8.1. Matrix representations of linear transformations

Matrix of a linear operator with respect to a basis.

Matrix of linear operators w.r.t. a basis

- One can use different representation of a transformation using basis.
- If one uses a right basis, the representation get simpler and easier to understand.
- x->Tx.
- [x]_B -> [Tx]_B = A[x]_B for some matrix A depending on B.
- How does one find A_B?
- This amounts to change of coordinates. (Coordinates are usually not canonical.)

Theorem 8.1.1 *Let $T : R^n \to R^n$ be a linear operator, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for R^n, and let*

$$A = \big[[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B \mid \cdots \mid [T(\mathbf{v}_n)]_B\big] \tag{4}$$

Then

$$[T(\mathbf{x})]_B = A[\mathbf{x}]_B \tag{5}$$

for every vector $\mathbf{x}$ in R^n. Moreover, the matrix A given by Formula (4) is the only matrix with property (5).

- Proof: If [x]_B=(c_1,c_2,..,c_n), then x=c_1v_1+…+c_nv_n.
  - T(x) = c_1T(v_1)+…+c_nT(v_n)
  - [T(x)]_B = c_1[T(v_1)]_B+…+c_n[T(v_n)]_B.

$$= \begin{bmatrix} [T(v_1)]_B & [T(v_2)]_B & \cdots & [T(v_n)]_B \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = A[x]_B$$

- A is called the matrix of t w.r.t. the basis B.
- $[T]_B = A = [[T(v_1)]_B, [T(v_2)]_B, \ldots, [T(v_n)]_B]$.
- $[T(x)]_B = [T]_B [x]_B$.
- If S is the standard basis, $[T]_S$ is the standard matrix for T.
- Example 1.
- Example 2. A matrix realized as a rotation….

Changing basis

- What is the relationship between [T]_B and [T]_B’ for two basis B and B’.
- [T]_B[x]_B = [T(x)]_B. [T]_B’[x]_B’=[T(x)]_B’.
- P_(B->B’)[T(x)]_B=[T(x)]_B’
- P_(B->B’)[x]_B=[x]_B’
- [T]_B’[x]_B’=[T(x)]_B’
- [T]_B’P[x]_B=P[T(x)]_B.
- (P^{-1}[T]_B’P)[x]_B=[T(x)]_B.
- Compare to [T]_B[x]_B=[T(x)]_B.
- Thus P^{-1}[T]_B’P=[T]_B.

Theorem 8.1.2 *If $T : R^n \to R^n$ is a linear operator, and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ are bases for R^n, then $[T]_B$ and $[T]_{B'}$ are related by the equation*

$$[T]_{B'} = P[T]_B P^{-1} \tag{12}$$

in which

$$P = P_{B \to B'} = \big[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'}\big] \tag{13}$$

is the transition matrix from B to B'. In the special case where B and B' are orthonormal bases the matrix P is orthogonal, so (12) *is of the form*

$$[T]_{B'} = P[T]_B P^T \tag{14}$$

- [T]_B=P^{-1}[T]_B'P.
- [T]_B=P^T[T]P if B, B' orthonormal basis.

S (standard basis)->B.

Theorem 8.1.3 *If $T : R^n \to R^n$ is a linear operator, and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for R^n, then $[T]$ and $[T]_B$ are related by the equation*

$$[T] = P[T]_B P^{-1} \tag{17}$$

in which

$$P = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n] \tag{18}$$

is the transition matrix from B to the standard basis. In the special case where B is an orthonormal basis the matrix P is orthogonal, so (17) is of the form

$$[T] = P[T]_B P^T \tag{19}$$

- Proof: P=P_(B->S) = [[v_1]_S,…,[v_n]_S] =[v_1,…,v_n]
- Some formula: [T]_B=P^{-1}[T]P. [T]_B=P^T[T]P.

- Example 3.
- Example 4. Any reflection can be made into a reflection on x-axis by changing basis or changing coordinates

Base changes for transformations T:R^n->R^m

- Suppose that we choose basis B for R^n and B' for R^m.
- x->T(x).
- [x]_B -> [T(x)]_B'
- A[x]_B=[T(x)]_B'. What is A?

Theorem 8.1.4 *Let $T : R^n \to R^m$ be a linear transformation, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be bases for R^n and R^m, respectively, and let*

$$A = \left[[T(\mathbf{v}_1)]_{B'} \mid [T(\mathbf{v}_2)]_{B'} \mid \cdots \mid [T(\mathbf{v}_n)]_{B'}\right] \tag{23}$$

Then

$$[T(\mathbf{x})]_{B'} = A[\mathbf{x}]_B \tag{24}$$

for every vector $\mathbf{x}$ *in* R^n*. Moreover, the matrix A given by Formula* (23) *is the only matrix with property* (24).

- Some formula
  [T]_B',B -> [[T(v_1)]_B',[T(v_2)]_B',…,[T(v_n)]_B']
  and
- [T(x)]_B' = [T]_B',B[x]_B
- Example 6.
- Remark: For operators T:R^n->R^n, [T]_B=[T]_B,B.

Effect of changing basis

- B_1,B_2 for R^n, B'_1,B'_2 for R^m.
- U transition matrix from B_2->B_1
- V transition matrix from B'_2->B'_1
- [T]_B'_1,B_1 = V[T]_B'_2,B_2U^{-1} (*)
- Proof: [T(x)]_B'_1 = [T]_B'_1,B_1[x]_B_1.
  - V[T(x)]_B'_2=[T]_B'_1,B_1U[x]_B_2
  - [T(x)]_B'_2=(V^{-1}[T]_B'_1,B_1U)[x]_B_2
  - % Use [w]_B'=P_{B->B'}[w]_B.

Representing Linear operators with two basis.

- Actually, we can use two basis for R^n as well.
- [T]_B',B.
- What we used was [T]_B=[T]_B,B. B'=B.
- So change of basis formula: [T]_B_1=P[T]_BP^{-1} for P=P_B->B_1.
- V,U=P in this case.
- Thus this follows from (*)