6.8. The primary decomposition theorem

Decompose into elementary parts using the minimal polynomials.

- Theorem 12. T in L(V,V). V f.d.v.s. over F. p minimal polynomial. $P=p_1^{r_1}\dots p_k^{r_k}$. $r_i > 0$. Let W_i= null $p_i(T)^{r_i}$.
  - Then
  - (i) $V = W_1 \oplus \dots \oplus W_k$.
  - (ii) Each W_i is T-invariant.
  - (iii) Let $T_i=T|W_i:W_i \to W_i$. Then minpolyT_i= $p_i(T)^{r_i}$

- Example:

$$T=\begin{bmatrix}1&2&0&0\\0&1&0&0\\0&0&2&1\\0&0&0&2\end{bmatrix}$$

  - Char.polyT=$(x-1)^2(x-2)^2$=min.polyT:
    - Check this by any lower degree does not kill T by computations.
  - null$(T-I)^2$ = null $\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}^2$ = null $\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&2\\0&0&0&1\end{bmatrix}$ = $\left\{\begin{bmatrix}x\\y\\0\\0\end{bmatrix} \mid x,y\in R\right\}$

  - Similarly null$(T-2I)^2$ =

$$\left\{\begin{bmatrix}0\\0\\x\\y\end{bmatrix} \mid x,y\in R\right\}$$

$$T_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

- Proof: idea is to get $E_1,..,E_k$.
  - Let $f_i = p/p_i^{r\_i} = p_1^{r\_1} \ldots p_{i-1}^{r\_i-1} p_{i+1}^{r\_i+1} \ldots p_k^{r\_k}$.
  - $f_1,\ldots,f_k$ are relatively prime since there are no common factors.
  - That is, $\langle f_1,\ldots,f_k \rangle = F[x]$.
  - There exists $g_1,\ldots,g_k$ in $F[x]$ s.t. $g_1 f_1 + \ldots + g_k f_k = 1$.
  - p divides $f_i f_j$ for $i \neq j$ since $f_i f_j$ contains all factors.
  - Let $E_i = h_i(T) = f_i(T) g_i(T)$, $h_i = f_i g_i$.

- Since $h_1+\ldots+h_k=1$, $E_1+\ldots+E_k=I$.
- $E_iE_j=0$ for $i\neq j$.
- $E_i = E_i(E_1 +\ldots+E_k)=E_i^2.$ Projections.
- Let Im $E_i = W_i$. Then $V = W_1 \oplus \ldots \oplus W_k$.
- (i) is proved.
- $T E_i= E_iT$. Thus Im $E_i = W_i$ is T-invariant.
- (ii) is proved.
- We show that Im E_i = null $p_i(T)^{r_i}$.
  - (⊂) $p_i(T)^{r_i} E_ia = p_i(T)^{r_i}f_i(T)g_i(T)a = p(T) g_i(T)a =0$.

- ($\supset$) a in null $p_i(T)^{r_i}$.
- If $j\neq i$, then $f_j(T)g_j(T)a = 0$ since $p_i^{r_i}$ divides f_j and hence f_jg_j.
- $E_ja=0$ for $j\neq I$. Since $a=E_1a+\ldots+E_ka$, it follows that $a=E_ia$. Hence a in Im E_i.

– (i),(ii) is completely proved.

– (iii) $T_i= T|W_i:W_i\to W_i$.

– $P_i(T_i)^{r_i} =0$ since W_i is the null space of $P_i(T)^{r_i}$.

– minpolyT_i divides $P_i^{r_i}$.

– Suppose g is s.t. $g(T_i)=0$.

- $g(T)f_i(T)=0$:
  - $f_i= p_1^{r_1} \dots p_{i-1}^{r_{i-1}} p_{i+1}^{r_{i+1}} \dots p_k^{r_k}$.
  - Im E_i=null $p_i^{r_i}$.
  - Thus Im $f_i(T)$ is in Im E_i since V is a direct sum of Im E_js.
- p divides gf_i.
- $p= p_i^{r_i} f_i$ by definition.
- Thus $p_i^{r_i}$ divides g.
- Thus, minpoly $T_i= p_i^{r_i}$.

- Corollary: $E_1,\ldots,E_k$ projections ass. with the primary decomposition of T. Then each E_i is a polynomial in T. If a linear operator U commutes with T, then U commutes with each of E_i and W_i is invariant under U.
- Proof: $E_i = f_i(T)g_i(T)$. Polynomials in T. Hence commutes with U.
  - $W_i = \mathrm{Im}\, E_i$. $U(W_i) = \mathrm{Im}\, U E_i = \mathrm{Im}\, E_i U$ in $\mathrm{Im}\, E_i = W_i$.

- Suppose that minpoly(T) is a product of linear polynomials. $p=(x-c_1)^{r\_1}\dots(x-c_k)^{r\_k}$. (For example F=C).
  - Let $D=c_1E_1+\dots+c_kE_k$. Diagonalizable one.
  - $T=TE_1+\dots+TE_k$
  - $N:=T-D=(T-c_1I)E1+\dots+(T-c_kI)E_k$
  - $N^2 = (T-c_1I)\ ^2E1+\dots+(T-c_kI)\ ^2E_k$
    - $$N^2 = \sum_{i,j}(T-c_iI)E_i(T-c_jI)E_j = \sum_i (T-c_iI)E_i(T-c_iI)E_i$$
      $$= \sum_i (T-c_iI)(T-c_iI)E_iE_i = \sum_i (T-c_iI)^2 E_i$$
  - $N^r = (T-c1I)\ ^rE1+\dots+(T-c_kI)\ ^rE_k$

- If $r \geq r_i$ for each I, $(T\text{-}c_iI)^r = 0$ on Im E_i.
- Therefore, $N^r = 0$. N=T-D is nilpotent.

- Definition. N in L(V,V). N is nilpotent if there is some integer r s.t. $N^r = 0$.
- Theorem 13. T in L(V,V). Minpoly T= prod.of 1st order polynomials. Then there exists a diagonalizable D and a nilpotent operator N s.t.
  - (i) T=D+N.
  - (ii) DN=ND.
  - D, N are uniquely determined by (i)(ii) and are polynomials of T.

- Proof: T=D+N. $E_i=h_i(T)=f_i(T)g_i(T)$.
  - $D=c_1E_1+\ldots+c_kE_k$ is a polynomial in T.
  - N=T-D a polynomial in T.
  - Hence, D,N commute.
- (Uniquenss) Suppose T=D'+N', D'N' commutes, D' diagonalizable, N nilpotent.
  - D' commutes T=D'+N'. D' commutes with any polynomials of T.
  - D' commutes with D and N.
  - D'+N'=D+N.
  - D-D'=N'-N. They commutes with each other.
  - Since D and D' commutes, they are simultaneously diagonalizable. (Section. 6.5 Theorem 8.)

– N’-N is nilpotent:

$$(N'-N)^r = \sum_{j=0}^{r} \binom{r}{j} (N')^{r-j} (-N)^j$$

- r is suff. large. (larger 2max of the degrees of N,N’) -> r-j or j is suff large.
- Thus the above is zero.

– D-D’=N’-N is a nilpotent operator which has a diagonal matrix. Thus, D-D’=0 and N’-N=0.

– D’=D and N’=N.

- Application to differential equations.
- Primary decompostion theorem holds when V is infinite dimensional and when p is only that p(T)=0. Then (i),(ii) hold.
- This follows since the same argument will work.
- A positive integer n.
- V = {f| n times continuously differentiable complex valued functions which satisfy ODE

$$\frac{d^n f}{d^n t} + a_{n-1}\frac{d^{n-1} f}{d^{n-1} t} + \ldots + a_1 \frac{df}{dt} + a_o f = 0, a_0, \ldots, a_{n-1} \in R$$

}
- C^n={n times continuously differentiable complex valued functions}

- Let $p=x^n+a_{n-1}x^{n-1}+\ldots+a_1x + a_0$.
- Let D differential operator,
- Then V is a subspace of C^n where $p(D)f=0$.
- $V=\text{null } p(D)$.
- Factor $p=(x-c_1)^{r\_1}\ldots(x-c_k)^{r\_k}$. $c_1,..,c_k$ in the complex number field C.
- Define $W_j := \text{null}(D-c_jI)^{r\_j}$.
- Then Theorem 12 says that $V = W_1\oplus\ldots \oplus W_k$
- In other words, if f satisfies the given differential operator, then f is expressed as $f =f_1+\ldots+f_k$, f_i in W_i.

- What are W_is? Solve $(D-cI)^r f=0$.
- Fact: $(D-cI)^r f=e^{ct}D^r(e^{-ct} f)$:
  - $(D-cI) f=e^{ct}D(e^{-ct} f)$.
  - $(D-cI)^2f= e^{ct}D(e^{-ct} e^{ct}D(e^{-ct} f))$….
- $(D-cI)^r f=0$ <-> $D^r(e^{-ct} f)=0$:
  - Solution: $e^{-ct} f$ is a polynomial of deg $< r$.
  - $f= e^{ct}(b_0+ b_1t +\ldots+ b_{r-1}t^{r-1})$.
- Here e^{ct} ,te^{ct} ,t^2e^{ct},…, $t^{r-1}e^{ct}$ are linearly independent.
- Thus $\{t^m e^{c_j t}|\ \ m=0,\ldots,r_j-1,\ j=1,\ldots,k\}$ form a basis for V.
- Thus V is finite-dimensional and has dim equal to deg. p.

7.1. Rational forms

- Definition: T in L(V,V), a vector *a*.
  T-cyclic subspace generated by *a* is
  $Z(a;T)=\{v=g(T)a \mid g \text{ in } F[x]\}$.
- $Z(a;T)=\langle a, Ta, T^2a, \ldots.\rangle$
- If $Z(a:T)=V$, then *a* is said to be a cyclic vector for T.
- Recall T-annihilator of *a* is the ideal
  $M(a:T)=\langle g \text{ in } F[x] \mid g(T)a=0\rangle=p_aF[x]$.
- p_a is the T-annihilator of *a.*

- Theorem 1. $a \neq 0$. p_a T-annihilator of a.
  - (i) deg p_a = dim Z(a;T).
  - (ii) If deg p_a =k, a, Ta,…,$T^{k-1}a$ is a basis of
  - (iii) Let U:=T|Z(a;T):Z(a:T)->Z(a;T). Minpoly U=p_a.
- Proof: Let g in F[x]. g=p_aq+r. deg(r) < deg(p_a). g(T)a=r(T)a.
  - r(T)a is a linear combination of a, Ta,…,$T^{k-1}a$.
  - Thus, this k vectors span Z(a;T).
  - They are linearly independent. Otherwise, we get another g of lower than k degree s.t. g(T)a =0.
  - (i),(ii) are proved.

- $U := T|Z(a;T) : Z(a:T) \to Z(a;T)$.
- g in F[x].
- $p_a(U)g(T)a = p_a(T)g(T)a$ (since g(T) *a* is in Z(*a*;T).)

  $= g(T)p_a(T)a = g(T)0 = 0$.
- $p_a(U)=0$ on Z(*a*;T) and p_a is monic.
- If h is a polynomial of lower-degree than p_a, then $h(U) \neq 0$. (since $h(U)a = h(T)a \neq 0$).
- Thus, p_a is the minimal polynomial of U.

- Suppose T:V->V has a cyclic vector *a*.
- deg minpolyU=dimZ(*a*;T)=dim V=n.
- minpoly U=minpoly T.
- Thus, minpoly T = char.poly T.
- We obtain:

T has a cyclic vector <-> minpoly T=char.polyT.

- Proof: (->) done above.
  - (<-) Later, we show for any T, there is a vector v s.t. minpolyT=annihilator v. (p.237. Corollary).
  - So if minpolyT=charpolyT. Then dimZ(v;T)=n and v is a cyclic vector.

- Study T by cyclic vector.
- U on W with a cyclic vector v. (W=Z(v:T) for example and U the restriction of T.)
- v, Uv, $U^2v,\ldots,U^{k-1}v$ is a basis of W.
- U-annihiltor of v = minpoly U by Theorem 1.
- Let $v_i=U^{i-1}v$. $i=1,\ldots,k$.
- Let $B=\{v_1,\ldots,v_k\}$.
- $Uv_i=v_{i+1}$. $i=1,\ldots,k-1$.
- $Uv_k=-c_0v_1-c_1v_2-\ldots-c_{k-1}v_k$ where minpolyU$=c_0+c_1x+\ldots+c_{k-1}x^{k-1}+x^k$.
    - $(c_0v+c_1Uv+\ldots+c_{k-1}U^{k-1}v+U^kv=0.)$

$$[U]_B = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \dots & 0 & -c_0 \\ 1 & 0 & 0 & 0 & \dots & \dots & 0 & -c_1 \\ 0 & 1 & 0 & 0 & \dots & \dots & 0 & -c_2 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 & -c_3 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 & -c_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & -c_{k-1} \end{bmatrix}$$

- This is called the companion matrix of pa. (defined for any *monic* polynomial.)

- Theorem 2. If U is a linear operator on a f.d.v.s.W, then U has a cyclic vector iff there is some ordered basis where U is represented by a companion matrix.
- Proof: (->) Done above.
- (<-) If we have a basis $\{v_1, \ldots, v_k\}$,
  - then v_1 is the cyclic vector.

- Corollary. If A is the companion matrix of a monic polynomial p, then p is both the minimal and the characteristic polynomial of A.
- Proof: Let *a*=(1,0,…0). Then a is a cyclic vector and Z(a;A)=V.
  – The annihilator of *a* is p. deg p=n also.
  – By Theorem 1(iii), the minimal poly for T is p.
  – Since p divides char.polyA. And p has degree n. p=char.polyA.