QR-DECOMPOSITION

HOUSEHOLDER TRANSFORMATIONS

QR-decomposition

- A mxk matrix with columns w_1,w_2,…,w_k. m-vectors.
- We find an orthonomal basis q_1,q_2,…,q_k,q_k+1,…,q_m of R^m.
- Then since q_i is orthogonal to w_1,…,w_k-1.
  - w_1=(w_1.q_1)q_1
  - w_2=(w_2.q_1)q_1+(w_2.q_2).q_2.
  - ….
  - w_k=(w_k.q_1)q_1+(w_k.q_2)q_2+…+(w_k.q_k)q_k.

- Use Theorem 3.1.8, A=QR

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix} \begin{bmatrix} (w_1.q_1) & (w_2.q_1) & \cdots & (w_1.q_k) \\ 0 & (w_2.q_2) & \cdots & (w_2.q_k) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (w_k.q_k) \end{bmatrix}$$

Theorem 7.10.1 (***QR-Decomposition***) *If A is an m × k matrix with full column rank, then A can be factored as*

$$A = QR \tag{4}$$

where Q is an m × k matrix whose column vectors form an orthonormal basis for the column space of A and R is a k × k invertible upper triangular matrix.

- A=QR.
- Assume A is a square matrix mxm.
- Then Q^TQ = I since $(Q^TQ)\_ij$= $q\_i^Tq\_j$ = 1 (i=j), 0 (i,j different) -> Q is orthogonal.
- Since the inverse of Q is Q^T, we have $R=Q^TA$.
- See Example 1.

QR-decompositions and the least square problem

- $Ax=b$. best approximate solution $x'=(A^TA)^{-1}A^Tb$.
- We write $A=QR$. $A^T=R^TQ^T$.
- $A^TAx'=A^Tb$.
- $R^TQ^TQRx'=R^TQ^Tb$.
- $R^TRx'=R^TQ^Tb$ and $x'=R^{-1}Q^Tb$.

Theorem 7.10.2 *If A is an $m \times k$ matrix with full column rank, and if $A = QR$ is a QR-decomposition of A, then the normal system for $A\mathbf{x} = \mathbf{b}$ can be expressed as*

$$R\mathbf{x} = Q^T\mathbf{b} \tag{9}$$

and the least squares solution can be expressed as

$$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b} \tag{10}$$

- Example 2.
- We will use Householder reflections to find Q instead since it has advantages in computer calculations.
- We obtain a formula for reflections:
  - Let a^c be the orthogonal hyperplane to span{a}
  - x-refl_a^c(x)=2proj_a(x).
  - Thus refl_a^c(x)=x-2proj_a(x)=x-2a(x.a)/||a||^2.

Definition 7.10.3 If $\mathbf{a}$ is a nonzero vector in R^n, and if $\mathbf{x}$ is any vector in R^n, then the ***reflection of*** $\mathbf{x}$ ***about the hyperplane*** $\mathbf{a}^\perp$ is denoted by $\text{refl}_{\mathbf{a}^\perp}\mathbf{x}$ and defined as

$$\text{refl}_{\mathbf{a}^\perp}\mathbf{x} = \mathbf{x} - 2\text{proj}_{\mathbf{a}}\mathbf{x} \tag{11}$$

The operator $T: R^n \to R^n$ defined by $T(\mathbf{x}) = \text{refl}_{\mathbf{a}^\perp}\mathbf{x}$ is called the ***reflection of*** $\boldsymbol{R^n}$ ***about the hyperplane*** $\mathbf{a}^\perp$.

- Thus the matrix H_a^c for refl_a^c is H_a^c=I-2aa^T/a^{T}a.
- If a is a unit vector u, then u^{T}u=||u||^2=1.
- refl_a^c=x-2(x.u)u and H_u^c=I-uu^T.
- See Example 3 and 4.

Definition 7.10.4 An $n \times n$ matrix of the form

$$H = I - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T \tag{16}$$

in which $\mathbf{a}$ is a nonzero vector in R^n is called a ***Householder matrix***. Geometrically, H is the standard matrix for the Householder reflection about the hyperplane $\mathbf{a}^\perp$.

Theorem 7.10.5 *Householder matrices are symmetric and orthogonal.*

- Proof: $H^T=I-(2/a^Ta)(aa^T)^T=H$.
  $HH=(I-2aa^T/a^Ta)\ (I-2aa^T/a^Ta)=I-4aa^T/a^Ta$
  $+(2aa^T/(a^Ta))(2aa^T/(a^Ta))=I$ (since $4(1/(a^Ta)^{2)}(aa^Taa^T) =$
  $4(1/(a^Ta)^{2)}((a^Ta)aa^T)=4(1/(a^Ta))(aa^T)$.)

Theorem 7.10.6 *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are distinct vectors in* R^n *with the same length, then the Householder reflection about the hyperplane* $(\mathbf{v}-\mathbf{w})^{\perp}$ *maps* $\mathbf{v}$ *into* $\mathbf{w}$*, and conversely.*

QR-decomposition using householder reflections

- The steps given A nxn matrix
- We apply a Householder matix Q_1 so that Q_1A has (||a_1||,0,…,0) as the first column by choosing Q_1 so that Q_1a_1=[||a_1||,0,…,0]^T.
- Now choose Q_2 which is 1 at (1,1)-entry and zero on elsewhere in the 1-st row and the 1-st column.
- Now concentrate on (n-1)x(n-1)-matrix in Q_1A removing 1st column and the 1st-row
- Q_2 sends the 2nd column a'_2 of Q_1A to [*,x,0,..,0]^T, nonzero x.
- We keep doing this… Q_n-1…Q_2Q_1A is upper triangular. We let it be R and Q=Q_1Q_2…Q_n-1. (notice the order change!)

- Example 7: