5.4. Additional properties

Cofactor, Adjoint matrix,

Invertible matrix, Cramers rule.

(Cayley, Sylvester….)

- det(A) = det(A^t), A nxn matrix
- Proof: $$A^t(i,\sigma i) = A(\sigma i, i)$$

$$\begin{aligned} det(A^t) &= \textstyle\sum_{\sigma\in S_n} \mathrm{sgn}\sigma A^t(1,\sigma 1)\cdots A^t(n,\sigma n) \\ &= \textstyle\sum_{\sigma\in S_n} \mathrm{sgn}\sigma A(\sigma 1,1)\cdots A(\sigma n,n) \end{aligned}$$

$$\begin{aligned} A(\sigma i, i) &= A(j,\sigma^{-1}j), j = \sigma_i \\ A(\sigma 1,1)\cdots A(\sigma n,n) &= A(1,\sigma^{-1}1)\cdots A(n,\sigma^{-1}n) \end{aligned}$$

$$sgn\sigma^{-1} = sgn\sigma$$

$$\begin{aligned} det(A^t) &= \textstyle\sum_{\sigma\in S_n} \mathrm{sgn}\sigma^{-1} A(1,\sigma^{-1}1)\cdots A(n,\sigma^{-1}n) \\ &= \textstyle\sum_{\sigma\in S_n} \mathrm{sgn}\sigma A(1,\sigma 1)\cdots A(n,\sigma n) \end{aligned}$$

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C$$

- A rxr-matrix B rxs matrix C sxs matrix
- Proof: Define D(A,B,C) = det $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$
  - Fix A,B, D(A,B,C) alternating s-linear for rows of C.
  - D(A,B,C)=(det C)D(A,B,I) by Theorem 2.
  - D(A,B,I)= det $\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$ = det $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ =D(A,0,I)
  - D(A,0,I)= det A D(I,0,I)= det A since alternating r-linear.
  - D(A,B,C)=detA det C

- Cofactors
  - Recall that $\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det A(i \mid j)$
  - Define $C_{ij} = (-1)^{i+j} \det A(i \mid j)$
    - (i,j)-cofactor of A.
  - Then $\det A = \sum_{i=1}^{n} A_{i,j} C_{i,j}$
  - We show if j≠k, then $0 = \sum_{i=1}^{n} A_{i,j} C_{i,k}$

- Proof: B obtained by replacing the j-th column of A by the k-th column.
  - det B=0 since det B= det B^t and two rows are same.
  - B(i,j)= A(i,k). B(i|j)=A(i|j)

$$\det B = \sum_{i=1}^{n} (-1)^{i+j} B_{i,j} \det B(i \mid j)$$

$$= \sum_{i=1}^{n} (-1)^{i+j} A_{i,k} \det A(i \mid j) = \sum_{i=1}^{n} A_{ik} C_{ij}$$

- Classical adjoint of A is the transpose of the cofactor matrix.
  - adj $A_{ij} = C_{ji} = (-1)^{i+j}$ det A(j|i)
- (adj A)A = (det A) I (*)

$$\sum_{i=1}^{n} A_{ik} C_{ij} = \sum_{i=1}^{n} C_{ij} A_{ik} = \sum_{i=1}^{n} (adjA_{ji}) A_{ik} = (\det A)\delta_{jk}$$

- A(adj A) = (det A)I
  - Proof: adj(A^t) = (adj A)^t since $(-1)^{i+j}$ det $A^t(i|j) = (-1)^{i+j}$ det A(j|i)
  - (adj A^t)A^t = det(A^t)I by (*).
  - (adj A)$^t A^t$ = det(A)I
  - A(adj A) = det(A) I.

- Invertible matrix
- Theorem 4. nxn matrix A over a commutative ring K with identity.
  - A is invertible (with an inverse with entries in K) iff det A in K is invertible in K .
  - $A^{-1}=(\det A)^{-1}\,\mathrm{adj}\,A$
- Proof:
- (->) $A.A^{-1}=I$. $\det A.A^{-1}=\det A \det A^{-1} =1$.
  - $\det A = (\det A^{-1})^{-1}$
- (<-) $(\mathrm{adj}\,A)\,A = (\det A)I$. $A(\mathrm{adj}\,A)=(\det A)I$.
  - $(\det A)^{-1}(\mathrm{adj}\,A)A = I$, $A\,(\det A)^{-1}(\mathrm{adj}\,A) = I$.
  - $A^{-1} = (\det A)^{-1}(\mathrm{adj}\,A)$

- Fact: An integer matrix has an integer inverse matrix iff determinant is ±1.
- Example: A = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ adj A = $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$
  - det A = 1.
  - A^{-1}=adj A
- See also examples 7, 8 on pages 160-161.

Cramers Rule

- A nxn-matrix, X nx1, Y nx1 matrices
- AX=Y.
  - (adj A) AX = (adj A)Y
  - (det A) X = (adj A)Y

$$(\det A)x_j = \sum_{i=1}^{n} (adjA)_{ji} y_i$$

$$= \sum_{i=1}^{n} (-1)^{i+j} \; y_i \det A(i \mid j)$$