8.3. Linear Functionals and adjoints

Many uses:
quantum mechanics…

- T:V->V. V finite dimensional inner product space.
- (Ta|b)=(a|T*b) for all a,b in V. T*:V->V is an adjoint linear transformation.
- Question: existence and uniqueness of T*.

- Theorem 6. Let f be in the dual space V*. Then there exists unique b in V s.t. f(.)=(.|b) :I.e.,f(a)=(a|b) for all a in V.
- Proof: $\{a_1,\ldots,a_n\}$ orthonormal basis of V.
  - Let $b=\sum^{n}_{j=1} f(a_j)^{-}a_j$.
  - Define f_b:V -> F by $f_b(a):=(a|b)$.
  - Then $f=f_b$: $f_b(a_k)=(a_k| \sum^{n}_{j=1} f(a_j)^{-}a_j)= f(a_k)(a_k,a_k)$ for all k=1,…,n. $f_b=f$.
  - Uniqueness:
    - Suppose (a|b)=(a|c).
    - Then (a|b-c)=0 for all a in V.
    - (b-c|b-c)=(b-c|b)-(b-c|c)=0. ||b-c||=0 -> b=c.

- Claim: b is in null $f^{\perp}$.
- Proof: Define W=null f.
  - $V=W\oplus W^{\perp}$. Let P be a projection to $W^{\perp}$.
  - Then f(a)=f(P(a)) for all a in V.
  - Dim $W^{\perp}=1$: (dim W=n-1 since rank f=1.)
  - Then $P(a)=((a|c)/||c||^2)c$ if $c \neq 0$ in $W^{\perp}$.
  - $f(a)=f(((a|c)/||c||^2)c)=(a|c)f(c)/||c||^2$
    $= (a|f(c)^{-}c/||c||^2)$.
  - Thus $b= f(c)^{-}c/||c||^2$ and is in $W^{\perp}$.

- Theorem 7: T in L(V,V). V f.d.v.s. Then there exists unique T* in L(V,V) s.t. (Ta|b)=(a|T*b) for all a, b in V.
- Proof: Let b be in V.
  - Define f:V->F by a-> (Ta|b).
  - There exists unique b' s.t. (Ta|b)=(a|b') for all a in V.
  - Define T*:V->V by sending b->b' as above (*).
  - Then (Ta|b)=(a|T*b) for all a,b in V.
  - We show T* is in L(V,V):

$$(a \mid T^*(gb+c)) = (Ta \mid gb+c) = (Ta \mid gb) + (Ta \mid c)$$
$$= \bar{g}(Ta \mid b) + (Ta \mid c) = \bar{g}(a \mid T^* b) + (a \mid T^* c)$$
$$= (a \mid gT^* b + T^* c).$$

– Thus, T*(gb+c)=gT*(b)+T*(c).
  - Rem: If (a|b)=(a|b’) for all a in V, then b=b’ :
    – (a|b-b’)=0 for all a. (b-b’ |b-b’)=0. b-b’ =0.

– Uniqueness. T*b is determined by (*).

- Definition: T in L(V,V). Then T* is called an adjoint of T.

- Example: Let $T:F^n \to F^n$ be defined by $Y=AX$ where A is an nxn-matrix.
  - Let F^n have the standard inner product.
  - Then $(TX|Z)=(AX|Z)=Z^*AX = (A^*Z)^*X= (X|A^*Z)=(X|T^*Z)$ for all Z,X.
  - Thus T^* is given by $Y=A^*X$.
- In fact if we have an orthogonal basis, this is always true:

- Theorem 8. $B=\{a_1,\ldots,a_n\}$ orthonormal basis of V. Let $A=[T]_B$. Then $A_{kj}=(Ta_j|a_k)$.
- Proof: $a=\sum_{k=1}^n (a|a_k)a_k$. --(*).
  - A_{kj} is defined by $Ta_j= \sum_{k=1}^n A_{kj}a_k$.
  - $Ta_j= \sum_{k=1}^n (Ta_j|a_k)a_k$ by (*).
  - By comparing the two, we obtain the result.
- Corollary. Matrix of T* = conjugate transpose of T. $[T^*]_B=[T]^*_B$.
- Proof: $[T^*]_{B,kj}=(T^*a_j|a_k)=(a_k|T^*a_j)^-=(Ta_k|a_j)^-=[T]_{B,jk}^-$.

- Example: E:V->W orthogonal projection. Then E*=E.
- Proof: (a|E*b)=(Ea|b)=(Ea|Eb+(I-E)b) =(Ea|Eb)=(Ea+(I-E)a|Eb)=(a|Eb) for all a,b in V. Thus, E*=E.
- If V is infinite-dimensional, an adjoint of an operator may not exist.
- Example: D:C[x]->C[x] differentiation. C[x]={f polynomials on [0,1] with values in C.}. $(f|g)=\int_0^1 f(t)g^{-}(t)dt$ defines an inner product.

- Suppose D* exists and find contradiction:
- (Df|g)=(f|D*g)
- (Df|g)= $\int_0^1 f'(t)\bar{g}(t)dt = f(t)g(t)|_0^1 - \int_0^1 f(t)\bar{g}'(t)dt$ =f(1)g(1)-f(0)g(0)-(f|Dg).
- Fix g, (f|D*g)= f(1)g(1)-f(0)g(0)-(f|Dg).
- (f|D*g+Dg)=f(1)g(1)-f(0)g(0).
- Define L(f):= f(1)g(1)-f(0)g(0). L is in L(V,F).
- L(f) can't be (f|h) for some h:
  - Define f=x(x-1)h.
  - L(x(x-1) h)=x(x-1)h(1)g(1)-x(x-1)h(0)g(0)=0.
  - Then (f|h)= $\int_0^1 (x(x-1))|h|^2 dt$ >0.
  - A contradiction.

- Theorem 9. V f.d. inner product space. T,U linear operators on V, c in F.
  1. $(T+U)^*=T^*+U^*$
  2. $(cT)^*=c^{-}T^*$.
  3. $(TU)^*=U^*T^*$
  4. $(T^*)^*=T$.
- Proof: 1,2. See book.
  - 3.$(a|(TU)^*b)=(TU(a)|b)=(Ua|T^*b)$ $=(a|U^*T^*b)$ for all a,b in V. Thus $(TU)^*=U^*T^*$.
  - 4. $(a|(T^*)^*b)=(T^*a|b)=(b|T^*a)^{-}=(Tb|a)^{-}=(a|Tb)$ for all a,b in V. Thus, $(T^*)^*=T$.

- Let T be in L(V,V). V f.d.complex inner product space. Then T=U+iV where U*=U and V*=V.
- Proof: Define U=(T+T*)/2. V=(T-T*)/2i.
  - U*=(T+T*)*/2=(T*+T)/2=U.
  - V*=(T-T*)*/(-2i)=(-T*+T)/2i=V.
  - (T+T*)*/2+i (T-T*)/2i=T.
- The operator s.t. T=T* is called a self-adjoint operator. $[T]_B=[T^*]_B=[T]_B^*$ for an orthogonal basis b.
- Many operators are self-adjoint and they are very useful (like real numbers.)