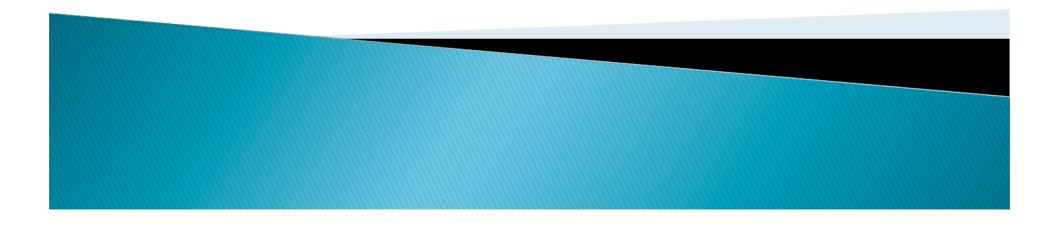
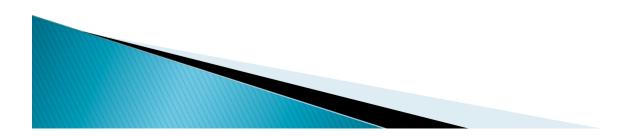
6.4 Composition and invertibility of linear transformations



Compositions of linear transformations

- A composition of functions: f:X->Y,g:Y->Z, we obtain g•f:X->Z.
- ▶ If f and g are linear, then g•f is also linear.
- To verify, we need to show + and scalar multiplications are preserved.

Theorem 6.4.1 If $T_1: \mathbb{R}^n \to \mathbb{R}^k$ and $T_2: \mathbb{R}^k \to \mathbb{R}^m$ are both linear transformations, then $(T_2 \circ T_1): \mathbb{R}^n \to \mathbb{R}^m$ is also a linear transformation.



- Recall T(x)=[T]x (p.272. (14))
- (T_2•T_1)(e_i)= T_2(T_1(e_i))=[T_2]([T_1](e_i))=([T_2][T_1])(e_i). (final step why?)
- Thus [T_2•T_1]=[T_2][T_1]. (why?)
- Conversely, given matrices A and B,
- ► T_B•T_A = T_BA. (Let T_2=T_B,T_1=T_A).
- Example 1. R_θ•R_Φ=R_(θ+Φ). Verify using computations
- Example 2. $H_\theta \cdot H_\Phi = R_2(\Phi \theta)$.
- Example 3. T•S may not equal S•T. We can see that from matrices T_A•T_B=T_AB. T_B• T_A=T_BA. They would be equal iff AB=BA.

Compositions of three or more linear transformations.

- T_1:Rⁿ->R^m,T_2:R^m->R^I,T_3:R^I->R^k We define T_3•T_2•T_1:Rⁿ->R^k by
- ▶ T_3•T_2•T_1(x)=T_3(T_2(T_1(x))).
- Since the compositions are associative, we have (T_3•T_2)•T_1=T_3•(T_2•T_1). Thus we can drop the paranthese.
- ▶ [T_3•T_2•T_1]=[T_3][T_2][T_1].
 - $[T_3 \bullet (T_2 \bullet T_1)] = [T_3][T_2 \bullet T_1] = [T_3]([T_2][T_1]).$
 - We use matrix multiplications are associative.
- T_C•T_B•T_A=T_CBA

• A classification:

- A rotation in $\mathbb{R}^3 < -> \det A = 1$.
- A reflection composed with a rotation in R³ <-> det A = 1.
- A product of series of rotations is a rotation.
- A product of series of reflections and rotations with an even number of reflections is a rotation.
- A product of series of reflections and rotations with an odd number of reflections is a reflection composed with a rotation.



Yaw, pitch and roll

- Yaw: z-axis (up direction), pitch:x-axis (wing direction), roll: y-axis (the direction of travel)
- Corresponding rotations are $R_z\alpha$, $R_y\beta$, $R_x\gamma$.
- A composition of R_zα, R_yβ, R_xγ can be achieved by a single rotation R_vδ in some direction of certain angle.
- Given these, we multiply them to get R_vδ, and then find the axis direction v and the rotation δ (between 0 and π).
- See Example 5.

 Conversely, any rotation can be factored into yaw, pitch, roll rotations.

Factoring linear operators ito compositions

- We wish to factor a matrix into elementary pieces so that we can understand it better.
- For example, a diagonal operator can be understood as a composition of contraction and expansion along individual axis. E
- We restrict to R² only.
- Example 7: There are five types of elementary matrices:



- (I) [[1,k],[0,1]] a shear in x-direction,
- (II) [[1,0],[k,1]] a shear in y-direction,
- (III) [[0,1],[1,0]] a reflection about x=y,
- (IV) [[k,0],[0,1]] compression or expansion for k≥0.
- (V) [[1,0],[0,k]] same. For k < 0, they are compression or expansion followed by a reflection.

Theorem 6.4.4 If A is an invertible 2×2 matrix, then the corresponding linear operator on R^2 is a composition of shears, compressions, and expansions in the directions of the coordinate axes, and reflections about the coordinate axes and about the line y = x.

Example 8: illustrates the factorization and how one can understand a linear transformation.



Inverse

- ▶ T:Rⁿ->R^m. Suppose it is one-to-one.
- Let w be in the range of T.
- Then there is a unque x in \mathbb{R}^n s.t. T(x)=w.
- ▶ Let T⁻¹(w) be defined as x.
- w=T(x) <-> x=T⁻¹(w) for w in range(T).
- ▶ T⁻¹: range(T) -> Rⁿ.
- TT⁻¹ = Id on range (T)
- ▶ T⁻¹T=Id on Rⁿ.

Theorem 6.4.5 If T is a one-to-one linear transformation, then so is T^{-1} .

Invertible linear operator

- If T is one-to-one and onto, then T⁻¹ exists on the codomain, and is linear and one-to-one and onto.
 (The linearity already shown above. Other is just from the function theory)
- ▶ The matrix of T⁻¹ is the inverse of the matrix of T.
 - $T^{-1}T(x)=[T^{-1}][T]x=x. [T^{-1}][T]=I.$

Theorem 6.4.6 If T is a one-to-one linear operator on \mathbb{R}^n , then the standard matrix for T is invertible and its inverse is the standard matrix for T^{-1} .



- ▶ [T⁻¹]=[T]⁻¹.
- ▶ (T_A)⁻¹=T_(A⁻¹).
- An inverse of a rotation in R² is a rotation with opposite angle.
- An inverse of a rotation in R³ is a rotation with the same axis with an opposite angle or an opposite axis with the same angle.
- An inverse of an expansion by k in an axis direction is a contraction by 1/k in the same axis direction.
- An inverse of a reflection is the same reflection. $H_{\theta}H_{\theta} = I.$

Inverse and linear system

- > y=Ax given by a linear system as in (18).
- ▶ We have x=A⁻¹y given by a linear system.
- We can obtain the second linear system by the first one by solving.
- Example 12.



Geometric properties of the invertible linear operators in R².

What happens to lines, segments, polygons after acting by T?

Theorem 6.4.7 If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear operator, then:

- (a) The image of a line is a line.
- (b) The image of a line passes through the origin if and only if the original line passes through the origin.
- (c) The images of two lines are parallel if and only if the original lines are parallel.
- (d) The images of three points lie on a line if and only if the original points lie on a line.
- (e) The image of the line segment joining two points is the line segment joining the images of those points.

Theorem 6.4.8 If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear operator, then T maps the unit square into a nondegenerate parallelogram that has a vertex at the origin and has adjacent sides $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. The area of this parallelogram is $|\det(A)|$, where $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is the standard matrix for T.