7.11. Coordinates with respect to a basis

Each basis gives you are coordinate system and conversely.

Nonrectangular coordinates

- Given a basis $v_1,v_2,..,v_n$, we can write each vector v as a unique linear combination.
  $v=c_1v_1+c_2v_2+\ldots+c_nv_n$.
- Fixing a basis, $v \to (c_1,c_2,..,c_n)$
- This is sensitive to the order of v_is.
- This gives us a coordinate system.
- Conversely, given any coordinate system $(1,0,..,0) \to v_1$, $(0,1,0,..,0) \to v_2$,..,$(0,0,..,1) \to v_n$. This forms a basis.

Definition 7.11.1 If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is an ordered basis for a subspace W of R^n, and if

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

is the expression for a vector $\mathbf{w}$ in W as a linear combination of the vectors in B, then we call

$$a_1, a_2, \dots, a_n$$

the ***coordinates of* w *with respect to*** B; and more specifically, we call a_j the $\mathbf{v}_j$***-coordinate of*** $\mathbf{w}$. We denote the ordered k-tuple of coordinates by

$$(\mathbf{w})_B = (a_1, a_2, \dots, a_k)$$

and call it the ***coordinate vector*** for $\mathbf{w}$ with respect to B; and we denote the column vector of coordinates by

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

and call it the ***coordinate matrix*** for $\mathbf{w}$ with respect to B.

- Example $B=\{(0,1),(1,0)\}$
  - $(a,b)=b(0,1)+a(1,0)$. $\to [(a,b)]_B=(b,a)$
- Example 1. $B=\{(2,1,2),(3,0,-1),(5,0,0)\}$.
  - $(3,1,4)=1(2,1,2)-2(3,0,-1)+(5,0,0)$
  - $[(3,1,4)]_B=(1,-2,1)$.
- Example 2. $B=\{e_1,e_2,..,e_n\}$
  - $w=(w_1,w_2,..,w_n)$ $=w_1e_1+w_2e_2+\ldots+w_ne_n$
  - $[w]_B=(w_1,w_2,\ldots,w_n)$

Coordinates with respect to orthonormal basis.

- Let B={v_1,v_2,..,v_n} be an orthonomal basis of R^{n}.
- We know w=(w.v_1)v_1+(w.v_2)v_2+…+(w.v_n)v_n.
  - [w]_B=((w.v_1),(w.v_2),…,(w.v_n))
- Example 3. B={(cos t,sint),(-sint, cos t)}
  - (a,b) = (acost + bsint)(cost,sint) +(-asint+bcost)(-sint, cost).
  - [(a,b)]_B=(acost+bsint,-asint+bcost)

Computing with coordinates w.r.t. orthonomal basis

- Dot product, norms are preserved under “coordinate changes”

Theorem 7.11.2 *If B is an orthonormal basis for a k-dimensional subspace W of R^n, and if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in W with coordinate vectors*

$$(\mathbf{u})_B = (u_1, u_2, \ldots, u_k), \quad (\mathbf{v})_B = (v_1, v_2, \ldots, v_k), \quad (\mathbf{w})_B = (w_1, w_2, \ldots, w_k)$$

then:

(*a*) $\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2 + \cdots + w_k^2} = \|(\mathbf{w})_B\|$

(*b*) $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_k v_k = (\mathbf{u})_B \cdot (\mathbf{v})_B$

Change of basis problems

The Change of Basis Problem If $\mathbf{w}$ is a vector in R^n, and if we change the basis for R^n from a basis B to a basis B', how are the coordinate matrices $[\mathbf{w}]_B$ and $[\mathbf{w}]_{B'}$ related?

- Solution: B={v_1,v_2,..,v_n}. B'={v'_1,v'_2,..,v'_n}.
  - v_1=p_11v'_1+p_21v'_2+…+p_n1v'_n.
  - v_2=p_12v'_1+p_22v'_2+…+p_n2v'_n.
  - …
  - v_n=p_1nv'_1+p_2nv'_2+…+p_nnv'_n.

- Let w be any vector in R^n.
- $w=a_1v_1+\dots+a_nv_n$. $[w]_B=(a_1,..,a_n)$
  - $w=a_1(p_{11}v'_1+p_{21}v'_2+\dots+p_{n1}v'_n)$ $+a_2(p_{12}v'_1+p_{22}v'_2+\dots+p_{n2}v'_n)+\dots$

  $+a_n(p_{1n}v'_1+p_{2n}v'_2+\dots+p_{nn}v'_n)$

  $=(a_1p_{11}+a_2p_{12}+\dots+a_np_{1n})v'_1$ $+(a_1p_{21}+a_2p_{22}+\dots+a_np_{2n})v'_2+\dots$ $+(a_1p_{n1}+a_2p_{n2}+\dots+a_np_{nn})v'_n$
- Since the entries equal P times column vector $(a_1,\dots,a_n)$
- $[w]_{B'}= P_{(B\to B')}[w]_B$.

Theorem 7.11.3 (***Solution of the Change of Basis Problem***) *If* $\mathbf{w}$ *is a vector in* R^n, *and if* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and* $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ *are bases for* R^n, *then the coordinate matrices of* $\mathbf{w}$ *with respect to the two bases are related by the equation*

$$[\mathbf{w}]_{B'} = P_{B \to B'}[\mathbf{w}]_B \tag{10}$$

where

$$P_{B \to B'} = \big[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'}\big] \tag{11}$$

This matrix is called the ***transition matrix*** (*or the* ***change of coordinates matrix***) *from* B *to* B'.

- Example 5. Let B={(1,0),(0,1)}, B'={(cos t,sint),(-sint, cos t)}
  - (1,0)=cost(cost,sint)+(-sint)(-sint,cost)
  - (0,1)=sint(cost,sint)+(cost)(-sint,cost).
  - Then P_(B->B')=[[cost,sint], [-sint,cost]].

Invertibility of transition matrices.

- B_1,B_2,B_3 three basis of R^n.
- Then
  - P_(B_2->B_3)P_(B_1->B_2)=P_(B_1->B_3).
  - We omit proof.
  - P_(B_2->B_1)P_(B_1->B_2)
  - =P_(B_1->B_1)=I.

Theorem 7.11.4 *If B and B' are bases for R^n, then the transition matrices $P_{B' \to B}$ and $P_{B \to B'}$ are invertible and are inverses of one another; that is,*

$$(P_{B' \to B})^{-1} = P_{B \to B'} \quad \text{and} \quad (P_{B \to B'})^{-1} = P_{B' \to B}$$

A Procedure for Computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$.

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

- Proof: To find [v_i]_B' we solve for [v'_1,v'_2,..,v'_n]x=v_i.
  - Form [v'_1,v'_2,…,v'_n|v_i] -> ref is [I|y] for some y. y=[v_i]_B' (Why?)
  - [v'_1,v'_2,…,v'_n|v_1,v_2,…,v_n]-> [I|P_(B->B')].
- Example 7.

Coordinate maps

- B a basis.
- x->(x)_B=[x]_B is a coordinate map.
- (cv)_B=cx_B since [cv]_B=c[v]_B
- (v+w)_B=v_B+w_B since [v+w]_B=[v]_B+[w]_B.

Theorem 7.11.5 *If B is a basis for R^n, then the coordinate map $\mathbf{x} \to (\mathbf{x})_B$ (or $\mathbf{x} \to [\mathbf{x}]_B$) is a one-to-one linear operator on R^n. Moreover, if B is an orthonormal basis for R^n, then it is an orthogonal operator.*

Theorem 7.11.6 *If A and C are $m \times n$ matrices, and if B is any basis for R^n, then $A = C$ if and only if $A[\mathbf{x}]_B = C[\mathbf{x}]_B$ for every $\mathbf{x}$ in R^n.*

Orthonomal basis and transition matrices

Theorem 7.11.7 *If B and B' are orthonormal bases for R^n, then the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are orthogonal.*

- Proof: [v_1]_B', [v_2]_B',…,[v_n]_B' is orthonormal also by Theorem 7.11.12.

- We can think of matrix as transformations. But we can also think of a nonsingular matrix as a transition matrix.
- Any nonsingular matrix can be considered a matrix of n column vectors forming a basis.
- See Example 9.
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Theorem 7.11.8 *If P is an invertible $n \times n$ matrix with column vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then P is the transition matrix from the basis $B = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ for R^n to the standard basis $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for R^n.*