8.4. Quadratic Forms

Quadratic forms generalize norm, lengths, inner-products,..

Definition of a quadratic forms

- Sum of $a_{ij}x_ix_j$ for $i.j=1,2,..,n$ ($i <j$ written usually)
- Example: $f(x_1,x_2,x_3)= a_{11}x_1^2+2a_{12}x_1x_2+2a_{13}x_1x_3+a_{22}x_2^2 +2a_{23}x_2x_3+a_{33}x_3^2$
- In matrix form $q(x)=x^TAx$ for A symmetric nxn-matrix.
- Note that $A_{ij}=a_{ij}=a_{ji}$… here
- If A=I, then $q(x)=x.Ix=x.x=||x||^2$.
- If A=D, diagonal, then $q(x)=l_1x_1^2+l_2x_2^2+\ldots+l_nx_n^2$.

Change of variables in a quadratic form.

- We can use substitution x=Py to simplify q(x).
- This will help us the solve many problems…
- Since A is symmetric, we can find P s.t. P^TAP is D.
- Then $x^TAx = y^TP^TAPy=y^TDy$.

Theorem 8.4.1 **(*The Principal Axes Theorem*)** *If A is a symmetric $n \times n$ matrix, then there is an orthogonal change of variable that transforms the quadratic form $\mathbf{x}^TA\mathbf{x}$ into a quadratic form $\mathbf{y}^TD\mathbf{y}$ with no cross product terms. Specifically, if P orthogonally diagonalizes A, then making the change of variable $\mathbf{x} = P\mathbf{y}$ in the quadratic form $\mathbf{x}^TA\mathbf{x}$ yields the quadratic form*

$$\mathbf{x}^TA\mathbf{x} = \mathbf{y}^TD\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the eigenvectors that form the successive columns of P.

- Example 2. Q(x)=-23/25x_1²-2/25x_2²+72/25x_1x_2

$$q(x) = x^T A x = [x_1, x_2] \begin{bmatrix} -23/25 & 36/25 \\ 36/25 & -2/25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- We find the eigenvectors to diagonalize it.
- Then make it into an orthonormal set.
- Use P=[v_1,v_2] eigenvectors. (One may need to orthogonalize it)

Quadratic forms in geometry

- $ax^2+2bxy+cy^2+dx+ey+f=0$.
- Set $d,e=0$.
- We wish to solve $ax^2+2bxy+cy^2+f=0$.
- We wish to turn it into $ax^2+cy^2+f=0$ by coordinate change.
- By dividing by $-f$, we obtain $a'x^2+b'y^2=1$.
- If $a,b>0$, then we obtain an ellipse or a circle.
- If $a>0,b<0$, or $a<0,b>0$ then we obtain a hyperbola
- If $a<0,b<0$, then an empty set.

Indentifying conic sections

- We identify minor and major axis. Thus, basically, we have to rotate.
- This amounts to finding P.
- For R^2, P is always a rotation. Find the rotation angle.
- Example 3.
- Remark: $ax^2+2bxy+cy^2=k$. Rotate by the angle t s.t $\cos 2t=(a-c)/2b$.
  Solution: Find eigenvectors for all a,b,c..

Positive definite quadratic forms

Definition 8.4.2 A quadratic form $\mathbf{x}^T A\mathbf{x}$ is said to be

positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$
negative definite if $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$
indefinite if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values

Theorem 8.4.3 *If A is a symmetric matrix, then:*

(a) $\mathbf{x}^T A\mathbf{x}$ *is positive definite if and only if all eigenvalues of A are positive.*
(b) $\mathbf{x}^T A\mathbf{x}$ *is negative definite if and only if all eigenvalues of A are negative.*
(c) $\mathbf{x}^T A\mathbf{x}$ *is indefinite if and only if A has at least one positive eigenvalue and at least one negative eigenvalue.*

- Positive semindefinite $x^T Ax \geq 0$ only if x is not O.
- Negative semidefinite $x^T Ax \leq 0$ only if x is not O.
- In higher dimensions, this is classified by the number of positive eigenvalues and negative eigenvalues and the multiplicity of O in the characteristic polynomial.

Classifying conics

- $x^TAx = 1$.
- A diagonalizes to [[l_1,0],[0,l_2]]
- If l_1>0 and l_2>0, then ellipse.
- If l_1<0 and l_2 <0, then no graph
- If l_1.l_2< 0, then a hyperbola.

Theorem 8.4.4 *If A is a symmetric* 2×2 *matrix, then:*

(a) $\mathbf{x}^T A\mathbf{x} = 1$ *represents an ellipse if A is positive definite.*

(b) $\mathbf{x}^T A\mathbf{x} = 1$ *has no graph if A is negative definite.*

(c) $\mathbf{x}^T A\mathbf{x} = 1$ *represents a hyperbola if A is indefinite.*

- Positive semidefinite case: two lines L union -L
- Negative semidefinite case: empty set.

Identifying positive definite matrices.

- k-th principal submatrix of an nxn-matrix consists of the first k-rows intersected with first k-columns of A.

Theorem 8.4.5 *A symmetric matrix A is positive definite if and only if the determinant of every principal submatrix is positive.*

Theorem 8.4.6 *If A is a symmetric matrix, then the following statements are equivalent.*

(a) *A is positive definite.*

(b) *There is a symmetric positive definite matrix B such that* $A = B^2$.

(c) *There is an invertible matrix C such that* $A = C^T C$.

- Proof: (a)->(b): A is positive definite. D has only positive eigenvalues. $D=D\_1^2$. $A=PD\_1^2P^T = PD\_1P^TPD\_1P^T$.
  - Let $B=PD\_1P^T$. B is symmetric.
  - Since D_1 has positive diagonals, B is positive definite.
- (b)->(c): $A=B^2$. B symmetric positive definite. B is invertible. Take C=B.
- (c)->(a): $A=C^TC$.
  - $x^TAx=x^TC^TCx=(Cx)^TCx = Cx.Cx=||Cx||^2 >0$ for x nonzero.
- Example 6.

Cholesky factorization

- $A = R^T R$. R is upper triangular and has positive entries in the diagonal.