

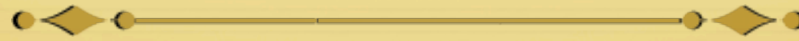


# QR-DECOMPOSITION

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HOUSEHOLDER TRANSFORMATIONS

# QR-decomposition



- ✦ A  $m \times k$  matrix with columns  $w_1, w_2, \dots, w_k$ .  $m$ -vectors.
- ✦ We find an orthonormal basis  $q_1, q_2, \dots, q_k, q_{k+1}, \dots, q_m$  of  $\mathbb{R}^m$ .
- ✦ Then since  $q_i$  is orthogonal to  $w_1, \dots, w_{k-1}$ .
  - ✦  $w_1 = (w_1 \cdot q_1)q_1$
  - ✦  $w_2 = (w_2 \cdot q_1)q_1 + (w_2 \cdot q_2)q_2$ .
  - ✦ ....
  - ✦  $w_k = (w_k \cdot q_1)q_1 + (w_k \cdot q_2)q_2 + \dots + (w_k \cdot q_k)q_k$ .

✦ Use Theorem 3.1.8,  $A=QR$

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix} \begin{bmatrix} (w_1 \cdot q_1) & (w_2 \cdot q_1) & \cdots & (w_1 \cdot q_k) \\ 0 & (w_2 \cdot q_2) & \cdots & (w_2 \cdot q_k) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (w_k \cdot q_k) \end{bmatrix}$$

**Theorem 7.10.1 (QR-Decomposition)** *If  $A$  is an  $m \times k$  matrix with full column rank, then  $A$  can be factored as*

$$A = QR \tag{4}$$

*where  $Q$  is an  $m \times k$  matrix whose column vectors form an orthonormal basis for the column space of  $A$  and  $R$  is a  $k \times k$  invertible upper triangular matrix.*

✦  $A=QR$ .

✦ Assume  $A$  is a square matrix  $m \times m$ .

✦ Then  $Q^T Q = I$  since  $(Q^T Q)_{ij} = q_i^T q_j = 1$  ( $i=j$ ),  $0$  ( $i, j$  different)  $\rightarrow Q$  is orthogonal.

✦ Since the inverse of  $Q$  is  $Q^T$ , we have  $R=Q^T A$ .

✦ See Example 1.



# QR-decompositions and the least square problem

- ✦  $Ax=b$ . best approximate solution  $x'=(A^T A)^{-1}A^T b$ .
- ✦ We write  $A=QR$ .  $A^T=R^T Q^T$ .
- ✦  $A^T A x'=A^T b$ .
- ✦  $R^T Q^T Q R x'=R^T Q^T b$ .
- ✦  $R^T R x'=R^T Q^T b$  and  $x'=R^{-1} Q^T b$ .

**Theorem 7.10.2** *If  $A$  is an  $m \times k$  matrix with full column rank, and if  $A = QR$  is a QR-decomposition of  $A$ , then the normal system for  $Ax = b$  can be expressed as*

$$Rx = Q^T b \tag{9}$$

*and the least squares solution can be expressed as*

$$\hat{x} = R^{-1} Q^T b \tag{10}$$

- ✦ Example 2.
- ✦ We will use Householder reflections to find  $Q$  instead since it has advantages in computer calculations.
- ✦ We obtain a formula for reflections:
  - ✦ Let  $a^\perp$  be the orthogonal hyperplane to  $\text{span}\{a\}$
  - ✦  $x\text{-refl}_{a^\perp}(x) = x - 2\text{proj}_a(x)$ .
  - ✦ Thus  $\text{refl}_{a^\perp}(x) = x - 2\text{proj}_a(x) = x - 2a(x \cdot a) / \|a\|^2$ .

**Definition 7.10.3** If  $\mathbf{a}$  is a nonzero vector in  $R^n$ , and if  $\mathbf{x}$  is any vector in  $R^n$ , then the *reflection of  $\mathbf{x}$  about the hyperplane  $\mathbf{a}^\perp$*  is denoted by  $\text{refl}_{\mathbf{a}^\perp} \mathbf{x}$  and defined as

$$\text{refl}_{\mathbf{a}^\perp} \mathbf{x} = \mathbf{x} - 2\text{proj}_{\mathbf{a}} \mathbf{x} \quad (11)$$

The operator  $T : R^n \rightarrow R^n$  defined by  $T(\mathbf{x}) = \text{refl}_{\mathbf{a}^\perp} \mathbf{x}$  is called the *reflection of  $R^n$  about the hyperplane  $\mathbf{a}^\perp$* .

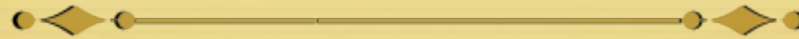
- ✦ Thus the matrix  $H_{\mathbf{a}^c}$  for  $\text{refl}_{\mathbf{a}^c}$  is  $H_{\mathbf{a}^c} = I - 2\mathbf{a}\mathbf{a}^T / \mathbf{a}^T\mathbf{a}$ .
- ✦ If  $\mathbf{a}$  is a unit vector  $\mathbf{u}$ , then  $\mathbf{u}^T\mathbf{u} = \|\mathbf{u}\|^2 = 1$ .
- ✦  $\text{refl}_{\mathbf{a}^c} = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u})\mathbf{u}$  and  $H_{\mathbf{u}^c} = I - \mathbf{u}\mathbf{u}^T$ .
- ✦ See Example 3 and 4.

**Definition 7.10.4** An  $n \times n$  matrix of the form

$$H = I - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T \quad (16)$$

in which  $\mathbf{a}$  is a nonzero vector in  $R^n$  is called a **Householder matrix**. Geometrically,  $H$  is the standard matrix for the Householder reflection about the hyperplane  $\mathbf{a}^\perp$ .

**Theorem 7.10.5** *Householder matrices are symmetric and orthogonal.*



✦ Proof:  $H^T = I - (2/a^T a)(aa^T)^T = H$ .

$$\begin{aligned} HH &= (I - 2aa^T/a^T a)(I - 2aa^T/a^T a) = I - 4aa^T/a^T a \\ &\quad + (2aa^T/(a^T a))(2aa^T/(a^T a)) = I \text{ (since } 4(1/(a^T a)^2)(aa^T aa^T) = \\ &\quad 4(1/(a^T a)^2)((a^T a)aa^T) = 4(1/(a^T a))(aa^T).) \end{aligned}$$

**Theorem 7.10.6** *If  $\mathbf{v}$  and  $\mathbf{w}$  are distinct vectors in  $R^n$  with the same length, then the Householder reflection about the hyperplane  $(\mathbf{v} - \mathbf{w})^\perp$  maps  $\mathbf{v}$  into  $\mathbf{w}$ , and conversely.*



# QR-decomposition using householder reflections

- ✦ The steps given A nxn matrix
- ✦ We apply a Householder matrix  $Q_1$  so that  $Q_1A$  has  $(\|a_1\|, 0, \dots, 0)$  as the first column by choosing  $Q_1$  so that  $Q_1a_1 = [\|a_1\|, 0, \dots, 0]^T$ .
- ✦ Now choose  $Q_2$  which is 1 at (1,1)-entry and zero on elsewhere in the 1-st row and the 1-st column.
- ✦ Now concentrate on  $(n-1) \times (n-1)$ -matrix in  $Q_1A$  removing 1<sup>st</sup> column and the 1<sup>st</sup>-row
- ✦  $Q_2$  sends the 2<sup>nd</sup> column  $a'_2$  of  $Q_1A$  to  $[\ast, x, 0, \dots, 0]^T$ , nonzero  $x$ .
- ✦ We keep doing this...  $Q_{n-1} \dots Q_2 Q_1 A$  is upper triangular. We let it be R and  $Q = Q_1 Q_2 \dots Q_{n-1}$ . (notice the order change!)

✦ Example 7:

