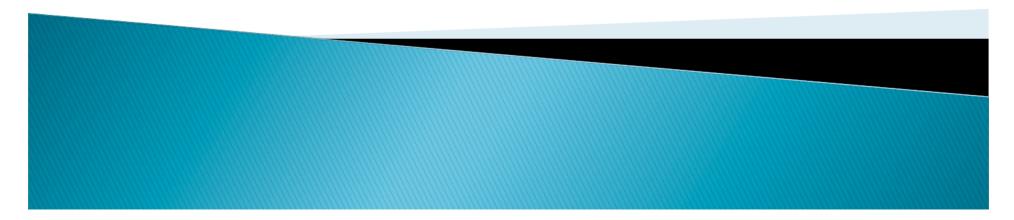
Chapter 6. Linear transformations

The purpose is to understand linear transformations, see various examples, kernel range, compositions and invertibility



6.1. Matrices as transformations

Definition 6.1.1 Given a set D of allowable inputs, a *function* f is a rule that associates a unique output with each input from D; the set D is called the *domain* of f. If the input is denoted by x, then the corresponding output is denoted by f(x) (read, "f of x"). The output is also called the *value* of f at x or the *image* of x under f, and we say that f *maps* x into f(x). It is common to denote the output by the single letter y and write y = f(x). The set of all outputs y that results as x varies over the domain is called the *range* of f.

- A function is a set {(x, f(x))|x in D} where x=y means f(x)=f(y)
- Example:
 - $T(x_1,x_2)=(x_1,x_2)$ or the identity map.
 - T(x_1, x_2)=(c_1,c_2) or a constant map.

- Example: T: R³->R³. T(x_1,x_2,x_3)=(x_1x_2,x_2x_3,x_3x_1).
- Example: Given 2x3 matrix A=[[1,0,1],[0,2,1]], define T(x_1,x_2,x_3) = (x_1+x_3, 2x_2+x_3). Or T_A(x)=Ax.
- Given a transformation T: Rⁿ->R^m. A domain is Rⁿ and codomain is R^m. The range is the actual set T(Rⁿ) in R^m which may or may not be the whole of R^m.
- ▶ An operator is a transformation Rⁿ->Rⁿ.



Matrix transformation

- Given A mxn matrix.
- We define $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by x -> Ax or T(x) = Ax.
- ► T_A: multiplication by A, or transformation A.
- A matrix transformation and the matrix itself is often considered a same object.
- Example: zero transformation T_O(x)=Ox=O.
- Identity operator T_I(x)=Ix=x.



Linear transformation

- The term linear was used to denote that the order of a polynomial was no more than one.
- Here, we will change meaning somewhat.
- A transformation will be linear if it sends O to O and each line to a line and planes to planes and so on.
- It turns out that this means that the transformation preserves addition and scalar multiplications and conversely.



Superposition principle: T(c_1v_1+c_2v_2+...+c_kv_k) = c_1T(v_1)+c_2T(v_2)+...+c_kT(v_k).

 Actually this is linearity. Physicists use it in different way also.

Definition 6.1.2 A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* from \mathbb{R}^n to \mathbb{R}^m if the following two properties hold for all vectors **u** and **v** in \mathbb{R}^n and for all scalars *c*:

- (i) $T(c\mathbf{u}) = cT(\mathbf{u})$ [Homogeneity property]
- (ii) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ [Additivity property]

In the special case where m = n, the linear transformation T is called a *linear operator* on R^n .



- Example: matrix transformations are linear. T_A(c_1x_1+c_2x_2)=A(c_1x_1+c_2x_2)= c_1Ax_1+c_2Ax_2=c_1T(x_1)+c_2T(x_2).
- Example: 2nd or higher order transformations are nonlinear. They do not preserve the scalar multiplication or additions sometimes.
 - $T(x_1,x_2,x_3)=(x_1x_2,x_2x_3,x_3x_1)$.
 - $T(2x_1,2x_2,2x_3)=4T(x_1,x_2,x_3)$.
 - $T(x_1+x'_1,x_2+x'_2,x_3+x'_3) = ((x_1+x'_1)(x_2+x'_2), (x_2+x'_2)(x_3+x'_3),(x_3+x'_3)(x_1+x'_1))$ is not $T(x_1,x_2,x_3)+T(x'_1,x'_2,x'_3)$ for arbitrary choices.



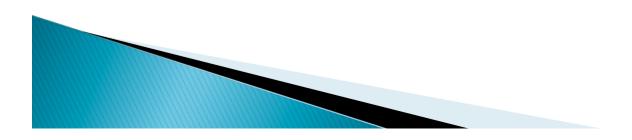
Properties

Theorem 6.1.3 If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then:

(*a*) $T(\mathbf{0}) = \mathbf{0}$

$$(b) T(-\mathbf{u}) = -T(\mathbf{u})$$

- (c) $T(\mathbf{u} \mathbf{v}) = T(\mathbf{u}) T(\mathbf{v})$
- Proof: (a) T(O)=T(0v)=0T(v)=O.
- Example: A translation is not linear.
 - T(x)=x+x_0. O-> x_0.



All linear transformations are matrix transformations

Suppose that T is liner: Rⁿ -> R^m.

- x=x_1e_1+x_2e_2+...+x_ne_n.
- T(x)=x_1T(e_1)+x_2T(e_2)+...+x_nT(e_n).
- $T(x)=[T(e_1),T(e_2),...,T(e_n)][x_1,x_2,...,x_n]^T$.
- Let A be [T(e_1),T(e_2),...,T(e_n)]. Then T(x)=Ax.

Theorem 6.1.4 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and suppose that vectors are expressed in column form. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors in \mathbb{R}^n , and if \mathbf{x} is any vector in \mathbb{R}^n , then $T(\mathbf{x})$ can be expressed as

$$T(\mathbf{x}) = A\mathbf{x} \tag{13}$$

where

 $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$

- A is a (standard) matrix corresponding to T.
- T is a transformation corresponding to A.
- T is a transformation represented by A.
- T is the transformation A.
- A=[T]=[T(e_1),T(e_2),...,T(e_n)].
- ▶ T(x)=[T]x.
- Example: T(x)=cx. c is some number. T is linear and is called a scaling operator.
- Then [T]=cl.



Representing transformations by equations....

- Rⁿ coordinates (x_1,x_2,...,x_n).
- R^m coordinates (w_1,w_2,..., w_m)
- Then (w_1,w_2,...,w_m)=T(x_1,x_2,...,x_n) can be written:
 - w_1= a_11 x_1+a_12 x_2+...+a_1n x_n
 - w_2= a_21 x_1+a_22 x_2+...+a_2n x_n
 - •••••
 - w_m=a_m1 x_1+a_m2 x_2+...+a_mn x_n.
- Conversely, this equation defines w=Ax and hence a linear transformation T_A.
- We can consider these identical definitions.

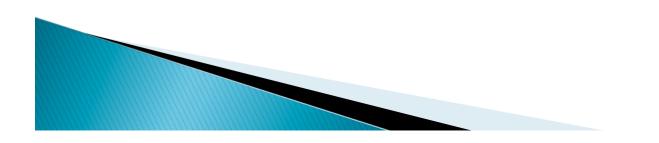
Rotations about the origin.

- Let us make a transformation that preserves length and send a vector to a vector rotated by an angle θ.
- e_1 -> (cosθ, sinθ), e_2 -> (-sinθ,cosθ).
- Thus let [T] =[Te_1,Te_2]
 = [[cosθ,-sinθ],[sinθ,cosθ]].
- Thus $R_{\theta x} = [[\cos\theta, -\sin\theta], [\sin\theta, \cos\theta]]x$.
- A rotation about nonorigin is not linear.



Reflection about a line through the origin.

- Take a line L through the origin having angle θwith the positive x-axis.
- T(e_1) is length 1 and has angle 20with the positive x-axis. T(e_1)=(cos20,sin20).
- T(e_2) is length 1 and has angle 2(π/2-θ) with the positive y-axis and has angle (π/2-2θ) with the positive x-axis.
 T(e_2)=(cos(π/2-2θ),sin(π/2-2θ))=(sin2θ,-cos2θ).
- H_θ(x)=[[cos2θ, sin2θ],[sin2θ,-cos2θ]]x



• Examples:

- (a) T(x,y)=(-y,x): reflection about the y-axis
- (b) T(x,y)=(x,-y): reflection about the x-axis.
- (c) T(x,y)=(y,x): reflection about y=x line.
- Example 13: θ=π/3.
 - H_π/3(x)
 - = [[$cos(2\pi/3)$, $sin(2\pi/3)$],[$sin(2\pi/3)$,- $cos(2\pi/3)$]
 - $= [[-1/2, 1/\sqrt{3}], [1/\sqrt{3}, 1/2]]x.$



Orthogonal projection onto the line through the origin.

- Define P_θ: R²->R² by sending a point x to a line L through O with angle θ with the positive x-axis.
- We find the formula by $P_{\theta}(x)-x=(H_{\theta}(x)-x)/2$.
- Thus, $P_{\theta(x)}=H_{\theta(x)}/2+x/2 = \frac{1}{2}(H_{\theta+1})(x)$.

P_θ=½(H_θ+I).

$$\begin{bmatrix} (1 + \cos 2\theta)/2 & (\sin 2\theta)/2 \\ (\sin 2\theta)/2 & (1 + \cos 2\theta)/2 \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix}$$

- The projection to the x-axis. Θ=0. Thus the matrix is [[1,0],[0,0]]. (x,y)->(x,0).
- The projection to the y-axis. Θ=π/2. Thus the matrix is [[0,0],[0,1]]. (x,y)-> (0, y).

