Row-equivalences again

Row spaces bases

computational techniques using row-equivalences

- The row space of A is the span of row vectors.
- The row rank of A is the dimension of the row space of A.
- Theorem 9: Row-equivalent matrices have the same row spaces.
  - proof: Check for elementary row equivalences only.
- Theorem 10: R nonzero row reduced echelon matrix. Then nonzero rows of R form a basis of the row space of R.

- proof: $\rho_1, \ldots, \rho_r$ row vectors of R.
  - $\rho_i = (R_{i1}, R_{i2}, \ldots, R_{in})$
  - $$\begin{aligned} R_{ij} = 0 \quad &\text{if} \quad j < k_i, \\ R_{i,k_j} \quad &= \quad \delta_{ij}, \\ k_1 < k_2 < \quad &\ldots \quad < k_r. \end{aligned}$$
  - Let $\beta = (b_1, \ldots, b_n)$ be a vector in the row space.

$$\beta = c_1\rho_1 + \cdots + c_r\rho_r$$

$$Claim: c_j = b_{k_j}$$

$$b_{k_j} = \sum_{i=1}^{r} c_i R_{i,k_j} = \sum_{i=1}^{r} c_i \delta_{ij} = c_j$$

$$\text{If } \beta = 0, \text{ then } c_j = 0, j = 0, \ldots, r.$$

  - $\rho_1, \ldots, \rho_r$ are linearly independent: basis

- Theorem 11: m, n, F a field. W a subspace of F^n. Then there is precisely one mxn r-r-e matrix which has W as a row space.
- Corollary: Each mxn matrix A is row equiv. to one unique r-r-e matrix.
- Corollary. A,B mxn.
  A and B are row-equiv iff A,B have the same row spaces.

Proof of Theorem 11

- R r.r.e. for any A whose rows span W.
- $r_1, r_2, \ldots, r_s$ row vectors of R.
- b any vector $b = c_1 r_1 + .. + c_r r_s$.
- $b = \sum_{i=1}^{s} b_{k_i} r_i \,, b_{k_i} \neq 0.$
- The first nonzero entry of b occurs at k_ith position.

- Given any other R’ from A’ with span A’=W,
- Let r'_j be the j-th row vector of R’.
- The last one $r'_{s'}$ has to coincide with r_s. (If not, we cannot write it as a linear combination.)
- In fact, each row r'_j has to coincide with r_k.

Summary of row-equivalences

- TFAE
  - A and B are row-equivalent
  - A and B have the same row-space
  - B= PA where P is invertible.
  - AX=0 and BX=0 has the same solution spaces.
- Proof: (i)-(iii) done before. (i)-(ii) above corollary. (i)->(iv) is also done. (iv)->(i) to be done later.

Computations

- Numerical problems:
  - 1. How does one determine a set of vectors $S=(a_1,..,a_n)$ is linearly independent. What is the dimension of the span W of S?
  - 2. Given a vector v, determine whether it belongs to a subspace W. How to write $v = c_1a_1+...+c_na_n$.
  - 3. Find some explicit description of W: i.e., coordinates of W. -> Vague...

- Let A be mxn matrix.
- r-r-e R
- dim W = r the number of nonzero rows of R.

$$W = \{\beta | \beta = c_1\rho_1 + \cdots + c_r\rho_r, c_i \in F\}$$

$$\begin{aligned}
\beta &= (b_1, \ldots, b_n) \\
\beta &= \textstyle\sum_{i=1}^{r} c_i R_{ij}, c_j = b_{k_j} \\
\beta &= \textstyle\sum_{i=1}^{r} b_{k_i}\rho_i \\
b_j &= \textstyle\sum_{i=1}^{r} b_{k_i} R_{ij}
\end{aligned}$$

$b_{k_1}, \ldots, b_{k_r}$ give a parametrization of W (also relations between coordinates of the row space).

- Example:

$$R = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b_{k_1}(1,2,0,0) + b_{k_2}(0,0,1,0) + b_{k_3}(0,0,0,1)$$

$$(b_{k_1}, 2b_{k_1}, b_{k_2}, b_{k_3})$$

- (1) can be answered by computing the rank of R. If rank R =m, then independent. If rank R < m, then dependent. (A=PR, P invertible.)

- (2): b given. Solve for AX = b.
- Second method: A=PR, P invertible.

$$\beta = x_1\alpha_1 + \cdots + x_m\alpha_m$$

$$\rho_i = \sum_{j=1}^{m} P_{ij}\alpha_j$$

$$\beta = \sum_{i=1}^{r} b_{k_i}\rho_i = \sum_{i=1}^{r}\ \sum_{j=1}^{m} b_{k_i}P_{ij}\alpha_j = \sum_{j=1}^{m}\ \sum_{i=1}^{r}\ b_{k_i}P_{ij}\alpha_j$$

$$x_j = \sum_{i=1}^{r}\ b_{k_i}P_{ij}$$

- In line 3, we solve for b_{k_i}
- Final equation is from comparing the first line with the second to the last line.

- $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$
- Find r-r-e R. Find a basis of row space
- Which vectors (b_1,b_2,b_3,b_4) is in W?
- coordinate of (b_1,b_2,b_3,b_4)?
- write (b_1,b_2,b_3,b_4) as a linear combination of rows of A.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & y_1 \\ 0 & 2 & 1 & 0 & y_2 \\ 0 & 1 & -1 & 1 & y_3 \end{array}\right) \quad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & y_1 \\ 0 & 1 & 1/2 & 0 & y_2/2 \\ 0 & 1 & -1 & 1 & y_3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 0 & y_1 - y_2/2 \\ 0 & 1 & 1/2 & 0 & y_2/2 \\ 0 & 0 & -3/2 & 1 & -y_2/2 + y_3 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & y_1 - y_2/3 - y_3/3 \\ 0 & 1 & 0 & 1/3 & y_2/3 + y_3/3 \\ 0 & 0 & 1 & -2/3 & y_2/3 - 2y_3/3 \end{array}\right)$$

$$Q = \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1/3 & 1/3 \\ 0 & 1/3 & -2/3 \end{bmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -2/3 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & -1/3 & -1/3 \\ 0 & 1/3 & 1/3 \\ 0 & 1/3 & -2/3 \end{pmatrix}$$

- R=QA.
- Basis of row spaces: rows above, dim=3 (note relations of b_4 in terms of other coordinates.)

$$\beta = b_1\rho_1 + b_2\rho_2 + b_3\rho_3 = (b_1, b_2, b_3, -b_1/3 + b_2/3 - 2b_3/3)$$

$$= [b_1, b_2, b_3]R = [b_1, b_2, b_3]QA$$

$$= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$$

$$x_i = [b_1, b_2, b_3]Q_i$$

- Q_i is the ith column of Q.

$$
\begin{aligned}
x_1 &= b_1 \\
x_2 &= -b_1/3 + b_2/3 + b_3/3 \\
x_3 &= -b_1/3 + b_2/3 - 2b_3/3
\end{aligned}
$$

These are coefficients of (b_1,b_2,b_3,b_4) written in terms of original rows of A.

Find description of solutions space V of AX=0.

- Basis of V?
- AX=0 ↔ RX=0.

$$y_1 = u/3, y_2 = -u/3, y_3 = 2u/3. \quad X = \begin{pmatrix} u/3 \\ -u/3 \\ 2u/3 \\ u \end{pmatrix}$$

- V is one-dimensional
- Basis of V: (1,-1,2,3).

For what Y, AX=Y has solutions?

- AX=Y for what Y? All Y? See page 63.
- Again, we find R and change Y in the same way. We consider 0 rows of R to obtain the relations for Y.
- Examples 21 and 22 must be thoroughly understood

A matrix and computations

- A some matrix.
- 1. Find invertible P so that PA = R r.r.e.
- 2. Basis of span W row space of A.
- 3. Characterize W. Parametrize by rows of R.
- 4. Write elements of W as linear combinations of rows of A. (technique we showed.)
- 5. AX=0Solution space; basis?, dim?
- 6. AX=Y. When Y has solutions? (multiply by P, PAX = PY. RX= PY. Consider 0 rows of RX.)