3.6. Double dual

Dual of a dual space

Hyperspace

- $(V^*)^* = V^{**}$ = ? (V is a v.s. over F.)
- = V.
- a in V. *l*: a -> L_a:V^*->F defined by $L_a(f)=f(a)$.
- Example: $V=R^2$. $L_{(1,2)}(f)= f(1,2)=a+2b$, if $f(x,y)=ax+by$.
- Lemma: If $a \neq 0$, then $L_a \neq 0$.
  - Proof: $B=\{a_1,\ldots,a_n\}$ basis of V s.t. $a=a_1$.
    - f in V^* be s.t. $f(x_1a_1+\ldots+x_na_n)=x_1$.
    - Then $L_{a1}(f) = f(a_1)= 1$. Thus $L_a \neq 0$.

- Theorem 17. V. f.d.v.s. over F. The mapping a -> L_a is an isomorphism V->V**
- Proof: *l:* a -> La is linear.

$$
\begin{array}{rcl}
L_\gamma(f) & = & f(\gamma) \\
& = & f(c\alpha+\beta) \\
\gamma = c\alpha+\beta \qquad & = & cf(\alpha)+f(\beta) \\
& = & cL\alpha(f)+L_\beta(f) \\
L_\gamma & = & cL_\alpha+L_\beta
\end{array}
$$

  - *l* is not singular. L_a =0 iff a =0. (-> above. <- obvious)
  - dim V = dim V*= dim V**.
  - Thus *l* is an isomorphism by Theorem 9.

- Corollary: V f.d.v.s. over F.
  If L:V->F, then there exists unique v in V s.t. $L(f)=f(a)=L_a(f)$ for all f in V*.
- Corollary: V f.d.v.s. over F.
  Each basis of V* is a dual of a basis of V.
- Proof: $B^*=\{f_1,\ldots,f_n\}$ a basis of V*.
  - By Theorem 15, there exists $L_1,\ldots,L_n$ for V** s.t. $L_i(f_j)= \delta_{ij}$.
  - There exists $a_1,\ldots,a_n$ s.t. $L_i=L_{ai}$.
  - $\{a_1,\ldots,a_n\}$ is a basis of V and B* is dual to it.

- Theorem: S any subset of V. f.d.v.s. $(S^0)^0$ is the subspace spanned by S in V=V**.
- Proof: W =span(S). $W^0=S^0$. $W^{00}=S^{00}$ Show $W^{00}=W$.
  - dim W+dim W^0= dim V.
  - dim W^0+dim W^{00}=dim V*.
  - dimW=dimW^{00}.
  - W is a subset of W^{00}.
    - v in W. L(v)=0 for all L in W^0. Thus v in W^{00}.
- If S is a subspace, then $S=S^{00}$.

- Example: $S=\{[1,0,0],[0,1,0]\}$ in R^3.
  - $S^0=\{cf_3|c \text{ in } F\}$. $f_3:(x,y,z)\text{->}z$
  - $S^{00}=\{[x,y,0]|x,y \text{ in } R\}=\text{Span}(S)$.
- A hyperspace is V is a maximal proper subspace of V.
  - Proper: N in V but not all of V.
  - Maximal.

$N \subset V$ is maximal if $N \subset W$ implies $W = N$ or $W = V$.

- Theorem. f a nonzero linear functional. The null space N_f of f is a hyperspace in V and every hyperspace is a null-space of a linear functional.
- Proof: First part. We show N_f is a maximal proper subspace.
  - v in V, $f(v) \neq 0$. v is not in N_f. N_f is proper.
  - We show that every vector is of form $w+cv$ for w in N_f and c in F.(*)
    - Let u in V. Let $c = f(u)/f(v)$. ($f(v) \neq 0$).
    - Let $w = u-cv$. Then $f(w)=f(u)-cf(v)=0$. w in N_f.

- N_f is maximal: N_f is a subspace of W.
- If W contains v s.t. v is not in N_f, then W=V by (*). Otherwise W=N_f.

- Second part. Let N be a hyperspace.
  - Fix v not in N. Then Span(N,v)= V.
    - Every vector u = w+cv for w in N and c in F.
    - w and c are uniquely determined:
      - u=w'+c'v. w' in N, c' in F.
      - (c'-c)v = w-w'.
      - If c'-c≠0, then v in N. Contradiction
      - c'=c. This also implies w=w'.
    - Define f:V->F by u = w+cv -> c. f is a linear function. (Omit proof.)

- Lemma. f,g linear functionals on V. $g=cf$ for c in F iff N_g contains N_f.
- Theorem 20. $g,f_1,\ldots,f_r$ linear functionals on V with null spaces $N_g,N_{f_1},\ldots,N_{f_r}$. Then g is a linear combination of $f_1,\ldots,f_r$ iff N contains $N_1\cap\ldots\cap N_r$.
- Proof: omit.