Abstract

Joint work with Gyeseon Lee

Abstract: Real projective structures are given as projectively flat structures on manifolds or orbifolds. Hyperbolic structures form examples. Deforming hyperbolic structures into a family of real projective structures might be interesting from some perspectives. We will try to find good projective invariants to deform projective 3-orbifolds with triangulations and obtain some deformations of reflection groups based on tetrahedra, pyramids, octahedra, and so on. This fixes mistaken examples in my paper JKMS 2003. (We will give some introduction to this area of research in the talk.)

Some references

- D. Sullivan, W. Thurston, Manifolds with canonical coordinate charts: some examples. Enseign. Math. (2) 29 (1983), no. 1-2, 15–25.

- E. Vinberg, V. Kac, Quasi-homogeneous cones. (Russian) Mat. Zametki 1 (1967) 347–354.

- S. Choi, W. Goldman,The deformation spaces of convex $\mathbb{RP}^2$-structures on 2-orbifolds. Amer. J. Math. 127 (2005), no. 5, 1019–1102.

- S. Choi, The deformation spaces of projective structures on 3-dimensional Coxeter orbifolds. Geom. Dedicata 119 (2006), 69–90

- S. Choi, Geometric structures on low-dimensional manifolds. J. Korean Math. Soc. 40 (2003), no. 2, 319–340.

Outline

Contents

1 Preliminary

1.1 Erlangen program

Some history

- Greeks started projective geometry and Mobius geometry.
- Klein proposed Erlangen program classifying all geometry together. (Many geometry appears as restricted form of projective geometry just as many Lie groups are subgroups of linear groups.)
- Many mathematicians became interested in lattice subgroups of Lie groups as they are useful in many parts of mathematics. (These subgroups are rigid and often unique in many cases particularly in rank two cases, and hence they are very good objects.)

Erlangen program of Klein

- A Lie group is a set of symmetries of some object forming a manifold.
- Klein worked out a general scheme to study almost all geometries...
- Klein proposed that a "geometry" is a space with a Lie group acting on it transitively.
- Essentially, the transformation group defines the geometry by determining which properties are preserved.

Erlangen program of Klein

- We extract properties from the pair:
  - For each pair (X, G) where G is a Lie group and X is a space.
  - Thus, Euclidean geometry is given by $X = \mathbb{R}^n$ and $G = Isom(\mathbb{R}^n) = O(n) \cdot \mathbb{R}^n$.
  - The spherical geometry: $G = O(n+1, \mathbb{R})$ and $X = \mathbf{S}^n$.
  - The hyperbolic geometry: $G = PO(n, 1)$ and X upper part of the hyperboloid $t^2 - x_1^2 - \cdots - x_n^2 = 1$. (Another representations: For $n = 2$, $G = PSL(2, \mathbb{R})$ and X the upper half-plane. For $n = 3$, $G = PSL(2, \mathbb{C})$ and X the upper half-space in $\mathbb{R}^3$.)

Erlangen program of Klein

- The above are rigid types with Riemannian metrics. We go to flexible geometries.
- Lorentzian space-times.... de Sitter, anti-de-Sitter,...
- The conformal geometry: $G = SO(n+1,1)$ and X the celestial sphere in $\mathbb{R}^{n+1,1}$.
- The affine geometry $G = SL(n,\mathbb{R}) \cdot \mathbb{R}^n$ and $X = \mathbb{R}^n$.
- The projective geometry $G = PGL(n+1,\mathbb{R})$ and $X = \mathbb{R}P^n$. (The conformal and projective geometries are "maximal" geometry in the sense of Klein; i.e., most flexible.)

1.2 Projective geometry

Some projective geometry notions.

- The complement of a hyperspace in $\mathbb{R}P^n$ can be identified with an affine space $\mathbb{R}^n$. (Moreover, The affine transformations extend to a projective automorphisms and projective automorphism acting on $\mathbb{R}^n$ restricts to affine transformations. Let us call the complement an *affine patch*.)
- A convex polytope in $\mathbb{R}P^n$ is a convex polytope in an affine patch.
- $\mathbb{R}P^n$ has homogeneous coordinates $[x_0, ..., x_n]$ so that

$$[\lambda x_0, ..., \lambda x_n] = [x_0, ..., x_n] \text{ for } \lambda \in \mathbb{R} - \{0\}.$$

- Projective subspaces always can be described as vector subspaces in homogeneous coordinates.

Some projective geometry notions.

- We can find homogeneous coordinates so that a simplex has vertices

$$[1, 0, ..., 0], .., [0, 0, .., 1].$$

- A projective automorphism is the map $x \mapsto Ax$ for $A \in GL(n,\mathbb{R})$. (Classically, they were known as collineations, or projectivities,...)
- The group of projective automorphism of $\mathbb{R}P^n$ is $PGL(n+1,\mathbb{R}) = GL(n+1,\mathbb{R})/\sim$.
- One looses notions such as angles, lengths... But these are replaced by other invariants such as cross ratios. We also gain duality.
- The oriented projective geometry $(\mathbf{S}^n, SL_\pm(n+1,\mathbb{R}))$ is better.

Projective geometry

- Many geometries are sub-geometries of projective (conformal) geometry. (The two correspond to maximal finite-dimensional Lie algebras acting locally on manifolds...)
  - Hyperbolic geometry: H^n imbeds in $\mathbb{R}P^n$ and $PO(n,1)$ in $PGL(n+1,\mathbb{R})$ in canonical way.
  - spherical, euclidean geometry, anti-de-Sitter geometry
  - The affine geometry $G = SL(n,\mathbb{R}) \cdot \mathbb{R}^n$ and $X = \mathbb{R}^n$, flat Lorentizan geometry, ...
  - Six of eight 3-dimensional geometries: euclidean, spherical, hyperbolic, nil, sol, $\tilde{SL}(2,\mathbb{R})$ (B. Thiel)
  - $\mathbf{S}^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$. (almost)

2 Manifolds and orbifolds with geometric structures

2.1 Definitions

Manifolds and orbifolds with geometric structures

- A manifold is a familar object.
- An *orbifold* can be thought of as a quotient space of manifold by a discrete group (infinite mostly). The action may have fixed points. The quotient topological space with local information form the orbifold.
- The orbifold has local charts based on open subsets of euclidean space with finite group actions. This defines the orbifold.

Manifolds and orbifolds with geometric structures

- An (X,G)-*geometric structure* on a manifold or orbifold M is given by a maximal atlas of charts to X where the transition maps are in G.
- This equips M with all of the local (X,G)-geometrical notions.
- If the geometry admits notions such as geodesic, length, angle, cross ratio, then M now has such notions...
- So the central question is: which manifolds and orbifolds admit which structures and how many and if geometric structures do not exist, why not? (See Sullivan-Thurston paper 1983 Enseign. Math.)

Deformation spaces of geometric structures on manifolds and orbifolds

- In major cases, $M = \Omega/\Gamma$ for a discrete subgroup Γ in the Lie group G and an open domain Ω in X.
- Thus, X provides a global coordinate system and the classification of discrete subgroups of G provides classifications of manifolds or orbifolds with (X, G)-structures.
- Here Γ"is" the fundamental group of M. Thus, a geometric structure can be considered a discrete representation of $\pi_1(M)$ in G up to conjugation by elements of G.
- Often, there are cases when Γ is unique up to conjugations and there is a unique (X, G)-structure. (Rigidity)
- Given an (X, G)-manifold (orbifold) M, the deformation space $D_{(X,G)}(M)$ is locally homeomorphic to the G-representation space $Hom(\pi_1(M), G)/G$ of $\pi_1(M)$. (M should be a closed manifold. Otherwise the geometric structures and homomorphisms should have boundary conditions.)

Deformation spaces of geometric structures

- Closed surfaces have either a spherical, euclidean, or hyperbolic structures depending on genus. We can classify these to form deformation spaces such as Teichmuller spaces.
  - A hyperbolic surface equals H^2/Γ for the image Γ of the representation $\pi \to PSL(2, \mathbb{R})$.
  - A Teichmuller space can be identified with a component of the space $\mathrm{Hom}(\pi, PSL(2, \mathbb{R}))/\sim$ of conjugacy classes of representations. The component consists of discrete faithful representations.

2.2 Manifolds and orbifolds with (real) projective structures

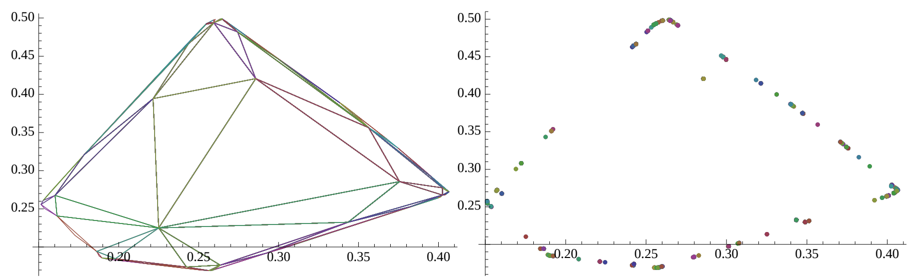
Real projective manifolds: how many?

- A manifold (orbifold) with a real projective structure is a manifold (orbifold) with $(\mathbb{R}P^n, PGL(n + 1, \mathbb{R}))$-structure.
- By the Klein-Beltrami model of hyperbolic space, we can consider H^n as a unit ball B in an affine subspace of $\mathbb{R}P^n$ and $\mathrm{PO}(n + 1, \mathbb{R})$ as a subgroup of $\mathrm{PGL}(n + 1, \mathbb{R})$ acting on B. Thus, a (complete) hyperbolic manifold (orbifold) has a projective structure. (In fact any closed surface has one.)
- 3-manifolds (orbifolds) with six types of geometric structures have real projective structures. (Up to coverings of order two, 3-manifolds (orbifolds) with eight geometric structures have real projective structures.)

Real projective manifolds: how many?

- *Here,* $M = \Omega/\Gamma$ *for some domain in* $\mathbb{R}P^n$ *where a discrete subgroup* Γ *of* $\mathrm{PGL}(n+1,\mathbb{R})$ *acts on.*
- A question arises whether these “induced" projective structures can be deformed to purely projective structures.
- Deformed projective manifolds from closed hyperbolic ones are convex in the sense that any path can be homotoped to a geodesic path. (Koszul’s openness)
- For open projective manifolds, this convexity issue is still unknown in some cases.

Real projective manifolds: how many?

- A $(3,3,5)$-turnover example. (A mathematica file in my homepage.)
- 

Manifolds with (real) projective structures

- Cartan first considered real projective structures on surfaces as projectively flat torsion-free connections on surfaces.
- Chern worked on general type of projective structures in the differential geometry point of view.
- The study of affine manifolds precedes that of projective manifolds.
  - There were extensive work on affine manifolds, and affine Lie groups by Chern, Auslander, Goldman, Fried, Smillie, Nagano, Yagi, Shima, Carriere, Margulis,... They are related to affine Lie groups, flat Lorentzian manifolds, de Sitter, anti-de-Sitter spaces,...
  - There are outstanding questions such as the Chern conjecture, Auslander conjecture,... I think that these areas are completely open... These questions might be related to bounded cohomology theory..
  - An aside: Conformal manifolds are extensively studied by PDE methods.

Projective manifolds: how many?

- In 1960s, Benzecri started working on strictly convex domain Ω where a projective transformation group Γ acted with a compact quotient. That is, $M = \Omega/\Gamma$ is a compact manifold (orbifold). (Related to convex cones and group transformations (Kuiper, Koszul,Vinberg,...)). He showed that the boundary is either C^1 or is an ellipsoid.
- In 1970s, Kac and Vinberg found the first nonhyperbolic example for $n = 2$ for a triangle reflection group associated with Kac-Moody algebra.
- Recently, Benoist found many interesting properties such as the fact that Ω is strictly convex if and only if the group Γ is Gromov-hyperbolic.
- The real projective structures on surfaces and 2-orbifolds are completely classified by Goldman and —. The deformations space is a countable disjoint union of cells. (converting maple files to mathematica.)

2.3 Projective 3-manifolds and deformations

Projective 3-manifolds and deformations

- We saw that many 3-manifolds, including hyperbolic ones, have projective structures.
- We wish to understand the deformation spaces for higher-dimensional manifolds and so on.
- There is a well-known construction of Apanasov using bending: deforming along a closed totally geodesic hypersurface.
- Johnson and Millson found deformations of higher-dimensional hyperbolic manifolds which are locally singular. (also generalized bending constructions)

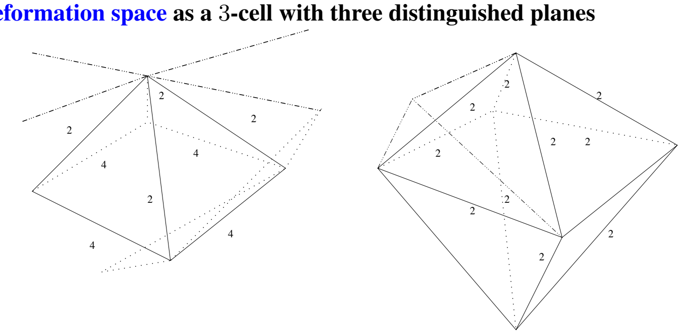
Projective 3-manifolds and deformations

- Cooper,Long and Thistlethwaite found a numerical (some exact algebraic) evidences that out of the first 1000 closed hyperbolic 3-manifolds in the Hodgson-Weeks census, a handful (10-15?) admit non-trivial deformations of their $SO^+(3,1)$-representations into $SL(4,\mathbb{R})$; each resulting representation variety then gives rise to a family of real projective structures on the manifold. (For example $vol3$.)
- Some class of 3-dimensional reflection orbifold that have positive dimensional smooth deformation spaces. These includes reflection group based on "orderable polytopes" some of which are compact hyperbolic polytopes. We worked on prism (Benoist), pyramid, and octahedrons. These are relatively easy, and some are solvable by geometry only.
- The flexibility indicates A. Weil type harmonic form arguments fail here. Also, it is uncertain if the deformations are from bending constructions.

Reflection orbifold.

- 3-orbifolds with base space is a 3-dimensional polytope and sides are silvered with edge orders integers ≥ 2.
- We can study the deformation space using classical geometry.
  - An order 2 edge means that each of the fixed points of reflection of the adjacent sides lies in the planes extending the other sides.
  - The order p for $p > 2$, edge means that the planes containing the fixed point of the reflections of the adjacent sides and the adjacent sides themselves has the cross ratio $2 + 2\cos 2\pi/p$.
  - Using these ideas, we can determine the deformation space for small orbifolds such as orbifolds based on tetrahedron, pyramid with edges orders 2 or 4, and the octahedron with edge orders 4.

The pyramid deformation space as a 2-cell with four distinguished arcs and the octahedron deformation space as a 3-cell with three distinguished planes



Projective 3-manifolds and deformations

- The polytope with edge orders given is *orderable* if faces can be given order so that each face contains less than 4 edges which are edges of order 2 or edges in a higher level face. We are trying to classify these types of polytopes which are hyperbolic.
- The reflection groups based on orderable polytopes: the dimension is $3f - e - e_2$ and that the local structure of the deformation spaces is smooth.
- Gyeseon Lee and I have worked out some numerical results for 3-dimensional reflection orbifolds based on cubes and dodecahedrons deformed from hyperbolic ones. (These are not orderable and Lee will talk about these in another talk.)

3 Cross ratios and projective invariant approach in 2-D

3.1 Cross-ratios and Goldman invariants of four adjacent triangles.

Cross-ratios and Goldman invariants of four adjacent triangles in $\mathbb{R}P^2$.

- A *cross ratio* is defined on a collection of four points x, y, z, t, at least three of which are distinct, on $\mathbb{R}P^1$ or on an 1-dimensional subspace of $\mathbb{R}P^n$.
- Suppose we can give a homogeneous coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1]$, then $t = [b, 1]$ for some b. b is the cross ratio $b(x, y, z, t)$.
- $$[y, z, u, v] = \frac{\bar{u} - \bar{y}}{\bar{u} - \bar{z}} \frac{\bar{v} - \bar{z}}{\bar{v} - \bar{y}}$$
- $b(x, y, z, t) = b(x, w, z, t)b(w, y, z, t)$, $b(x, y, z, t) = b(y, x, t, z) = b(z, t, x, y) = b(t, z, y, x) = 1/b(x, y, t, z)$.

Cross-ratios and Goldman invariants of four adjacent triangles in $\mathbb{R}P^2$.

- Consider now a triangle T_0 with three triangles T_1, T_2, T_3 adjacent to it. Suppose their union is a domain in $\mathbb{R}P^2$. Then we can regard T_0 to have vertices $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$. (See Goldman JDG 1990).
- To obtain projective invariants, we act by positive diagonal matrices and find which are invariant.
- The complete invariants are cross ratios and two other *Goldman σ-invariants*:

  $$\rho_1 = b_3c_2, \rho_2 = a_3c_1, \rho_3 = a_2b_1, \sigma_1 = a_2b_3c_1, \sigma_2 = a_3b_1c_2$$

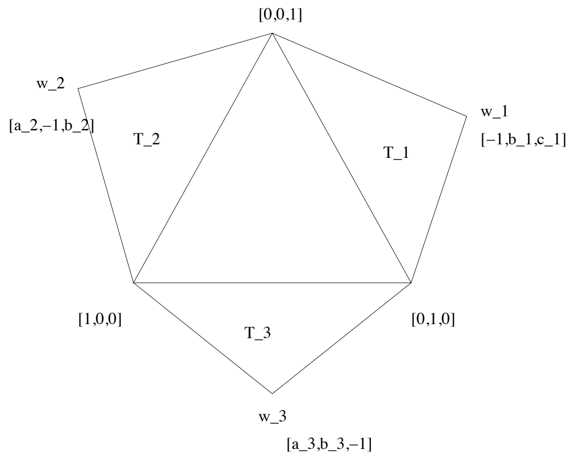
  with complete relations

  $$\rho_1\rho_2\rho_3 = \sigma_1\sigma_2. \tag{1}$$

  No coordinates of w_i are zero. All the projective invariants are cyclic invariants first given by Kac and Vinberg.

Cross ratios and other invariants

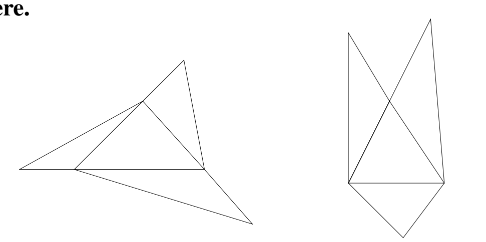
The coordinates can be considered slopes in appropriate affine chart with coordinate system

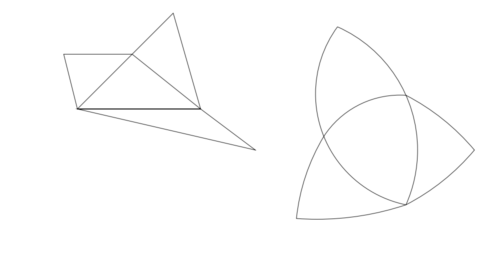


Cross-ratios and Goldman σ-invariants of four adjacent triangles in $\mathbb{R}P^2$.

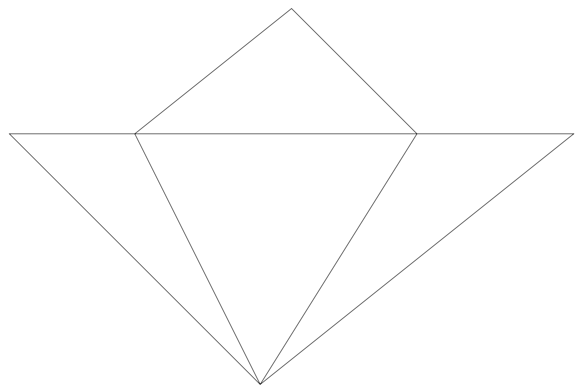
- There is a one-to-one correspondence between the space of invariants satisfying equation 1 and the space of projectively equivalent class of the four triangles with nonzero conditions on coordinates. (*projective invariant coorespondence.*)
- Note that if the edges of T_i do not extend one of T_0, then the invariants are never 0. But if they do, then cross ratios and some Goldman σ-invariant will be zero.
- There is one configuration, where the invariants do not determined the configuration.

Some "allowable configurations". We can extend projective invariance correspondence here.





The exceptional case. This does not occur if we restrict that the directions at vertex be either four or two and not three.



3.2 Surfaces and 2-orbifolds with real projective structures.

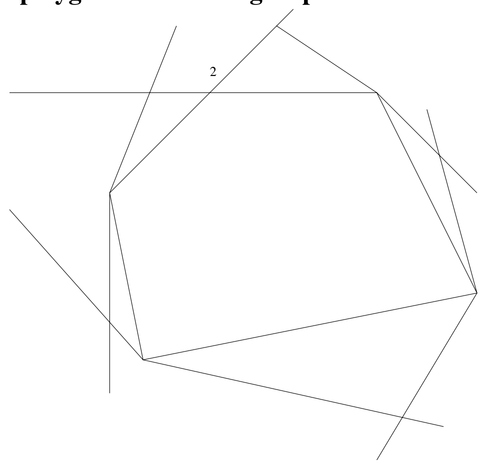
Surfaces and 2-orbifolds with real projective structures.

- To classify convex real projective structures on a orientable closed surface of genus > 1, Goldman decomposed the surface into pairs of pants.
- Each pair of pants decomposes into a union of two triangles.
- In the universal cover, the union of four adjacent triangles determine the pair of pants completely. Actually, a pair of four adjacent triangles determine everything.
- After this step, we glue back the results. The gluing parameters contribute to the dimension.

Surfaces and 2-orbifolds with real projective structures.

- For 2-orbifolds, we again classified convex projective structures using projective invariants by decomposing into elementary 2-orbifolds. Then we determined the deformation space for each piece and glued back the result. (See Goldman,—)
- If $p, q, r \geq 2$ and $1/p + 1/q + 1/r < 1$, a p, q, r turn-over has as a real projective deformation space diffeomorphic to $\mathbb{R}^2$. (A mathematica file will be uploaded.)
- A reflective polygonal orbifold with at least four vertices has a real projective deformation space diffeomorphic to $\mathbb{R}^{v-v_2}$ where v is the number of vertices and v_2 is the number of vertices of order 2.
- A reflective triangular orbifold of p, q, r order has a real projective deformation space diffeomorphic to $\mathbb{R}$.

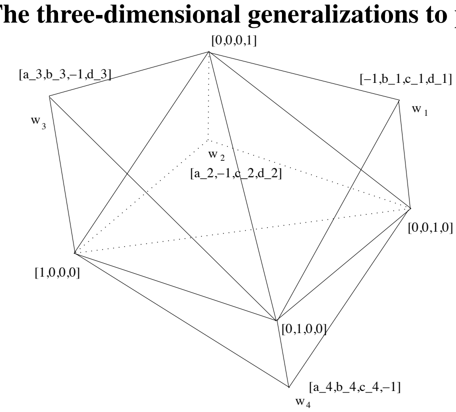
The polygonal reflection group



4 Cross ratios and projective invariant approach in 3-D

4.1 The three-dimensional generalizations of projective invariants

The three-dimensional generalizations to projective invariants



- Suppose now that we have a tetrahedron T_0 and four other tetrahedrons T_1, T_2, T_3, T_4. We can put T_0 to have vertices $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, and $[0,0,0,1]$.
- To obtain projective invariants, we act by positive 4×4-diagonal matrices and find which are invariant.
- Assuming these coordinates are nonzero. Cross ratios satisfying $\rho_{ij} = \rho_{ji}$ are

invariants.

$$
\begin{aligned}
\rho_{12} &= c_4 d_3 & \rho_{13} &= b_4 d_2 & \rho_{14} &= b_3 c_2 \\
\rho_{23} &= d_1 a_4 & \rho_{24} &= c_1 a_3 & \rho_{21} &= c_4 d_3 \\
\rho_{34} &= a_2 b_1 & \rho_{31} &= d_2 b_4 & \rho_{32} &= d_1 a_4 \\
\rho_{41} &= b_3 c_2 & \rho_{42} &= a_3 c_1 & \rho_{43} &= a_2 b_1
\end{aligned} \tag{2}
$$

ρ_{ij} is the cross ratio of four planes through an edge ij.

- For each vertex v_i of T_0, we can form a projective plane $\mathbb{R}P_i^2$ considering lines through it. Then the configuration gives us a central triangle and three adjacent triangles at each vertex. Thus, we have σ-invariants σ_i, σ_i' satisfying

$$\sigma_i \sigma_i' = \rho_{ij}\rho_{ik}\rho_{il} \text{ for each } i. \tag{3}$$

- We compute

$$
\begin{aligned}
\sigma_1 &= b_3 c_4 d_2 & \sigma_1' &= b_4 c_2 d_3 \\
\sigma_2' &= a_3 c_4 d_1 & \sigma_2 &= a_4 c_1 d_3 \\
\sigma_3 &= a_2 b_4 d_1 & \sigma_3' &= a_4 b_1 d_2 \\
\sigma_4' &= a_2 b_3 c_1 & \sigma_4 &= a_3 b_1 c_2
\end{aligned} \tag{4}
$$

.

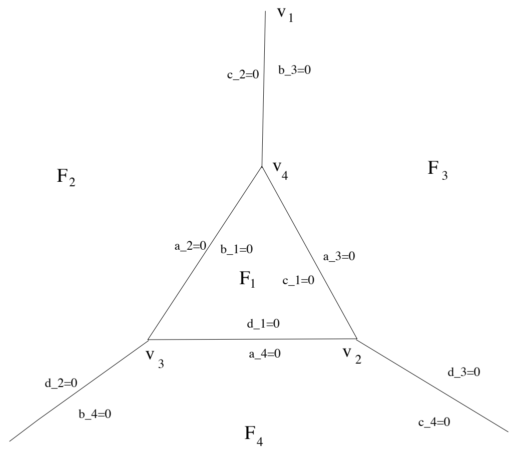
- We finally have a relation

$$\sigma_1\sigma_2\sigma_3\sigma_4 = \rho_{14}\rho_{24}\rho_{34}\rho_{23}\rho_{13}\rho_{23} = \sigma_1'\sigma_2'\sigma_3'\sigma_4' \tag{5}$$

- There is a one-to-one correspondence between the space of projective invariants satisfying the equations 3 and 5 and the space of projective equivalence classes of five tetrahedron with nonzero coordinate conditions. (The *holographic correspondence*)

- Note that if the edges of T_i do not extend one of T_0, then the invariants are never 0. But if they do, then cross ratios and some Goldman σ-invariant will be zero.

The various possibilites for our configurations

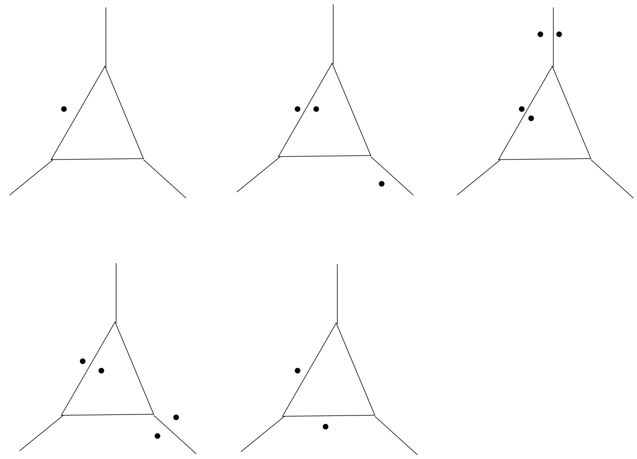


4.2 Non-generic cases.

Non-generic cases.

- We can let some coordinates be zero. Then some or all of the projective invariants can become zero.
- One example is setting $c_4 = d_3 = 0$ and $a_2 = b_1 = 0$ and other variables nonzero. Then two cross ratios are zero and four cross ratios are not zero. The Goldman invariants are zero. Fixing the four cross ratios and Goldman invariants zero, we obtain a one-dimensional projectively inequivalent configurations. Thus, these invariants are not enough in this case.
- Such cases are called *exceptional* cases or *nonholographic* cases.
- There are other situations, where the 2-dimensional configuration seen from a vertex is exceptional.
- There are many different types of configurations up to hundreds. The question is how to classify "holographic ones" and "nonholographic ones".

Some holographic (top three) and nonholographic (bottom two) examples



4.3 3-manifolds with real projective structures.

Projective 3-manifolds described by projective invariants

- We will only look at triangulations so that each edge meets more than four faces.
- We triangulate a compact projective 3-manifold. Look at the vertex link sphere.
- The spheres are triangulated. We can read invariants of any central triangle with four adjacent ones: the cross ratios and two Goldman σ-invariants.
- They satisfy equations 1, 4, and 5. (We should avoid nonholographic situations.)
- If the 3-manifold has a hyperbolic structure, the corresponding projective structure will be described by invariants.

- Denoting $v_{i,j}$ the vertex of the link at vertex v_i meeting with the edge e_j and ρ_{ijk} the cross ratios around the vertex for $k = 1, \ldots, f_j$, we have $\rho_{ijk} = \rho_{jik}$.
- Denoting the triangle T_{ik}, the intersection of a tetrahedron T_i with a linking sphere at v_k. Let $v_{k_1}, v_{k_2}, v_{k_3}, v_{k_4}$ the vertex of T. We have

$$\sigma_{T_{i,k_1}}\sigma_{T_{i,k_2}}\sigma_{T_{i,k_3}}\sigma_{T_{i,k_4}} = \sigma'_{T_{i,k_1}}\sigma'_{T_{i,k_2}}\sigma'_{T_{i,k_3}}\sigma'_{T_{i,k_4}} = \rho_1\rho_2\rho_3\rho_4\rho_5\rho_6$$

  the product of six cross ratios of the edges of T_i.
- Thus, we have a 2-dimensional description. (holographic representation)
- We can drop σ' from consideration in holographic situations.

The converse

- Suppose M be a 3-manifold with a triangulation.
- We are given geometric triangulations of vertex spheres which satisfy the above equations.
- Assume that these configurations of tetrahedrons are all holographic. Then there exists a real projective structure on M.
- Solving the equations, we can parameterize the small neighborhood of the deformation space of real projective structures on M. Hence, the deformation from hyperbolic structures can be understood.
- The question is how to avoid the zero invariants. In manifold cases, this seems to be avoidable. (We wish to be in holographic situations.)

Example: the figure eight knot complement.

- We can extend above to the case with ideal triangulation exists. The complement of the union of ideal vertices is an open manifold. Then the ideal vertex can have a torus as a link.
- The figure eight knot complement is triangulated by two ideal tetrahedra. Here, there is only a torus which is the link of the unique ideal vertex.
- The torus has a deformation space of dimension 4. The torus is triangulated to eight triangles. There are three free vertices.
- We (J.R. Kim) showed that there is six cross ratio equations associated with two edges and two Goldman invariant equations. One equation is redundant.
- This describes the deformation from the hyperbolic structure. However, this does not include the Dehn surgery.

- The number of variables is $4 + 2 \times 3$.
- We determined that the local dimension of deformation is $3 = 10 - 7$ using numerical computations of Jacobians.
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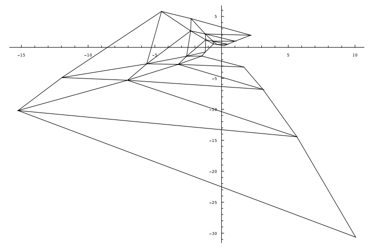


Figure-eight knot complement

4.4 3-orbifolds with real projective structures.

The generalization to orbifolds

- Now we try to extend this to 3-orbifolds. Let us triangulate the orbifold M so that singularities of orbifold lies in the union of right dimensional faces. (equivariant trianguations)
- Assume that every vertex of the triangulation has linking orbifold which is a quotient orbifold $\mathbf{S}^2/F$ or T^2/F.
- Given a real projective 3-orbifold with a triangulation (possibly with ideal vertices), we obtain a triangulation of linking 2-orbifolds.
- Looking at the universal cover of the 2-orbifolds. We recover cross-ratios and invariants. They satisfy equations as above.
- Here, we might not be able to avoid zero invariants. (We wish to be in holographic situations.)
- The hyperbolic structure will be described by these invariants.

The converse

- Conversely, given triangulations of linking 2-orbifolds satisfying the equations so that we have only holographic five tetrahedral configurations, we can construct the real projective structures.
- Computing the deformation spaces in theory:
  - First determine the real projective deformation space of the vertex linking 2-orbifolds.
  - Put in vertices. Find the total configuration space with points.
  - Find all projective invariants from the quadruples of triangles.
  - Find complete independent projective invariants and find all relations. (Finding good parameterization coordinates)
  - Now write 3-dimensional equations and solve them.

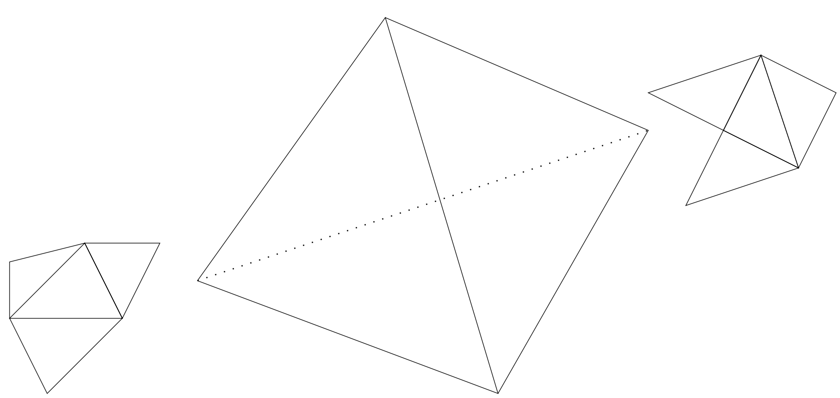
- Thus, we can determine the local and global deformation spaces in this manner.
- We can understand the local deformation from hyperbolic structures.

The case of triangulated-disk vertex-link orbifolds with relations.

- If a triangle has two edge in the singular locus. Then the vertex has to have a cross ratio $2 + 2\cos 2\pi/n$ where n is the vertex order.
- If two triangles divides the angle with dihedral group action, then the two cross ratios satisfy $(\rho_1 - 2)(\rho_2 - 2) = 2 + 2\cos 2\pi/n$.
- If there are more triangles at a vertex, there are more complicated relations.
- If a vertex of three triangles lies in a silvered line, then there are more complicated relations between their cross ratios.
- The Goldman invariants often satisfy relations.
- Since all these are very complex, you should try to do numerical study instead. Or is there a simpler theory?

Testing out the theory: example 1: the tetrahedral orbifolds

- Tetrahedrons with edge orders.
  - The cross ratio $(1 + cos(2\pi/p))/2$ is determined by order p.
  - Here the link orbifolds are triangular disk orbifolds of corresponding orders with no interior vertices. Thus, their deformation space is a real line parameterized by Goldman σ-invariants or a singleton when they have some order 2.
  - Suppose that all edge orders are ≥ 3. Thus, four Goldman σ-invariants satisfying one equation parameterize the space. Hence our space is a three-dimensional cell.

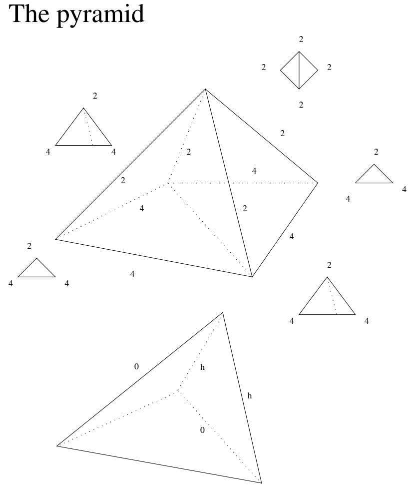


- Suppose that there are two edges opposite each other of order 2 and others have higher orders. (Other edge orders in $3, 4, 5, 6$. This is a compact hyperbolic reflective orbifold)
- Then these correspond to a nonholographic situation. The cross ratios of four other edges are determined by orders. The two cross ratios and the Goldman σ-invariants are zero.
- The deformation space is a real line parameterized by a 3-dimensional projective invariant (a cylic invariant of Kac and Vinberg).
- In all other cases, there is a unique orbifold and the deformation space is a point. They are all holographic.

Example 2: Pyramid

- Pyramid with orders $2, 2, 2, 2$ on edges from the top vertex and orders 4 on the opposite edges.
- Here we can divide into two tetrahedrons.
- Each of T_1 and T_2 has two edges with a cross ratio $= 0$. Other edges have cross ratios given by order 4 or have harmonic cross ratio 2. There are only two nonzero σs which are independent. The two σ-invariants are complete invariant variables. (Thus here there are no equations to study)
- In generic situation, they are holographic.
- Thus, the choice of points in the triangle edges determine the deformation space. Hence, the deformation space is a union of two 2-disks.
- For nongeneric choice, we get 1-dimensional line. Their union is again a disk with a distinguished line through it.

The pyramid

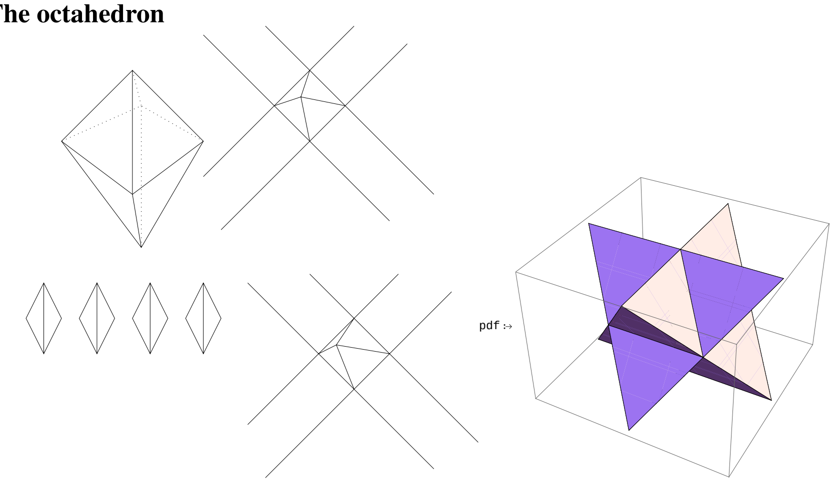


- If pyramid is given edge orders all > 2, then vertex orbifolds varies in a space of dimension $4 + 4 * 1$ and the two points can move $+2$ and hence there are 10 variables. The number of equations are two cross ratio equations for two edges from the top vertex and two tetrahedral equations. Thus, $10 - 4 = 6$ should be the "dimension". We have to show that differentials of the equations is of full rank.

The example 4: Octahedrons

- *The octahedron with all edge orders* 2. Then there is a three-dimensional space of deformations of the octahedrons in projective 3-space.
- The reflection group is determined by the stellations. Hence, the deformation space is a cell of dimension 3.
- Here we can divide into four tetrahedrons.
- There are six quadrilateral reflection orbifolds. The top and the bottom ones are divided into four triangles with an interior vertex. The side ones are all divided into two. The deformation spaces are all rigid.
- Hence, the configuration space is a cell of dimension four. Thus, the cross ratio at the center point and σ invariants are important.

The octahedron



Octahedrons

- We can take one σ from the top and the bottom orbifolds. The two cross ratios and two σ-invariants from the top and the bottom is the complete invariant variables for 2-dimensional considerations.
- The only equation is the cross ratio equation for two center vertices. (There is a 0 cross ratio for each holographic tetrahedron)

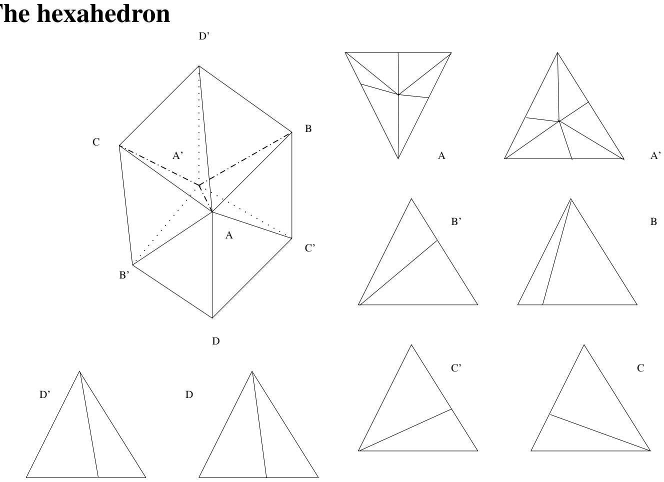
- Hence, we obtain 3-cells as the deformation space. This is for the generic situation.
- The total space is a 3-cell with three distinguished planes meeting transversally. The planes form the nongeneric locus.

- It the edge orders are all > 2, then each quadrilateral orbifold has the deformation space a cell of dimension 4. Thus, the total configuration space is a cell of dimension $24 + 4 = 28$.
- The relations are from cross ratios of edges from the top and the bottom vertices and the sigma equations of four tetrahedron and the cross ratio matching condition for the central vertex.
- Thus, our space should be a cell of dimension $28 - 8 - 4 - 1 = 15$. (This can be computed geometrically also.)

The example 5: Hexahedron

- Consider the hexahedron with all edge orders equal to 3.
- We decompose into six tetrahedrons sharing a central diagonal.
- Then the vertex correspond to $3, 3, 3$-triangle groups. Two A, A' are triangulated into six triangles (barycentric subdivision form) and the rest is divided into two triangles.
- The projective structure on each triangle group is classified by a line. The total dimension is 8 for the triangle group deformations, 4 for two central vertices in A and A' and 6 for vertices in the six edges of A and A' and 6 for vertices in the edges of the rest of the triangles. Total dimension is 24.

The hexahedron



The hexahedron

- The equations are 3 from the diagonal AA' and 6 from the cross ratio equation from six sides of the hexahedron and 6 from σ invariants of the six tetrahedra.
- Thus, we should have $24 - 15 = 9$ dimensional space.
- This agrees with $3f - e = 18 - 12 = 6$ with 3 dimensional space of hexahedra.
- But we do need to investigate the matrix rank.