

7.1. Dimension and structure

We will study subspaces through bases. Although there are many, they can be used to make a subspace like the Euclidean space.

This enables us to study abstract vector spaces later.

Bases for subspaces

- Consider $V = \text{Span}\{v_1, v_2, \dots, v_l\}$.
- If v_i is a linear combination of other vectors, we can drop v_i . $V = \text{Span}\{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_l\}$.
- To obtain a minimal set for a given V , we need to get $V = \text{Span}\{v_1, \dots, v_s\}$ so that v_1, \dots, v_s are linearly independent.

Definition 7.1.1 A set of vectors in a subspace V of R^n is said to be a *basis* for V if it is linearly independent and spans V .

- Example: $\{0\}$ no basis.
 - \mathbb{R}^n itself is a subspace and has a standard basis.
 - A line through 0 has a basis consisting of only one unique vector. (One can choose any such.)
 - A plane through 0 has a basis consisting of two nonzero vectors tangent to the plane. Any two nonparallel and nonzero will form a basis.
 - To make the independence test easier, we use the following. That is we will only need to consider first i ones to understand independence.

Theorem 7.1.2 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a set of two or more nonzero vectors in \mathbb{R}^n , then S is linearly dependent if and only if some vector in S is a linear combination of its predecessors.*

- The nonzero row vectors in the ref are linearly independent.
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- Proof: Given a row with a leading 1 at j th position, the linear combinations of the previous rows, will give you a nonzero entry at entries below the j -th position. By theorem 7.1.2, we are done.
 - Given an independent set of vectors $\{v_1, v_2, \dots, v_s\}$, suppose v is a nonzero vector which is not a linear combinations of the given ones, then one can add v to the list and the list is still independent. Why?
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The existence of basis

Theorem 7.1.3 (*Existence of a Basis*) *If V is a nonzero subspace of R^n , then there exists a basis for V that has at most n vectors.*

- Proof: V is not $\{O\}$. Let v_1 be a nonzero vector. (It exists.)
 - If $V = \text{Span}\{v_1\}$, we are done.
 - If V is not $\text{Span}\{v_1\}$. Choose v_2 not in $\text{Span}\{v_1\}$. $\{v_1, v_2\}$ are independent (why?). If $V = \text{Span}\{v_1, v_2\}$, then we are done.
 - Suppose we did this continuously, V has an independent set $S = \{v_1, v_2, \dots, v_s\}$. If $V = \text{Span}S$, then we are done. Otherwise, choose v_{s+1} not in the span.
 - By Theorem 3.4.8, s cannot be greater than n .
 - Thus we must stop at some s to get $V = \text{Span}S$ and S is independent.
- Basis is not unique for V .

Theorem 7.1.4 *All bases for a nonzero subspace of R^n have the same number of vectors.*

- Proof: $\{v_1, \dots, v_k\}, \{w_1, \dots, w_m\}$ bases. Show $k=m$.
 - Assume $k < m$ without loss of generality.
 - We can write w_i as linear combination of v_1, \dots, v_k .
 - Let A be $k \times m$ matrix doing this.
 - $w_i = \sum_j A_{ji} v_j$ (*)
 - Then $Ax=0$ has a nontrivial solution since $k < m$.
 - Let (c_1, \dots, c_m) be the nontrivial solution.
 - Then $c_1 a_{11} + \dots + c_m a_{m1} = 0$ for a_i i th row of A .
 - Then $c_1 w_1 + \dots + c_m w_m = 0$ by computations.
 - This follows by plugging in (*) to the equation and collecting over v_i s.

Dimension

Definition 7.1.5 If V is a nonzero subspace of R^n , then the *dimension* of V , written $\dim(V)$, is defined to be the number of vectors in a basis for V . In addition, we define the zero subspace to have dimension 0.

- Example: R^n has dimension n .
- Example: Solution space has dimension equal to the number f of free variables.
 - Setting i -th free variable 1 and the rest 0 gives us a column vector v_i . (canonical solutions)
 - Then $\{v_1, v_2, \dots, v_f\}$ spans the solution space.
 - $\{v_1, v_2, \dots, v_f\}$ is linearly independent since the positions of 1 and 0 for free variable positions in v_i s are different.
 - Thus $\{v_1, v_2, \dots, v_f\}$ is a basis.
- See Example 7.

Dimension of a hyperplane

- $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$.
- $[1, *, *, \dots, *]$ or $[0, 1, *, \dots, *], \dots$
- It has $n-1$ free variables.

Theorem 7.1.6 *If \mathbf{a} is a nonzero vector in R^n , then $\dim(\mathbf{a}^\perp) = n - 1$.*
