Chapter 4 Determinants

SECTION 4.1. DETERMINANTS; COFACTOR EXPANSIONS

DETERMINANTS

- Determinants are useful because it gives us invariant. Related to volume change.
- Invariants are like the essential properties.
- Important properties of a person is his character. In fact, character determines a person and not the reverse is true.
- In fact, the properties of the determinants makes it useful and not its formula.

DETEMINANTS FOR 2X2, 3X3 MATRICES

- Determinants for 2x2 case were discovered by solving equations.
- u=ax+by, v=cx+dy. -> x=(du-bv)/(ad-bc), y=(av-cu)/(ad-bc).
- det A= |{a,b},{c,d}|=ad-bc
- For 3x3 case:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

ELEMENTARY PRODUCTS

- 3x3 case: formula consists of a_1?a_2?a_3?.
- The ? were obtained by permuting 1,2,3,
- How do we get the signs?
- An interchange: exchange two but leave everything else fixed.
- Given a permutation {j_1, j_2, j_3}, we can put this back to (1,2,3) by interchanges.
- This is done by bringing 1 to the first position by interchanges and then 2 to the second position and so on.

There may be many ways to do this.

- However, the number interchanges is either odd or even.
- Hence if the number of interchanges is even, then we use +. If the number of interchanges is odd, then we use -.
- A signed elementary product is an elementary product with a sign given as above.

Definition 4.1.1 The ***determinant*** of a square matrix A is denoted by $\det(A)$ and is defined to be the sum of all signed elementary products from A.

- Formula

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \sum \pm a_{1j\_1} a_{2j\_2} \dots a_{nj\_n}$$

- The summation is over all permutations {j_1, j_2,…,j_n}.

EVALUATION

- Evaluation may not be so easy from this formula since the number of terms is n!
- This grows exponentially fast.
- We use Gaussian eliminations and LU-decompositions to obtain it much much faster.

DETERMINANTS WITH A ZERO ROW

Theorem 4.1.2 *If A is a square matrix with a row or a column of zeros, then* $\det(A) = 0$.

- Proof: Every signed elementary product is zero.

DETERMINANTS OF TRIANGULAR MATRICES

Theorem 4.1.3 *If A is a triangular matrix, then* $\det(A)$ *is the product of the entries on the main diagonal.*

Proof: Each elementary product get a unique entry from each column and each row.

- The diagonal clearly survive. Given any permutation.
- Any other elementary product will have 0s.

Gaussian elimination can produce this.

MINOR, COFACTOR

- A a square matrix
- The minor of a_ij: Remove i-th row and j-th column
  and take its determinant: M_ij.
- The cofactor of a_ij: C_ij=$(-1)^{i+j}$M_ij.
- Example 3.

COFACTOR EXPANSIONS

Theorem 4.1.5 *The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i \le n$ and $1 \le j \le n$,*

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(*cofactor expansion along the* jth *column*)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(*cofactor expansion along the* ith *row*)

See Example 5:

EX SET 4.1

- 1-10 Determinant using formula
- 11,12 permutation
- 13-18 determinants
- 19,20 inspection determinants
- 21-32 Cofactor expansions
- 33-36 a bit harder