on M has lens-shaped or horospherical radial ends only.

We need the end classification results from Chapters 3, 5, and 7 to prove the following. Let $g \in \pi_1(\mathscr{O})$. Using the choice of representing matrix of g as in Remark 1.1.5, we let $\lambda_x(g)$ denote the eigenvalue of holonomy of g associated with the vector in direction of x if x is a fixed point of g.

The holonomy group of $\pi_1(\mathscr{O})$ can be lifted to $\mathrm{SL}_{\pm}(n+1,\mathbb{R})$ so that $\lambda_{\mathrm{v}_{\tilde E}}(g) = 1$ for the holonomy of every $g \in \pi_1(\tilde E)$ where $\mathrm{v}_{\tilde E}$ is a p-end vertex of a p-end $\tilde E$ corresponding to E. Then we say that E or $\tilde E$ satisfies the *unit-middle-eigenvalue condition* with respect to $\mathrm{v}_{\tilde E}$ or the R-p-end structure.

Suppose that E is a T-end. If the hyperspace containing the ideal boundary component $\tilde S_{\tilde E}$ of p-end $\tilde E$ of E corresponds to 1 as the eigenvalue of the dual of the holonomy of every $g \in \pi_1(\tilde E)$, then we say we say that E or $\tilde E$ satisfies the *unit middle eigenvalue condition* with respect to $\tilde S_{\tilde E}$ or the T-p-end structure.

LEMMA 12.1.2. *Suppose that $\mathscr{O}$ is a strongly tame convex real projective orbifold with radial ends. Assume that the end fundamental group $\pi_1(E)$ of an end E satisfies* (*NS*). *Let E be an R-end, or is a T-end. Suppose that one of the following holds*:

- *$\pi_1(E)$ is virtually generated by finite order elements or is simple, or*
- *the end holonomy group of E satisfies the unit middle eigenvalue condition.*

Then the following hold:

- *the end E is either a properly convex generalized lens-shaped R-end or a lens-shaped T-end, or is horospherical.*
- *If the end E furthermore has a virtually abelian end holonomy group, then E is a lens-shaped R-end, a lens-shaped T-end, or is a horospherical end.*

PROOF. We suppose that $\tilde{\mathscr{O}}$ is a convex domain in of $\mathbb{S}^n$. First, let E be an R-end. The map

$$g \in \Gamma_{\tilde E} \mapsto \lambda_{\mathrm{v}_{\tilde E}}(g) \in \mathbb{R}_+$$

is a homomorphism. Thus, $\lambda_{\mathrm{v}_{\tilde E}}(g) = 1$ for $g \in \Gamma_{\tilde E}$ since the end holonomy group is simple or virtually generated by the finite order elements.

Each R-end is either complete, properly convex, or is convex but not properly convex and not complete by Section 3.1.6.

Suppose that $\tilde E$ is complete. Then Theorem 8.1.4 shows that either $\tilde E$ is horospherical or each element g, $g \in \pi_1(\tilde E)$ has at most two norms of eigenvalues where two norms for an element are realized. Since the multiplication of all eigenvalues equals 1, we obtain $\lambda_1^{n+1-r}(g)\lambda_{\mathrm{v}_{\tilde E}}(g)^r = 1$ for some integer $r, 1 \leq r \leq n$ and the other norm $\lambda_1(g)$ of the eigenvalues. The second case cannot happen.

Suppose that $\tilde E$ is properly convex. Then the uniform middle eigenvalue condition holds by Remark 5.3.2 since $\lambda_{\mathrm{v}_{\tilde E}}(g) = 1$ for all g. (See Definition 5.1.2.) By Theorem 5.1.5, $\tilde E$ is of generalized lens-type.

Finally, Corollary 8.2.2 rules out the case when $\tilde E$ is convex but not properly convex and not complete.

Now, let E be a T-end. By dualizing the above, E satisfies the uniform middle eigenvalue condition (see Definition 5.1.3). Theorem 5.5.4 implies the result. [$\mathbb{S}^n$S] □

THEOREM 12.1.3. *Suppose that $\mathscr{O}$ is a strongly tame properly convex real projective orbifold with R-ends or T-ends. Suppose that each end fundamental group satisfies property* (*NS*) *and is virtually generated by finite order elements, or is simple or satisfies the unit middle eigenvalue condition. Then the holonomy is in*

$$\mathrm{Hom}^s_{\mathscr{E},u,\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

PROOF. Suppose that E is an R-end. Let $\tilde E$ be a p-end corresponding to E and $\mathrm{v}_{\tilde E}$ be the p-end vertex. By Lemma 12.1.2, we obtain the R-end is lens-type or horospherical.

We prove the uniqueness of the fixed point under $h(\pi_1(\tilde E))$: Suppose that x is another fixed point of $h(\pi_1(\tilde E))$. Since $\pi_1(\tilde E)$ is as in the premise, the eigenvalue $\lambda_x(g)$ for every $g \in \pi_1(\tilde E)$ associated with x is always 1. In the horospherical case $x = \mathrm{v}_{\tilde E}$ since the cocompact lattice action on a cusp group fixes a unique point in $\mathbb{RP}^n$.

Now consider the lens case. The uniform middle eigenvalue condition with respect to $\mathrm{v}_{\tilde E}$ and x holds by Remark 5.3.2 since $\lambda_x(g) = 1$ for all g. Lemma 12.1.2 shows that $\pi_1(\tilde E)$ acts on a lens-cone with vertex at x. Proposition 5.4.4 implies the uniqueness of the p-end vertex.

Suppose that E is a T-end. The proof of Proposition 5.5.1 shows that the hyperspace containing $\tilde S_{\tilde E}$ corresponds to $\mathrm{v}_{\tilde E^*}$ for the R-p-end $\tilde E^*$ corresponding to the dual of the T-p-end $\tilde E$ and vice verse. Hence, the result follows from the R-end part of the proof. □

This was proved by Marquis in Theorem A of [127] when the orbifold is a Coxeter one.

Theorems 12.1.3, 11.0.8, and 6.0.4 imply the following:

COROLLARY 12.1.4. *Let $\mathscr{O}$ be a noncompact strongly tame SPC n-dimensional orbifold with R-ends and T-ends and satisfies* (*IE*) *and* (*NA*). *Suppose that each end fundamental group is generated by finite order elements or is simple. Suppose each end fundamental group satisfies* (*NS*). *Assume $\partial\mathscr{O} = \emptyset$, and that the nilpotent normal subgroups of every finite-index subgroup of $\pi_1(\mathscr{O})$ are trivial. Then*

$$\mathrm{CDef}_{\mathscr{E}}(\mathscr{O}) = \mathrm{CDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$$

and hol *maps the deformation space* $\mathrm{CDef}_{\mathscr{E}}(\mathscr{O})$ *of SPC-structures on* $\mathscr{O}$ *homeomorphic to a union of components of*

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

which is also a union of components of

$$\mathrm{rep}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})) \text{ and } \mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R})).$$

The same can be said for

$$\mathrm{SDef}_{\mathscr{E}}(\mathscr{O}) = \mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}).$$

These type of deformations from structures with cusps to ones with lens-shaped ends are realized in our main examples as stated in Section 2.2. We need the restrictions on the target space since the convexity of $\mathscr{O}$ is not preserved under the hyperbolic Dehn surgery deformations of Thurston, as pointed out by Cooper at ICERM in September 2013.

Virtually abelian groups satisfy (NS) clearly. Since finite-volume hyperbolic n-orbifolds satisfy (IE) and (NA) (see P.151 of [124] for example), strongly tame properly convex orbifolds admitting complete hyperbolic structures end fundamental groups generated by finite order elements will satisfy the premise. Hence, $2h\_1\_1$ and the double of the simplex orbifold discussed in Section 12.2 do also.

Since Coxeter orbifolds satisfy the above properties, we obtain a simple case:

COROLLARY 12.1.5. *Let* $\mathscr{O}$ *be a strongly tame Coxeter* n*-dimensional orbifold,* $n \geq 3$*, with only* $\mathscr{R}$*-ends. admitting a finite-volume complete hyperbolic structure. Then*

$$\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$$

is homeomorphic to a union of components of

$$\mathrm{rep}^s_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

which is also a union of components of

$$\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(n+1,\mathbb{R}))$$

Finally,

$$\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) = \mathrm{SDef}_{\mathscr{E}}(\mathscr{O}).$$

12.2. Two specific examples

The example of S. Tillmann is an orbifold on a 3-sphere with singularity consisting of two unknotted circles linking each other only once under a projection to a 2-plane and a segment connecting the circles (looking like a linked handcuff) with vertices removed and all arcs given as local groups the cyclic groups of order three. (See Figure 1.) This is one of the simplest hyperbolic orbifolds in the list of Heard, Hodgson, Martelli, and Petronio [97] labeled $2h\_1\_1$. The orbifold admits a complete hyperbolic structure since we can start from a complete hyperbolic tetrahedron with four dihedral angles equal to $\pi/6$ and two equal to $2\pi/3$ at a pair of opposite edge e_1 and e_2. Then we glue two faces adjacent to e_i by an isometry fixing e_i for $i = 1,2$. The end orbifolds are two 2-spheres with three cone points of orders equal to 3 respectively. These end orbifolds always have induced convex real projective structures in dimension 2, and real projective structures on them have to be convex. Each of these is either the quotient of a properly convex open triangle or a complete affine plane as we saw in Lemma 12.1.2.

Porti and Tillmann [139] found a two-dimensional solution set from the complete hyperbolic structure by explicit computations. Their main questions are the preservation of

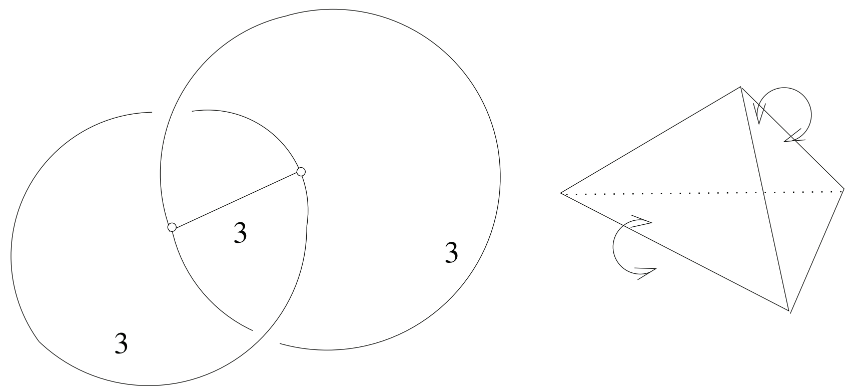

FIGURE 1. Handcuff orbifold.

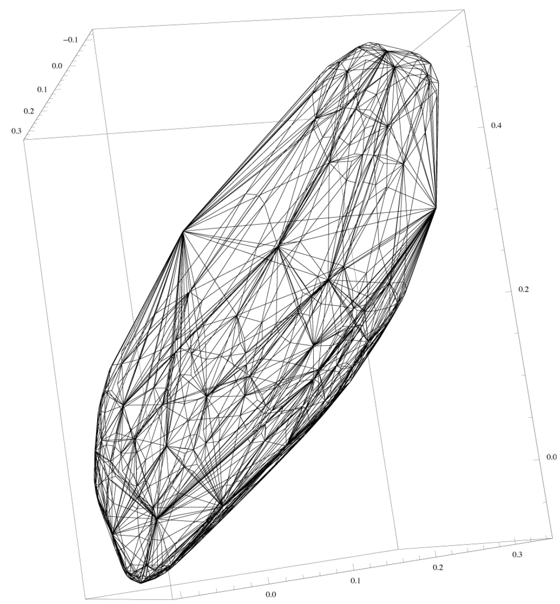

FIGURE 2. A convex developing image example of a tetrahedral orbifold of orders $3,3,3,3,3,3$.

convexity and realizability as convex real projective structures on the orbifold. Corollary 12.1.4 answers this since their deformation space identifies with $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) = \mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$.

Another main example can be obtained by doubling a complete hyperbolic Coxeter orbifold based on a convex polytope. We take a double D_T of the reflection orbifold based on a convex tetrahedron with orders all equal to 3. This also admits a complete hyperbolic structure since we can take the two tetrahedra to be the regular complete hyperbolic tetrahedra and glue them by hyperbolic isometries. The end orbifolds are four 2-spheres with three singular points of orders 3. Topologically, this is a 3-sphere with four points removed and six edges connecting them all given order 3 cyclic groups as local groups.

THEOREM 12.2.1. *Let $\mathscr{O}$ denote the hyperbolic 3-orbifold D_T. We assign the $\mathscr{R}$-type to each end. Then* $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$ *equals* $\mathrm{SDef}_{\mathscr{E},\mathrm{u,lh}}(\mathscr{O})$ *and* hol *maps* $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$ *as an onto-map to a component of characters*

$$\mathrm{rep}_{\mathscr{E}}(\pi_1(\mathscr{O}), \mathsf{PGL}(4,\mathbb{R}))$$

containing a hyperbolic representation which is also a component of

$$\mathrm{rep}_{\mathscr{E},\mathrm{u,lh}}(\pi_1(\mathscr{O}), \mathsf{PGL}(4,\mathbb{R})).$$

In this case, the component is a cell of dimension 4*.*

PROOF. A solvable subgroup of $\mathsf{PSO}(n,1)$ fixes a point of the boundary of the Klein ball model B. Since $\pi_1(\mathscr{O})$ is not elementary, a finite-index subgroup of $\pi_1(\mathscr{O})$ has only trivial normal solvable subgroups. The end orbifolds have zero Euler characteristics, and all the singularities are of order 3. For each end E, $\pi_1(E)$ is virtually abelian. Hence, $\pi_1(E)$ satisfies (NS).

Since $\mathscr{O}$ admits a complete hyperbolic structure with finite volume, $\pi_1(\mathscr{O})$ is relatively hyperbolic with respect to its end fundamental groups. (This follows from Theorem 0.1 of Yaman [157] since the group acts on the sphere of infinity of the hyperbolic 3-space accordingly.) By Corollary 10.3.7, any properly convex structures on $\mathscr{O}$ with $\mathscr{R}$- or $\mathscr{T}$-ends are strictly convex. By Corollary 12.1.4, $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$ equals $\mathrm{SDef}_{\mathscr{E},u,\mathrm{lh}}(\mathscr{O})$. Each of the ends has to be either horospherical or lens-shaped or totally geodesic radial type. Let $\partial_{\mathscr{E}}\mathscr{O}$ denote the union of end orbifolds of $\mathscr{O}$.

In [48], we showed that the triangulated real projective structures on the ends determined the real projective structure on $\mathscr{O}$. First, there is a map $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O}) \to \mathrm{CDef}(\partial_{\mathscr{E}}\mathscr{O})$ given by sending the real projective structures on $\mathscr{O}$ to the real projective structures of the ends. (Here if $\partial_{\mathscr{E}}\mathscr{O}$ has many components, then $\mathrm{CDef}(\partial_{\mathscr{E}}\mathscr{O})$ is the product space of the deformation space of all components.) Let J be the image.

Let μ be a projective structure corresponding to an element of $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$. The universal cover $\tilde{\mathscr{O}}$ is identified to a properly convex domain in $\mathbb{S}^3$. Each singular geodesic arc in $\tilde{\mathscr{O}}$ connects one of the p-end vertices to the another. The developing image of $\tilde{\mathscr{O}}$ is a convex open domain and the developing map is a diffeomorphism. The developing images of singular geodesic arcs form geodesics meeting at vertices transversely. There exists two convex tetrahedra T_1 and T_2 with vertices removed from which decomposes $\tilde{\mathscr{O}}$. They are adjacent and their images under $\pi_1(\mathscr{O})$ tessellate $\tilde{\mathscr{O}}$.

The end orbifold is so that if given an element of the deformation space, then the geodesic triangulation is uniquely obtained. Hence, there is a proper map from $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$ to the space of invariants of the triangulations as in [48], i.e., the product space of cross-ratios and Goldman-invariant spaces.

Now $\mathscr{O}$ is the orbifold obtained from doubling a tetrahedron with edge orders $3,3,3$. We consider an element of $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$. Since it is convex, we triangulate $\mathscr{O}$ into two tetrahedra, and this gives a triangulation for each end orbifold diffeomorphic to $S^{(i)}_{3,3,3}$, $i = 1,2,3,4$, corresponding to four ends, each of which gives us triangulations into two triangles. We can derive from the result of Goldman [88] and Choi-Goldman [55] that given projective invariants $\rho^{(i)}_1, \rho^{(i)}_2, \rho^{(i)}_2, \sigma^{(i)}_1, \sigma^{(i)}_2$ for each of the two triangles satisfying $\rho^{(i)}_1\rho^{(i)}_2\rho^{(i)}_3 = \sigma^{(i)}_1\sigma^{(i)}_2$, we can determine the structure on $S^{(i)}_{3,3,3}$ for $i = 1,2,3,4$ completely.

For each $S^{(i)}_{3,3,3}$ with a convex real projective structure and divided into two geodesic triangles, we compute respective invariants $\rho^{(i)}_1, \rho^{(i)}_2, \rho^{(i)}_2, \sigma^{(i)}_1, \sigma^{(i)}_2$ for one of the triangles corresponding to the link of T_1:

$$\begin{gathered} s_i^2+s\tau+1, s_i^2+s_i\tau+1, s_i^2+s_i\tau+1, \\ t_i\left(s_i^2+s_i\tau+1\right), \tfrac{1}{t_i}\left(s_i^2+s_i\tau+1\right)\left(s_i^2+s_i\tau+1\right) \end{gathered} \tag{12.2.1}$$

and for the other triangle corresponding to the link of T_2 the respective invariants are

$$\begin{gathered} \tfrac{1}{s_i^2}(s_i^2+s_i\tau+1), \quad \frac{1}{s_i^2}(s_i^2+s_i\tau+1), \frac{1}{s_i^2}(s_i^2+s_i\tau+1), \\ \tfrac{t_i}{s_i^3}\left(s_i^2+s_i\tau+1\right), \quad \frac{1}{s_i^3t_i}\left(s_i^2+s_i\tau+1\right)\left(s_i^2+s_i\tau+1\right) \end{gathered} \tag{12.2.2}$$

where s_i, t_i, $i=1,2,3,4$, are Goldman parameters and $\tau = 2\cos 2\pi/3$. (See [38].)

Since $\partial_{\mathscr{E}}\mathscr{O}$ is a disjoint union of four spheres with singularities $(3,3,3)$, $\mathrm{CDef}(\partial_{\mathscr{E}}\mathscr{O})$ is parameterized by s_i, t_i and hence is a cell of dimension 8. (This can be proved similarly to [56].)

The set J is given by projective invariants of the $(3,3,3)$ boundary orbifolds satisfying some equations. By the method of [48] developed by the author, we obtain the equations that J satisfies. These are

$$\begin{aligned} s_i^2+s_i\tau+1 &= s_j^2+s_j\tau+1, i,j=1,2,3,4 \\ \frac{1}{s_i^2}(s_i^2+s_i\tau+1) &= \frac{1}{s_j^2}(s_j^2+s_i\tau+1), i,j=1,2,3,4 \\ t_1t_2t_3t_4\prod_{i=1}^4\left(s_i^2+s_i\tau+1\right) &= \frac{1}{t_1t_2t_3t_4}\prod_{i=1}^4\left(s_i^2+s_i\tau+1\right)^2 \\ \prod_{i=1}^4\frac{t_i}{s_i^3}\left(s_i^2+s_i\tau+1\right) &= \prod_{i=1}^4\frac{1}{s_i^3t_i}\left(s_i^2+s_i\tau+1\right)^2. \end{aligned}$$

The first and second lines of equations are from matching the cross ratios $\rho_l^{(i)}$ with $\rho_l^{(j)}$ for any pair i,j corresponding to an edge connecting the i-th vertex to the j-vertex in D_T and four faces containing the edge. (See (5) of [48]). The third and fourth lines of equations are from the equations matching the products $\prod_{i=1}^4\sigma_1^{(i)} = \prod_{i=1}^4\sigma_2^{(i)}$ for T_1 and T_2 respectively.

The equation is solvable:

$$s_1=s_2=s_3=s_4=s, t_1t_2t_3t_4=C(s) \text{ for a constant } C(s)>0 \text{ depending on } s.$$

Thus J is contained in the solution subspace C, a 4-dimensional cell in $\mathrm{CDef}(\partial_{\mathscr{E}}\mathscr{O})$.

Conversely, given an element of C, we can assign invariants at each edge of the tetrahedron and the Goldman σ-invariants at the vertices if the invariants satisfy the equations. This is given by starting from the first convex tetrahedron and gluing one by one using the projective invariants (see [48] and [42]): Let the first one by always be the standard tetrahedron with vertices

$$[1,0,0,0],[0,1,0,0],[0,0,1,0], \text{ and } [0,0,0,1]$$

and we let T_2 a fixed adjacent tetrahedron with vertices

$$[1,0,0,0],[0,1,0,0],[0,0,1,0] \text{ and } [2,2,2,-1].$$

Then projective invariants will determine all other tetrahedron triangulating $\tilde{\mathscr{O}}$. Given any deck transformation γ, T_1 and $\gamma(T_1)$ will be connected by a sequence of tetrahedrons related by adjacency, and their pasting maps are wholly determined by the projective invariants, where cross-ratios do not equal 0. Therefore, as long as the projective invariants are bounded, the holonomy transformations of the generators are bounded. Corollary 12.1.5 shows that these corresponds to elements of $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O}) = \mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$. (This method was spoken about in our talk in Melbourne, May 18, 2009 [42].) Hence, we showed that $\mathrm{SDef}_{\mathscr{E},\mathrm{u},\mathrm{lh}}(\mathscr{O})$ is parameterized by the solution set C. Thus $J = C$ since each element of C gives us an element of $\mathrm{SDef}_{\mathscr{E}}(\mathscr{O})$. [$\mathbb{S}^n$S] □

The dimension is one higher than that of the deformation space of the reflection 3-orbifold based on the tetrahedron. Thus we have examples not arising from reflection ones here as well. See the Mathematica files [39] for a different explicit method of solutions. Also, see [41] to see how to draw Figure 2.

We remark that the above theorem can be generalized to orders ≥ 3 with hyperideal ends with similar computations. See [39] for examples to modify orders and so on.

APPENDIX A

Projective abelian group actions on convex domains

We will explore some theories of projective abelian group actions on convex domains, which is not necessarily properly convex. In Section A.1.1, we show that a free abelian group action decomposes the space into joins. In Section A.1.2, we discuss convex projective orbifolds with free abelian holonomy groups. In Lemma A.1.6, we will show a decomposition similar to the Benoist decomposition for the divisible projective actions on properly convex domains. We also show that a parameter of orbits of free abelian groups geometrically converges to an orbit in Lemma A.1.7. In Section A.1.3, we will show that such actions always immediately deform to the abelian group actions on properly convex domains. In Section A.1.4, we prove geometric convergences of convex real projective orbifolds slightly more general than that explored by Benoist. In Section A.2, we give some justification of why we are using the weak middle eigenvalue conditions.

A.1. Convex real projective orbifolds

We will explore a class of convex real projective orbifolds a little bit more general than the properly convex ones. Also see Leitner [119], [118], and [120] for a similar work, where she explores representations of abelian groups; however, these do not act cocompactly on convex domains.

For our purposes in the monograph, we will mostly work on $\mathbb{S}^{n-1}$ but sometimes with $\mathbb{RP}^{n-1}$.

Recall the Cartan decomposition $\mathsf{SL}_{\pm}(n,\mathbb{R}) = KTK$ where $K = \mathsf{O}(n,\mathbb{R})$ and T is the group of positive diagonal matrices. Note that the endomorphisms in $M_n(\mathbb{R})$ may have null spaces.

A Cartan decomposition $g = k_{1,g}A_g k_{2,g}$ for $k_{1,g}, k_{2,g} \in \mathsf{O}(n,\mathbb{R})$ and a diagonal matrix A_g with nonnegative nonincreading set of diagonals for each element g of $M_n(\mathbb{R})$ exists since each element is a limit of elements of $\mathsf{GL}(n,\mathbb{R})$ admitting a Cartan decomposition.

Each induced projective endomorphism g for $\mathbb{S}^{n-1}$ may have a nonempty subspace V_g where it is not defined. We call the projectivization of the null space the *undefined subspace* of g. It could be an empty set.

Let N_g be the projectivization of the null space of A_g. Then $V_g := k_{2,g}^{-1}N_g$ is the undefined subspace of f.

Let M_g be the matrix of g written as $k_{1,g}\hat{A}_g k_{2,g}$ where $\hat{A}_g$ is the diagonal matrix with the maximal entry being 1. We call this the normalization of g.

Use the Riemannian metric of $\mathbb{S}^{n-1}$ to compute the norms of differentials. We will call these **d**-norms.

A.1.1. A connected free abelian group with positive eigenvalues only. Recall that for a matrix A, we denote by $|A|$ the maximum of the norms of entries of A.

We can deform the unipotent abelian representation to diagonalizable ones that are arbitrarily close to the original one.

LEMMA A.1.1. *Let $h : \mathbb{Z}^l \to \mathsf{SL}_\pm(n,\mathbb{R})$ be a representation to unipotent elements. Let $g_1,\dots,g_l$ denote the generators. Then given $\varepsilon > 0$ there exists a positive diagonalizable representation $h' : \mathbb{Z}^l \to \mathsf{SL}_\pm(n,\mathbb{R})$ with matrices satisfying $|h'(g_i) - h(g_i)| < \varepsilon, i = 1,\dots,l$. Furthermore, we may choose a continuous parameter of h'_t so that $h'_0 = h$ and h'_t is positive diagonalizable for $t > 0$.*

PROOF. First assume that every $h(g_i)$, $i = 1,\dots,l$, has matrices that are upper triangular matrices with diagonal elements equal to 1 since the Zariski closure is in a nilpotent Lie group and Theorem 3.5.4 of [150].

Let $\varepsilon > 0$ be given. We will inductively prove that we can find h' as above with eigenvalues of $h'(g_1)$ are all positive and mutually distinct. For $n = 2$, we can simply change the diagonal elements to positive numbers not equal to 1. Then the group embeds in $\mathbf{Aff}(\mathbb{R}^1)$. We choose positive constant a_i so that $|a_i - 1| < \varepsilon$. Let g_i be given as $x \mapsto a_i x + b_i$. The commutativity reduces to equations $a_j b_i = a_i b_j$ for all i, j. Then the solution are given by $b_i = a_1^{-1} a_i b_1$ for any given b_1. We can construct the diagonalizable representations.

Suppose that the conclusion is true for dimension $k-1$. We will now consider a unipotent homomorphism $h : \mathbb{Z}^l \to \mathsf{SL}_\pm(k,\mathbb{R})$. We conjugate so that every $h(g_i)$ is upper-triangular. Since $h(g_i)$ is upper triangular, let $h_1(g_i)$ denote the upper-left $(k-1)\times(k-1)$-matrix. By the induction hypothesis, we find a positive diagonalizable representation $h'_1 : \mathbb{Z}^l \to \mathsf{SL}_\pm(k-1,\mathbb{R})$. Also assume $|h'_1(g_i) - h_1(g_i)| < \varepsilon/2$ for $i = 1,\dots,l$. We change

$$h(g_i) = \begin{pmatrix} h_1(g_i) & b(g_i) \\ 0 & 1 \end{pmatrix} \text{ to } h'(g_i) = \begin{pmatrix} \frac{1}{\lambda'(g_i)^{\frac{1}{k-1}}} h'_1(g_i) & b'(g_i) \\ 0 & \lambda'(g_i) \end{pmatrix}$$

for some choice of $h'_1(g), b'(g), \lambda'(g_i) > 0$ for $i, j = 1,\dots,l$. For commutativity, we need to solve for $b'(g_i)$ for $i, j = 1,\dots,l$,

$$\left(\frac{1}{\lambda'(g_i)^{\frac{1}{k-1}}} h'_1(g_i) - \lambda'(g_j)\mathrm{I}\right) b'(g_j) = \left(\frac{1}{\lambda'(g_j)^{\frac{1}{k-1}}} h'_1(g_j) - \lambda'(g_i)\mathrm{I}\right) b'(g_i).$$

We denote by

$$M_i := \left(\frac{1}{\lambda'(g_i)^{\frac{1}{k-1}}} h'_1(g_i) - \lambda'(g_j)\mathrm{I}\right).$$

Note $M_i M_j = M_i M_j$. By generic choice of $\lambda'(g_i)$s, we may assume that M_i are invertible. The solution is given by

$$b'(g_i) = M(g_1)^{-1} M(g_i) b'(g_1)$$

for an arbitrary choices of $b'(g_1)$. We choose $b'(g_1)$ arbitrarily near $b(g_1)$. Here, $\lambda(g_1)$ has to be chosen generically to make all the eigenvalues distinct and sufficiently near 1 so that $|h'(g_i) - h(g_i)| < \varepsilon$, $i = 1,\dots,l$. We can check the solution by the commutativity. Hence, we complete the induction steps.

To find a parameter denoted h'_t, we simply repeat the induction process building a parameter of h'_t. ☐

LEMMA A.1.2. *Let L be a connected projective abelian group acting on a properly convex domain K cocompactly and faithfully. Then L is positive diagonal and the domain is a simplex.*

PROOF. L contains a cocompact lattice L'. By the Hilbert metric of K^o, L' acts properly discontinuously on K^o. Proposition 1.4.10 applies now. Since L is the Zariski closure of positive diagonalizable L', we are done. ☐

Let us fix $g \in A, g \neq \mathrm{I}$. The minimal polynomial of g is of form $\prod_{i=1}^{m}(x-\lambda_i)$ where each nonreal λ_i pairs with exactly one other λ_j equal to $\bar{\lambda}_i$. We can write it as

$$(x-\lambda_1)^{r_1}\cdots(x-\lambda_s)^{r_s}\prod_{t=0}^{(m-s)/2-1}(x^2-2\Re\lambda_{s+2t+1}x-|\lambda_{s+2t+1}|^2)^{r_{s+2t+1}}. \tag{A.1.1}$$

We define a primary decomposition subspace C_i to be the kernel of

$$M_i(g) := (g-\lambda_i\mathrm{I})^{r_i} \text{ for } i = 1,\dots,s$$

and C_t to be the kernel of

$$M_{s+2t+1}(g) := (g^2-2\Re\lambda_{s+2t+1}x-|\lambda_{s+2t+1}|^2\mathrm{I})^{r_{s+2t+1}} \text{ for } t = 0,\dots,(m-s)/2-1.$$

(See [101].)

LEMMA A.1.3. *Let $g \in \mathrm{SL}_{\pm}(n+1,\mathbb{R})$ be a nonidentity element.*

- *Given a primary decomposition space C_i of g, we have $hC_i = C_i$ for any h commuting with g.*
- *Given a primary decomposition subspace C of g and D of h for $g,h \neq \mathrm{I}$, $C \cap D$ are both h and g invariant provided h and g commute with each other.*
- *Given a free abelian group A of finite-rank, there exists a maximal collection of invariant subspaces*

$$C_1,\dots,C_m \text{ satisfying } \mathbb{R}^{n+1} = C_1 \oplus \cdots \oplus C_m$$

where each C_j is g-invariant in a primary decomposition space of every g, $g \in A$.

PROOF. Corollary to Theorem 12 of Section 6.8 in [101] implies the first statement. Let $g_1,\dots,g_k$ denote the generators of A. We obtain $C_1,\dots,C_m$ by taking a primary decomposition space $C_{i,j}$ for g_j and taking intersections of the arbitrary collections of $C_{i,j}$ for all i, j. □

A *scalar group* is a group acting by $s\mathrm{I}$ for $s \in \mathbb{R}$ and $s > 0$. A *scalar unipotent group* is a subgroup of the product of a scalar group with a unipotent group. Hence, on each $A|C_i$ is a scalar unipotent group for each i.

LEMMA A.1.4. *Let A be a connected free abelian group acting on $\mathbb{R}^n$ with positive eigenvalues only. Then there exists a decomposition $\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ where A acts as a positive diagonalizable group on V_0 and acts as a positive scalar unipotent group on each V_i for $1 \leq i \leq m$.*

PROOF. We obtain $C_1,\dots,C_m$ by Lemma A.1.3. On C_i, A acts as a scalar group acting on a one-dimensional space or a scalar unipotent group since the corresponding factor of the minimal polynomial is $(x-\lambda_i\mathrm{I})^{r_i}$. □

PROPOSITION A.1.5. *Suppose that Γ is a discrete free abelian group whose Zariski closure is A. Suppose that elements of Γ have only positive eigenvalues. Then A/Γ is compact.*

PROOF. By Lemma A.1.4, there is a decomposition $\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ where A acts as a positive diagonalizable group on V_0 and acts as a positive scalar unipotent group on each V_i for $1 \leq i \leq m$.

Let $q = \dim V_0$. Let $x_1(g),\dots,x_q(g)$ denote eigenvalues of g, $g \in A$, for V_0 and $x_{q+1}(g),\dots,x_{q+m}(g)$ denote respective ones for $V_1,\dots,V_m$. Let D be a positive diagonalizable group acting as a scalar group on each $V_1,\dots,V_m$ and positive diagonalizable group on V_0 defined as a subgroup of $\mathbb{R}_+^{q+m}$ given by the equation $x_1(g)\cdots x_{q+m}(g) = 1$

There is a homomorphism $c : A \to D$ given by sending g to the tuples of eigenvalues respectively associated with V_0 and V_i for $i = 1, \dots, m$. Then Γ has a cocompact image in $c(A)$ under c since $c(A)$ is a connected diagonalizable group that is a Zariski closure of $c(\Gamma)$ and c is continuous.

Let K be the kernel of $c : A \to D$ which is an algebraic group. This is a unipotent subgroup of A and contains $K \cap \Gamma$. Let K_1 be the Zariski closure of $K \cap \Gamma$ in K. Since $K \cap \Gamma$ is normal in Γ, and K_1 is the minimal algebraic group containing $K \cap \Gamma$, K_1 is normalized by Γ and hence by A.

If K_1 is a proper subgroup of K, then there is a proper algebraic subgroup of A containing Γ since A is a product of K and a group isomorphic to $c(A)$. This is a contradiction.

Since K is unipotent, $K \cap \Gamma$ is cocompact in K. Hence, Γ is cocompact in A. □

A.1.2. Convex real projective structures. One can think of the following lemma as a classification of convex real projective orbifolds with abelian fundamental groups. Benoist [16], [19] investigated these in a more general setting.

LEMMA A.1.6. *Let Γ be a finitely generated free abelian group acting on a convex domain Ω of* $\mathbb{S}^{n-1}$ (resp. $\mathbb{RP}^{n-1}$) properly and cocompactly. Then the following hold:

- (i) the Zariski closure L of a finite index subgroup Γ' of Γ is so that L/Γ' is compact, and L has only positive eigenvalues (resp. a lift of L to $\mathsf{SL}_\pm(n,\mathbb{R})$ has).
- (ii) Ω is an orbit of the abelian Lie group L acting properly and freely on it.
- (iii) $\Omega = (A_1 * \cdots * A_p * \{p_1\} * \cdots * \{p_q\})^o$ for a complete affine subspace A_i, $i = 1, \dots, p$, and points p_j, $j = 1, \dots, q$. Here, $\langle A_1\rangle, \dots, \langle A_p\rangle, p_1, \dots, p_q$ are independent.
- (iv) L contains a central Lie subgroup Q of rank $p+q-1$ acting trivially on A_j and p_k for $j = 1, \dots, p, k = 1, \dots, q$.

PROOF. We will prove for the case $\Omega \subset \mathbb{S}^{n-1}$. The other case is implied by this. If Ω is properly convex, then Proposition 1.4.10 gives us a diagonal matrix group L acting on a simplex. (i) to (iv) follow in this case.

(i) Assume that Ω is not properly convex.

Now, Γ has no invariant lower-dimensional subspace P meeting Ω: otherwise, Γ acts on $P \cap \Omega$ properly so that $P \cap \Omega/\Gamma$ is virtually homeomorphic to a lower-dimensional manifold homotopy equivalent to a cover of Ω/Γ by Theorem 1.1.19. This is a contradiction.

The positivity of the eigenvalue will be proved: Let $C_1, \dots, C_{p+q}$ denote the subspaces of Γ obtained by Lemma A.1.3 where $\dim C_1, ..., \dim C_p \geq 2, \dim C_{p+1} = \cdots = \dim C_{p+q} = 1$. We also denote $C_j = \{p_{j-p}\}$ for $j = p+1, \dots, p+q$. Let $\lambda_1(g), \dots, \lambda_p(g)$ denote the norms of eigenvalues of each element g of L restricted to $C_1, \dots, C_p$ of dimension ≥ 2 respectively. The eigenvalues associated with $C_{p+1}, \dots, C_{p+q}$ of dimension 1 are clearly positive.

Define

$$\hat{\mathbb{S}}_j := \mathbb{S}(C_1) * \cdots * \mathbb{S}(C_j) * \mathbb{S}(C_{j+1}) * \cdots * \mathbb{S}(C_p).$$

Since L acts transitively on Ω,

$$\Omega \cap \hat{\mathbb{S}}_j = \emptyset.$$

Let

$$\Pi_j : \mathbb{S}^{n-1} - \hat{\mathbb{S}}_j \to \mathbb{S}(C_j), j = 1, \dots, p$$

denote the projection. Consider Ω_j denote the image under Π_i. The image is a convex subset of $\mathbb{S}(C_j)$ since convex segments go to convex segments or a point under Π. The image is open otherwise the dimension of $\Omega < n$. Now Γ acts cocompactly on Ω_j since Γ acts so on Ω the domain of Π_i. Then $\Omega \subset \Omega_1 * \cdots * \Omega_p$.

Since Ω is contained in an open hemisphere, the corresponding cone is contained in a half-space in $\mathbb{R}^n$, and it follows that its image Ω_j is contained in a hemisphere and is a convex open domain for each j.

We can consider the action of Γ on C_j have the norm of the eigenvalue equal to 1 only by multiplying by a representation $\Gamma \to \mathbb{R}_+$ and we are working on $\mathbb{S}^{n-1}$. The action of Γ on C_j is orthopotent by Theorem 1.3.7. By Conze-Guivarc'h [62] or Moore [132], there is an orthopotent flag in $\mathbb{S}_j$ and hence a proper Γ-invariant subspace. Let Γ_j denote the image of Γ by the restriction homomorphism to $\mathbb{S}(C_j)$. Since Γ_j is abelian, Γ_j contains a uniform lattice L'_j in the Zariski closure of Γ_j. Since L'_j is discrete, Theorem 1.3.7 shows that L'_j is virtually unipotent and so is its Zariski closure. Hence Γ_j is virtually unipotent. (See Theorem 3 of Fried [80].) Let Γ'_j be the unipotent subgroup of Γ_j of finite index. We can take $\Gamma' := \bigcap_{j=1}^{p} \Pi_j^{*-1}(\Gamma'_j)$. The finite index subgroup Γ' of Γ has only positive eigenvalue at $\mathbb{S}(C_j)$ for each j, $j = 1, \dots, p$. Also, the Zariski closure Z_j of Γ'_j is isomorphic to $\mathbb{R}^{n_j}$ for some n_j.

We assume that Γ' is torsion-free by takine a finite index subgroup by Selberg's lemma, i.e, Theorem 1.1.19. The Zariski closure L' of Γ is in $Z_1 \times \cdots \times Z_p$ and hence is free abelian. L'/Γ is a closed manifold by Proposition A.1.5. We take a connected component L of L' and let $\Gamma' = L \cap \Gamma$. Now, L/Γ' is a manifold, and Ω/Γ' is a closed manifold. Since they are both $K(\Gamma', 1)$-spaces, it follows that $\dim L = \dim \Omega = n-1$. This proves (i).

(ii) We will now let Γ to be Γ' above without loss of generality. Suppose that $p = 1$ and $q = 0$, or suppose that the associated eigenvalue of each $g \in \Gamma$ in C_j is independent of j. Since Ω is a convex domain in an affine subspace in $\mathbb{S}^{n-1}$, Ω is in a complete affine subspace. We can change Γ to be unipotent by changing scalars. A unipotent group acts on a half-space in $\mathbb{R}^n$ since its dual must fix a point in $\mathbb{R}^{n*}$ being solvable. Thus Γ acts on an affine subspace $\mathbb{A}^{n-1}$ in $\mathbb{S}^{n-1}$, and Γ acts as an affine transformation group of $\mathbb{A}^{n-1}$. Proposition T of [90] proves our result.

Otherwise, it must be that Ω is not complete affine but not properly convex. There exists a great sphere $\mathbb{S}^{i-1}$ in the boundary of Ω where L acts on and is the common boundary of i-dimensional affine spaces foliating Ω by Proposition 1.1.4 as in Section 7.1.1. There is a projective projection

$$\Pi_{\mathbb{S}^{i-1}} : \mathbb{S}^{n-1} - \mathbb{S}^{i-1} \to \mathbb{S}^{n-i-1}.$$

Then the image Ω_1 of Ω is properly convex. and Ω is the inverse image $\Pi_{\mathbb{S}^{i-1}}^{-1}(\Omega_1)$. Since L acts on Ω_1, it follows that L acts on Ω. Since $\dim L = \dim \Omega$ and Γ acts properly with a compact fundamental domain, L acts properly and cocompactly on Ω. (See Section 3.5 of [149].)

Let N denote the kernel of L going to a connected Lie group L_1 acting on Ω_1 properly and cocompactly.

$$1 \to N \to L \to L_1 \to 1.$$

By Lemma A.1.2, Ω_1 is a simplex. Hence, L_1 is a positive diagonalizable group. Since Ω_1/L_1 is compact, L_1 acts simply transitively on Ω_1 by Lemma 2.5 of [21]. $\dim L_1 = n - i - 1$. Thus, $\dim N = i$ and the abelian group N acts on each complete affine i-dimensional affine space A_l that is a leaf. Since the action of N is proper, N acts on A_l transitively by the proof of Lemma 2.5 of [21]. The action is simple since $\dim A_l \leq \dim l = i$. Hence, L acts transitively on Ω. Since the action of L is proper, L acts simply transitively by the dimension count.

(iv) By Lemma A.1.4 and (i), we decompose $\mathbb{R}^n$ into subspaces $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_p \oplus V_0$ where V_j corresponds to C_j for $j = 1, \dots, p$, V_0 corresponds to $C_{p+1} * \cdots * C_{p+q}$, L acts on V_0 as a positive diagonalizable linear group and L acts on V_j as elements of an abelian positive scalar unipotent group for $j = 1, \dots, p$. (Here V_0 can be 0 and V_i for $i \geq 1$ equals C_j for some j.)

Since L acts on V_0 as a positive diagonalizable group, it fixes points $p_1, \dots, p_q$ in general position in $\mathbb{S}^{q-1} := \mathbb{S}(V)$ with $q = \dim V_0$. We claim that L contains an abelian Lie subgroup Q of rank $p+q-1$ acting trivially on each Ω_j, $j = 1, \dots, p$, and fixing p_i, $i = 1, \dots, q$. Suppose that Ω is properly convex. Then Ω is the interior of a simplex. The cocompactness of a lattice of L shows that L contains a discrete free abelian central group of rank $p+q-1$ by the last part of Proposition 1.4.10. The central group is contained in Q. Since L is the Zariski closure of the lattice, Q is a subgroup of L.

Suppose that Ω is not properly convex. We deform a lattice of L to a diagonalizable one by a generalization of Lemma A.1.1 to the direct sum of scalar unipotent representations, and use the limit argument.

Actually, Q is the maximal diagonalizable group with over vectors in directions of $p_1, \dots, p_q$ and eigenspaces V_j, $j = 1, \dots, p$. This again follows by the limit argument. This proves (iv).

(iii) We choose a generic point $(\!(\vec{x})\!)$, $(\!(\vec{x})\!) \in \Omega$, in the complement of $\mathbb{S}(V_0)$, $\mathbb{S}(V_j)$ for $j = 1, \dots, m$. Since these are independent spaces, $\vec{x} = \vec{x}_0 + \sum_{i=j}^m \vec{x}_j$ where $\vec{x}_0 \in V_0$, $\vec{x}_j \in V_j$. We choose a parameter of element η_t of Q fixing V_0 or V_j for some j with largest norm eigenvalues and $\{\eta_t\}$ converging to 0-maps on other subspaces as $t \to \infty$. By Theorem 1.3.13, we obtain a projection to V_0 or V_j for each j as a limit in $\mathbb{S}(M_n(\mathbb{R}^n))$, and $(\!(\vec{x}_0)\!)$, $(\!(\vec{x}_j)\!)$ are in the closure of $L((\!(x)\!))$.

Since $\Pi_j(L(x)) = L(\Pi_j(x))$, we obtain

$$L((\!(\vec{x})\!)) \subset L((\!(\vec{x}_0)\!)) * L((\!(\vec{x}_1)\!)) * \cdots * L((\!(\vec{x}_m)\!)). \tag{A.1.2}$$

From the above paragraph, we can show that $L((\!(\vec{x}_j)\!))$ is contained in $\mathrm{Cl}(L((\!(\vec{x})\!)))$. Hence,

$$\mathrm{Cl}(L((\!(x)\!))) \supset L((\!(\vec{x}_0)\!)) * L((\!(\vec{x}_1)\!)) * \cdots * L((\!(\vec{x}_m)\!)). \tag{A.1.3}$$

Since L acts transitively on Ω, L acts so on the projection Ω_j under Π_j. Hence, $L((\!(\vec{x})\!)) = \Omega$ and $L((\!(\vec{x}_j)\!)) = \Omega_j$. By convexity of the domain $\mathrm{Cl}(\Omega)$, $(\mathrm{Cl}(\Omega))^o = \Omega$. We obtain

$$\Omega = (\{p_1\} * \cdots * \{p_q\} * \Omega_1 * \cdots * \Omega_m)^o.$$

Recall from above that $L|\Omega_j$ is a unipotent abelian group and hence has a distal action. The proof of Theorem 2 of [80] applies here since $L|\Omega_j$ contains a cocompact lattice, and it follows that $L(\vec{x}_j)$ is a complete affine space in $\mathbb{S}(V_j)$. This proves (iii). [$\mathbb{S}^n$T] $\square$

LEMMA A.1.7. *Let $t_0 \in I$ for an interval I. Suppose that we have a parameter of compact convex domains $\triangle_t \subset \mathbb{S}^{n-1}$ for $t < t_0$, $t \in I$, and a transitive group action $\Phi_t : L \times \triangle_t^o \to \triangle_t^o, t \in I$ by a connected free abelian group L of rank $n-1$ for each $t \in I$. Suppose that Φ_t depends continuously on t and Φ_t is given by a continuous parameter of homomorphisms $h_t : L \to \mathsf{SL}_{\pm}(n, \mathbb{R})$. Then $\{\triangle_t\} \to \triangle_{t_0}$ geometrically where $\triangle_{t_0}$ is a convex domain, and L acts transitively on $\triangle_{t_0}^o$.*

PROOF. Let $L \cong \mathbb{R}^{n-1}$ have coordinates $(z_1, \ldots, z_{n-1})$. $\Phi_t(g, \cdot) : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is represented as a matrix

$$h_t(g) = \exp(H_t(\sum_{i=1}^{n-1} z_i(g)e_i)) \tag{A.1.4}$$

where $\{H_t : \mathbb{R}^{n-1} \to \mathfrak{sl}(n, \mathbb{R})\}$ is a uniformly bounded collection of linear maps.

Since Φ_{t_0} is an isomorphism, we may assume that some open set is contained in an orbit of $\Phi_{t_0}(L)$ for a point x_0. Since $\Phi_t \to \Phi_{t_0}$ algebraically, we may assume without loss of generality that $\bigcap_{t\in I} \triangle_t^o \neq \emptyset$ and contains an open neighborhood of x_0 by taking a smaller I containing t_0.

Let Δ_{t_0} denote the interior of a geometric limit of $\mathrm{Cl}(\Delta_{t_i})$ for some subsequence $t_i \to t_0$ as we can see from Section 1.1.2. By Proposition 1.1.9, Δ_{t_0} is a convex open set.

Any point $x \in \triangle_{t_0}^o$ equals $\Phi_{t_0}(g, x_0)$ for $g \in L$. Therefore,

$$\{\Phi_t(g, x_0)\} \in \triangle_t^o \to \Phi_{t_0}(g, x_0) \text{ as } t \to t_0.$$

Hence, every point of $\triangle_{t_0}$ is the limit of a path $\gamma(t) \in \triangle_t^o$ for $t < t_0$. Hence, $\Phi_{t_0}(L)(x_0)$ is contained in Δ_{t_0} by (1.1.1).

Now we show that $\Delta_{t_0}^o$ is a unique open orbit of L under Φ_{t_0}.

First suppose that Δ_t is properly convex for $t \in I - \{t_0\}$. Suppose that Δ'_{t_0} contains more than two open orbits under Φ_{t_0}. Then there exists a point y in the interior of Δ'_{t_0} and in an orbit of dimension $< n$. By (iv) of Lemma A.1.6, there exists a one parameter group of element of form $g^s \in L, s \in \mathbb{R}$ fixing each point of a hyperplane P passing y and having a largest norm eigenvalue at another point x of multiplicity one outside P. (To see, we just need to consider the diagonalizable group acting trivially on each of the subspaces and find the kernel acting trivially on the join producing P.)

Let $B(y)$ denote a compact ball which is contained in Δ'_{t_0} and in Δ_t for sufficiently close t to t_0. For sufficiently close t to t_0, g still has the largest a largest norm eigenvalue of multiplicity one and a g-invariant plane P_t meeting the interior of $B(y)$. Then acting by $g^s, s \in \mathbb{R}$, we see that Δ_t cannot be properly convex. This is a contradiction. Hence, $\Delta_{t_0}^o$ is an orbit.

Suppose now that Δ_t is not properly convex for t in $J - \{t_0\}$ for some interval $J \subset I$. Then $\mathrm{Cl}(\Delta_t) = \mathbb{S}_t^{i-1} * K_t$ for a properly convex domain K_t by Proposition 1.1.4. L acts on a great sphere $\mathbb{S}_t^{i-1}$ in the boundary of Δ_t. Now, $\mathbb{S}_t^{i-1}$ is the common boundary of i-dimensional affine spaces foliating Δ_t by Proposition 1.1.4 as in Section 7.1.1. We may assume without loss of generality that $\mathbb{S}_t^{i-1}$ is a fixed sphere $\mathbb{S}^{i-1}$ acting by an element g_t where $\{g_t\}$ converging to I as $t \to t_0$. There is a projection

$$\Pi_{\mathbb{S}^{i-1}} : \mathbb{S}^{n-1} - \mathbb{S}^{i-1} \to \mathbb{S}^{n-i-1}.$$

Then we consider $\Pi_{\mathbb{S}^{i-1}}(K_t) \subset \mathbb{S}^{n-i-1}$. Now, the discussion reduces to the above by taking a subgroup $L' \subset L$ acting transitively on the interior of $\Pi_{\mathbb{S}^{i-1}}(K_t)$ for $t \in J - \{t_0\}$.

For each i-hemisphere in Δ_t^o for t near t_0 with boundary $\mathbb{S}^{i-1}$, L acts transitively. We may assume that this is true for $t = t_0$ by the limit argument. Hence, we show that Δ_{t_0} is an orbit. $\square$

A.1.3. Deforming convex real projective structures.

LEMMA A.1.8. *Let μ be a real projective structure on a closed orbifold M with a developing map* $\mathbf{dev} : \tilde{M} \to \mathbb{S}^{n-1}$ (*resp.* $\mathbb{RP}^{n-1}$) *is not injective. Then for any structure μ' sufficiently close to μ, its developing map* $\mathbf{dev}'$ *is not injective.*

PROOF. We prove for $\mathbb{S}^{n-1}$. We take two open sets U_1 and U_2 respectively containing two points $x, y \in \tilde{M}$ with $\mathbf{dev}(x) = \mathbf{dev}(y)$ where $\mathbf{dev}|U_i$ is an embedding for each $i = 1,2$. Then for any developing map $\mathbf{dev}'$ for μ' perturbed from $\mathbf{dev}$ under the C^r-topology, $\mathbf{dev}'(U_1) \cap \mathbf{dev}(U_2) \neq \emptyset$. Hence, $\mathbf{dev}'$ is not injective. [$\mathbb{S}^n$S] □

A convex real projective structure μ_0 on an orbifold Σ is *virtually immediately deformable to* a properly convex real structure if there exists a parameter μ_t of real projective structures on a finite cover $\hat{\Sigma}$ of Σ so that $\hat{\Sigma}$ with induced structures $\hat{\mu}_t$ is properly convex for $t > 0$.

PROPOSITION A.1.9. *A convex real projective structure on a closed $(n-1)$-orbifold M with virtually free abelian holonomy subgroup of a finite index is always virtually immediately deformable to a properly convex real projective structure.*

PROOF. Again, we prove for $\mathbb{S}^{n-1}$. Let $\mathbb{Z}^l$ denote the fundamental group of a finite cover M' of M. Let $h \in \mathrm{Hom}(\mathbb{Z}^l, \mathsf{SL}_{\pm}(n,\mathbb{R}))$ be the restriction of the holonomy homomorphism to $\mathbb{Z}^l$. Nearby every h, there exists a positively diagonalizable holonomy $h' : \mathbb{Z}^l \to \mathsf{SL}_{\pm}(n,\mathbb{R})$ by Lemmas A.1.1 and A.1.6. By the deformation theory of [49], h'' is realized as a holonomy of a real projective manifold M'' diffeomorphic to M'. Also, the universal cover of M' is a union of orbits of an abelian Lie group L by Benoist [16]. Here, $h'(\mathbb{Z}^l)$ is a lattice in L by Lemma A.1.6.

By premise, h' is deformable to h'' where $h''(\mathbb{Z}^l)$ acts on properly convex domain cocompactly. By Lemma A.1.10, M'' is a properly convex real projective orbifold. [$\mathbb{S}^n$T] □

Let us recall a work of Benoist: Let M be a real projective $(n-1)$-orbifold with nilpotent holonomy. (Here we are not working with orbifolds.) Let N be the nilpotent identity component of the Zariski closure of the holonomy group, Benoist [16], [19] showed that $\tilde{M}$ decomposes into a union of connected open submanifolds D_i, $i \in I$ for an index set I, so that $\mathbf{dev} : D_i \to \mathbf{dev}(D_i)$ is a diffeomorphism to an orbit of a nilpotent Lie group N. The brick number of M is the number of the $(n-1)$-dimensional open orbits that map to mutually distinct connected open strata in M.

LEMMA A.1.10. *Let M be a closed $(n-1)$-orbifold. Let (M,μ) be a convex real projective orbifold with a virtually abelian fundamental group. Suppose μ is deformed to a continuous parameter μ_t of real projective structures so that $\mu_0 = \mu$. Let $h_t : \pi_1(M) \to \mathsf{SL}_{\pm}(n,\mathbb{R})$ (resp. $\mathsf{PGL}(n,\mathbb{R})$), $t \in [0,1] = I$, be a continuous family of the associated holonomy homomorphism with $\mathbf{dev}_t : \tilde{M} \to \mathbb{S}^{n-1}$ (resp. $\mathbb{RP}^n$). Suppose μ_t has the holonomy group $h_t(\pi_1(M))$ with following properties for $t > 0$:*

- **(A):** *it is virtually diagonalizable for or, more generally, it acts on some properly convex domain D_t, or*
- **(B):** *it acts properly on a complete affine space D_t*

where D_t has no proper $h_t(\pi_1(M))$-invariant open domain.

Then (M,μ_t) is properly convex or is complete affine for $t > 0$.

PROOF. Again, we prove for $\mathbb{S}^{n-1}$. For $t = 0$, $\mathbf{dev}_0$ is a diffeomorphism to a complete affine space or a properly convex domain by our conditions.

Define the following sets:

- A is the subset of t satisfying (A) and μ_t is a properly convex structure, and
- B is the subset of t for (B).

We will decompose $[0,1] = A \cup B$.

By Lemma A.1.7, the set $\hat{A}$ of points of I satisfying (A) is open. By Koszul [115], the set A is open and $A \subset \hat{A}$. We have $\hat{A} \cup B = [0,1]$.

(A) Suppose that we have an open connected subset U with $U \subset \hat{A}$ and $t_0 \in \mathrm{Cl}(U)$ with properly convex or complete affine μ_{t_0}. Then we claim $U \subset A$.

Let $t \in U$. Since the holonomy group $h_t(\pi_1(M))$ is virtually abelian, a finite cover M' of M is octantizable by Proposition 2 of [16]. $\mathbf{dev}_t$ maps to a union of orbits of an abelian Zariski closure Δ_t of a finite index abelian subgroup H_t of $h(\pi_1(M'))$ in $\mathbb{S}^{n-1}$ by [19]. He also shows that Δ_t acts on $\tilde{M}$.

By our assumption for h_t, $h_t(\pi_1(M'))$ acts on a properly convex domain in $\mathbb{S}^{n-1}$. Δ_t is positive diagonalizable by Lemma A.1.2. Now orbits of Δ_t are convex simplexes in $\mathbb{S}^{n-1}$ by Section 3.1 of Benoist [19] where he explains the classification of such structures by Smillie [144] and [145].

We claim that $\mathbf{dev}_t$ cannot map to a more than one orbit of Δ_t: Suppose not. We take a finite cover M'' of M so that M'' with induced μ_t has a brick number > 1. We can find a parameter $\{\mu_t\}$ for induced real projective structures of μ_t on M'' converging to a real projective structure μ_{t_0} on M' in the C^r-sense, and a compact fundamental domain F_t of $\tilde{M}$ obtained by perturbing a compact fundamental domain F of $\tilde{M}$ for μ_{t_0}. Since F maps into an open orbit, F_t maps into an open orbit for sufficiently small t by Lemma A.1.7. However, this means that F_t is in an orbit of $\tilde{M}$. (In other words, a sequence of real projective structures with more than one bricks cannot converge to a properly convex structure or a complete affine structure.) Since the orbits of Δ_t on $\mathbb{S}^{n-1}$ are properly convex, we conclude that $\mu_t \in A$ for a sufficiently small $|t - t_0|$ by Theorem 1.4.15. Hence, $U \cap A$ is a nonempty open set.

Also, we claim that U is in A: for any sequence $\{t_i\}$ converging to t_0' in U in $\hat{A}$, choose a point x_0 in the developing image of $\mathbf{dev}_{t_i}$ for sufficiently large i since μ_{t_i} are sufficiently C^r-close. Then $\Delta_{t_i}(x_0) \to \Delta_{t_0'}(x_0)$ by Lemma A.1.7. Since $\tilde{M}$ with μ_{t_i} develops into $\Delta_{t_i}(x_0)$, the fundamental domain of $\tilde{M}$ with $\mu_{t_0'}$ develops into $\Delta_{t_0'}(x_0)$, and hence $\mathbf{dev}_{t_0'}$ develops into $\Delta_{t_0'}(x_0)$. The developing map is a diffeomorphism to $\Delta_{t_0'}(x_0)$ by Theorem 1.4.15. Hence, $t_0' \in A$ also. Thus, $U \cap A$ is open and closed and hence $U \subset A$.

Hence, for connected open U, $U \subset \hat{A}$, with $t_0 \in \mathrm{Cl}(U)$ for properly convex or complete affine μ_{t_0}, we have $U \subset A$.

(B) Let I' denote the subset of t in $[0,1]$ consisting of t with μ_t satisfying conclusions of the lemma. Since μ_0 is properly convex or complete affine, I' is not empty. Also, I' contains all components of $\hat{A}$ meeting it by the above argument. Also, $A' = I' \cap \hat{A}$ is open.

We will show that I' is open and closed in $[0,1]$ and hence $I' = [0,1]$:

First, we show that I' is open. Also, for $t_0 \in I' \cap B$, we claim that a neighborhood is in I': Otherwise, there is a sequence $\{t_i\}$ for $t_i \notin I'$ converging to t_0 with $t_i \in I$ and $\mathbf{dev}_{t_i}$ is not a diffeomorphism to a complete affine space where h_{t_i} acts on.

- If $t_i \in \hat{A}$ for infinitely many i, then $t_i \in A \subset I'$ for sufficiently large i by the same argument as the fourth paragraph above. This is a contradiction.
- Assume now $t_i \in B$ for sufficiently large i. Proposition 2 of [16] shows that $\tilde{M}$ decomposes into orbits of an abelian group that is the Zariski closure of $h_{t_i}(\pi_1(M))$. By our condition, open orbits are open hemispheres. Since $\mathbf{dev}_{t_i}$ is not a diffeomorphism to an orbit, $\tilde{M}$ has more than two open orbits. Also, $\tilde{M}$ does not have a compact fundamental domain contained in an orbit since otherwise by $\mathbf{dev}_{t_i}$ must map into an orbit. We choose a compact fundamental domain F for $\mathbf{dev}_{t_0}$,

which can be perturbed to a compact fundamental domain F_t which is inside an open orbit. Since $\tilde{M}$ is a union of images of F_t, $\tilde{M}$ is inside one orbit. This is a contradiction.

Hence, we showed that I' is an open subset of I.

Now, we show that I' is closed: For any boundary point t_0 of I' in $\hat{A}$, we have $t_0 \in I'$ since I' contains a component of $\hat{A}$ it meets.

For any boundary point t_0 of I' that is in B, we have $\mathbf{dev}_{t_0}$ must be injective by Lemma A.1.8 since $\mathbf{dev}_t$ is injective for $t < t_0$ or $t > t_0$ for t in an open interval with boundary point t_0. Hence, the image of $\mathbf{dev}_{t_0}$ meets a complete affine subspace invariant under the action of $h_{t_0}(\pi_1(M))$. Since the image of $\mathbf{dev}_t$ for $t \in I'$ is in a hemisphere or a compact properly convex domain D_i, the image of $\mathbf{dev}_{t_0}$ is in a hemisphere or a properly convex domain that is the geometric limit of D_i up to a choice of subsequences. Hence, $t_0 \in I'$.

Thus, I' is open and closed. This completes the proof. $[\mathbb{S}^n\mathrm{T}]$ □

LEMMA A.1.11. *Let Σ be a properly convex closed orbifold with a structure $\mu_t, t \in [0,1]$. Let $\mathbf{dev}_t$ be a continuous parameter of developing maps for μ_t. Then for t in a open subset of $[0,1]$,*

$$\mathrm{Cl}(\mathbf{dev}_t(\tilde{\Sigma})) = S_1 * \cdots * S_{m_t}$$

for properly convex domains $S_1, \dots, S_{m_t}$ where each $S_{j,t}$ span a subspace $P_{j,t}$. The finite-index subgroup of $h_t(\pi_1(\Sigma))$ acting on $P_{j,t}$ acts strongly irreducible for each t. Furthermore, $m_t, \dim S_{j,t}$ are constant and $P_{j,t}, j = 1, \dots, m_t$ are always independent, and $P_{j,t}$ forms a continuous parameter in the Grassmannian spaces $G(n, \dim P_{j,t})$ up to reordering.

PROOF. For an open susbet O of $[0,1]$, $\mathbf{dev}_t, t \in O$, is a diffeomorphism by Porposition 5.3.11. The decomposition follows from Proposition 1.4.10. There is a virtual center Z, a free abelian group of rank m_1, mapping to a positive diagonalizable group Z_t acting trivially on each $S_{j,t}$. By Theorem 1.1 of [21], an infinite-order virtually central element cannot have nontrivial action on $S_{j,t}$ since otherwise the Zariski closure of the subgroup of $h_t(\pi_1(\Sigma))$ acting on $P_{j,t}$ cannot be simple as claimed immediately after that theorem. Hence, any infinite-order virtually central elements are in a maximal free abelian group of rank m_1. Hence, for each t, there are subspaces $S_{j,t}$ for $j = 1, \dots, m_1$ so that the above decomposition hold. Now, we need to show that the dimensions are constant.

We decompose I into mutually disjoint subsets $I_{n_1, \dots, n_{m_1}}$ where

$$\dim S_{j,t} = n_j \text{ where } n_1 + \cdots + n_{m_1} + m_1 = n + 1 \text{ for } n_1 \leq n_2 \leq \cdots \leq n_{m_1}$$

by reordering the indices. Then each of these sets is closed as we can see from a sequence argument since the above rank argument shows that there cannot be further decomposition in the limit increasing the rank of the virtual center. Since I is connected, there is only one such set equal to I. Now, the conclusion follows up to reordering. □

We generalize Proposition A.1.10:

COROLLARY A.1.12. *Suppose that a real projective orbifold Σ is a closed $(n-1)$-orbifold with the structure μ. Let μ_t, $t \in [0,1]$, be a parameter of projective structures on Σ so that μ_0 is properly convex or complete affine and $\mu_1 = \mu$. Let h_t denote the associated holonomy homomorphisms.*

- *Suppose that the holonomy group $h_t(\pi_1(\Sigma))$ in $\mathsf{PGL}(n,\mathbb{R})$ (resp. $\mathsf{SL}_\pm(n,\mathbb{R})$) acts on a properly convex domain or a complete affine subspace D_t.*
- *Suppose D_t is the minimal $h_t(\pi_1(\Sigma))$-invariant convex open domain.*

- *We require that $\pi_1(\Sigma)$ to be virtually abelian if D_{t_0} is complete affine for at least one t_0.*

Then Σ is also properly convex or complete affine where the following hold:

- *a developing map* $\mathbf{dev}_t$ *is a diffeomorphism to $R_t(D_t)$ for every $t \in [0,1]$*
- *where R_t is a uniformly bounded projective automorphism for each t and is a composition of reflections commuting with one another.*
- *Furthermore, Σ is properly convex if so is D_t.*

PROOF. Again, we prove in $\mathbb{S}^{n-1}$. If $\pi_1(\Sigma)$ is virtually abelian, then it follows from Proposition A.1.10

Now, suppose that $\pi_1(\Sigma)$ is not virtually abelian. Then D_t is properly convex for every $t, t \in I$, by the premise.

The set A where μ_t is properly convex is open by Koszul [114]. Let t_0 be the supremum of the connected component A' of A containing 0. There is a developing map $\mathbf{dev}_t : \tilde{\Sigma} \to \mathbb{S}^{n-1}$ for $t \in A$ is a diffeomorphism to a properly convex domain D'_t where $D'_t = R_t(D_t)$ for a projective automorphism R_t by Lemma 1.4.16.

Since associated developing map $\mathbf{dev}_t$ maps into D'_t for $t < t_0$, $\mathbf{dev}_t|K$ for a compact fundamental domain $F \subset \tilde{\Sigma}$ maps into a compact subset of D'^o_t for $t < t_0$. Since $\mathbf{dev}_t : \tilde{\Sigma}$ is injective, $\mathbf{dev}_{t_0} : \tilde{\Sigma} \to \mathbb{S}^{n-1}$ is also injective by Lemma A.1.8. By the injectivity and the invariance of domain, $\mathbf{dev}_{t_0}$ is a diffeomorphism to an open domain Ω. Since every point of the image of $\mathbf{dev}_{t_0}$ is approximated by the points in the image D'^o_t of $\mathbf{dev}_t$ for $t < t_0$. Hence, $\mathrm{Cl}(\Omega)$ is contained in the geometric limit of a convergent subsequence of any sequence $\mathrm{Cl}(D'_{t_i})$ by Proposition 1.1.7. By Lemma 1.4.16, Ω is a properly convex domain since the holonomy group acts on a properly convex domain D_{t_0} and $\Omega = R_{t_0}(D_{t_0})$ for a projective automorphism R_{t_0} that is a composition of reflections commuting with one another.

Hence, A is also closed, and $A = [0,1]$. By Lemma A.1.11 the uniform boundedness of R_t follows since the subspaces $P_{j,t}$ are continuous and R_t are either I or $\mathscr{A}$ on it. [$\mathbb{S}^n$T] $\square$

A.1.4. Geometric convergence of convex real projective orbifolds. Note that the third item of the premise below is automatically true by Theorem 8.1.2 if Σ is an end-orbifold of a properly convex affine n-orbifold for any t.

COROLLARY A.1.13. *Suppose that Σ is a closed $(n-1)$-orbifold. We are given a path μ_t, $t \in [0,1]$, of convex $\mathbb{RP}^{n-1}$-structures on Σ equipped with the C^r-topology, $r \geq 2$. Suppose that μ_0 is properly convex or complete affine.*

- *Suppose that the holonomy group $h_t(\pi_1(\Sigma))$ in $\mathsf{PGL}(n,\mathbb{R})$ (resp. $\mathsf{SL}_\pm(n,\mathbb{R})$) acts on a properly convex domain or a complete affine subspace D_t.*
- *Suppose D_t is the minimal holonomy invariant domain.*
- *We require that if μ_t is complete affine for at least one t, then the holonomy group is virtually abelian.*

Then the following holds:

- *We can find a family of developing maps* $\mathbf{dev}_t$ *to $\mathbb{RP}^{n-1}$ (resp. in $\mathbb{S}^{n-1}$) continuous in the C^r-topology and a continuous family of holonomy homomorphisms $h_t : \Gamma \to \Gamma_t$ so that $K_t := \mathrm{Cl}(\mathbf{dev}_t(\tilde{\Sigma}))$ is a continuous family of convex domains in $\mathbb{RP}^{n-1}$ (resp. in $\mathbb{S}^{n-1}$) under the Hausdorff metric topology of the space of closed subsets of $\mathbb{RP}^{n-1}$ (resp. $\mathbb{S}^{n-1}$).*
- *In other words, given $0 < \varepsilon < 1/2$ and $t_0, t_1 \in [0,1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $K_{t_1} \subset N_\varepsilon(K_{t_0})$ and $K_{t_0} \subset N_\varepsilon(K_{t_1})$.*

- *Also, given $0 < \varepsilon < 1/2$ and $t_0, t_1 \in [0,1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $\partial K_{t_1} \subset N_\varepsilon(\partial K_{t_0})$ and $\partial K_{t_0} \subset N_\varepsilon(\partial K_{t_1})$ where we assume $\mathbf{dev}_t$ maps to $\mathbb{S}^{n-1}$ exclusively here.*
- *Finally, μ_t is always virtually immediately deformable to a properly convex structure.*

PROOF. We will prove for $\mathbb{S}^{n-1}$. Suppose first that $\pi_1(\Sigma)$ is not virtually abelian. Then D_t is never a complete affine space and hence is always properly convex by the premise. By Lemma A.1.10, $\mathbf{dev}_t$ is an embedding to the interior of K_t for each t.

First, for any sequence $\{t_i\}$ converging to t_0, we can choose a subsequence $\{t_{i_j}\}$ so that $\{K_{t_{i_j}}\}$ converges to a compact convex set K_∞ in the Hausdorff metric. $h_{t_0}(\pi_1(\Sigma'))$ acts on K_∞ by Corollary 1.4.17.

By Lemma A.1.11, we define $K_{l,t} := P_{l,t} \cap K_t$ where $h_t(\pi_1(\Sigma))$ acts strongly irreducibly on $P_{l,t}$, and $K_t = K_{1,t} * \cdots * K_{m_1}$. We obtain $K_{l,t_{i_j}} \to K_l$ as $j \to \infty$ for a compact convex set K_l in a subspace P_{l,t_0} where a finite-index subgroup of $h_{t_0}(\pi_1(\Sigma))$ acts on by Lemma A.1.11 and Proposition 1.1.7. Proposition 1.4.10 shows that the action on P_{l,t_0} by $\pi_1(\Sigma')$ is irreducible. Hence, K_l must be properly convex by Proposition 1.4.1. We have $K_\infty \subset K_1 * \cdots * K_m$ by Proposition 1.1.7 since $K_{t_{i_j}} \subset K_{1,t_{i_j}} * \cdots * K_{m,t_{i_j}}$ for each j. Hence, K_∞ is properly convex.

Now, for any sequence $\{t'_i\}$ covering to t_0, suppose that a convergent subsequence $\{K_{t'_{i_j}}\}$ converges to K'_∞. Then we claim that $K_\infty = K'_\infty$: Now, K'_∞ is properly convex also. Choose a torsion-free finite-index subgroup Γ' of $h_{t_0}(\pi_1(\Sigma))$ by Theorem 1.1.19. K^o_∞/Γ' and K'^o_∞/Γ' are homotopy equivalent. Since $\mathbf{dev}_{t_i}$ and $\mathbf{dev}_{t'_i}$ are close, we may assume that $K^o_\infty \cap K'^o_\infty \neq \emptyset$. Lemma 1.4.16 shows that $K^o_\infty = K'^o_\infty$. This implies the first item for this case.

Suppose now that Γ is virtually abelian. Then Ω_t is determined by the generators of the free abelian subgroup Γ' of a finite index with only positive eigenvalues by Lemma A.1.6. Γ' determines the connected abelian Lie group Δ_t containing $h_t(\Gamma')$ and Ω_t is an orbit of Δ_t by Lemma A.1.6. Now Lemma A.1.7 implies the first item.

The second item follows from the first one. The third one can be deduced by Theorem 1.1.11. The fourth item follows by Proposition A.1.9. [$\mathbb{S}^n$T] □

REMARK A.1.14. Of course, we wish to generalize Lemma A.1.10 and Corollaries A.1.12 and A.1.13 for fully general cases without the restriction on the domains of actions starting from any properly convex projective orbifold and show the similar results. Then we can allow NPNC-ends into the discussions. We leave this as a question of whether one can achieve such results.

A.2. The justification for weak middle eigenvalue conditions

THEOREM A.2.1. *Let $\mathscr{O}$ be a properly convex real projective orbifold with ends. Let $\Sigma_{\tilde E}$ be an end orbifold of an R-p-end $\tilde E$ of $\mathscr{O}$ with the virtually abelian end-fundamental group $\pi_1(\tilde E)$.*

- *Suppose that μ_i be a sequence of properly convex structure on an R-p-end neighborhood $U_{\tilde E}$ of $\tilde E$ corresponding to a generalized lens-shaped R-p-ends satisfying the uniform middle eigenvalue conditions.*
- *Suppose that μ_i limits to μ_∞ in the C^r-topology, $r \geq 2$.*

Then μ_∞ satisfies the weak middle eigenvalue condition for $\tilde E$. Furthermore, the holonomy group for μ_∞ virtually satisfies the transverse weak middle eigenvalue condition for $\tilde E$ if it is NPNC and $\pi_1(\tilde E)$ is virtually abelian.

PROOF. Assume first that $\tilde{\mathscr{O}} \subset \mathbb{S}^n$. We may assume that the p-end vertex $\mathrm{v}_{\tilde{E}}$ is independent of μ_i by conjugation of the holonomy homomorphism. Let $h_i : \pi_1(\Sigma_{\tilde{E}}) \to \mathrm{SL}_{\pm}(n+1,\mathbb{R})_{\mathrm{v}_{\tilde{E}}}$. Since these satisfy the uniform middle eigenvalue conditions, we have

$$C^{-1}\mathrm{cwl}(g) < \log\left(\frac{\lambda_1(h_i(g))}{\lambda_{\mathrm{v}_{\tilde{E}}}(h_i(g))}\right) < C\mathrm{cwl}(g), C > 1, g \in \pi_1(\Sigma_{\tilde{E}})$$

where C is a constant which may depend on h_i. Let h_∞ denote the holonomy homomorphism for μ_∞, which is an algebraic limit of h_i. By taking limits, we obtain that h_∞ satisfies the weak middle eigenvalue condition.

Suppose now that μ_∞ is NPNC. For convenience, we may assume without loss of generality that $\pi_1(\tilde{E})$ is free abelian. Let Γ denote $h_\infty(\pi_1(\tilde{E}))$.

By Lemma A.1.6, $\tilde{\Sigma}_E$ is the interior of a strict join of hemispheres and a properly convex domain

$$H_1 * \cdots * H_m * K_0 \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$$

where

- $\Gamma|H_j$, $j = 1,\dots,m$, has the Zariski closure a unipotent Lie group for a finite index subgroup Γ of $h_\infty(\pi_1(\tilde{E}))$,
- $\Gamma|K_0$ is a diagonalizable group acting so, and
- Γ has a center Q of rank $m + \dim K_0 - 1$ acting trivially on each H_i, $i = 1,\dots,m$ and fixing the vertices of K_0.

Given any i-dimensional hemisphere V of $\mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ for $0 \le i \le n-1$, there exists unique $i+1$-dimensional hemisphere $\hat{V}$ in $\mathbb{S}^n$ in the direction of V from $\mathrm{v}_{\tilde{E}}$ and containing $\mathrm{v}_{\tilde{E}}$ in $\partial\hat{V}$

We denote by $\hat{H}_i$ the hemispheres in $\mathbb{S}^n$ corresponding to the directions of H_i for $i = 1,\dots,m$ and $\hat{p}_i$ the great segments in $\mathbb{S}^n$ corresponding to the directions vertices $p_1,\dots,p_k$ of K_0. Let $g \in h(\pi_1(\tilde{E}))$. Since a CA-lens for $h_i(\pi_1(\tilde{E}))$ contains the points affiliated with the largest norm of eigenvalues for $h_j(g)$ for each $g \in \pi_1(\tilde{E})$, a limiting argument shows that points in $\hat{H}_i^o$ or $\hat{p}_i$ must be affiliated with the largest norm $\lambda_1(h(g))$ of the eigenvalues. (Of course, these are not all such points necessarily)

By Proposition 1.1.4, the maximal dimensional great sphere $\mathbb{S}^{i_0-1}_\infty$ in $\mathrm{bd}\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathrm{v}_{\tilde{E}}}$ corresponding the boundary of complete affine leaves in $\tilde{\Sigma}_{\tilde{E}}$ equals $\partial\hat{H}_1 * \cdots * \partial\hat{H}_m$. Since these points are not in the directions of $\partial\hat{H}_1 * \cdots * \partial\hat{H}_m$, the desired inequality $\lambda^{Tr}_{\mathrm{v}_{\tilde{E}}}(g) \ge \lambda_{\mathrm{v}_{\tilde{E}}}(g)$ holds. [$\mathbb{S}^n$T] □

Index of Notations

These are not the definitions. Please see the pages to find precise definitions.

$\overline{pzq}$**:** the geodesic segment in $\mathbb{S}^n$ connecting p and q antipodal to p and passing z. p.4

$()^*$**:** Duals of vector spaces or convex sets or linear groups. p.36

$()^\dagger$**:** The proper-subspace dual of a properly convex domain in a subspace. p.37

$()_{\mathscr{E}}$**:** The subscript denotes that the representation space or the character space is restricted by the condition that each end holonomy group to have a fixed point for R-ends or to have a holonomy group invariant hyperspace satisfying the lens-condition for T-ends. p.233

$()_{ce}$**:** The subscript denotes that the representation space or the character space or the deformation space is restricted by the condition that the ends be lens-shaped R-ends or lens-shaped T-ends only or the corresponding condition for the end holonomy groups. p.238

$()_{\mathrm{u}}$**:** The subscript denotes that the representation space or the character space or the deformation space is restricted by the end holonomy group having a unique fixed point for R-ends or having a unique end holonomy invariant hyperplane satisfying the lens-condition for T-ends. p.234

$()^s$**:** The superscript denotes that the representation space or the character space or the deformation space is restricted by the stability condition. p.238

$()_{s_{\mathscr{V}}}$**:** The subscript denotes that the deformation has holonomy in an open subset $\mathscr{V}$ of the character space and the end is determined by the fixing section $s_{\mathscr{V}}$. p.276

Bibliography

[1] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and stringy topology*, volume 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.

[2] J. Anderson and R. Canary. Algebraic limits of Kleinian groups which rearrange the pages of a book. *Invent. Math.*, 126(2):205–214, 1996.

[3] E. Andreev. Convex polyhedra in Lobačevskiĭ spaces. *Mat. Sb. (N.S.)*, 81 (123):445–478, 1970.

[4] S. Ballas. Deformations of noncompact projective manifolds. *Algebr. Geom. Topol.*, 14(5):2595–2625, 2014.

[5] S. Ballas. Finite volume properly convex deformations of the figure-eight knot. *Geom. Dedicata*, 178:49–73, 2015.

[6] S. Ballas, J. Danciger, and G.-S. Lee. Convex projective structures on nonhyperbolic three-manifolds. *Geom. Topol.*, 22(3):1593–1646, 2018.

[7] Samuel Ballas and Ludovic Marquis. Properly convex bending of hyperbolic manifolds. *Groups Geom. Dyn.*, 14(2):653–688, 2020. arXiv:1609.03046.

[8] Samuel A. Ballas, Daryl Cooper, and Arielle Leitner. Generalized cusps in real projective manifolds: classification. *J. Topol.*, 13(4):1455–1496, 2020.

[9] Samuel A. Ballas, Daryl Cooper, and Arielle Leitner. The moduli space of marked generalized cusps in real projective manifolds. *Conform. Geom. Dyn.*, 26:111–164, 2022.

[10] T. Barbot. Variétés affines radiales de dimension 3. *Bull. Soc. Math. France*, 128(3):347–389, 2000.

[11] T. Barbot. Globally hyperbolic flat space-times. *J. Geom. Phys.*, 53(2):123–165, 2005.

[12] D. Barden. *The structure of manifolds*. PhD thesis, Cambridge University, Cambridge, England, 1963.

[13] V.V. Batryrev. Toric varieties and smooth convex approximations of a polytope. *RIMS Kokyuroku*, 776:20–20, 1992.

[14] A. Ben-Tal and M. Teboulle. A smoothing technique for nondifferentiable optimization problems. In *Optimization (Varetz, 1988)*, volume 1405 of *Lecture Notes in Math.*, pages 1–11. Springer, Berlin, 1989.

[15] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[16] Y. Benoist. Nilvariétés projectives. *Comment. Math. Helv.*, 69(3):447–473, 1994.

[17] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.

[18] Y. Benoist. Automorphismes des cônes convexes. *Invent. Math.*, 141(1):149–193, 2000.

[19] Y. Benoist. Tores affines. In *Crystallographic groups and their generalizations (Kortrijk, 1999)*, volume 262 of *Contemp. Math.*, pages 1–37. Amer. Math. Soc., Providence, RI, 2000.

[20] Y. Benoist. Convexes divisibles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(5):387–390, 2001.

[21] Y. Benoist. Convexes divisibles. II. *Duke Math. J.*, 120(1):97–120, 2003.

[22] Y. Benoist. Convexes divisibles. I. In *Algebraic groups and arithmetic*, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004.

[23] Y. Benoist. Convexes divisibles. III. *Ann. Sci. École Norm. Sup. (4)*, 38(5):793–832, 2005.

[24] Y. Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. *Invent. Math.*, 164(2):249–278, 2006.

[25] J.-P. Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[26] M. Berger. *Geometry I*. Universitext. Springer-Verlag, Berlin, 2009. Translated from the 1977 French original by M. Cole and S. Levy, Fourth printing of the 1987 English translation [MR0882541].

[27] Marc A. Berger and Yang Wang. Bounded semigroups of matrices. *Linear Algebra Appl.*, 166:21–27, 1992.

[28] J. Bochi. Inequalities for numerical invariants of sets of matrices. *Linear Algebra Appl.*, 368:71–81, 2003.

[29] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[30] B. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[31] E. Breuillard and C. Sert. The joint spectrum. arXiv:1809.02404.

[32] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[33] R. Canary, D. Epstein, and P. Green. Notes on notes of Thurston [mr0903850]. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 1–115. Cambridge Univ. Press, Cambridge, 2006. With a new foreword by Canary.

[34] Y. Carrière. Flots Riemanniens. *Astérisque*, (116):31–52, 1984. Transversal structure of foliations (Toulouse, 1982).

[35] Y. Carrière. Feuilletages Riemanniens à croissance polynômiale. *Comment. Math. Helv.*, 63(1):1–20, 1988.

[36] Y. Chae, S. Choi, and C. Park. Real projective manifolds developing into an affine space. *Internat. J. Math.*, 4(2):179–191, 1993.

[37] S. Choi. The classification of radial or totally geodesic ends of real projective orbifolds. II: properly convex ends and totally geodesic ends. arXiv:1501.00352.

[38] S. Choi. The mathematica file to compute real projective structures on $S^2(3,3,3)$. http://mathsci.kaist.ac.kr/~schoi/triangle4reversal.nb.

[39] S. Choi. The mathematica files to compute tetrahedral real projective orbifolds. http://mathsci.kaist.ac.kr/~schoi/exampletet3.nb, http://mathsci.kaist.ac.kr/~schoi/equations.nb.

[40] S. Choi. The mathematica files to drawing the end neighorhood of the quasi-joined end. http://mathsci.kaist.ac.kr/~schoi/quasijoin2ii.nb.

[41] S. Choi. The mathematica files to drawing the tetrahedral real projective orbifolds. http://mathsci.kaist.ac.kr/~schoi/coefficientsol8.nb.

[42] S. Choi. Real projective structures on 3-orbifolds and projective invariants. the presentation for Melbourne Talk, May 18, 2009. http://mathsci.kaist.ac.kr/~schoi/melb2009handout.pdf.

[43] S. Choi. Convex decompositions of real projective surfaces. I. π-annuli and convexity. *J. Differential Geom.*, 40(1):165–208, 1994.

[44] S. Choi. Convex decompositions of real projective surfaces. II. Admissible decompositions. *J. Differential Geom.*, 40(2):239–283, 1994.

[45] S. Choi. The Margulis lemma and the thick and thin decomposition for convex real projective surfaces. *Adv. Math.*, 122(1):150–191, 1996.

[46] S. Choi. *The convex and concave decomposition of manifolds with real projective structures*, volume 78 of *Mém. Soc. Math. Fr.* (*N.S.*). Soc. Math. France, Paris, 1999.

[47] S. Choi. The decomposition and classification of radiant affine 3-manifolds. *Mem. Amer. Math. Soc.*, 154(730):viii+122, 2001.

[48] S. Choi. Geometric structures on low-dimensional manifolds. *J. Korean Math. Soc.*, 40(2):319–340, 2003.

[49] S. Choi. Geometric structures on orbifolds and holonomy representations. *Geom. Dedicata*, 104:161–199, 2004.

[50] S. Choi. The deformation spaces of projective structures on 3-dimensional Coxeter orbifolds. *Geom. Dedicata*, 119:69–90, 2006.

[51] S. Choi. *Geometric structures on 2-orbifolds*: *exploration of discrete symmetry*, volume 27 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2012.

[52] S. Choi. A classification of radial or totally geodesic ends of real projective orbifolds. I: a survey of results. In *Hyperbolic Geometry and Geometric Group Theory*, Advanced Studies in Pure Mathematics, pages 69–134, Tokyo, 2017. Mathematical Society of Japan.

[53] S. Choi. The convex real projective orbifolds with radial or totally geodesic ends: a survey of some partial results. In A. Reid A. Basmanjian, Y. Minsky, editor, *In the Tradition of Ahlfors–Bers, VII*, volume 696 of *Contemp. Math.*, pages 51– 86, Providence, RI., 2017. Amer. Math. Soc.

[54] S. Choi and W. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.

[55] S. Choi and W. Goldman. The classification of real projective structures on compact surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 34(2):161–171, 1997.

[56] S. Choi and W. Goldman. The deformation spaces of convex $\mathbb{RP}^2$-structures on 2-orbifolds. *Amer. J. Math.*, 127(5):1019–1102, 2005.

[57] S. Choi, R. Green, G. Lee, and L. Marquis. Projective deformations of hyperbolic coxeter 3-orbifolds of finite volume. in preparation.

[58] S. Choi, C. Hodgson, and G. Lee. Projective deformations of hyperbolic Coxeter 3-orbifolds. *Geom. Dedicata*, 159:125–167, 2012.

[59] S. Choi, C. Hodgson, and G.-S. Lee. Projective deformations of hyperbolic Coxeter 3-orbifolds. *Geom. Dedicata*, 159:125–167, 2012.

[60] S. Choi and G. Lee. Projective deformations of weakly orderable hyperbolic Coxeter orbifolds. *Geom. Topol.*, 19(4):1777–1828, 2015.

[61] S. Choi, G. Lee, and L. Marquis. Convex real projective structures on manifolds and orbifolds. In S.-T. Yau L. Ji, A. Papadopoulos, editor, *The Handbook of Group Actions III*, chapter 10, pages 263–310. Higher Education Press and International Press, Boston, 2018. arXiv:1605.02548.

[62] J.-P. Conze and Y. Guivarc'h. Remarques sur la distalité dans les espaces vectoriels. *C. R. Acad. Sci. Paris Sér. A*, 278:1083–1086, 1974.

[63] D. Cooper. The Heisenberg group acts on a strictly convex domain. *Conform. Geom. Dyn.*, 21:101–104, 2017.

[64] D. Cooper, D. Long, and M. Thistlethwaite. Computing varieties of representations of hyperbolic 3-manifolds into $\mathrm{SL}(4,\mathbb{R})$. *Experiment. Math.*, 15(3):291–305, 2006.

[65] D. Cooper, D. Long, and M. Thistlethwaite. Flexing closed hyperbolic manifolds. *Geom. Topol.*, 11:2413–2440, 2007.

[66] D. Cooper, D. Long, and S. Tillman. Deforming convex projective manifolds. *Geometry & Topology*, 22:1349–1404, 2018. arXiv:1511.06206.

[67] D. Cooper, D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

[68] M. Crampon and L. Marquis. Finitude géométrique en géométrie de Hilbert. *Ann. Inst. Fourier (Grenoble)*, 64(6):2299–2377, 2014.

[69] M. Crampon and L. Marquis. Le flot géodésique des quotients géométriquement finis des géométries de Hilbert. *Pacific J. Math.*, 268(2):313–369, 2014.

[70] J. Danciger, F. Guéritaud, and F. Kassel. Convex cocomapct acts on real projective geometry. preprint, arXiv: 1704.08711, 2021.

[71] J. de Groot and H. de Vries. Convex sets in projective space. *Compositio Math.*, 13:113–118, 1958.

[72] P. de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. Cambridge Univ. Press, Cambridge, 1993.

[73] G. de Rham. *Differentiable manifolds*, volume 266 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[74] C. Druţu and M. Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[75] C. Druţu. Relatively hyperbolic groups: geometry and quasi-isometric invariance. *Comment. Math. Helv.*, 84(3):503–546, 2009.

[76] C. Druţu and M. Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.

[77] D. B. A. Epstein. Transversely hyperbolic 1-dimensional foliations. *Astérisque*, (116):53–69, 1984. Transversal structure of foliations (Toulouse, 1982).

[78] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[79] D. Fried. The geometry of cross sections to flows. *Topology*, 21(4):353–371, 1982.

[80] D. Fried. Distality, completeness, and affine structures. *J. Differential Geom.*, 24(3):265–273, 1986.

[81] D. Fried. Choi's lenses. informal note, 2012.

[82] D. Fried and W. Goldman. Three-dimensional affine crystallographic groups. *Adv. in Math.*, 47(1):1–49, 1983.

[83] H. Garland and M. S. Raghunathan. Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups. *Ann. of Math. (2)*, 92:279–326, 1970.

[84] M. Ghomi. Optimal smoothing for convex polytopes. *Bull. London Math. Soc.*, 36(4):483–492, 2004.

[85] W. Goldman. Affine manifolds and projective geometry on surfaces. senior thesis, Princeton University, 1977.

[86] W. Goldman. Geometric structures on manifolds. in preparation (version June 28, 2020), formerly titled "Projective geometry on manifolds", http://www.math.umd.edu/~wmg.

[87] W. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.

[88] W. Goldman. Convex real projective structures on compact surfaces. *J. Differential Geom.*, 31(3):791–845, 1990.

[89] W. Goldman. Geometric structures on manifolds. 2022.

[90] W. Goldman and M. Hirsch. Affine manifolds and orbits of algebraic groups. *Trans. Amer. Math. Soc.*, 295(1):175–198, 1986.

[91] W. Goldman, F. Labourie, and G. Margulis. Proper affine actions and geodesic flows of hyperbolic surfaces. *Ann. of Math. (2)*, 170(3):1051–1083, 2009.

[92] W. Goldman and J. Millson. The deformation theory of representations of fundamental groups of compact Kähler manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (67):43–96, 1988.

[93] R. Greene. *The deformation theory of discrete reflection groups and projective structures*. PhD thesis, Ohio State University, 2013.

[94] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[95] O. Guichard. Sur la régularité Hölder des convexes divisibles. *Ergodic Theory Dynam. Systems*, 25(6):1857–1880, 2005.

[96] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.

[97] D. Heard, C. Hodgson, B. Martelli, and C. Petronio. Hyperbolic graphs of small complexity. *Experiment. Math.*, 19(2):211–236, 2010.

[98] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[99] M. Heusener and J. Porti. Infinitesimal projective rigidity under Dehn filling. *Geom. Topol.*, 15(4):2017–2071, 2011.

[100] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[101] K. Hoffman and R. Kunze. *Linear algebra*. Second edition. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971.

[102] J. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[103] M. Islam and A. Zimmer. Convex co-compact actions of relatively hyperbolic groups. arXiv:1910.08885.

[104] M. Islam and A. Zimmer. A flat torus theorem for convex co-compact actions of projective linear groups. arXiv:1907.03277.

[105] J. Jenkins. Growth of connected locally compact groups. *J. Functional Analysis*, 12:113–127, 1973.

[106] D. Johnson and J. Millson. Deformation spaces associated to compact hyperbolic manifolds. In *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, volume 67 of *Progr. Math.*, pages 48–106. Birkhäuser Boston, Boston, MA, 1987.

[107] V. Kac and È. Vinberg. Quasi-homogeneous cones. *Mat. Zametki*, 1:347–354, 1967.

[108] M. Kapovich. *Hyperbolic manifolds and discrete groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 2001 edition.

[109] F. Kassel and R. Potrie. Eigenvalue gaps for hyperbolic groups and semigroups. arXiv:2002.07015.

[110] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[111] I. Kim. Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds. *J. Differential Geom.*, 58(2):189–218, 2001.

[112] S. Kobayashi. Projectively invariant distances for affine and projective structures. In *Differential geometry (Warsaw, 1979)*, volume 12 of *Banach Center Publ.*, pages 127–152. PWN, Warsaw, 1984.

[113] B. Kostant and D. Sullivan. The Euler characteristic of an affine space form is zero. *Bull. Amer. Math. Soc.*, 81(5):937–938, 1975.

[114] J.-L. Koszul. Déformations de connexions localement plates. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 1):103–114, 1968.

[115] J.-L. Koszul. Trajectoires convexes de groupes affines unimodulaires. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages 105–110. Springer, New York, 1970.

[116] N. H. Kuiper. On convex locally-projective spaces. In *Convegno Internazionale di Geometria Differenziale, Italia, 1953*, pages 200–213. Edizioni Cremonese, Roma, 1954.
[117] F. Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure Appl. Math. Q.*, 3(4, part 1):1057–1099, 2007.
[118] A. Leitner. A classification of subgroups of $SL(4,\mathbb{R})$ isomorphic to $\mathbb{R}^3$ and generalized cusps in projective 3 manifolds. *Topology Appl.*, 206:241–254, 2016.
[119] A. Leitner. Conjugacy limits of the diagonal Cartan subgroup in $SL_3(\mathbb{R})$. *Geom. Dedicata*, 180:135–149, 2016.
[120] A. Leitner. Limits under conjugacy of the diagonal subgroup in $SL_n(\mathbb{R})$. *Proc. Amer. Math. Soc.*, 144(8):3243–3254, 2016.
[121] W. Lok. *Deformations of locally homogeneous spaces and Kleinian groups*. PhD thesis, Columbia University, New York, New York, 1984. pp. 1–178.
[122] A. Malcev. On isomorphic matrix representations of infinite groups. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):405–422, 1940.
[123] A. I. Malcev. On a class of homogeneous spaces. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:9–32, 1949.
[124] A. Marden. *Outer circles*. Cambridge University Press, Cambridge, 2007.
[125] L. Marquis. Espace des modules de certains polyèdres projectifs miroirs. *Geom. Dedicata*, 147:47–86, 2010.
[126] L. Marquis. Surface projective convexe de volume fini. *Ann. Inst. Fourier (Grenoble)*, 62(1):325–392, 2012.
[127] L. Marquis. Coxeter group in Hilbert geometry. *Groups Geom. Dyn.*, 11(3):819–877, 2017.
[128] Barry Mazur. Relative neighborhoods and the theorems of Smale. *Ann. of Math. (2)*, 77:232–249, 1963.
[129] G. Mess. Lorentz spacetimes of constant curvature. *Geom. Dedicata*, 126:3–45, 2007.
[130] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.
[131] P. Molino. *Riemannian foliations*, volume 73 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.
[132] C. Moore. Distal affine transformation groups. *Amer. J. Math.*, 90:733–751, 1968.
[133] J. Morgan. On Thurston's uniformization theorem for three-dimensional manifolds. In *The Smith conjecture (New York, 1979)*, volume 112 of *Pure Appl. Math.*, pages 37–125. Academic Press, Orlando, FL, 1984.
[134] J. Morgan and P. Shalen. Degenerations of hyperbolic structures. III. Actions of 3-manifold groups on trees and Thurston's compactness theorem. *Ann. of Math. (2)*, 127(3):457–519, 1988.
[135] D. Morris. *Introduction to arithmetic groups*. Deductive Press, [place of publication not identified], 2015.
[136] J. Munkres. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.
[137] T. Nagano and K. Yagi. The affine structures on the real two-torus. I. *Osaka J. Math.*, 11:181–210, 1974.
[138] J Porti. Dimension of representation and character varieties for two- and three-orbifolds. *Algebraic & Geometric Topology*, 22:1905—-1967, 2020.
[139] J. Porti and S. Tillman. Projective structures on a hyperbolic 3–orbifold. *Acta Mathematica Vietnamica*, 46(2):347–355, 2021.
[140] R. Roeder, J. Hubbard, and W. Dunbar. Andreev's theorem on hyperbolic polyhedra. *Ann. Inst. Fourier (Grenoble)*, 57(3):825–882, 2007.
[141] D. Ruelle and D. Sullivan. Currents, flows and diffeomorphisms. *Topology*, 14(4):319–327, 1975.
[142] A. Selberg. On discontinuous groups in higher-dimensional symmetric spaces. In *Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960)*, pages 147–164. Tata Institute of Fundamental Research, Bombay, 1960.
[143] H. Shima. *The geometry of Hessian structures*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
[144] J. Smillie. Affine manifolds with diagonal holonmy. preprint, 1977.
[145] J. Smillie. *Affinely flat manifolds*. PhD thesis, University of Chicago, Chicago, USA, 1977.
[146] D. Sullivan. Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):57–73, 1982.
[147] D. Sullivan and W. Thurston. Manifolds with canonical coordinate charts: some examples. *Enseign. Math. (2)*, 29(1-2):15–25, 1983.
[148] W. Thurston. *The geometry and topology of three-manifolds*. Lecture notes. Princeton University, 1980. Available at http://library.msri.org/books/gt3m/.
[149] W. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[150] V. S. Varadarajan. *Lie groups, Lie algebras, and their representations*, volume 102 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Reprint of the 1974 edition.

[151] J. Vey. Sur les automorphismes affines des ouverts convexes saillants. *Ann. Scuola Norm. Sup. Pisa (3)*, 24:641–665, 1970.

[152] È. Vinberg. The theory of homogeneous convex cones. *Trudy Moskov. Mat. Obšč.*, 12:303–358, 1963.

[153] È. Vinberg. Discrete linear groups that are generated by reflections. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:1072–1112, 1971.

[154] A. Weil. On discrete subgroups of Lie groups. II. *Ann. of Math. (2)*, 75:578–602, 1962.

[155] T Weisman. Dynamical properties of convex cocompact actions in projective space. *J. Topology*, 16, 2023.

[156] D. Witte. Superrigidity of lattices in solvable Lie groups. *Invent. Math.*, 122(1):147–193, 1995.

[157] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[158] H. Zassenhaus. Beweis eines satzes über diskrete gruppen. *Abh. Math. Sem. Univ. Hamburg*, 12(1):289–312, 1937.

[159] A. Zimmer. A higher-rank rigidity theorem for convex real projective manifolds. *Geometry & Topology*, 23(7):2899–2936, 2023.

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