Linear Algebra: Final Exam (2005 Spring)

Justify your answers fully.

1. State whether each of the following statements is true or false. If false, provide a counter-example.

   (a) (10 pts.) The minimal polynomial divides the characteristic polynomial.

   (b) (10 pts.) Let T be a linear operator on a finite dimensional vector space V^n over the real numbers R. Then T is diagonalizable if the characteristic polynomial factors into 1st order polynomials.

   (c) (10 pts.) For any two $n \times n$-matrices A and B with entries in R such that $AB = BA$, there exists an $n \times n$-matrix P with entries in R such that $P^{-1}AP$ and $P^{-1}BP$ become upper triangular.

2. (20 pts.) Let A and B be 3×3-matrices with entries in R. Prove that A and B are similar over R if and only if A and B have the same characteristic polynomials and the same minimal polynomials.

3. (30 pts.) Let W be the subspace of R^3 spanned by the unit vector (a_1, a_2, a_3). Using the standard inner product, let E be the orthogonal projection of R^3 onto W.

   (a) (6 pts.) Find the formula of $E(x_1, x_2, x_3)$.

   (b) (7 pts.) Find the matrix of E in the standard basis of R^3.

   (c) (7 pts.) Find the orthogonal basis in which E is represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (d) (10 pts.) Generalize (a),(b) and (c) to R^n for any dimension $n \geq 3$.

4. Let A be the matrix

$$\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

   (a) (10 pts.) Find a primary decomposition

$$W_1 \oplus \cdots \oplus W_k.$$

   That is, find each W_i and associated polynomials p_is.

(b) (10 pts.) Write $A = T + N$ where T is a diagonal matrix and N is a nilpotent matrix such that $NT = TN$. That is, find T and N.

(c) (10 pts.) Find the Jordan form of A.

(d) (10 pts.) Find a cyclic decomposition of A

$$Z(\alpha_1, T) \oplus \cdots \oplus Z(\alpha_k, T)$$

and the respective T-annihilators.

5. Let A be the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

(a) (10 pts.) Find the unitary 3×3-matrix P such that $P^{-1}AP$ is diagonal.

(b) (10 pts.) Is there a real orthogonal P such that PAP^{-1} is diagonal? If not, explain why not.

(c) (10 pts.) Find a real 3×3-matrix with real entries so that for no real orthogonal matrix P, PAP^{-1} is diagonal.