Ch 4: Polynomials

Polynomials

Algebra

Polynomial ideals

Polynomial algebra

- The purpose is to study linear transformations. We look at polynomials where the variable is substituted with linear maps.
- This will be the main idea of this book to classify linear transformations.

- F a field. A linear algebra over F is a vector space A over F with an additional operation AxA -> A.
  - (i) a(bc)=(ab)c.
  - (ii) a(b+c)=ab+ac,(a+b)c=ac+bc ,a,b,c in A.
  - (iii) c(ab)=(ca)b= a(cb), a,b in A, c in F
  - If there exists 1 in A s.t. a1=1a=a for all a in A, then A is a linear algebra with 1.
  - A is commutative if ab=ba for all a,b in A.
  - Note there may not be a^{-1}.

- Examples:
  - F itself is a linear algebra over F with 1. (R, C, Q+iQ,…) operation = multiplication
  - M_{nxn}(F) is a linear algebra over F with 1=Identity matrix. Operation=matrix mutiplication
  - L(V,V), V is a v.s. over F, is a linear algebra over F with 1=identity transformation. Operation=composition.

- We introduce infinite dimensional algebra (purely abstract device)

$$F^\infty = \{(f_0, f_1, f_2, ...)|f_i \in F\}$$

$$\begin{aligned} f &= (f_0, f_1, f_2, ...) \\ g &= (g_0, g_1, g_2, ...) \\ af + bg &= (af_0 + bg_0, af_1 + bg_1, ...) \\ (fg)_n &= \sum_{i=0}^n f_i g_{n-i}, n = 0, 1, 2... \end{aligned}$$

$$\begin{aligned} fg &= gf \\ (gf)_n &= \sum_{i=0}^n g_i f_{n-i} = \sum_{j=1}^n f_j g_{n-j} = (fg)_n \end{aligned}$$

- (fg)h=f(gh)
- Algebra of formal power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

$$F[x] \subset F^{\infty}, F[x] = Span(1, x, x^2, x^3, ...)$$

- deg f:

$$f(x) = f_0 x^0 + f_1 x^1 + \cdots + f_n x^n, \deg f = n$$

- Scalar polynomial cx^0
- Monic polynomial $f_n = 1$.

- Theorem 1: f, g nonzero polynomials over F. Then
  1. fg is nonzero.
  2. deg(fg)=deg f + deg g
  3. fg is monic if both f and g are monic.
  4. fg is scalar iff both f and g are scalar.
  5. If f+g is not zero, then deg(f+g) $\leq$ max(deg(f),deg(g)).
- Corollary: F[x] is a commutative linear algebra with identity over F. $1=1.x^0$.

- Corollary 2: f,g,h polynomials over F. $f \neq 0$. If fg=fh, then g=h.
  - Proof: f(g-h)=0. By 1. of Theorem 1, f=0 or g-h=0. Thus g=h.
- Definition: a linear algebra A with identity over a field F. Let $a^0=1$ for any a in A. Let $f(x) = f_0x^0+f_1x^1+\ldots+f_nx^n$. We associate f(a) in A by $f(a)=f_0a^0+f_1a^1+\ldots+f_na^n$.
- Example: $A = M_{2x2}(C)$. $B= \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $f(x)=x^2+2$.

$$f(B) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 3 & 0 \\ -3 & 6 \end{bmatrix}$$

- Theorem 2: F a field. A linear algebra A with identity over F.
  - 1. $(cf+g)(a)=cf(a)+g(a)$
  - 2. $fg(a) = f(a)g(a)$.
- Fact: $f(a)g(a)=g(a)f(a)$ for any f,g in F[x] and a in A.
- Proof: Simple computations.
- This is useful.

Lagrange Interpolations

- This is a way to find a function with preassigned values at given points.
- Useful in computer graphics and statistics.
- Abstract approach helps here: Concretely approach makes this more confusing. Abstraction gives a nice way to view this problem.

- $t_0, t_1, \ldots, t_n$ n+1 given points in F. (char F=0)
  - V={f in F[x]| deg f ≤n } is a vector space.
  - $L_i(f) := f(t_i)$. L_i: V -> F. i=0,1,…,n. This is a linear functional on V.
  - $\{L_0, L_1, \ldots, L_n\}$ is a basis of V*.
  - We find a dual basis in V=V**:
    - We need $L_i(f_j) = \delta_{ij}$. That is, $f_j(x_i) = \delta_{ij}$.
    - Define $$P_i(x) = \prod_{j \neq i} \left(\frac{x - t_j}{t_i - t_j} \right)$$

$$P_2(x) = \frac{x - t_0}{t_2 - t_0} \frac{x - t_1}{t_2 - t_1} \frac{x - t_3}{t_2 - t_3} \frac{x - t_4}{t_2 - t_4}, n = 4, i = 2$$

- Then $\{P_0, P_1, \ldots, P_n\}$ is a dual basis of V** to $\{L_0, L_1, \ldots, L_n\}$ and hence is a basis of V.
- Therefore, every f in V can be written uniquely in terms of P_is.

$$\begin{aligned} f(x) &= \textstyle\sum_{i=0}^{n} L_i(f) P_i \\ &= \textstyle\sum_{i=0}^{n} f(t_i) P_i \end{aligned}$$

- This is the Lagrange interpolation formula.
  - This follows from Theorem 15. P.99. (a->f, L_i->f_i,a_i->P_i)

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

- Example: Let f = x^j. Then

$$x^j = \sum_{i=1}^{n} (t_i)^j P_i$$

- Bases $\{x^0, x^1, \ldots, x^n\}, \{P_0.P_1, \ldots, P_n\}$

$$\begin{bmatrix} 1 & t_0 & t_0^2 & \ldots & t_0^n \\ 1 & t_1 & t_1^2 & \ldots & t_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \ldots & t_n^n \end{bmatrix}$$

- The change of basis matrix is invertible

(The points are distinct.) Vandermonde matrix

- Linear algebra isomorphism I: A->A'
  - I(ca+db)=cI(a)+dI(b), a,b in A, c,d in F.
  - I(ab)=I(a)I(b).
  - Vector space isomorphism preserving multiplications,
  - If there exists an isomorphism, then A and A' are isomorphic.
- Example: L(V) and $M_{nxn}(F)$ are isomorphic where V is a vector space of dimension n over F.
  - Proof: Done already.

- Useful fact:

$$
\begin{aligned}
f &= \sum_{i=0}^{n} c_i x^i \\
f(U) &= \sum_{i=0}^{n} c_i U^i \\
[f(U)]_{\mathcal{B}} &= \sum_{i=0}^{n} c_i [U^i]_{\mathcal{B}} \\
[T_1 T_2]_{\mathcal{B}} &= [T_1]_{\mathcal{B}} [T_2]_{\mathcal{B}} \\
[U^i]_{\mathcal{B}} &= [U]_{\mathcal{B}}^i \\
[f(U)]_{\mathcal{B}} &= f([U]_{\mathcal{B}})
\end{aligned}
$$