Spherical triangles and the two components of the SO(3)-character space of the fundamental group of a closed surface of genus 2

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We use geometric techniques to explicitly find the topological structure of the space of SO(3)-representations of the fundamental group of a closed surface of genus 2 quotient by the conjugation action by SO(3). There are two components of the space. We will describe the topology of both components and describe the corresponding SU(2)-character spaces by parametrizing them by spherical triangles. There is the sixteen to one branch-covering for each component, and the branch locus is a union of 2-spheres or 2-tori. Along the way, we also describe the topology of both spaces. We will later relate this result to future work into higher-genus cases and the SL(3, ℝ)-representations.

It is well-known that the set of homomorphisms of a finitely presented group Γ to a Lie group G has a topology as a subspace of G^n, where n is the number of generators of Γ, defined by a number of polynomial equations corresponding to the relations. Let π be the fundamental group of a compact surface and G a compact Lie group. Then the space of homomorphisms $\mathrm{Hom}(\pi, G)$ is an algebraic set for which G acts by conjugation. Generally, the quotient space by this action $\mathrm{Hom}(\pi, G)/G$ is a semi-algebraic set, called the *G-character space* of π.

The study of this space constitutes an interesting subject with a long history and involves many other principal fields such as algebraic geometry, topology, geometry, and mathematical physics, particularly when the Lie group is the special unitary group SU(2). (See Goldman [17] for a part of the beginning of the subject.)

We denote by $\mathrm{SL}(3, \mathbb{R})$ the special linear group, and SO(3) the special orthogonal group for $\mathbb{R}^3$ with a fixed Euclidean metric. The SO(3)-character space will embed inside the corresponding $\mathrm{SL}(3, \mathbb{R})$-character space. The two components will embed in the two components of the later space respectively and the Hitchin-Teichmuller component will not contain any of the SO(3)-characters. We plan to use our work in this paper to describe the topology of the $\mathrm{SL}(3, \mathbb{R})$-character space.

In this paper, we will find a detailed cell-decomposition of the SO(3)-character space of the fundamental group $\pi_1(\Sigma)$ of a closed orientable genus 2-surface Σ by parametrizing the representations by spherical triangles. The cell-decomposition is

one adopted to the later study of the $SL(3, \mathbb{R})$-character space for $\pi_1(\Sigma)$, which is the purpose of this paper. This cellular decomposition allows us to obtain the topological characterization of the SO(3)-character space of $\pi_1(\Sigma)$. The topology of the two components of this space is described.

The classical work of Narashimhan, Ramanan, Seshadri, Newstead and so on [28], [29],[31] shows that the space of SU(2)-characters for a genus-two closed surface is diffeomorphic to CP^3. We will show that this space 16 to 1 branch-covers the component of our space containing the identity representation. The other component is also branch-covered by the space of SU(2)-characters for a punctured genus two surface with the puncture holonomy $-I$ in a 16 to 1-manner. We will show that this second space is a real 6-dimensional manifold, so-called an octahedral manifold, obtained by a completion of a toric variety over the octahedron. We will obtain the precise $\mathbb{Z}_2^4$-group action inducing the branch coverings.

Newstead [30] and others worked on determining cohomology rings of moduli spaces arising in this manner. Huebschmann, Jeffrey and Weitsmann [24] [25] researched extensively on the spaces of characters to SU(2), and showed that they have toric structures by finding the open dense sets where the n-dimensional torus act on. Huebshmann [22] also showed that this space branch-covers the SO(3)-character spaces;

One can study this area using the Yang-Mills and Higgs bundle techniques as initiated by Atiyah-Bott [1], Donaldson [12], [13], Hitchin [14], Simpson [32], and Corlette [10]. There are now extensive accomplishments in this area using these techniques. (See Bradlow, Garcia-Prada, and Gothen [5] and [6].)

If we compare this paper to the papers using algebraic geometry or Yang-Mills fields, our method is more elementary and geometric. However, the steps and the details to check seem more here since we are not using already established theories, and also the arguments are not totally geometrical yet. The main point of our method seems to be that we have more direct way to relate the SO(3)-character spaces with the SU(2)-character spaces with cell-structures preserved under the branch-covering map.

For the G-character spaces of free groups for many Lie groups G, see Florentino and Lawton [20]. In some cases, the cohomology groups and explicit topological structures are known when the fundamental group are free groups of rank two and three due to T. Goodwillie [19] (see Baird [2] also).

We also remark that the branching property is of course known by Ramanan [33] and Huebshmann [22]; but we find the exact branch locus here and the finite group action. One can find a much shorter proof but with no results on the cell structures or the branch loci using a generalization of the arguments of the recent paper of Florentino and Lawton [20] for the free-group representations; however, the author is not able to obtain the proof.

A major disadvantage of our method is that we need to make use of the smoothness result of Huebshmann [22]. However, we have worked on a geometric way to

avoid this problem and directly prove our theorems. We will continue our geometric methods and present the method if successfully accomplished. Our method seems to be able to replicate these results. We are currently interested in working on free groups of rank four and so on.

By a *representation*, we mean a group homomorphism into a specified Lie group such as $SO(3)$ or $SU(2)$. By a *character*, we mean an equivalence class of the representation by conjugations by elements of the Lie group. Also, any adjective modifying a representation modifies the corresponding character and vice versa in our definitions by the convention that a character has a property if and only if every representation in the character has the property.

Let Σ be a closed surface of genus 2 and $\pi_1(\Sigma)$ its fundamental group. The space of homomorphisms $\pi_1(\Sigma) \to SO(3)$ admits an action by $SO(3)$ given by

$$h(\cdot) \mapsto g \circ h(\cdot) \circ g^{-1}, \text{ for } g \in SO(3) \text{ and a homomorphism } h.$$

We can realize the set of homomorphisms $\mathrm{Hom}(\pi_1(\Sigma), SO(3))$ as an algebraic subset of $SO(3)^4$ where 4 is the number of generators of $\pi_1(\Sigma)$. We denote by $\mathrm{rep}(\pi_1(\Sigma), SO(3))$ the topological quotient space of the space under the action of $SO(3)$. This is called the *space of* $SO(3)$-*characters* of $\pi_1(\Sigma)$. Sometimes, we say that a representation *belong to* it if the corresponding character belongs to it.

There are two topological components of $\mathrm{rep}(\pi_1(\Sigma), SO(3))$. There is the identity component $\mathcal{C}_0$ containing the equivalence class of the identity representation, i.e., the one sending every element of $\pi_1(\Sigma)$ to the identity. The representations in the other component $\mathcal{C}_1$ correspond to the flat $SO(3)$-bundles with the Stiefel-Whitney numbers equal to $1 \in \mathbb{Z}_2$.
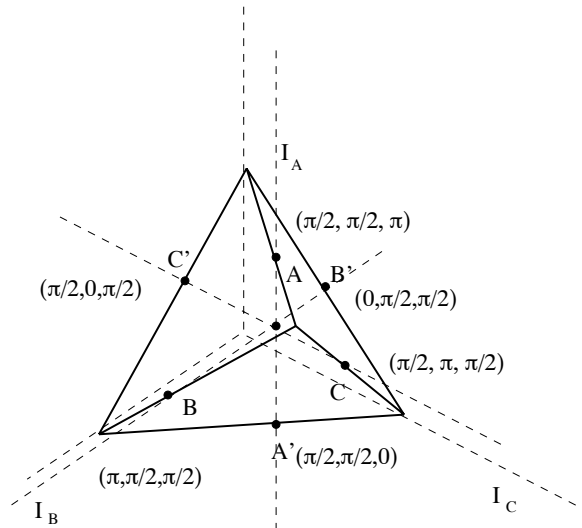

Fig. 1.   The tetrahedron and the Klein four-group-symmetries. The three edges in front are labeled A, B, and C in front and the three opposite edges are labeled A', B', and C'.

4 *The* $\mathrm{SO}(3)$*-character spaces*

We define a solid tetrahedron Δ in the positive octant of $\mathbb{R}^3$ by the equation $x + y + z \geq \pi$, $x \leq y + z - \pi$, $y \leq x + z - \pi$, and $z \leq x + z - \pi$. There is a natural action of the Klein four-group on Δ by isometries generated by three involutions each fixing a maximal segment in Δ (See Figure 1.)

For a natural number n, let T^n denote the n-dimensional torus, i.e., a product of n circles $\mathbf{S}^1$, and let $\mathbb{Z}_2^n$ denote the direct sum of n copies of the cyclic group $\mathbb{Z}_2$ of order 2. For each n, let $\mathbf{S}^n$ denote the unit sphere in Euclidean space $\mathbb{R}^{n+1}$.

We will denote the Klein four-group by V whenever possible. Actually V is isomorphic to $\mathbb{Z}_2^2$, which will be called a Klein four-group as well. By a *double Klein four-group*, we mean a product of two copies of the Klein four-group, which is isomorphic to $\mathbb{Z}_2^4$.

We will realize CP^3 as a torus fibration over Δ where fibers over the interior points are 3-dimensional tori, fibers over the interior of faces 2-dimensional tori, fibers over the interior of edges circles, and the fiber over a vertex a point. (See Section 6.1 for more details.)

Theorem A . Let $\pi_1(\Sigma)$ the fundamental group of a closed surface Σ of genus 2.

  (i) The identity component $\mathcal{C}_0$ of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ is homeomorphic to the quotient space of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ by a double Klein four-group action with nonempty fixed points.
  (ii) $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ is homeomorphic to CP^3.
  (iii) The quotient by the double Klein four-group induces a 16-to-1 branch-covering of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ onto $\mathcal{C}_0$.
  (iv) $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ has an orbifold structure with singularities in a union of six 2-spheres meeting transversally.

More precisely, we will see $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ as CP^3 by inserting into CP^3 the four 3-balls corresponding to the vertices and inserting solid tori at the circles over the interior of edges. The parameters of solid tori over the open edges will converge to 3-balls as they approach the fibers above the vertices. The subspace of abelian characters consist of 2-tori over the interior of faces and the boundary 2-tori of the solid tori over edges and the boundary sphere of the vertex 3-balls.

Remark 0.1. We can describe the branch locus exactly with two Klein four-groups acting differently. The two generates the double Klein four-group action.

  (i) The Klein four-group $\mathbb{Z}_2^2$ is acting on each of the torus fibers with the branch locus equal to the union of six 2-spheres over the six edges of the tetrahedron Δ. Two of the six 2-spheres meet at a point if and only if their edges do so at a vertex. Three of them meet at a unique point if and only if their edges do so.
  (ii) The next Klein four-group $\mathbb{Z}_2^2$ acts on the tetrahedron Δ and inducing the corresponding action on $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ preserving the torus fibration. The branch locus is a union of six 2-spheres over three mutually disjoint

maximal segments in Δ. There are two disjoint 2-spheres over each maximal segment. Over the same segment, the 2-spheres are disjoint, and the spheres over distinct axes meet at a point if their segments do so.

(iii) The spheres in (i) and the spheres in (ii) meet at two points if and only if the corresponding edges of Δ and the maximal segments meet. This gives the complete intersection pattern.

(iv) The images of the spheres are three for the type (ii) spheres and three for the type (iii) ones.

See Remark 7.1 for more details.

Consider a solid octahedron in $\mathbb{R}^3$. Let B^3 be the 3-dimensional unit ball and $\mathbf{S}^3$ the unit sphere in $\mathbb{R}^4$. An *octahedral manifold* is a manifold obtained from the toric variety over the solid octahedron by removing the six singularities and gluing in six submanifolds homeomorphic to $B^3 \times \mathbf{S}^3$ to make it into a manifold.

The octahedral manifold is a torus fibration over an octahedron so that over the interior of the octahedron the fibers are 3-dimensional tori and over the interior of faces the fibers are 2-dimensional tori and over the interior of the edges the fibers are circles and the over the vertex the fibers are 3-spheres. We will define this precisely later.

Let I be the identity matrix in SU(2) and define $-I = (-1)I$. Let Σ_1 denote a surface of genus two with one puncture, and $\mathrm{rep}_{-I}(\pi_1(\Sigma), \mathrm{SU}(2))$ be the quotient space of the subspace of $\mathrm{Hom}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ determined by the condition that the holonomy of the boundary curve $-I$ under the usual conjugation action.

Theorem B .

(i) $\mathcal{C}_1$ is homeomorphic to the double Klein four-group quotient of an octahedral manifold.

(ii) $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ is homeomorphic to the octahedral manifold and branch-covers $\mathcal{C}_1$ in a 16 to 1 manner by an action of $\mathbb{Z}_2^4$ and has a cell structure.

(iii) There is a Klein four-group $\mathbb{Z}_2^2$-action preserving each of the torus fibers of the space, where the branch locus is a union of three mutually disjoint 2-tori. There is another Klein four-group $\mathbb{Z}_2^2$-action on the space where the branch locus is the union of the three 2-tori. They generate a double Klein group action with the branch locus the union of six 2-tori. (See Section 10.2.)

(iv) The quotient of $\mathrm{rep}_{-I}(\pi_1(\Sigma_1, \mathrm{SU}(2))$ under the double Klein four-group action gives us $\mathcal{C}_1$ and has an equivariant T^3-cell decomposition.

The component $\mathcal{C}_1$ was originally studied by Ramanan (See Theorem 1 of [33]) following Newstead's work [31]. Ramanan described the spaces as a finite quotient of the intersections of two quadrics but without details of the topology of the space as was pointed out by Oscar Garcia-Prada to the author. For the SU(2)-character variety $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$, Newstead [31] studied it as the space of stable bundles

of rank two and odd degrees over a curve of genus 2 and he showed that it is home-omorphic to the intersection of two quadrics. (See also Desale and Ramanan [11].) Miles Reid communicated to the author by an e-mail that the author's description as a toric variety with six singular points replaced by three dimensional spheres is homeomorphic to the intersection description. (See his thesis [34] to understand the issues here.)

We briefly outline this paper: We start with $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ for a pair-of-pants P. We show that there is a dense set where the characters are obtainable by gluing two isometric triangles. Every character is obtained by gluing two isometric triangles which may be degenerate ones. The choices involved give us the V-action.

The surface Σ decomposes into two pairs-of-pants. The space $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ is then constructed using the gluing maps between two pairs-of-pants. We obtain the two components in this manner and realize them as quotient spaces of T^3-bundles over a blown-up tetrahedron or a blown-up octahedron under a V-action.

In Section 1, we compactify the space of isometry classes of spherical triangles by adding generalized triangles including hemispheres, segments, and points. This space can be viewed as a tetrahedron truncated at vertices and edges in a parallel manner.

In Section 2, we will describe the SO(3)-character space of the fundamental group of a pair-of-pants by relating spherical triangles with the SO(3)-characters. Each of them is a tetrahedron with the above described action by V. (This is an old idea which also appeared in Thurston's book [35] and similar to Goldman [18].)

In Section 3, we discuss the relationship of SU(2) with SO(3) and find a useful way to describe elements. We redescribe the SO(3)- and SU(2)-character spaces of a free group of rank 2 for the purposes of using the idea in the later sections.

In Section 4, we describe the SO(3)-character space for the fundamental group of the surface Σ of genus 2, which has two components $\mathcal{C}_0$, containing the identity representation, and the other component $\mathcal{C}_1$.

In Section 5, we show how we can consider $\mathcal{C}_0$ as a quotient space of the T^3-bundle over the blown-up tetrahedron above. We will put the details in Appendix A.

In Section 6, we move to the SU(2)-character space of the fundamental group of Σ and we consider the geometric representations of such characters using the spherical triangles again. The character space is again considered a quotient space of the T^3-bundle over the blown-up tetrahedron. The details will be in Appendix B. We determine the topology using the quotient relation. Here, we make use of the smoothness result of this space. (See Huebshmann [22].)

In Section 7, we determine the topology of $\mathcal{C}_0$ and describe the $\mathbb{Z}_2^4$-action on the SU(2)-character space of the fundamental group of Σ to branch-cover $\mathcal{C}_0$.

In Section 8, we now turn our attention to $\mathcal{C}_1$. Here, we show that our space is the quotient space of an octahedron blown-up at vertices times T^3. We describe the equivalence relations.

In Section 9, we show that the space $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ of SU(2)-characters

of the fundamental group $\pi_1(\Sigma_1)$ of the once-punctured genus two surface Σ_1 with the puncture holonomies equal to $-I$ is homeomorphic to an octahedral manifold.

In Section 10, we determine the topology of $\mathcal{C}_1$ and describe the $\mathbb{Z}_2^4$-action on the above manifold to branch-cover $\mathcal{C}_1$.

In Appendix A, we describe the explicit quotient relations for $\mathcal{C}_0$ by going over each of the faces of the blown-up tetrahedron. This will be a long section; however, the ideas here are very repetitive and we write them for the convenience of the readers and for future uses.

In Appendix B, we describe the equivalence relations on for the SU(2)-character space in detail.

1. The geometric limit configuration space

In this section, we will define the solid tetrahedron Δ precisely and the Klein four-group action on it. The above tetrahedron is not enough for our purpose. We need to consider the sequence of spherical triangles converging to the boundary. We will use the geometric-limit configuration spaces for the next section when we are discussing the surface of genus 2.

First, we describe a tetrahedron Δ and the blown-up tetrahedron $\tilde{\Delta}$. Next, we will describe the geometric-limit configuration space of generalized triangles and its metric and topology. We show that the geometric-limit configuration space can be identified with a blown-up tetrahedron by describing each face of $\tilde{\Delta}$ with explicit space of generalized triangles. We show that the Klein four-group V acts on $\tilde{\Delta}$ extending the action on Δ^o. Finally, we present the standard position for generalized triangles.

We remark that the author cannot find a way to identify $\tilde{\Delta}$ as a blown-up tetrahedron in a direct and more algebraic way without using the topological methods below. We believe that there should naturally be a known real algebraic blow-up construction along some divisers to see $\tilde{\Delta}$ as being blown-up from Δ.

1.1. *A blown-up tetrahedron*

By a *rotation*, we mean a transformation which can be locally conjugated to be an orientation-preserving isometry in the unit n-sphere or a Euclidean n-space with a set of fixed point a subspace of codimension two for $n \geq 2$. A *rotation angle* is the angle around the fixed subspace in counterclockwise direction provided the fixed subspace is given an orientation.

We define a tetrahedron T in the positive octant of $\mathbb{R}^3$ by the equation $x+y+z \geq \pi$, $x \leq y + z - \pi$, $y \leq x + z - \pi$, and $z \leq x + z - \pi$. This has vertices $(\pi, 0, 0)$, $(0, \pi, 0)$,$(0, 0, \pi)$, and (π, π, π) and is a regular tetrahedron with edge lengths $\sqrt{2}\pi$.

Recall that the *Klein four-group* V is a four element group $\{I, i, j, k\}$ where $ij = k, jk = i, ki = j$ isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We can identify each element of V with an involutionary rotation with an axis through $(\pi/2, \pi/2, \pi/2)$ parallel to one of the axis (see Figure 1). (Here an *axis* means a maximal segment fixed by an

involution) We denote by I_A the involution about the axis parallel to the z-axis, I_B the one about the axis parallel to the x-axis, and I_C the one about the axis parallel to the y-axis.

By a *double Klein four-group*, we mean a product of two Klein four-groups, which is isomorphic to $\mathbb{Z}_2^4$.

We will realize CP^3 as a torus fibration over Δ where fibers over the interior points are 3-dimensional tori, fibers over the interior of faces 2-dimensional tori, fibers over the interior of edges circles, and the fiber over a vertex a point. (See Section 6.1 for more details.)

A *blown-up tetrahedron* $\tilde{T}$ is combinatorially obtained by first truncating small tetrahedron T at each of the vertices of T with base parallel to the respective opposite side of the vertex and then truncating a thin prism along each of the remaining six edges by a plane parallel to one of the xy-, yz-, or zx-plane so that a vertex of one of the six newly formed rectangles is a vertex of another rectangle:

$$x + y + z \geq \pi, y \leq x + z - \pi, x \leq y + z - \pi, z \leq x + z - \pi,$$
$$\epsilon \leq x \leq \pi - \epsilon, \epsilon \leq y \leq \pi - \epsilon, \epsilon \leq z \leq \pi - \epsilon$$
$$x + y + z \leq 3\pi - 4\epsilon, x \leq y + z + \pi - 4\epsilon, y \leq x + z + \pi - 4\epsilon, z \leq x + y + \pi - 4\epsilon$$
$$\tag{1.1}$$

where $\epsilon > 0$ can be any number less than say $1/10$. We label as follows (see Figure 2):

$$a : x + y + z = \pi, b : y = x + z - \pi, c : x + y + z - \pi, d : z = x + y - \pi$$
$$A : z = \pi - \epsilon, A' : z = \epsilon, B : x = \pi - \epsilon, B' : x = \epsilon, C : y = \pi - \epsilon, C' : y = \epsilon$$
$$I : x+y+z = 3\pi-4\epsilon, II : x = y+z+\pi-4\epsilon, III : y = x+z+\pi-\epsilon, IV : z = x+y+\pi-4\epsilon$$
$$\tag{1.2}$$

The faces are given by $\tilde{T}$ intersected with the planes of the respective equations. (Here of course the geometry does not matter and only the combinatorial structure is needed for our purposes. More precisely, $\tilde{T}$ is the combinatorial polytope obtained from the region)

We can think of $\tilde{T}$ as T with each vertex replaced by a triangle and each point of the interior of edge replaced by a segment. This replacement is reminiscent of algebraic blowing-up operations.

We will use the term blown-up tetrahedron. (See Figure 2.) By a *region*, we will mean the subdomain in the boundary of $\tilde{T}$ or $\tilde{T}$ times a three-dimensional torus and their quotient images under the quotient maps that we will introduce later.

1.2. *Generalized spherical triangles and geometric-limit configuration spaces*

We will be using the Hausdorff distances between closed subsets of $\mathbf{S}^2$: Denote by $N_\epsilon(A)$ the ϵ-neighborhood of a subset A for a real number $\epsilon > 0$. Let K_1 and K_2 be
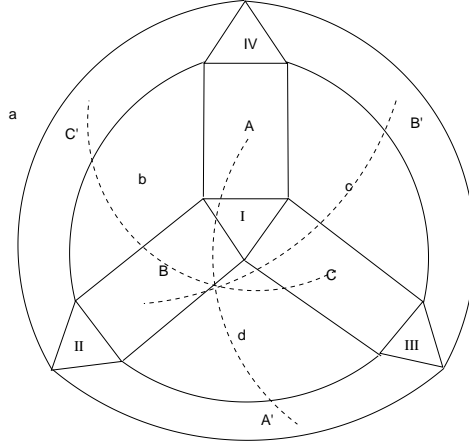
Fig. 2.   The face diagram of blown-up solid tetrahedron and regions to be explained later.

the two closed subsets of $\mathbf{S}^2$, and hence they are compact. We define the *Hausdorff distance* between K_1 and K_2 by the condition $d_H(K_1, K_2) < \epsilon$ if $K_1 \subset N_\epsilon(K_2)$ and $K_2 \subset N_\epsilon(K_1)$.

A geodesic in $\mathbf{S}^2$ is an arc in a great circle in $\mathbf{S}^1$. A *short geodesic* is a geodesic of length $\leq \pi$. A *great segment* is a geodesic segment of length equal to π.

By a *triangle* in $\mathbf{S}^2$, we mean a disk bounded by three short geodesics and with vertices to be denoted by v_0, v_1, v_2 and edges l_0, l_1, l_2 opposite to these respectively. They are spherical triangles in the usual terminology outside this paper. We will always assume that v_is and l_is appear in the clockwise direction.

A *lune* is the closed domain in $\mathbf{S}^2$ bounded by two segments connecting two antipodal points forming an angle $< \pi$. A *hemisphere* is the closed domain bounded by a great circle in $\mathbf{S}^2$.

We call ordinary triangles *nondegenerate triangles*. We now define pointed-lunes, pointed-hemispheres, pointed segments, and pointed-points, generally referred to as *degenerate triangles*.

- A *pointed-lune* is a lune with three ordered points where two of them are antipodal vertices of the lune, and the third one is either on an edge or identical with one of the vertices. If, in the boundary, the three points appear in clockwise direction, then the pointed lune is positively oriented. If two points are identical, the pointed-lune is considered *degenerate*.
- A *pointed-hemisphere* is a hemisphere with three ordered points on the boundary great circle where a segment between any two not containing the other is of length $\leq \pi$. Again, a *pointed-hemisphere* is positively oriented if three points appear in the clockwise direction. Note that two of the points could be identical and now two distinct points are antipodal. In this case, the pointed-hemisphere is said to be *degenerate*.

- A *pointed-segment* is a segment of length $\leq \pi$ with three ordered points where two are the endpoints and one is on the segment. Here again, the third point could be identical with one of the endpoint, and the *pointed-segment* is *degenerate*.
- A *pointed-point* is a point with three identical vertices.

Sometimes, we will drop the "pointed" in using the terminology.

A nondegenerate or degenerate triangle is a *generalized triangle*. A *triangle* is just a generalized triangle from now on. The three ordered points are said to be *vertices* and are denoted by v_0, v_1, v_2.

The notion of edges for nondegenerate triangles is the same as in the ordinary spherical geometry. One can define notions of edges of the degenerate triangles as the closure of the appropriate component of the complement of the pair of vertices if the pair of vertices is distinct, and the set of the pair if the pair is not distinct. Thus, we can form generalized edges l_0, l_1, and l_2 as being opposite v_0, v_1, and v_2 respectively. Again, the orderings must follow the clockwise direction.

The notion of angles for nondegenerate triangles is the same as the ordinary notions. We now associate angles to each of the three vertices of degenerate triangles by the following rules, where one can assign more than one set of angles to a given degenerate triangle. The angles are numbers in $[0, \pi]$. Let us use indices in $\mathbb{Z}_3$:

- If a vertex v_i has two nonzero length edges l_{i-1} and l_{i+1} ending at v_i, then we define the angle θ_i at v_i to be the interior angle between the edge vectors oriented away from v_i.
- If a vertex v_i is such that exactly one of l_{i-1} or l_{i+1} has a zero length, say l_{i-1} without loss of generality, then we choose an arbitrary great circle $\mathbf{S}^1_{i-1}$ containing v_i so that the generalized triangle is contained in the closure of one of the hemisphere H bounded by it. We take the counter-clockwise unit tangent vector for $\mathbf{S}^1_{i-1}$, to be called the *direction vector* at v_i for l_{i-1}, and we take the unit tangent vector for l_{i+1} at v_i and measure the angle between them. The degenerate segment l_{i-1} with the projective class of the direction vector is said to be an *infinitesimal edge*. If l_{i+1} has a zero length, then we choose $\mathbf{S}^1_{i+1}$ in the above manner and the direction vector at v_i is chosen to be in the clockwise direction.
- If a vertex v_i is such that both of l_{i-1} or l_{i+1} are zero lengths, then we have a pointed-point, and the angles to the three vertices are given arbitrarily so that they sum up to π.

The following are the consequences of the above angle assignment method:

- For the pointed-lune with three distinct vertices, the angles at the vertices of the lune are the ones given by the segments, i.e., the angle of the lune itself, and the vertex in the interior of a segment is given the angle π.
- For the pointed-lune with two vertices equal, the vertex which are distinct from others are given the angle of the lune, and the two vertices are given

angles greater than or equal to the lune-angle by the above procedure. We can show that the three angles have a sum equal to π plus 2 times the angle of the lune.

- For the pointed hemisphere, the vertices are given angles π.
- For the pointed segment with three distinct vertices, the angles $0, 0, \pi$ are given to the endpoints of the segment and the vertex in the middle.
- For the pointed segment with two vertices equal, the vertex which are distinct from the others is given the angle 0 and the other two are given angles which sums to π.

Let $\hat{\Delta}$ denote the space of all triangles, pointed-lunes, pointed-hemispheres, pointed-segments and pointed-points on $\mathbf{S}^2$ with angles assigned. Let $\hat{\Delta}$ be given a metric defined by letting $D(L, M)$ to be maximum of the Hausdorff distance between the closed domains of elements L and M of $\hat{\Delta}$ and the Hausdorff distances between corresponding points and segments of L and M and the absolute values of the differences between the corresponding angles respectively.

Proposition 1.1. $\hat{\Delta}$ *is compact under the metric, and the subspace of nondegenerate triangles are dense in* $\hat{\Delta}$.

Proof. The space $\mathcal{K}$ of all compact subsets of $\mathbf{S}^2$ with the Hausdorff metric is compact. $\hat{\Delta}$ is embedded as a closed subset of the compact metric space $\mathcal{K} \times (\mathbf{S}^2)^3 \times \mathcal{K}^3 \times [0, \pi]^3$ by a map sending an element to a tuple of itself in $\mathcal{K}$ and vertices in $(\mathbf{S}^2)^3$ and edges in K^3 and the three angles in $[0, \pi]^3$. Thus it is compact.

Given a degenerate triangle T', we take three great circles $\mathbf{S}^1_i$ containing l_i for $i = 0, 1, 2$ and by a perturbation, we can obtain a nondegenerate triangle T'' in any neighborhood of T'. □

Definition 1.1. Given a generalized triangle, each edge, degenerate or not, is contained in a unique great circle determined by the direction vector and the vertices of the edge. When we say an edge of a triangle is *contained* in a great circle, we mean only this situation.

An element of $\hat{\Delta}$ can be put using an orientation-preserving isometry to a *polar position*: Given a generalized triangle with vertices $v_0, v_1,$ and v_2, if $v_0 \neq v_1$, then by setting v_0 be the north pole $[0, 0, 1]$ and the edge l_2 connecting v_0 to v_1 in the boundary to lie on the xz-plane intersected with $\mathbf{S}^2$. If $v_0 = v_1$, and not equal to v_2, then we move v_0 to the north pole and move the edge l_1 to lie in the yz-plane intersected with $\mathbf{S}^2$ by isometries. If $v_0 = v_1 = v_2$, then we simply put every thing to the north pole. This agrees with the clock-wise orientation of $v_0, v_1,$ and v_2 on the boundary of the triangle.

The isometry group SO(3) acts properly on $\hat{\Delta}$. The quotient topological space is denoted by $\tilde{\Delta}$. This is a compact metric space with metric induced from $\hat{\Delta}$ by taking the Hausdorff distances between the orbits. We will denote by Δ^o the quotient space of the space of nondegenerate triangles by the SO(3)-action.

Proposition 1.2. *Let* Δ^o *be the set of isometry classes of nondegenerate triangles. Then there is a map sending a spherical triangle to its angles:* $f(t) = (\theta_1, \theta_2, \theta_3)$ *for the angles* θ_1, θ_2, *and* θ_3 *of* v_1, v_2, *and* v_3. *The space* Δ^o *of isometry classes of nondegenerate triangles is homeomorphic with interior of* T *by* f.

Proof. This map is one-to-one since angles determine a triangle up to isometry by classical spherical geometry. Furthermore, f is continuous since the metric D contains the angle differences.

Also, the image of f is in T^o: Let us choose an element in Δ^o and realize it by a nondengenerate triangle t. We know that the angles θ_1, θ_2, and θ_3 sum to a number $\geq \pi$ by spherical geometry.

Also, constructing a lune L whose sides contain two sides l_1 and l_2 of the triangle t, we see that the closure of $L - t$ is another nondegenerate triangle t' with angles $\theta_0, \pi - \theta_1, \pi - \theta_2$. Thus $\theta_0 + \pi - \theta_1 + \pi - \theta_2 > \pi$ or $\theta_0 < \theta_1 + \theta_2 - \pi$. By considering the indices to be in $\mathbb{Z}_3$ and the symmetry of indices, we see that the image is in T^o.

Conversely, given an element of T^o, we first construct a lune L with angle θ_0 at a point and let the point be the vertex v_0. Then we choose a point v'_1 on a right edge e_1 of L and take a maximal segment s_1 in L from v'_1 of angle θ_1. Let v'_2 to be the endpoint of s_1 in e_2 and let θ'_2 be the angle at v'_2 in the triangle with vertices v_0, v'_1, v'_2.

We first assume $\theta_0 \geq \theta_1$ by symmetry. We temporarily assume that $\theta_0 + \theta_1 < \pi$ Moving v_1 on e^o_1, we can realize the variable θ'_2 can realize any value in the open interval $(\pi - \theta_0 - \theta_1, \pi - \theta_0 + \theta_1)$. We now suppose that $\theta_0 + \theta_1 \geq \pi$. Moving v_1 on e^o_1, we can realize the variable θ'_2 can realize any value in $(-\pi + \theta_0 + \theta_1, \pi - \theta_0 + \theta_1)$. We conclude that over the region $[0, \pi] \times [0, \pi]$ with $\theta_0 \geq \theta_1$ θ'_2 can achieve any angle in T^o.

By symmetry between 0 and 1, we obtain that any point of T^o is realizable as a triangle. Also, our argument constructs a continuous map $T^o \to \Delta^o$ by construction of a triangle from angles. This is a homeomorphism as can be verified by the above construction. $\square$

As discussed above, each triangle or degenerate triangle has an angle or length associated with each vertex and edge. We denote by $v(0), v(1), v(2)$ the angles at the vertices v_0, v_1, v_2 respectively and $l(0), l(1), l(2)$ the length of the edges l_0, l_1, l_2 respectively.

Proposition 1.3. *A sequence of elements of* $\tilde{\Delta}$ *converges to an element of* $\tilde{\Delta}$ *if and only if the corresponding sequences of angles and lengths converge to those of the limit. Hence, we can embed* $\tilde{\Delta}$ *as the compact closure of the image of* Δ^o *in* $\mathbb{R}^6$ *by the function* $\iota : \tilde{\Delta} \to \mathbb{R}^6$ *given by sending an isometry class of a generalized triangle to the associated angles and lengths* $(v(0), v(1), v(2), l(0), l(1), l(2))$.

Proof. If a sequence of 6-tuples of angles and lengths converges and the corresponding sequence of lengths denoted by $l(2)$ is bounded below by a small positive

number, then the corresponding elements of $\hat{\Delta}$ can be put in $\mathbf{S}^2$ in polar positions so that the sequence of 6-tuples converges geometrically to an element of $\hat{\Delta}$ by spherical geometry.

If the sequence of lengths denoted $l(2)$ is going to zero but the sequence of lengths denoted $l(1)$ is not, then we change the polar position to ones where the segments denoted by $l(1)$ are put on a great circle defined by $x = 0$ and putting the vertex denoted by v_0 on the north pole. The arguments are the same. If the sequences of lengths $l(1)$ and $l(2)$ both go to zero, then the sequence of triangles converges to a pointed point, and the sequences of angles and lengths all converge. Hence, we have a convergence in $\tilde{\Delta}$.

Conversely, if a sequence of elements of $\tilde{\Delta}$ converges to say e, then we can again put the elements to polar forms. Since $\hat{\Delta}$ is compact, the sequence has a subsequence converging in $\hat{\Delta}$. Since SO(3) is compact, the limit is in the equivalence class of the limit e. The corresponding sequences of lengths and angles converge also clearly.

The map ι is continuous because of the above paragraph. ι is injective since two degenerate or nondegenerate triangles are clearly not isometric if they have different angles or different edge lengths. Since $\tilde{\Delta}$ is compact and $\mathbb{R}^6$ is Hausdorff, the map ι is an embedding. $\qquad\square$

The main aim of the section is to prove the following proposition.

Theorem 1.1. *The geometric-limit configuration space $\tilde{\Delta}$ is homeomorphic to a blown-up solid tetrahedron.*

We prove this in Section 1.4. But first, we need to describe the 2-dimensional domains classifying the above degenerate triangles:

1.3. *Parameterizing the degenerate triangles*

We will classify the degenerate triangles according to their types and show that the collection form nice topology of triangles and rectangles, i.e., 2-cells. (See Figures 3, 4, and 5 as reference.) Let us denote by $l(i)$ the coordinate function measuring length of l_i for $i = 0, 1, 2$, and $v(j)$ the coordinate function measuring the angle of v_i for $i = 0, 1, 2$.

First, we discuss regions a, b, c, and d. (See Figure 3.) These are formed by degenerate triangles where two vertex are the same and the other is either equal to the first ones or antipodal to them. These are pointed points or pointed lunes. There are four classes given by $l(0) = l(1) = l(2) = 0$ (the face a), $l(0) = l(2) = \pi$ and $l(1) = 0$ (the face b), $l(1) = l(2) = \pi$ and $l(0) = 0$ (the face c), and $l(0) = l(1) = \pi$ and $l(2) = \pi$ (the face d).

Each point of region a corresponds to a pointed-point with three vertices which are all equal to one another. Here lengths $l(0), l(1)$, and $l(2)$ are all zero and vertices v_0, v_1, and v_2 are given angles $v(0), v(1)$, and $v(2)$ summing up to π. Two of the angles parametrize the space as we can see from the definition of our metric D above. They satisfy $v(0) + v(1) + v(2) = \pi$ and $0 \le v(i) \le \pi$ for $i = 0, 1, 2$.
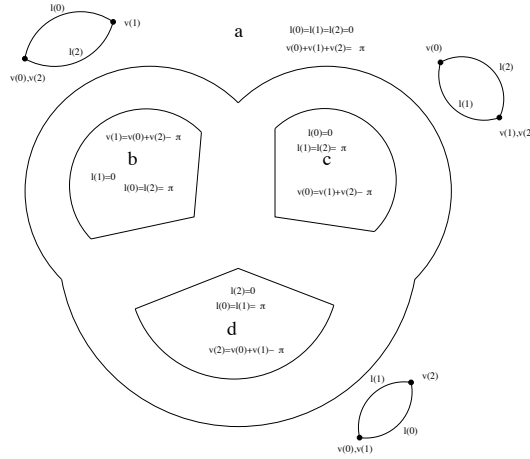
Fig. 3.   Regions $a, b, c,$ and d

Each point of region b corresponds to a lune with vertices v_1 and $v_0 = v_2$. The vertices are assigned angles $v(0), v(1), v(2)$ satisfying $v(1) = v(0) + v(2) - \pi$ with $0 \leq v(i) \leq \pi$ for $i = 0, 1, 2$.

Regions c and d are described in the similar way in Figure 3.

The regions $a, b, c,$ and d are still said to be *faces*.

The angles parameterize each of a, b, c, d and produce a one-to-one correspondences. Since the parametrizing rectangles are compact and $\tilde{\Delta}$ is Hausdorff, we conclude that they are closed embedded 2-dimensional disks in $\tilde{\Delta}$ and are mutually disjoint.

For regions $A, B,$ and C, we classify lunes with one vertex of angle π and the lune angles in $(0, \pi)$; a segment with one vertex of angle π and two antipodal vertices or; a hemisphere with one angle π and the other two vertices are antipodal. For regions A', B' and C', we classify a segment with one vertex of angle 0 and two other vertices identical, including a pointed point with one vertex angle 0 or a segment of length π, with one end vertex angle 0 and the other two vertices equal (see Figure 4):

Each point of A corresponds to a lune with vertices v_0 and v_1 and the vertex v_2 is on a segment of length π between them. Thus $l(2) = \pi$ and $v(2) = \pi$ here. For region A, there are two coordinate functions $l(0)$ increasing from 0 to π as indicated by the top dotted arrow and $v(0)$ increasing from 0 to π indicated by the down dotted arrow in the left of the square A. The values of $l(0)$ and $v(0)$ determine the configuration uniquely. The coordinate system is given by $(v(0), l(0))$.

A *tie* is a segment in the edge region defined by the lune angles set constant. For example, in A, it is given by the equation $v(0) = \theta$ for $\theta \in [0, \pi]$ and $l(0)$ can take any values in $[0, \pi]$. An *end tie* is a tie in the end of the edge region. A *mid-tie* is a tie given by setting the angle parameter exactly in the middle. For A, it is given by $v(0) = \pi/2$.
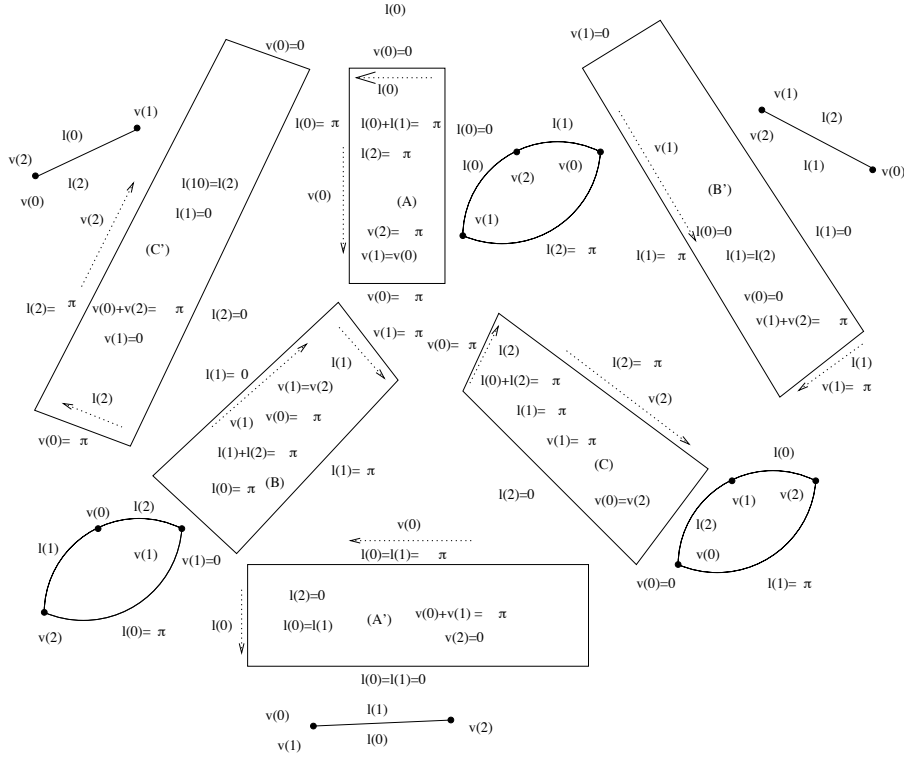
Fig. 4.   Regions A, A', B, B', C, and C'

For region A', each point on the square corresponds to a segment of length $l(0)$ with vertices $v_0 = v_1$ and v_2. We have $l(2) = 0$ and the two coordinate functions are $l(0)$ and $v(0)$ represented by top dotted arrow pointing right and the left dotted arrow pointing up. They take values in $[0, \pi]$ as above. v_0 is given an angle $v(0)$ and v_1 given the angle $v(1)$ where $v(0) + v(1) = \pi$ and v_2 is given the angle 0. Here l_2 is a degenerate segment, i.e., a point. The coordinate system is given by $(v(0), l(0))$.

There is actually a one-to-one correspondence between A and A' by taking the lune and moving v_0 to v_1 and taking the segment between v_0 and v_2. This changes the associated angle at v_0 to be π minus the original angle and similarly for the angle at v_2.

For region B, each point of the rectangle corresponds to a lune with vertices v_1 and v_2 with v_0 on the segment between v_1 and v_2 as a generalized triangle. We have $v(0) = \pi$. We have two parameterizing functions $l(1)$ and $v(1)$.

For region B', each point of the rectangle, we associate a segment with vertices v_0 and $v_1 = v_2$ of length $l(1)$. For regions B and B', the coordinate systems are given by $(v(1), l(1))$.

There is a one-to-one correspondence between B and B' given by sending v_1 to v_2 in the lune for B and changing the angles at v_1 and v_0 to π minus the original

angles respectively.

The analogous descriptions hold for regions C and C'. The coordinates are given by $(v(2), l(2))$. Equivalently, we can use $(v(0), l(0))$.

These domains from the edges of Δ are said to be *edge regions*.

We note that the edge regions are disjoint except at the vertices. For each of the three cycles in A, B, C, A', B', C', we collapse six vertices to three vertices as indicated in Figure 2. This can be easily verified using the metric D and looking at the configurations.

For region I, pointed-hemispheres are classified here. Here we have $v(0) = \pi, v(1) = \pi, v(2) = \pi$. We identify each point $(l(0), l(1), l(2))$ with a closed hemisphere with vertices $v(0), v(1), v(2)$ where $l(i)$ is the distance between $v(i+1)$ and $v(i+2)$ for each $i = 0, 1, 2$. The functions $l(0), l(1)$, and $l(2)$ are extended as in the Figure 5 satisfying relations $l(0) + l(1) + l(2) = 2\pi$ where $0 \leq l(i) \leq \pi$ for each $i = 0, 1, 2$. Actually, the equation gives a plane in the $(l(0), l(1), l(2))$-space intersected with the closed positive octant.

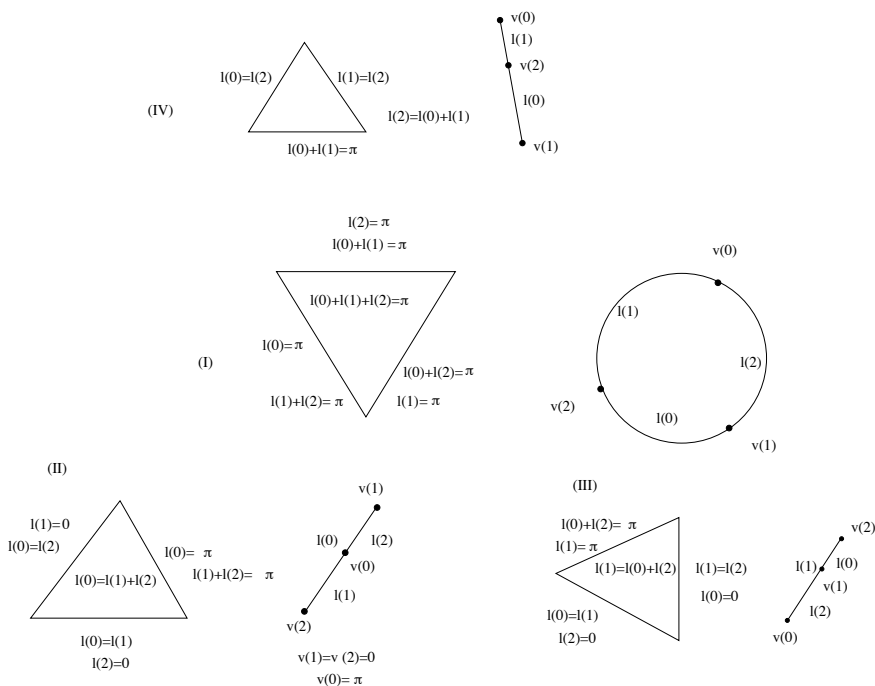Fig. 5.   Regions I, II, III, and IV with configurations in the right sides

For region II, we have $v(0) = \pi, v(1) = v(2) = 0$ and $l(0) = l(1) + l(2)$ and the bounds as above for each $l(i)$. Each point $(l(0), l(1), l(2))$ is identified with a segment of length $l(0)$ with vertices $v(1), v(2)$ and $v(0)$ on it. Clearly $l(1)$ is the distance between $v(0)$ and $v(2)$ and $l(2)$ one between $v(0)$ and $v(1)$.

For region III, we have $v(1) = \pi, v(0) = v(2) = 0$ and $l(1) = l(0) + l(2)$. Each point $(l(0), l(1), l(2))$ is identified with a segment of length $l(1)$ with vertices $v(0), v(2)$ and v_1 on it.

For region IV, we have $l(2) = l(0) + l(1)$. Each point $(l(0), l(1), l(2))$ is identified with a segment of length $l(2)$ with vertices v_0, v_1 and v_2 on it.

Notice for regions I, II, III, and IV, the points are coordinatized by $(l(0), l(1))$ for $0 \leq l(0), l(1) \leq \pi$. These are said to be *vertex regions*. The regions I, II, III, and IV are mutually disjoint disks as we can see from the metric D.

Finally, the union of the regions, $a, b, c, d, A, B, C, A', B', C', I, II, III$, and IV, is a 2-sphere being identified as in Figure 2: This follows since each region is homeomorphic to disks and the identifications are along arcs as in Figure 2. The union is homeomorphic to the identification space of the regions. This fact can be seen by looking at the metric D and the configurations themselves as some of the pointed degenerate triangles are identical.

1.4. *The proof of Theorem 1.1*

Proof. Since $\tilde{\Delta}$ is compact and $\mathbb{R}^6$ is Hausdorff and ι is continuous, $\tilde{\Delta}$ is embedded as the closure of $\iota(\Delta^o)$. We only need to show that $\tilde{\Delta}$ has the structure of a blown-up solid tetrahedron. To do that, we show that $\tilde{\Delta}$ is a manifold with boundary consisting of degenerate triangles or the union of the above regions.

First, the union of $a, b, c, d, A, B, C, A', B', C', I, II, III$, and IV is homeomorphic to $\mathbf{S}^2$. Each point of the interior of each region has a neighborhood homeomorphic to an open ball intersected with closed upper half-space. For region I, we can use coordinate functions $l(0), l(1)$, and $l(2)$ now satisfying $l(0) + l(1) + l(1) \leq 2\pi$, $l(0) \geq 0, l(1) \geq 0, l(2) \geq 0, l(i) + l(i+1) \geq l(i+2)$ for a cylic index $i = 0, 1, 2$ mod 3. Then we see that they parameterize a sufficiently small neighborhood N of a point of I. More precisely, $(l(0), l(1), l(2))$ form an injective continuous map η from a neighborhood N to an open subset N' of $\mathbb{R}^3$ defined by $l(0) + l(1) + l(2) \leq 2\pi$, $l(0) \geq 0, l(1) \geq 0, l(2) \geq 0, l(i) + l(i+1) \leq l(i+2)$ for a cyclic index $i = 0, 1, 2$ mod 3. (Here, $\mathbb{R}^3$ has coordinates $(l(0), l(1), l(2))$.) Conversely, from a sufficiently small open set in N', we can construct elements of $\tilde{\Delta}$ in N, using given the length informations and polar positions. This forms a continuous inverse map to η as we can see from the polar position conditions.

Hence a neighborhood of I^o is a manifold. The similar consideration holds for each of II^o, III^o, or IV^o.

For the region A^o, there are coordinate functions $l(0)$ and $v(0)$. We use coordinate functions $l(0), v(1), v(2)$. Then these functions parameterize a neighborhood of an interior point of A^o. Let $\mathbb{R}^3$ have coordinates $(l(0), v(1), v(2))$. More precisely, $(l(0), v(1), v(2))$ defines a homeomorphism η from a neighborhood N of a point of A^o to an open subset in the subset of $\mathbb{R}^3$ defined by $0 < v(2) \leq \pi, 0 < l(0) < \pi, 0 < v(1) < \pi$ since $l(0), v(1), v(2)$ determine a generalized triangle uniquely. This

shows that a neighborhood of A^o is a manifold. (Here, $\mathbb{R}^3$ has coordinates denoted by $l(0), v(1), v(2)$. The homeomorphism property of η again follows as above by geometric constructions to obtain η^{-1}.)

Similar considerations shows that some respective neighborhoods of $B^o, C^o, A'^o, B'^o, C'^o$ are manifolds.

For the region a^o, we use coordinate functions $(v(0), v(1), v(2))$ to find a function η from a neighborhood of a point of a^o to a small open subset of the subspace of $\mathbb{R}^3$ defined by $v(0) + v(1) + v(2) \geq \pi$, $0 < v(i) < \pi$ for $i = 0, 1, 2$. This implies as above that a^o has a manifold neighborhood. (Here $\mathbb{R}^3$ has coordinates denoted by $v(0), v(1), v(2)$. The homeomorphism property of η is similar to above.)

Similar consideration shows that some respective neighborhoods of b^o, c^o, d^o are manifolds.

We now discuss the interiors of the intersecting edges and show that they have manifold neighborhoods. First, consider the edge $I_{A,b} := A \cap b$. Let $p \in I^o_{A,b}$. Here we have $l(0) = l(2) = \pi$ and $l(1) = 0$ and $v(1) = v(0) + v(2) - \pi$ from b and $l(0) + l(1) = \pi$, $l(2) = \pi$, $v(0) = v(1)$, $v(2) = \pi$ from A. Thus, we have $l(1) = 0, l(0) = l(2) = \pi$ and $v(0) = v(1), v(2) = \pi$ on the edge. We use $(l(1), v(0), v(2))$ as coordinate functions in a sufficiently small neighborhood N of p. That is, they define an injective function η from N to a small neighborhood of a corner point of a subset $\mathbb{R}^3$ defined by $l(1) \geq 0$ and $v(2) \leq \pi$. Then $b \cap N$ is characterized by $l(1) = 0$, and $A \cap N$ is characterized by $v(2) = \pi$. (Here $\mathbb{R}^3$ has coordinates denoted by $l(1), v(0), v(2)$. The homeomorphism property of η is similar to above.) This implies that $I^o_{A,b}$ has a manifold neighborhood.

Next, consider the edge $I_{A,I} := I \cap A$. Here, we use coordinate functions $l(0), l(1), l(2)$ on a small neighborhood N of a point of $I^o_{A,I}$. A is determined by $l(2) = \pi$ and I is determined by $l(0) + l(1) + l(2) = \pi$. There is a homeomorphism η from N to a small open subset of $\mathbb{R}^3$ defined by $l(2) \leq \pi, l(0) + l(1) + l(2) \leq 2\pi$.

For all other edges, similar considerations show that there are some respective manifold neighborhoods.

Next, we consider 12 vertices, i.e., points where three regions meet. Remove these points from $\tilde{\Delta}$. For each point, we can find a disk D with ∂D in $\tilde{\Delta} - \Delta^o$ to obtain a contactible manifold N homeomorphic to compact 3-ball B^3 with one point removed. Hence, with the point added back, the manifold N becomes a compact manifold homeomorphic to B^3. The disks can be chosen to be mutually disjoint and each containing one of 12 vertices in one side. Doing these for each point, we have shown that $\tilde{\Delta}$ is homeomorphic to B^3 and has the boundary structure of a blown-up tetrahedron. $\qquad\square$

Remark 1.1. From now on, we can identify the blown-up tetrahedron T with $\tilde{\Delta}$ and the interior of T with the interior Δ^o. Thus, when we say $\tilde{\Delta}$, we will think of a blown-up tetrahedron.

Remark 1.2. Let us give some "intuitive idea" here.

Given each smooth arc $\alpha : [0, 1) \to \Delta^o$ starting from a point of Δ^o and does not have a compact closure in Δ^o and has well-defined limits of derivatives as $t \to 1$, it follows that α ends at a unique point of $\tilde{\Delta}$. This can be seen from the angle and length formulas of spherical geometry and L'Hospital's rules. Setting α as a half-open segment is most illustrative here.

This gives us a natural way to view $\tilde{\Delta}$ as a blown-up tetrahedron. Each ray ending at a common point the interior of an edge of Δ will have different limits in $\tilde{\Delta}$ according to its direction. Hence, we see that a point is replaced by an arc, i.e. a tie. Hence, an open edge is replaced by an open rectangle.

Each ray ending at a vertex of Δ will have different limits in $\tilde{\Delta}$ according to its direction. Here, we see that a triangle replaces a point.

The author is not yet able to find natural way to use these facts to prove Theorem 1.1 but probably our process is a natrual blow-up construction in real algebraic scheme theory.

1.5. *An extension of the Klein four-group action*

We now extend the discussions to the boundary and study identifications in the boundary: The map I_A in $\tilde{\Delta}^o$ can be described as first find an element μ in $\tilde{\Delta}^o$ and representing it as a triangle with vertices v_0, v_1, v_2 and taking a triangle with vertices $v_0' = -v_0$ and $v_1' = -v_1$ and $v_2' = v_2$. That is, we change our triangle with vertices v_0, v_1, and v_2 and edges l_0, l_1, and l_2 to one with vertices $-v_0, -v_1$, and v_2 and edges l_0' with endpoints $-v_1, v_2$ in the great circle containing l_0 and l_1' endpoints $-v_0, v_2$ in the one containing l_1 and $l_2' = -l_2$ with endpoints $-v_0, -v_1$ in the one containing l_2. (Here, the word "contain" is according to Definition 1.1.) Then take the isometry class of the second triangle as the image $I_A(\mu)$. This gives the changes in parameter as follows

$$(v(0), v(1), v(2), l(0), l(1), l(2)) \to (\pi - v(0), \pi - v(1), v(2), \pi - l(0), \pi - l(1), l(2)).$$

Similarly, the map I_B changes the triangle with vertices v_0, v_1, and v_2 to one with $v_0, -v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2)) \to (v(0), \pi - v(1), \pi - v(2), l(0), \pi - l(1), \pi - l(2)).$$

Similarly, the map I_C changes the triangle with vertices v_0, v_1, and v_2 to one with $-v_0, v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2)) \to (\pi - v(0), v(1), \pi - v(2), \pi - l(0), l(1), \pi - l(2)).$$

For our geometric degenerate triangles, we do the same. For regions a, b, c, and d, the transformations are merely the linear extensions or equivalently extensions with respect to the metrics.

We first describe I_A:

A, A': Regions A, A' are preserved under I_A, and $(l(0), v(0))$ is mapped to $(\pi - l(0), \pi - v(0))$.

B, B': The region B is mapped to B' and vice versa. $(l(1), v(1))$ is mapped to $(\pi - l(1), \pi - v(1))$.

C, C': The region C is mapped to C' and vice versa. $(l(2), v(2))$ is mapped to $(l(2), v(2))$.

I, IV: The region I is mapped to IV and vice versa. $(l(0), l(1), l(2))$ is mapped to $(\pi - l(0), \pi - l(1), l(2))$.

II, III: The region II is mapped to III and vice versa. $(l(0), l(1), l(2))$ is mapped to $(\pi - l(0), \pi - l(1), l(2))$.

The transformations I_B and I_C are analogous. In fact, recall that one can identify $\tilde{\Delta}$ by the tetrahedron that closure of $\tilde{\Delta}^o$ truncated by planes parallel to sides at vertices and edges: more precisely, at a vertex v, we take a plane parallel to the opposite side and passing through the tetrahedron of distance ϵ to v for a sufficiently small $\epsilon > 0$. We cut off the neighborhood of v by the plane. We do the same steps at each vertex for the same ϵ. At an edge e, we take a plane parallel to e and the edge opposite e which is at the distance $\epsilon/3$ from the line containing e. We truncated every edge. This gives us an embedding of $\tilde{\Delta}$ with incorrect coordinates.

Each face of $\tilde{\Delta}$ has a metric induced by the parameterization maps into $\mathbb{R}^2$ or $\mathbb{R}^3$ with Euclidean metrics. In this identification, I_A is realized as an involutive isometry of Δ^o and each of the faces of $\tilde{\Delta}$ of order two with the set of fixed points the axis trough the center of A and A'. I_B is realized as one with the fixed axis though the center of B and B' and I_C one with the fixed axis though the center of C and C'.

1.6. *Standard positions for generalized triangles*

For later purposes, we introduce *standard positions* for generalized triangles. Heuristically speaking, although the edges maybe degenerate, we can still read angles with respect to the degenerate edges with infinitesimal directions only. We use the idea:

Let H be the hemisphere containing $[1, 0, 0], [0, 1, 0]$ in the boundary great circle $\mathbf{S}^1$ and H^o containing $[0, 0, 1]$.

We are given a generalized triangle with vertices v_0, v_1, v_2, and we find the standard position to put this triangle in $\mathbf{S}^1$: By an isometry, the vertex v_0 is to be put on $[1, 0, 0]$ and v_1 on the $\mathbf{S}^1$ in the closure L of the component of $\partial H - \{[1, 0, 0], [-1, 0, 0]\}$ containing $[0, -1, 0]$, and v_2 on H. As a consequence, the interior of the triangle to be in H^o and $l_2 \subset L$. (Denote by $-L$ the antipodal segment to L, i.e., the set of antipodes to points of L.) (We remark that the triangle is then put in the right orientation.)

For any nondegenerate triangle, our process determines the triangle in a standard position uniquely. For generalized triangles in the regions I, II, III, and IV, the standard position is determined by the above conditions as well. In fact, in III, IV, the triangle is a subset of L and in II, our triangle is a segment in ∂H with a point of it $[1, 0, 0]$.

For a generalized triangle in the regions a, b, c, and d, if the vertices satisfy

$v_0 = v_1 = v_2$, then we let these be $[1, 0, 0]$ all (the case a).

Suppose that $v_0 = v_2$ and are antipodal to v_1 (the case b). Then we let $v_0 = v_2 = [1, 0, 0]$ and $v_1 = [-1, 0, 0]$ and let l_1 be the degenerate segment equal to $\{v_0 = v_2\}$ and l_0 is the segment of length π determined by angle θ_0 with L at v_0 and $l_2 = L$.

Suppose $v_1 = v_2$ and are antipodal to v_0 (the case c). Then we let $v_0 = [1, 0, 0]$ and $v_1 = v_2 = [-1, 0, 0]$. We let $l_2 = L$ and l_1 be the segment of length π determined by the angle θ_0 from L at v_0 and let l_0 be the degenerate segment equal to $\{v_1 = v_2\}$.

Finally, suppose that $v_0 = v_1$ and v_2 is antipodal to these (the case d). Then we let $v_0 = v_1 = [1, 0, 0]$ and $v_2 = [-1, 0, 0]$ and let the edge l_1 be the segment of length π determined by the angle θ_0 from L at v_0, l_0 be one determined by the angle $\pi - \theta_1$ from L at v_1, and l_2 is the degenerate segment equal to $\{v_0 = v_1\}$.

Now we consider the interiors of regions A, A', B, B', C, C': for triangles there we assign $v_0 = [1, 0, 0]$.

(A) We put $v_1 = [-1, 0, 0]$ and $l_2 = L$, and $l_0 \cup l_1$ is the segment of length π determined by θ_0 with L at v_0.

(A') We put $v_1 = [1, 0, 0]$ and $l_2 = \{v_0 = v_1\}$, and $l_1 = l_0$ is the segment of length $\leq \pi$ determined by angle θ_0 from L at v_0 as above.

(B) We put $v_1 \in L$, $v_2 = -v_1 \in -L$, and $l_2 \subset L$, and $l_1 \subset -L$. l_0 is determined by θ_1 at v_1.

(B') We put $v_1 = v_2$ and $l_1 = l_2$ are in L and $l_0 = \{v_1 = v_2\}$.

(C) v_1 is on L and $v_2 = [-1, 0, 0]$ and l_0 and l_2 are in L, and l_1 is the segment connecting $[1, 0, 0]$ to $[-1, 0, 0]$ of angle θ_0 with L at v_0.

(C') We have $v_2 = v_0$, and $l_0 = l_2$ are in L, and $l_1 = \{v_0 = v_2\}$.

Let $\mathcal{S} \subset \hat{\Delta}$ denote the subspace of generalized triangles in standard positions.

Proposition 1.4. *There is a section* $\tilde{\Delta} \to \hat{\Delta}$*, which is a homeomorphism to its image* $\mathcal{S}$*.*

Proof. The injectivity is clear. The continuity follows by showing that puting a generalized triangle t to one in $\mathcal{S}$, we need an isometry i_t that depends continuously on t. A sequence argument will show this. □

2. The character space of the fundamental group of a pair-of-pants

The character space of the fundamental group of a pair-of-pants will be analyzed geometrically using triangles in this section and will be shown to be the quotient of a solid tetrahedron with some Klein four-group V axial action. That is, for each representation we obtain a triangle determined up to the V-action. The basic idea is similar to that of Goldman [18] modified to our setting and is related to matrix-multiplication by geometry. (In the next Section 3, we generalize these results to SU(2).)

2.1. *Matrix-multiplications by geometry*

We will coordinatize $\mathbf{S}^1$ by $[0, 2\pi]$ with 0 and 2π identified. In fact, we will think of a number in $[0, 2\pi]$ as a number mod 2π if our group is SO(3) and without mod 2π if our group is SU(2).

An isometry ι in SO(3) fixes at least one point v in $\mathbf{S}^2$. Its antipodal point $-v$ is also fixed and with respect to a coordinate system with v as one of the frame vector, a matrix of ι equals

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, 0 \leq \theta \leq 2\pi.$$

Here, θ is the counter-clockwise rotation angle of ι and is determined by the choice of the fixed point v and will be $2\pi - \theta$ if we chose the other fixed point. Notice that the rotation angle is measured in the counter-clockwise direction with respect to the chosen fixed point.

An element of SO(3) can be written as $R_{x,\theta}$ where x is a fixed point and θ is the angle of rotation in the counter-clockwise direction where $0 \leq \theta \leq 2\pi \mod 2\pi$. For the identity element, x is not determined but $\theta = 0$. For any nonidentity element, x is determined up to antipodes: $R_{x,\theta} = R_{-x,2\pi-\theta}$.

We begin with "multiplication by geometry": Let w_0, w_1, and w_2 be vertices of a triangle oriented in the clockwise direction. Let e_0, e_1, and e_2 denote the opposite edges. Let θ_0, θ_1, and θ_2 be the respective angles. Then

$$R_{w_2,2\theta_2} R_{w_1,2\theta_1} R_{w_0,2\theta_0} = \mathrm{I}:$$

Draw three adjacent isometric triangles obtained by reflecting isometrically on the great circles containing edges e_0, e_1, and e_2 respectively. (See Definition 1.1.) Call them T_0, T_1, and T_2 while they are obtained from reflecting. Since their edge lengths are the same, they are congruent. Moreover, by considering the congruent edges and the angles between them, we obtain that $R_{w_0,2\theta_0}$ fixes w_0 and sends T_1 to T_2, $R_{w_1,2\theta_1}$ fixes w_1 and sends T_2 to T_0, and $R_{w_2,2\theta_2}$ fixes w_2 and sends T_0 to T_1. The composition the three isometries sends each vertex to itself and hence is the identity map.

Finally, denoting the rotation at w_0, w_1, w_2 by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively, we obtain

$$\mathcal{C}\mathcal{B}\mathcal{A} = \mathrm{I}, \mathcal{C}^{-1} = \mathcal{B}\mathcal{A}, \mathcal{C} = \mathcal{A}^{-1}\mathcal{B}^{-1}. \tag{2.1}$$

We will generalize the idea when the triangles are degenerate. (See Section 2.2.)

Of course, careful considerations of orientations are important: the triangle is ordered clockwise and the rotation directions are counter-clockwise.

2.2. *Mutiplications by geometry extended*

We now extend the idea of "multiplication by geometry" to degenerate cases (which is applicable in SU(2) as well): Let w_0, w_1, and w_2 be vertices of a degenerate triangle
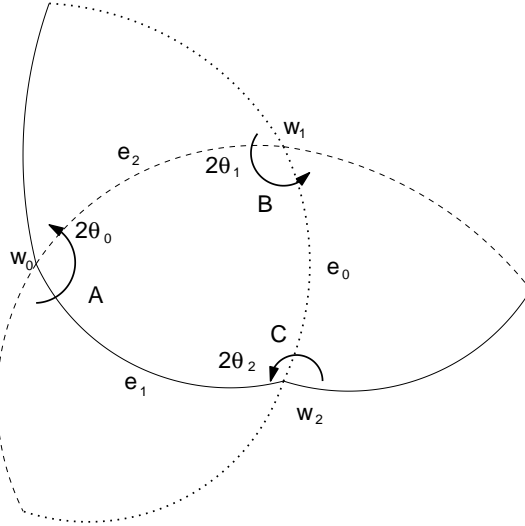
Fig. 6.   Triangular representations

oriented in the clockwise direction. Let e_0, e_1, and e_2 denote the opposite edges. Even in a degenerate case, since we can put the generalized triangle in the standard position, we have a well-defined great circle containing each of the edges for any generalized triangle. Thus, we can reflect these by the great circles containing the edges and obtain the same constructions as above. (For a pointed-point case, we simply use the rotations of the double the angles.)

Let θ_0, θ_1, and θ_2 be the respective angles in $[0, \pi]$. Then

$$R_{w_2, 2\theta_2} R_{w_1, 2\theta_1} R_{w_0, 2\theta_0} = \mathrm{I}$$

will hold. This is checked by using the same argument as above.

Using this equation, we can compute

$$R_{w_2, 2\pi - 2\theta_2} = R_{w_1, 2\theta_1} R_{w_0, 2\theta_0},$$

for example. Finally, denoting the rotation at w_0, w_1, w_2 by $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively, we obtain

$$\mathcal{CBA} = \mathrm{I}, \mathcal{C}^{-1} = \mathcal{BA}, \mathcal{C} = \mathcal{A}^{-1} \mathcal{B}^{-1}. \tag{2.2}$$

We should remark that when $\theta_2 = 0, \pi$, then $R_{x, 2\theta}$ is the identity map. However, the equation still holds.

When the angle at w_2 is 0 or π so that $\mathcal{C}$ equals I (respectively I or $-I$ in SU(2)), the vertex w_2 can be chosen arbitrarily on the boundary of the triangle.

We note that under quotient relation, $\mathcal{C}$ is a smooth function of $\mathcal{A}$ and $\mathcal{B}$.

As an illustration, we give some examples here. The readers can fill in all remaining cases using permutations. The triangle is clockwise oriented in the boundary by the order of v_0, v_1, and v_2 appearing with respective angles θ_0, θ_1, and θ_2.

- The triangle is a point with $v_0 = v_1 = v_2$. Then $\theta_2 = \pi - \theta_0 - \theta_1$ and $\mathcal{C}$ is a rotation at v_2 of the angle $2\theta_2$.
- The triangle is a segment with $v_0 = v_1$ and v_2 with nonzero angles θ_0, θ_1 and $\theta_2 = 0$. Then $\theta_1 = \pi - \theta_0$ naturally. Thus $\mathcal{B} = \mathcal{A}^{-1}$ and $\mathcal{C} = I$. Actually, we can choose v_2 to be any point in $\mathbf{S}^2$.
- The triangle which is a lune with v_0 and v_2 as two vertices of the lune and v_1 in the interior of an edge. Then the lune angle θ equals θ_0 and $\mathcal{C}$ has a fixed point $v_2 = -v_0$ and the angle of rotation is $2\theta_0$.
- The triangle which is a lune with v_0 and v_1 as the two vertices of the lune with lune angle θ. Then the lune angle θ equals smaller of θ_0 or θ_1 which can be seen from the classification of generalized triangles.

  – If $\theta_0 = \theta$ and $\theta_1 = \theta$, then v_2 can be any point on the segment of a lune and $\theta_2 = \pi$ and $\mathcal{C}$ is I.
  – If $\theta_0 = \theta$ and $\theta_1 > \theta$, then $v_2 = v_1$ and $\theta_2 = \theta_0 + \pi - \theta_1$.
  – If $\theta_1 = \theta$ and $\theta_0 > \theta$, then $v_2 = v_0$ and $\theta_2 = \theta_1 + \pi - \theta_0$.

- If $\theta_0 = \theta_1 = \pi$, then the triangle is a hemisphere, and v_2 can be any point on a segment with endpoints $-v_0, -v_1$ and $\theta_2 = \pi$. In this case, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are all identity.

(In SU(2) case, we only need some obvious sign changes.)

From above, we obtain the following rules.

Proposition 2.1. *Suppose that we have a generalized triangle with vertices v_0, v_1, and v_2 occurring in the clockwise order with angles ϕ_0, ϕ_1, ϕ_2 respectively. Let $\mathcal{A}$ be a transformation with a fixed point v_0 and rotation angle ϕ_0, $\mathcal{B}$ be one with v_1 and ϕ_1, and $\mathcal{C}$ be one with v_2 and ϕ_2 respectively satisfying $\mathcal{C}\mathcal{B}\mathcal{A} = \mathrm{I}$.*

- *Suppose that v_0 and v_1 do not coincide and do not form an antipodal pair. Then $\pm v_0, \pm v_1$ form four points on a great circle.*

  – *Suppose that ϕ_0, ϕ_1 are both not zero nor π. Then so is ϕ_2 and v_0, v_1, and v_2 form the vertices of a nondegenerate triangle.*
  – *Suppose that $\phi_0 = 0$ and $\phi_1 \neq 0, \pi$. Then $v_2 = v_1$ and $\phi_2 = \pi - \phi_1$.*
  – *Suppose that $\phi_0 = \pi$ and $\phi_1 \neq 0, \pi$. Then $v_2 = -v_1$ and $\phi_2 = \phi_1$.*
  – *Suppose that $\phi_0 = 0$ and $\phi_1 = 0$. Then $\mathcal{C} = \mathrm{I}$ and v_2 can be any point on the segment $\overline{v_0 v_1}$ and $\phi_2 = \pi$.*
  – *Suppose that $\phi_0 = 0$ and $\phi_1 = \pi$. Then $\mathcal{C} = \mathrm{I}$ and v_2 can be any point on the segment $\overline{v_0(-v_1)}$ and $\phi_2 = 0$.*
  – *Suppose that $\phi_0 = \pi$ and $\phi_1 = 0$. Then $\mathcal{C} = \mathrm{I}$ and v_2 can be any point on the segment $\overline{(-v_0)v_1}$ and $\phi_2 = 0$.*
  – *Suppose that $\phi_0 = \pi, \phi_1 = \pi$. Then $\mathcal{C} = \mathrm{I}$ and v_2 can be any point on the segment $\overline{(-v_1)(-v_2)}$, $\phi_2 = \pi$, and v_0, v_1, and v_2 are the vertices of a pointed hemisphere.*

- *Suppose that $v_0 = v_1$. Then $v_2 = v_0 = v_1$ and the angle $\phi_2 = \pi - \phi_0 - \phi_1$ if $\phi_0 \neq 0, \pi, \phi_1 \neq 0, \pi$. If $\phi_0 = 0, \phi_1 = \pi$ or $\phi_0 = \pi, \phi_1 = 0$, then $\phi_0 = 0$ and $\mathcal{C} = \mathrm{I}$ and v_2 can be any point of $\mathbf{S}^2$. If $\phi_0 = \phi_1 = 0$, then $\phi_2 = \pi$ and $v_2 = v_0$. if $\phi_0 = \phi_1 = \pi$, then $\phi_2 = \pi$ and $v_2 = -v_0$.*
- *Suppose that $v_0 = -v_1$. Then*
    - *If $\phi_0 > \phi_1$, then $v_2 = v_0$ and $\phi_2 = \pi - \phi_0 + \phi_1$.*
    - *If $\phi_1 > \phi_0$, then $v_2 = v_1$ and $\phi_2 = \pi + \phi_0 - \phi_1$.*
    - *If $\phi_0 = \phi_1$, then $\mathcal{C} = \mathrm{I}$ and $\phi_2 = \pi$ and v_2 can be any point of $\mathbf{S}^2$.*

2.3. The $\mathrm{SO}(3)$-character space of the fundamental group of a pair-of-pants

Let P be a pair-of-pants and let $\tilde{P}$ be the universal cover. Let c_0, c_1, and c_2 denote three boundary components of P oriented using the boundary orientation. Let $\pi_1(P)$ denote the fundamental group of P seen as a group of deck transformations generated by three elements $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ parallel to the boundary components of P satisfying $\mathcal{CBA} = I$. $\mathcal{A}$ acts on a lift $\tilde{c}_0$ of c_0 in $\tilde{P}$ and $\mathcal{B}$ on $\tilde{c}_1$ a lift of c_1 and $\mathcal{C}$ on $\tilde{c}_2$ a lift of c_2.

Take a triangle on the sphere $\mathbf{S}^2$ with geodesic edges so that each edge has length $< \pi$ so that the vertices are ordered in a clockwise manner in the boundary of the triangle. Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \tag{2.3}$$

$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i = 0, 1, 2, \tag{2.4}$$

where the indices are considered to be in $\mathbb{Z}_3$. The region gives us an open tetrahedron in the positive octant of $\mathbb{R}^3$ with vertices

$$(\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, \pi, \pi)$$

and thus we have $0 < \theta_i < \pi$. This is a regular tetrahedron with edge lengths all equal to $\sqrt{2}\pi$.

Let $\triangle_1$ be a nondegenerate spherical triangle with vertices removed and $\triangle_2$ an isometric triangle with vertices also removed. We can glue $\triangle_1$ and $\triangle_2$ along corresponding open edges and obtain an open pair-of-pants P' with an elliptic metric. We can attach the boundary component circles to P' with labeling $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ corresponding to the vertices. This gives us a geometric structure on P' modeled on $(\mathbf{S}^2, \mathrm{SO}(3))$. Hence, there is an associated representation $h : \pi_1(P') \to \mathrm{SO}(3)$ determined up to conjugation. Such a representation is said to be a *triangular representation*. This approach was used in Thurston's book [35] and Goldman [18].

Lemma 2.1. $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *contains a dense open set where each character is triangular. In fact, one can always find a path from any character to triangular characters where each point except the beginning point of the path is triangular.*
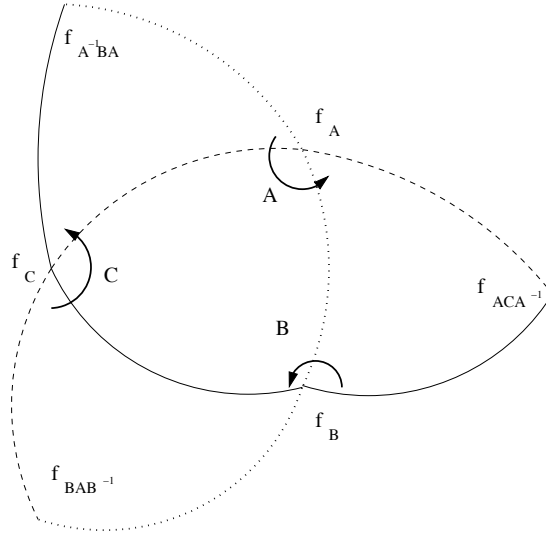
Fig. 7.   Triangular representations

Proof. Let h be a representation $\pi_1(P) \to \mathrm{SO}(3)$. Then $h(\mathcal{A}), h(\mathcal{B})$, and $h(\mathcal{C})$ are elements of $\mathrm{SO}(3)$ and hence contains a fixed point in $\mathbf{S}^2$. Let $f_{\mathcal{A}}$ and $-f_{\mathcal{A}}$ denote the fixed points of $h(\mathcal{A})$ and $f_{\mathcal{B}}$ and $-f_{\mathcal{B}}$ those of $h(\mathcal{B})$ and $f_{\mathcal{C}}$ and $-f_{\mathcal{C}}$ those of $h(\mathcal{C})$. The fixed points are chosen arbitrarily if there are more than one fixed points up to antipodal transformations, i.e., only when the matrix is the identity matrix.

We can choose one among each of the three antipodal pairs and obtain a triangle which might be degenerate. (The choice of the degenerate triangle can be arbitrary. See Section 1 for example of degenerate triangles. Essentially, if the holonomy of one of the generators is the identity, we fall into the edge regions, and if two or more are so, then we fall into to the vertex regions, and finally if the holonomy is abelian with no identity for generators, then we fall into the face regions.) Let $\triangle(f_{\mathcal{A}}, f_{\mathcal{B}}, f_{\mathcal{C}})$ denote such a triangle, and the vertices are oriented in a clockwise direction (See Figure 7).

- $h(\mathcal{A})$ fixes $f_{\mathcal{A}}$ and send $f_{\mathcal{C}}$ to $h(\mathcal{A})(f_{\mathcal{C}})$ chosen to be $f_{\mathcal{ACA}^{-1}}$ and $h(\mathcal{A})^{-1}(f_{\mathcal{B}})$ chosen to be $f_{\mathcal{A}^{-1}\mathcal{BA}}$ to $f_{\mathcal{B}}$. Hence, $h(\mathcal{A})$ send $\triangle(f_{\mathcal{A}}, f_{\mathcal{C}}, f_{\mathcal{A}^{-1}\mathcal{BA}})$ to $\triangle(f_{\mathcal{A}}, f_{\mathcal{ACA}^{-1}}, f_{\mathcal{B}})$.
- $h(\mathcal{B})$ fixes $f_{\mathcal{B}}$ and send $f_{\mathcal{A}}$ to $h(\mathcal{B})(f_{\mathcal{A}})$ chosen to be $f_{\mathcal{BAB}^{-1}}$ and send $f_{\mathcal{ACA}^{-1}}$ to $h(\mathcal{B})(f_{\mathcal{ACA}^{-1}}) = h(\mathcal{B})h(\mathcal{A})(f_{\mathcal{C}}) = h(\mathcal{C})^{-1}(f_{\mathcal{C}}) = f_{\mathcal{C}}$ as $\mathcal{BA} = \mathcal{C}^{-1}$ and hence sends $\triangle(f_{\mathcal{B}}, f_{\mathcal{A}}, f_{\mathcal{ACA}^{-1}})$ to $\triangle(f_{\mathcal{B}}, f_{\mathcal{BAB}^{-1}}, f_{\mathcal{C}})$.
- $h(\mathcal{C})$ fixes $f_{\mathcal{C}}$ and sends $f_{\mathcal{B}}$ to $h(\mathcal{C})(f_{\mathcal{B}}) = h(\mathcal{A})^{-1}h(\mathcal{B})^{-1}(f_{\mathcal{B}}) = h(\mathcal{A})^{-1}(f_{\mathcal{B}}) = f_{\mathcal{A}^{-1}\mathcal{BA}}$ as $\mathcal{C} = \mathcal{A}^{-1}\mathcal{B}^{-1}$ and sends $f_{\mathcal{BAB}^{-1}}$ to $h(\mathcal{C})(f_{\mathcal{BAB}^{-1}}) = h(\mathcal{C})h(\mathcal{B})(f_{\mathcal{A}}) = h(\mathcal{A})^{-1}(f_{\mathcal{A}}) = f_{\mathcal{A}}$ and hence sends $\triangle(f_{\mathcal{C}}, f_{\mathcal{B}}, f_{\mathcal{C}^{-1}\mathcal{BC}})$ to $\triangle(f_{\mathcal{C}}, f_{\mathcal{A}^{-1}\mathcal{BA}}, f_{\mathcal{A}})$.

Thus we obtain the diagram in Figure 7.

Since $h(\mathcal{A}), h(\mathcal{B})$, and $h(\mathcal{C})$ are isometries, we see that the three adjacent triangles are congruent triangles. By the length consideration, the four triangles are congruent to one another.

If one of the triangle is nondegenerate, then all are not and is congruent to $\triangle(f_{\mathcal{A}} f_{\mathcal{B}} f_{\mathcal{C}})$.

Thus, if one of the triangle is degenerate, then all other are degenerate also. h corresponds to an abelian representation and it fixes a pair of antipodal points $x, -x$. A nonidentity abelian representation is determined by a three angles θ_0, θ_1, and θ_2 at a fixed point x of $h(\mathcal{A}), h(\mathcal{B})$, and $h(\mathcal{C})$ where $\theta_0 + \theta_1 + \theta_2 = 2n\pi$ for $n = 1, 2$. If $\theta_0 + \theta_1 + \theta_2 = 2\pi$, then we see that the region a gives us all such abelian representation. If $\theta_0 + \theta_1 + \theta_2 = 4\pi$, then we may assume without loss of generality that $\theta_0 > \pi$ and we take as the fixed point of $h(\mathcal{A})$ as $-x$. Then we see that a point of the face c represents it. Thus, every representation comes from a generalized triangle construction.

If the triangle corresponding to a representation is degenerate, then we can continuously change the lengths and angles of the triangles to make them nondegenerate by Proposition 1.1 and the deformation creates a representation near h as possible by the density of the nondegenerate triangles in $\tilde{\Delta}$. Hence, the triangular representations are dense.

Also, if the triangles are not degenerate, then the representation is triangular by definition. A sufficiently nearby representation to any triangular representation has nondegenerate triangles, and hence, they are triangular. Thus the set of triangular characters are open. $\qquad\square$

Proposition 2.2. $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *is homeomorphic to the quotient of the tetrahedron* Δ *by a* $\{I, I_A, I_B, I_C\}$-*action. (See Figure 1.)*

Proof. The set of triangular representations up to conjugacy are classified by the open tetrahedron of angle parameters determined by equation (2.4). The ambiguity is given by the choice of fixed points. In the interior, this is precisely the ambiguity. We can choose three noncollinear fixed points of $h(\mathcal{A}), h(\mathcal{B})$, and $h(\mathcal{C})$ respectively as long as the vertices in the boundary appear in the clockwise-direction. Thus, we obtain that V is generated by the maps

$$I_A : (v, w, z) \mapsto (v, -w, -z), \tag{2.5}$$

$$I_B : (v, w, z) \mapsto (-v, w, -z), \tag{2.6}$$

$$I_C : (v, w, z) \mapsto (-v, -w, z). \tag{2.7}$$

Respectively, these maps make the angles change by

$$I_A : (\theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \pi - \theta_2, \pi - \theta_3) \tag{2.8}$$

$$I_B : (\theta_1, \theta_2, \theta_3) \mapsto (\pi - \theta_1, \theta_2, \pi - \theta_3) \tag{2.9}$$

$$I_C : (\theta_1, \theta_2, \theta_3) \mapsto (\pi - \theta_1, \pi - \theta_2, \theta_3). \tag{2.10}$$

In the boundary of the tetrahedron, all fixed points with angles not equal to π, are either identical or antipodal. Since the angle π, corresponds to the identity transformation, we can regard any two such transformations with same angles up to the V-action to be the same. As they are abelian characters, we see that there are no more identification other than ones obtained from the V-action. $\square$

3. SO(3) and SU(2): geometric relationships

Here, we discuss some well-known geometric relationships between SO(3) and SU(2). Using them, we will show that the SU(2)-character space of the fundamental group of a pair-of-pants is a tetrahedron branch-covering the SO(3)-character space of the same group. This is a result known to Ramanan [33] and Hubueschmann [22] and recently written up by Florentino and Lawton [20] in some generality. Finally, for later purposes, we reprove these by considering the fundamental group as a free group of rank two. This has a different flavor and will be used later.

3.1. *The covering group*

We can characterize an element of SO(3) by noting its antipodal pair of fixed points and the rotation angles at the points related by θ to $2\pi - \theta$. Thus, SO(3) can be identified with $\mathbb{R}P^3$ in the following way: Take B^3 of radius π in $\mathbb{R}^3$ and parameterize each segment from the origin to the boundary by the distance to the origin. Then for each nonidentity $g \in$ SO(3), we choose the fixed point with angle $\theta < \pi$ and take the point in the ray to the point in B^3 of distance θ from the origin. If $\theta = \pi$, then we take both points in the boundary $\mathbf{S}^1$ of B^3 in the direction and identified them.

Taking the quotient space, we obtain $\mathbb{R}P^3$. This correspondence gives a diffeomorphism SO(3) $\rightarrow \mathbb{R}P^3$.

The space of quaternions can be described as the space of matrices

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}, x, y \in \mathbb{C}.$$

Elements of SU(2) can be represented as elements of the unit 3-sphere in the space of quaternions and the action is given by left-multiplications. There is a unit sphere in the purely quaternionic subspace in the space of quaternions, i.e.,

$$\begin{pmatrix} i\beta & \gamma + i\delta \\ -\gamma + i\delta & -i\beta \end{pmatrix}, \beta, \gamma, \delta \in \mathbb{R}, \beta^2 + \gamma^2 + \delta^2 = 1.$$

SU(2) acts on $\mathbf{S}^2$ by conjugation. This gives a homomorphism from SU(2) to SO(3) with kernels $\{I, -I\}$ where I is a 2×2 identity matrix, which provides us with a double cover SU(2) $\cong \mathbf{S}^3 \rightarrow$ SO(3) $\cong \mathbb{R}P^3$.

Since SU(2) double-covers SO(3), the Lie group SU(2) is diffeomorphic to $\mathbf{S}^3$. We realize the diffeomorphism in the following way: we take the ball B_2^3 of radius 2π so that the boundary is identified with a point. Hence, we obtain $\mathbf{S}^3$. Let $||v||$

denote the norm of a vector v in B_2^3. Take the map from $B_2^3 \to B^3$ given by sending a vector v to v if $||v|| \leq \pi$ or to $(\pi - ||v||)v$ if $||v|| > \pi$. This is a double-covering map clearly.

By lifting, we obtain an identification of SU(2) to $B_2^3/\sim$ by collapsing the sphere at radius 2π to a point to be regarded as $-I$. Thus, every element of SU(2) unless I or $-I$ can be identified by a choice of a fixed point in $\mathbf{S}^2$ and the angle of rotation between 0 and 2π. Hence, it is to be regarded as an element of SO(3) with a choice of a fixed point and the angle of rotation as seen from that point. Furthermore, I can be considered a rotation with any chosen fixed point and 0 rotation angle and $-I$ a rotation with any fixed point and a 2π rotation angle.

Since we have $R_{x,\theta} = R_{-x,4\pi-\theta}$, an element of SU(2) can be considered as a fixed point of $\mathbf{S}^2$ with angles in $[-2\pi, 2\pi]$ where -2π and 2π are identified or with angles in $[0, 4\pi]$ where 0 and 4π are identified.

Each one parameter family of a commutative subgroup of SU(2) double-covers a one-parameter family of a commutative subgroup in SO(3). Hence, we have a circle $\mathbf{S}_2^1$ of twice the circumference in SU(2).

We denote by $R_{x,\theta}$ for $x \in \mathbf{S}^2$ and $\theta \in [-2\pi, 2\pi] \mod 4\pi$ or $\theta \in [0, 4\pi] \mod 4\pi$, the element of $B_2^3/\sim$ represented as above. This element lifts $R_{x,\theta'}$ for $\theta' = \theta \mod 2\pi$. Given an element of SU(2) with a fixed vector and a rotation angle in $(0, 2\pi)$, the multiplication has the effect of taking the antipodal vector as the fixed vector:

$$-IR_{w,\theta} = R_{w,2\pi}R_{w,\theta} = R_{w,2\pi+\theta} = R_{-w,4\pi-2\pi-\theta} = R_{-w,2\pi-\theta}. \qquad (3.1)$$

Note that this also holds when $\theta = 0, \pi$.

(Clearly, the meaning of $R_{x,\theta}$ changes depending on the context whether we are in SO(3) or in SU(2).)

Definition 3.1. By choosing θ to be in $(0, 2\pi)$, $R_{x,\theta}$ is now a point in $B_2^{3,o} - \{O\}$. Thus, each point of $\mathbf{S}^3 - \{I, -I\}$, we obtain a unique rotation $R_{x,\theta}$ for $\theta \in (0, 2\pi), x \in \mathbf{S}^2$ and conversely. Hence, we have a normal representation of each element of SU(2) $- \{I, -I\}$ as a rotation with angles in $[0, 2\pi]$ with a fixed point in $\mathbf{S}^2$.

The "multiplication by geometry" is also available for SU(2): Let w_0, w_1, and w_2 be vertices of a triangle, possibly degenerate, oriented in the clockwise direction. Let e_0, e_1, and e_2 denote the opposite edges. Let θ_0, θ_1, and θ_2 be the respective angles for $0 \leq \theta \leq 2\pi$. Then

$$R_{w_2,\theta_2}R_{w_1,\theta_1}R_{w_0,\theta_0} = -I.$$

Here the minus sign is needed. We can show this by continuity since we can let $w_0 = w_1 = w_2$ and this holds (That is, in this case, θ_i are angles of a Euclidean triangle. The sum of two times the angles of a Euclidean triangle is 2π.)

We can extend angles further. An *immersed* triangle is a disk bounded by three arcs meeting at three-points mapping with injective differential to $\mathbf{S}^2$ so that the

arcs map as geodesics. Here the angle at the vertex is measured by breaking it into smaller angles so that the angles inject and summing up the values. Actually, we restrict it so that the angles are always $\leq 2\pi$.

If none of the angles equal to 2π or 0, then the disk is either

- a hemisphere H with a nondegenerate triangle T, $T \subset H$, with an edge in the boundary of the hemisphere removed,
- a hemisphere H with a nondegenerate triangle T, $T \subset \mathbf{S}^2 - H^o$, with an edge in the boundary of the hemisphere added, or
- the exterior of a nondegenerate triangle or a pointed lune,

In the first case two angles are $< \pi$ and one $> \pi$, in the second case two angles are $> \pi$ and one $< \pi$, and in the last case, all three angles are $> \pi$ or two $> \pi$ and one $= \pi$. If one of the angles is zero or 2π, we do not have a definition at this stage. (There are more we can include as "immersed triangles" but we do not need them in this paper)

Thus, for general θ_i arising from an immersed triangle, we see that

$$R_{w_2, 2\theta_2} R_{w_1, 2\theta_1} R_{w_0, 2\theta_0} = -\mathrm{I}.$$

if θ_is are angles of a triangle oriented counter-clockwise and w_0, w_1, and w_2 are ordered in the clockwise orientation from the disk. (Of course, the above equation holds with I replacing $-\mathrm{I}$ in the SO(3)-cases.)

To summarize, denoting the rotation at w_0, w_1, and w_2 by $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ respectively, where w_0, w_1, w_2 are in clockwise orientation as seen from inside of an immersed triangle, we obtain

$$\mathcal{CBA} = -\mathrm{I}, \mathcal{C}^{-1} = -\mathcal{BA}, \mathcal{C} = -\mathcal{A}^{-1}\mathcal{B}^{-1}. \tag{3.2}$$

Thus, $\mathcal{BA}$ is obtained from multiplying $-I$ to the geometrically constructed one. That is $\mathcal{BA}$ can be considered as the element of SU(2) with fixed point the antipode of the vertex of the triangle and the angle 2π minus the constructed angle. (See Section 2.2.)

Of course, if one is considering only representations, we can always find an embedded triangle with all edge length $\leq \pi$ giving us same elements of SU(2).

Proposition 3.1. *Suppose that we have a generalized triangle with vertices v_0, v_1, and v_2 occurring in the clockwise order with angles ϕ_0, ϕ_1, ϕ_2 respectively. Let $\mathcal{A}$ be a transformation with a fixed point v_0 and rotation angle ϕ_0, $\mathcal{B}$ be one with v_1 and ϕ_1, and $\mathcal{C}$ be one with v_2 and ϕ_2 respectively satisfying $\mathcal{CBA} = -\mathrm{I}$. Then the conclusions of Proposition 2.1 hold.*

By Proposition 2.2, $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ can be identified with the tetrahedron: Clearly, $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ maps onto $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$. In the interior of the second space, each element corresponds to the triangle with angles between 0 and π. This gives a unique character with a representation h in the $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ by taking $h(c_0)$ and $h(c_1)$ as given in the triangle and taking as $h(c_2)$ the $-\mathrm{I}$ times the

one given by the triangle. The tetrahedron Δ parametrizing $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ now lifts to $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ by lifting triangles to the $\mathrm{SU}(2)$-characters. (We need a bit of care when some of $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ correspond to I or $-I$.) The map ι_Δ from the tetrahedron to $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ is one-to-one since the distinct pair of angles in $[0, 2\pi]$ in the normal representation imply different $\mathrm{SU}(2)$-elements up to conjugations. (See Definition 3.1) The map from the tetrahedron Δ to $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ is a branch-covering map with the branch locus equal to the union of the three axes. Removing the union of the axes from the tetrahedron, we obtain a map from the space Δ' to the open subspace S of $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ where the angles are all distinct from π or $\pi/2$. We know that S is a cover of the open subset of $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ where all the angles are distinct from π or $\pi/2$. Since so is Δ', we see that the map ι_Δ is onto. Since this correspondence is one-to-one map from a compact space to a Hausdorff space, we obtain the following:

Proposition 3.2. $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ *is homeomorphic to the tetrahedron, and map to* $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ *as a four to one branched covering map induced by the Klein four-group* V*-action.* $\square$

We note here that given boundary components c_0, c_1, and c_2 of P, by letting c_0 and c_1 be regarded as generators of $\pi_1(P)$, we see that $-h(c_2)$ is obtained from triangle constructions, i.e., $-h(c_2)^{-1} = -h(c_1)h(c_0)$. Thus, if we use, the angles of the fixed point of $h(c_i)$ in $\mathrm{SU}(2)$, then we must use 2π minus two times the angle of third vertex of the triangle since we need to take the antipodes.

Hence, by true angle parameterization, $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ is parametrized by the tetrahedron Δ' given by sending $(\theta_0, \theta_1, \theta_2) \to (2\theta_0, 2\theta_1, 2\pi - 2\theta_2)$ where Δ' lies in in the positive octant of $\mathbb{R}^3$ and is given by the inequalities $x + y \le z$, $x + z \le y$, $y + z \le x$, and $x + y + z \le 4\pi$. (See Proposition 3.5 of [24] and also [25].)

Remark 3.1. However, we will use our tetrahedron Δ to parameterize $\mathrm{rep}(\pi_1(P), \mathrm{SU}(2))$ because one can read off the characters to $\mathrm{SO}(3)$ directly with the understanding that $h(c_2)$s are the antipodal ones to what are obtained from the triangle constructions.

It will be an important point in this paper that we will be "pretending" that our $\mathrm{SU}(2)$-characters correspond to triangles. This will not cause any trouble in fact.

3.2. *The* SO(3)*- and* SU(2)*-character spaces of the free group of rank two*

This will be a short section: Let F_2 denote a free group of two generators. We will define $H(F_2)$ as the space of homomorphisms $F_2 \to \mathrm{SO}(3)$ quotient by the conjugation action by $\mathrm{SO}(3)$. That is,

$$H(F_2) = \mathrm{Hom}(F_2, \mathrm{SO}(3))/\mathrm{SO}(3).$$

Of course F_2 is isomorphic to $\pi_1(P)$ for a pair-of-pants P so the above subsections covered what we intend to redo. (See Propositions 2.2 and 3.2). The reason we *redo*

this is since we definitely need some modification of the description in this subsection later. (The descriptions as $\pi_1(P)$ using triangles is also needed at other places.)

In this section, we will use the generators directly:

Let g_0 and g_1 be the two generators of F_2. Consider a representation $h : F_2 \to$ SO(3). The $h(g_i)$ has a pair of fixed points f_i and $-f_i$ antipodal to each other for each $i = 0, 1$. Then f_0, $-f_0$, f_1, and $-f_1$ lie on a great circle which is unique if $\{f_0, -f_0\}$ are distinct from $\{f_1, -f_1\}$. Since up to SO(3)-action, the configuration is determined by the distance between the pair, we chose a maximal arc in the complement of $\{f_0, -f_0, f_1, -f_1\}$. Then this pair is described by the length l of the arc and the counter-clockwise respective angles θ_0 and θ_1 of rotations of g_0 and g_1 at the corresponding endpoints of l. Thus $0 \leq l \leq \pi$ and $0 \leq \theta_i \leq 2\pi$ for $i = 0, 1$. Since the choice of the arc is not unique, there are maps

$$T_0 : (l, \theta_0, \theta_1) \mapsto (\pi - l, 2\pi - \theta_0, \theta_1)$$
$$T_1 : (l, \theta_0, \theta_1) \mapsto (\pi - l, \theta_0, 2\pi - \theta_1)$$
$$T_0 T_1 = T_1 T_0 : (l, \theta_0, \theta_1) \mapsto (l, 2\pi - \theta_0, 2\pi - \theta_1) \tag{3.3}$$

(This is again called a *Klein four-group action* and is identical with the Klein four-group action for the character spaces of a pair-of-pants. We will need the maps later in Appendix A.3 and Section 8.3.) Also, if $\theta_i = 0$, then l is not determined. Therefore, there is another equivalence relation

$$(l, 0, \theta_1) \sim (l', 0, \theta_1), (l, \theta_0, 0) \sim (l', \theta_0, 0), 0 \leq l, l' \leq \pi, \tag{3.4}$$

for any θ_0 and θ_1. It is straightforward to see that these are all possible equivalence classes under the conjugation action of SO(3).

Proposition 3.3. $H(F_2)$ *is homeomorphic to a* 3-*ball. The boundary of the* 3-*ball corresponds to precisely the subspace of abelian characters of* F_2.

Proof.

Let $I = [0, \pi]$. We consider the space $I \times \mathbf{S}^1 \times \mathbf{S}^1$ parameterizing the above configurations of arc and isometries with fixed points at the end. There is a surjective (continuous) map $Q : I \times \mathbf{S}^1 \times \mathbf{S}^1 \to H(F_2)$ defined by sending the configuration to the equivalence class of representations sending g_0 and g_1 to the isometries with the fixed points at the end of the arc.

Let $\sim$ be given by equations (3.3) and (3.4). We see that the quotient space $B = I \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim$ is mapped to $H(F_2)$ in a one-to-one and onto manner. There is an inverse map defined on $H(F_2)$ by sending each character to (l, θ_0, θ_1) by doing the procedure before this proposition. Hence B is homeomorphic to $H(F_2)$.

We now show that the quotient space B is homeomorphic to B^3.

All points of B have equivalent points on $[0, \pi/2] \times \mathbf{S}^1 \times \mathbf{S}^1$. We take a domain $D = [0, \pi/2] \times [0, \pi] \times [0, \pi]$. Then by the equivalence relation generated by $T_0 T_1 = T_1 T_0$ acting fiberwise on $[0, \pi/2] \times \mathbf{S}^1 \times \mathbf{S}^1$, the quotient image D' of D is homeomorphic to $[0, \pi/2] \times \mathbf{S}^2$

The equivalence relation generated by T_0 is the same as that of T_1. Only the point in $\{\pi/2\} \times \mathbf{S}^2$ are involved here. A circle S^1 in $\{\pi/2\} \times \mathbf{S}^2$ contains the four singular points. The equivalence can be described by an isometric reflection on a circle with some metric on a sphere of constant curvature 1. This last identification gives us a ball B^3.

The final identification corresponding to equation (3.4) collapses two rectangular disks in B_3 with one edge in the boundary sphere ∂B_3 meeting each other at a connected boundary edge. The rectangular disks are foliated by arcs in I-directions, and hence can be considered strips. The two disks are called the I-collapsing quadrilaterals. The collapsing the strips in I-direction gives us the final result.

Since abelian characters correspond to the configurations where one of the angle is zero, they are in the I-collapsed part, or in where the segment is of length 0 or π. We see that these points correspond to the points of boundary of the ball B_3 when collapsed as above. $\square$

The last collapse is said to be the I-collapse.

3.3. *The* SU(2)-*character space*

Now we go over to the SU(2) case. Define

$$H_{\mathrm{SU}(2)}(F_2) := Hom(F_2, \mathrm{SU}(2))/\mathrm{SU}(2).$$

Then since F_2 is isomorphic to $\pi_1(P)$, the space is homeomorphic to a tetrahedron Δ.

To understand in term of the above picture, we look at two points in $\mathbf{S}^2$ with angles in the range $(0, 2\pi)$. This determines a unique pair of points and the distance between them if the angles are not 0 or 2π. Parameterizing by the distance and the angle, we obtain $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ by picking only the fixed points with angle in $[0, 2\pi]$. The equivalence relation is given by $(l, 0, \theta_2) \sim (l', 0, \theta_2)$ for any l, l' and $(l, 2\pi, \theta_2) \sim (l', 2\pi, \theta_2)$ for any l, l' and $(l, \theta_1, 0) \sim (l', \theta_1, 0)$ for any pair l, l' and $(l, \theta_1, 2\pi) \sim (l', \theta_1, 2\pi)$ for any pair l, l'.

These lifts the I-collapsing above and there are no other nontrivial equivalence relations. This contracts sides of the cube in the l direction. Thus, we obtain a 3-ball.

Since $[0, \pi] \times [0, 2\pi] \times [0, 2\pi]$ represents $[0, \pi] \times \mathbf{S}^1 \times \mathbf{S}^1$ up to identification, we see that there is a four-to-one branch covering map to $H(F_2)$ as generated by transformations of forms T_0 and T_1 as in equation (3.3).

The following is clear:

Proposition 3.4. $H_{\mathrm{SU}(2)}(F_2)$ *is homeomorphic to a 3-ball. The boundary of the 3-ball precisely corresponds to the subspace of abelian characters in $H(F_2)$.*

4. The character space of a closed surface of genus 2.

We will now discuss the character space of the fundamental group of a closed orientable surface Σ of genus 2.

First, we discuss the two-components of the character space. Next, we discuss how to view a representation as two related representations of the fundamental groups of two pairs-of-pants glued by three pasting maps.
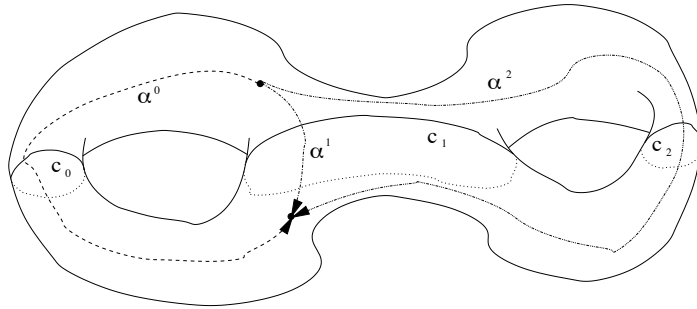

Fig. 8. Σ and closed curves.

4.1. *Two components*

We take three simple closed curves c_0, c_1, and c_2 on Σ so that the closures of the two components of the complement of their union are two pairs-of-pants S_0 and S_1 so that $S_0 \cap S_1 = c_0 \cup c_1 \cup c_2$. We assume that there is an order two diffeomorphism ϕ acting on c_0, c_1, and c_2 respectively and sending S_0 to S_1 and vice versa. We assume that c_i are oriented with boundary orientation from S_0.

There are two components of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ as shown by Goldman [15]. To summarize, each representation corresponds to a flat $\mathbb{R}^3$-bundle over Σ. The Stiefel-Whitney class in $H^2(\Sigma, \pi_1(\mathrm{SO}(3))) = \mathbb{Z}_2$ of the flat bundle classifies the component. A component $\mathcal{C}_0$ contains the identity representation, i.e., a representation which send every element to the identity. Let c_1, d_1, c_2, and d_2 are the simple closed curves which represents a fixed set of the standard generators of $\pi_1(\Sigma)$ so that c_1, c_2 are obtained from the boundary components of S_1 by conjugating by simply disjoint paths in S_1. (There is a slight notational abuse here.) The other component to be denoted by $\mathcal{C}_1$ contains the representation h^* which sends generators corresponding to c_1 and d_1 a simple closed curve meeting c_1 once transversely and disjoint from c_2, d_2 to the following matrices respectively

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } B_1 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

And h^* sends c_2 and a simple closed curve d_2 meeting c_2 once transversely and disjoint from c_1, d_1 to the identity matrices respectively

Let this representation be named the *other-component basis representation*.

Remark 4.1. The representation belongs to $\mathcal{C}_1$:

The above matrices A_1 and B_1 respectively correspond to

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

in SU(2) whose commutator equals $-I$. This does not give a representation $\pi_1(\Sigma) \to$ SU(2) but will induce a representation $\pi_1(\Sigma) \to$ SO(3) whose Stiefel-Whitney class is not trivial.

4.2. *Representations considered with pasting maps*

Let us choose a base point x^* of Σ in the interior of S_0. Given a representation $h : \pi_1(\Sigma) \to$ SO(3), we obtain a representations $h_0 : \pi_1(S_0) \to$ SO(3) and $h_1 : \pi_1(S_1) \to$ SO(3). The second one is determined only up to conjugation. A representation h defined on $\pi_1(\Sigma)$ is *triangular* if the restrictions to $\pi_1(S_0)$ and $\pi_1(S_1)$ are triangular. Let us restrict ourselves to triangular representation in rep$(\pi_1(\Sigma), $SO(3)$)$ for now.

Let c_0^0, c_1^0, and c_2^0 denote the oriented simple closed curves on S_0 with base point x_0^* that are freely homotopic to c_0, c_1, and c_2 respectively, and bounding annuli with the curves respectively. Let us choose a base point x_1^* in S_1 and oriented simple closed curves c_0^1, c_1^1, and c_2^1 homotopic to c_0, c_1, and c_2. Let α^i denote a simple arc from x_0^* to x_1^* whose interior is disjoint from c_j for $j \neq i$ and meets c_i exactly once. We can choose so that $\alpha^0 * c_0^1 * \alpha^{0,-1}$ is homotopic to c_0^0.

Let d_1 denote the closed loop based at x_0^* meeting c_0 and c_1 only once transversally and disjoint from c_2 oriented from c_0 to c_1 direction. Let d_2 denote the closed loop based at x_0^* meeting c_0 and c_2 only once transversally and disjoint from c_1 oriented from c_2 to c_0 direction. We assume that $\alpha^1 * \alpha^{0,-1}$ is homotopic to d_1, $\alpha^2 * \alpha^{0,-1}$ is homotopic to d_2, and c_1^0, c_2^0, d_1, and d_2 generate the fundamental group $\pi_1(\Sigma)$ and they satisfy the relation

$$[c_1^0, d_1][c_2^0, d_2] = 1.$$

(We identify $\pi_1(S_1)$ with $\alpha_0^{-1} * i_*(\pi_1(S_1)) * \alpha_0$.)

If $h_0 : \pi(S_0) \to$ SO(3) and $h_1 : \pi_1(S_1) \to$ SO(3) are induced from a representation $h : \pi_1(\Sigma) \to$ SO(3), then $h_0(c_i^0)$ are conjugate to $h_1(c_i^1)$ by an element P_i of SO(3), i.e.,

$$P_i h_0(c_i^0) P_i^{-1} = h_1(c_i^1) \text{ for } i = 0, 1, 2.$$

We call P_i the pasting map for c_i for $i = 0, 1, 2$.

If h_0 and h_1 are obtained from restricting h, then $P_0 = I$. If we change $h_1(\cdot)$ to be $h_1'(\cdot) = k \circ h_1(\cdot) \circ k^{-1}$, then the new pasting maps become

$$P_0' = kP_0, P_1' = kP_1 \text{ and } P_2' = kP_2. \tag{4.2}$$

so that

$$P_i' h_0(c_i^0) P_i'^{-1} = h_1'(c_i^1) \text{ for } i = 0, 1, 2. \tag{4.3}$$

Proposition 4.1. *We have*

$$h(d_1) = P_0^{-1} P_1, h(d_2) = P_0^{-1} P_2.$$

Proof. When $k = \mathrm{I}$, the equation holds. By equation (4.2), it follows. □

The space $\mathrm{rep}(\pi_1(P), \mathrm{SO}(3))$ is classified as a quotient space of a tetrahedron. Thus, a character is determined by angles θ_0, θ_1, and θ_2 in $[0, \pi]$ as the halves of the rotation angles. The equation (4.3) shows that if h_0 is represented by triple $(\theta_0, \theta_1, \theta_2)$, the h_1 is represented by $(\theta_0', \theta_1', \theta_2')$ so that $\theta_i' = \theta_i$ or $\theta_i' = \pi - \theta_i$ for $i = 0, 1, 2$.

We showed:

Proposition 4.2. *Let h_0 and h_1 be representations of the fundamental groups of pairs-of-pants S_0 and S_1 from a $\mathrm{SO}(3)$-representation h. The angles $(\theta_0, \theta_1, \theta_2)$ of h_0 and $(\theta_0', \theta_1', \theta_2')$ of h_1 satisfy the equation*

$$\theta_i' = \theta_i \ or \tag{4.4}$$

$$\theta_i' = \pi - \theta_i \ for \ i = 0, 1, 2. \tag{4.5}$$

5. Establishing equivalence relations on $\tilde{\Delta} \times T^3$ to make it equal to $\mathcal{C}_0$.

In this major section, we will study the identity component $\mathcal{C}_0$ by finding the equivalence relation on $\tilde{\Delta} \times T^3$ giving us the same representations up to conjugacy, i.e., to find a point of $\mathcal{C}_0$.

This will be done by considering each of the regions of $\tilde{\Delta}$:

$$\Delta^o, a, b, c, d, A, A', B, B', C, C', I, II, III, IV$$

individually and considering the action of the Klein four-group $\{I, I_A, I_B, I_C\}$ extended to $\tilde{\Delta} \times T^3$.

We begin by discussing pasting angles. Next, we define a map $\mathcal{T}$ from $\tilde{\Delta} \times T^3$ to $\mathcal{C}_0$ corresponding the finding holonomy for the pasting angles. We will show that the map is onto by showing the characters corresponding to nondegenerate triangles are dense.

Then we discuss the equivalence relation $\sim$ on $\tilde{\Delta} \times T^3$ so that the quotient map induces a homeomorphism $\tilde{\Delta} \times T^3 / \sim \to \mathcal{C}_0$. Next, we discuss the quotient relation for the interior and move to the boundary regions denoted by a, b, c, and d, I, II, III, and IV and finally regions B, B', A, A', C and C'. Mainly, we will induce the quotient relations to make the map $\mathcal{T}$ injective. The final result of the section is Theorem 5.1 establishing the homeomorphism of the appropriate quotient space of $\tilde{\Delta} \times T^3$ to $\mathcal{C}_0$.

In this section, we will not determine the topology of the quotient spaces but will only explain the equivalence relations. The geometric nature of equivalence relations instead of algebraic equivalences are emphasized here to gain more a geometric point of view of the subject instead of algebraic one.

The equivalence relation *is* the equivalence relation to obtain the identical SO(3)-characters. However, we chose not to simply state this algebraic fact to preserve the geometric flavor of the paper and to gain the topological characterizations of the subspaces in the next section. The major point is that the equivalence relations over each of the faces and edges can be described separately. We will put most of the argument in the Appendix A.

5.1. *The density of triangular characters in* $\mathcal{C}_0$.

Lemma 5.1. *For* $i = 0, 1, 2$, *define a function* $f_i : \operatorname{rep}(\pi_1(\Sigma), \operatorname{SO}(3)) \to \mathbb{R}$ *by* $f_i : [h] \mapsto \operatorname{tr}h(c_i)$. *Then the triangular subset* D_0 *of* $\mathcal{C}_0$ *is not empty and* f_i *is not constant.*

Proof. The triangular part of $\mathcal{C}_0$ can be constructed by taking two identical triangles with identity to construct representations corresponding to $\pi_1(S_0)$ and $\pi_1(S_1)$. Here clearly, c_i has nonconstant rotation angles in D_0. Thus f_i is not constant. $\square$

Proposition 5.1. *The subset* D_0 *of* $\mathcal{C}_0$ *of triangular characters is dense and open and moreover, any character can be deformed to a triangular one by an analytic path consisting of triangular character except at the end.*

Proof. The subset where $h(c_i) = \mathrm{I}$ for the associated representations h is given by an analytic subset of $\mathcal{C}_0$. Define a real analytic map $f_i : [h] \mapsto \operatorname{tr}h(c_i)$ for each $i = 0, 1, 2$ which is well-defined. Since the function is not constant on each component of $\mathcal{C}_0$, by the standard result in real algebraic geometry, the set of points where $f_i = 3$ is an analytic subset whose complement is dense and open. Finally, the set of abelian ones are of lower dimension 4 and is an analytic subset, and the result follows. $\square$

5.2. *Pasting angles*

Proposition 5.2. *If* h *is in the identity component* $\mathcal{C}_0$ *of* $\operatorname{rep}(\pi_1(\Sigma), \operatorname{SO}(3))$, *and* h_0 *and* h_1 *are obtained as above by restrictions to* S_0 *and* S_1, *then*

   (a) *We can conjugate* h_1 *so that* $h_0 = h_1$ *and corresponding angles are equal.*
   (b) *For each representation* h *in* $\mathcal{C}_0$, *we can associate a pair of identical degenerate or nondegenerate triangles and an element of* $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, *i.e., the parameter space of the pasting angles. (This is not normally a unique association for the degenerate triangle cases.)*

Proof. (a) This is proved by a continuity argument. We look at the dense open set $D_0 \subset \mathcal{C}_0$ of triangular characters from Proposition 5.1. Taking a triangular representation $h_0 : \pi_1(S_0) \to \mathrm{SO}(3)$ associated with a triangle of angles all equal to $2\pi/5$ and $h_1(S_1) \to \mathrm{SO}(3)$ associated with the same triangle, then $h_0 = h_1$. Let $h^0 : \pi_1(\Sigma) \to \mathrm{SO}(3)$ be the representation obtained from h_0 and h_1 with pasting maps P_is equal to I.

There is an open subset of D_0 where induced h_0 and h_1 have angles θ_i' and θ_i so that $\theta_i' = \theta_i$ for $i = 0, 1, 2$: At h^0, we see that $\theta_i' = \theta_i$ for $i = 0, 1, 2$ by construction. As a function of D_0, θ_i and θ_i' form continuous functions since these are obtainable as traces of holonomies of boundary curves. So by continuity and equation (4.4), it follows that there exists a neighborhood where the angles are equal. Since triangular representations are determined by angles, we can conjugate so that $h_0 = h_1$.

By continuity argument again, the equation holds for all of $\mathcal{C}_0$.

(b) From (a), the equation is true for elements of D_0 as the elements of $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ are determined by the rotation angles of pasting maps P_0, P_1, P_2 at the respective vertices of the triangle.

For the element d in the boundary of D_0, we take a sequence $d_i \in D_0$ converging to d. Associated to each d, we obtain a triangle, of $\tilde{\Delta}$ and an elements P_i with pasting angles $(\phi_0^i, \phi_1^i, \phi_2^i)$ in $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. We can take a subsequence so that $(\tilde{T}_i, \phi_0^i, \phi_1^i, \phi_2^i)$ is convergent. Let $(\tilde{T}_\infty, \phi_0^\infty, \phi_1^\infty, \phi_2^\infty)$ denote the limit. The associated triangle $\tilde{T}_i$ with angles assigned converges a generalized pointed triangle $\tilde{T}^\infty$ with assigned angles that are respective limits of the sequences of angles of P_i.

Since $\mathrm{SO}(3)$ is compact, each pasting map P_j^i, $j = 0, 1, 2$, for d_i converges to an automorphism P_j^∞ with angle θ_j^∞ for $j = 0, 1, 2$. It is straightforward to see that d can be realized as a pair of generalized triangles $\tilde{T}^\infty$ and its copy and pasting maps P_j^i, $j = 0, 1, 2$. □

5.3. *The definition of the map* $\mathcal{T}$.

We will now define a map from $\tilde{\Delta} \times T^3$ to $\mathcal{C}_0$ which is almost everywhere 4 to 1. Let us define a map

$$\mathcal{T} : \tilde{\Delta} \times T^3 \to \mathcal{C}_0 \subset \mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3)).$$

Take an element of $\tilde{\Delta}$. This gives us a triangle perhaps degenerate with angles assigned. Actually, the map is induced from a map

$$\mathcal{T}' : \hat{\Delta} \times T^3 \to C_0' \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SO}(3))$$

where C_0' is the component corresponding to $\mathcal{C}_0$ and there is a commutative diagram:

$$
\begin{array}{ccc}
\hat{\Delta} \times T^3 & \xrightarrow{\mathcal{T}'} & C_0' \subset \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SO}(3)) \\
{\scriptstyle q_0} \downarrow & & {\scriptstyle q_1} \downarrow \\
\tilde{\Delta} \times T^3 & \xrightarrow{\mathcal{T}} & \mathcal{C}_0 \subset \mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3)).
\end{array}
$$

Here, q_0 is the quotient map under the isometric SO(3)-action on the metric space of generalized triangles and q_1 one under the conjugation action of the space of representations.

We define the map $\mathcal{T}'$ from $\hat{\Delta} \times T^3$. If the triangle is nondegenerate, then we can glue two copies with vertices removed along the open edges and obtain a pair-of-pants, from which we obtain a representation of $\pi_1(S_0) \to$ SO(3) for a pair-of-pants S_0. We obtain the second pair-of-pants S_1 using the exactly the same method. We choose base points x_i^* for on S_i respectively for $i = 0, 1$. Let S_i to be a union of two triangles T_i^0, T_i^1 with vertices removed and where T_i^0 has the base-point of S_i for $i = 1, 2$. We put T_i^1 in the standard position and obtain the elements of SO(3) corresponding to c_j^i. We see that T_i^0 are identical with T_i^1 and so are the representations. Using $(\phi_0, \phi_1, \phi_2) \in T^3$, we assign the gluing maps to be the rotations of those angles respectively at the fixed points of holonomies of the boundary components of S_0 and S_1 as above.

If the triangle is degenerate, i.e., a pointed hemisphere, a pointed lune, a pointed segment, or a pointed singleton, we can do the similar construction as above considering them as having infinitesimal edges if necessary. From the constructions, one obtains the full-representations.

The map $\mathcal{T}'$ induces a map $\mathcal{T}$ since the isometry SO(3)-action on $\hat{\Delta}$ and pasting map results in conjugation of corresponding holonomies in $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SO}(3))$.

5.4. *Extending* I_A, I_B*, and* I_C

In fact, I_A, I_B, and I_C extend to self-homeomorphisms of $\tilde{\Delta} \times T^3$ since our action is just taking different fixed point for two vertices of the triangle and then the pasting angle changes correspondingly also.

For example, I_A can be realized given a generalized triangle with vertices v_0, v_1, v_2 as taking the opposite triangle at the vertex v_0 and I_B at v_1 and I_C at v_2. We can assume that our triangle is always one of the four possible ones.

The pasting maps themselves are independent of the choice of triangles among the four. However, the fixed point used to measure the angle become antipodal in our picture. In this case, the rotation angle becomes 2π minus the rotation angle as measured originally because of the orientation consideration.

We describe the action below where I_A, I_B, I_C inside are the transformations on $\tilde{\Delta}$ described above:

$$I_A : (x, \phi_0, \phi_1, \phi_2) \mapsto (I_A(x), \phi_0, 2\pi - \phi_1, 2\pi - \phi_2)$$
$$I_B : (x, \phi_0, \phi_1, \phi_2) \mapsto (I_B(x), 2\pi - \phi_0, \phi_1, 2\pi - \phi_2)$$
$$I_C : (x, \phi_0, \phi_1, \phi_2) \mapsto (I_C(x), 2\pi - \phi_0, 2\pi - \phi_1, \phi_2). \tag{5.1}$$

Since the actions correspond to changing the fixed points of c_i and hence does not change the associated representations, we have $\mathcal{T} \circ I_A = \mathcal{T} \circ I_B = \mathcal{T} \circ I_C = \mathcal{T}$.

In Sections 5.5, Appendix A.1, A.2, and A.3, we will introduce equivalence relation based on making $\mathcal{T}$ injective on facial regions times T^3.

5.5. *The interior equivalence relation*

By the following proposition, a representation or a character is *over* a subset W of $\tilde{\Delta}$ if it comes from a point of the subset $W \times T^3$.

Proposition 5.3.

$$\mathcal{T} : \tilde{\Delta} \times T^3 \to \mathcal{C}_0 \subset \mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$$

is onto and 4 to 1 branch-covering from the interior of $\tilde{\Delta} \times T^3$ to the dense open subset of $\mathcal{C}_0$, i.e., the triangular subspace given by the action of the Klein four-group described by equation (5.1).

Proof. First, $\mathcal{T}|\tilde{\Delta}^o \times T^3$ is onto the set of triangular characters in $\mathcal{C}_0$ since a nondegenerate triangle and three pasting angles determine a triangular character $\pi_1(\Sigma) \to \mathrm{SO}(3)$ up to conjugations and hence a point in $\mathcal{C}_0$ as a consequence by Proposition 5.2. Since the set of triangular characters in $\mathcal{C}_0$ are dense, $\mathcal{T}$ is onto $\mathcal{C}_0$.

A triangular holonomy of a pair-of-pants determines a triangle only up to the action of the group of order 4 generated by I_A, I_B, I_C. Given a triangle realized by the holonomy, the other three are given as follows: Extend the three sides into great circles and divide the sphere into 8 triangles. Take only four with same orientation as the given triangle. These are the ones obtainable by the action of the group. We take the identical triangles for $\pi_1(S_0)$ and $\pi_1(S_1)$. □

Remark 5.1. By above, the set of triangular characters and $\tilde{\Delta}^o \times T^3 / \{I, I_A, I_B, I_C\}$ are in one-to-one correspondence.

But if h is nontriangular, then there may be more than one way to conjugate $h|\pi_1(S_0)$ and $h|i_*\pi_1(S_1)$ so that they correspond to generalized triangles in standard position.

5.6. *The summary of the equivalence relations*

We will summarize the equivalence relation over each region because of the length. The details can be found in the Appendix A.

Over the each of the faces a, b, c, d, we have abelian characters only. The gluing maps are in the T^3-parameter space. The equivalence relation is given by the $\mathbf{S}^1$-action in the direction orthogonal to the faces. We show that the quotient under this action gives us a T^2-fibration over each of the faces identified by I_A, I_B, or I_C over the union of its boundary edges. (See Appendix A.1 for details. Note that the space of characters above a is same as the space of those above b, c, or d by the transitive action of the Klein four group by equation 5.1.)

The characters over each of the regions I, II, III, IV are simply ones for representation $F_2 \to \mathrm{SO}(3)$. The space of characters over each of the region is homeomorphic to a 3-ball.

Define $U = I \cup II \cup III \cup IV$. Next, we will consider the space of characters over each of the regions $A - U, A' - U, B - U, B' - U, C - U$, and $C' - U$. Here, the

equivalence relations are given by circle actions or by the equivalence relations for the boundary. We do not obtain the topology of each of the spaces. We will also go over the space of characters over $A \cap U, A' \cap U$, $B \cap U$, $B' \cap U$, $C \cap U$, and $C' \cap U$. We show that in fact the space of characters of each component is homeomorphic to the space of characters of one of I, II, III, or IV that it meets as it maps onto the later space.

5.7. *Obtaining the homeomorphism*

We will now find the topological structure of $\mathcal{C}_0$ as an identification space obtained from $\tilde{\Delta} \times T^3$. The equivalence relation is namely the ones we obtained for a, b, c, d, I, II, III, IV, A, A', B, B', C, and C' and by the action of $\{I, I_A, I_B, I_C\}$.

One can characterize the characters over regions I, II, III, and IV, as ones which have factors into the free-group-of-rank two characters with canonical maps sendings c_is to the identity element and the characters over regions A, A', B, B', C, and C' as one where the representations of two of c_is share the fixed points, and characters over regions a, b, c, and d as abelian characters.

We will describe the topology in the next section.

Theorem 5.1. *The identity component $\mathcal{C}_0$ of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$ is homeomorphic to $\tilde{\Delta} \times T^3/ \sim$. Thus $\mathcal{C}_0$ is a topological complex consisting of the 3-dimensional $H(F_2)$, the 4-dimensional complexes C^A, C^B, and C^C, the space of abelian representations, coming from the boundary of the 3-ball $\tilde{\Delta}$, and the 6-dimensional complex from the interior of $\mathcal{C}_0$*

Proof.

The proof is divided into three parts. We show that

$$\mathcal{T} : \tilde{\Delta} \times T^3 \to \mathcal{C}_0 \subset \mathrm{rep}(\pi_1(\Sigma), \mathrm{SO}(3))$$

induces one-to-one onto map from the quotient space of $\tilde{\Delta} \times T^3$ by the equivalence relation described above.

We showed that the map $\mathcal{T}$ is onto in Proposition 5.3. We now verify the injectivity which was partially done in the above arguments.

First, let us look at the interior. By Proposition 5.3, $\mathcal{T}|\tilde{\Delta}^o \times T^3/ \sim$ is injective onto the set of conjugacy classes of triangular characters.

We now restrict our attention to $\mathcal{T}|\partial\tilde{\Delta} \times T^3/ \sim$. The characters coming from $\partial\tilde{\Delta} \times T^3/ \sim$ do not coincide with any from $\tilde{\Delta}^o \times T^3/ \sim$ since the former ones are not triangular.

The injectivity over each of $a \times T^3/ \sim, b \times T^3/ \sim, c \times T^3/ \sim$, and $d \times T^3/ \sim$ to the set of abelian characters was shown above. The injectivity over each of $I \times T^3/ \sim$, $II \times T^3/ \sim$, $III \times T^3/ \sim$, and $IV \times T^3/ \sim$ to $H(F_2)$ is proved above also. The injectivity over each $A \times T^3/ \sim$, $A' \times T^3/ \sim$, $B \times T^3/ \sim$, $B' \times T^3/ \sim$, $C \times T^3/ \sim$, and $C' \times T^3/ \sim$ to the space of degenerate characters is also clear and done above. (Up to the action of the Klein four-group, these are only three distinct maps.)

Recall $U = I \cup II \cup III \cup IV$. The characters coming from the regions $I \times T^3, II \times T^3, III \times T^3, IV \times T^3$ do not coincide with ones coming from the elements in $(A - U) \times T^3, (A' - U) \times T^3, (B - U) \times T^3, (B' - U) \times T^3, (C - U) \times T^3$, and $(C' - U) \times T^3$. The reason is that two of $h(c_i)$ are not identity for the later regions. Also, the region $(A \cup A' - U) \times T^3$, the region $(B \cup B' - U) \times T^3$, and the region $(C \cup C' - U) \times T^3$ map to mutually disjoint subspaces of the character space. This follows since the characters have $h(c_i) = I$ and $h(c_j) \neq I$ for $j \neq i$ and the index denoted by i are different for A, A' and B, B' and C, C'.

Also, $a^o \times T^3, b^o \times T^3, c^o \times T^3$, and $d^o \times T^3$ do not produce characters from the union of the regions I, II, III, IV, A, A', B, B', C, and C' times T^3. This is so since all $h(c_i)$ is not identity in the former case while in the other $h(c_i)$ is identity for at least one i.

Because of the identification by the $\{I, I_A, I_B, I_C\}$-action, we need to consider maps from $a \times T^3$, $I \times T^3$, $A \times T^3, B \times T^3, C \times T^3$, and $\Delta^o \times T^3$. We can verify that the respective regions $I^o \times T^3, (A - U) \times T^3, (B - U) \times T^3, (C - U) \times T^3$, and $a^o \times T^3$ correspond to disjoint character spaces by above. Since we know that over each of these the map is injective, this completes the proof of injectivity.

Since SO(3) is a compact Lie group, the space $\mathrm{rep}(\pi_1(\Sigma), SO(3))$ is Hausdorff. Since $\tilde{\Delta} \times T^3$ is compact, it follows that $\mathcal{T}$ is a homeomorphism. $\square$

6. The SU(2)-character space

We will first find the topological type of the SU(2)-character space of π and then in the next section do the same for the SO(3)-character space. The following is the classical result of Narashimhan, Ramanan, and Seshadri [28], [29] which we prove using a different method. However, we will need the smoothness result of Hubueschmann [22].

We first study $\mathbb{CP}^3$ as a quotient space of a tetrahedron times a 3-torus. Then we represent each SU(2)-character by a generalized triangle and pasting angles as in the SO(3)-case. An SU(2)-character of a pair-of-pants corresponds to a generalized triangle in a one-to-one manner except for the degenerate ones. The space of pasting maps in SU(2) is still homeomorphic to $\mathbf{S}^1$ but it double covers the space of pasting maps in SO(3). Thus, we parametrize the pasting angles by $[0, 4\pi]$ where $0 = 4\pi$ correspond to I and 2π represents $-I$ for $I \in$ SU(2). We denote by $\mathbf{S}_2^1$ the quotient of $[0, 4\pi]$ by identifying the endpoints.

We denote by T_2^3 the product of the three copies of $\mathbf{S}_2^1$. We find the equivalence relations on $\tilde{\Delta} \times T_2^3$ to make it homeomorphic to the character space. These equivalence relations "lift" the equivalence relations in the SO(3)-case. We discuss abelian characters and discuss the characters over the fattened edges of $\tilde{\Delta}$ showing that they form solid torus fibrations. The main result of this section Theorem 6.1 will be proved finally. We will see that our space is homeomorphic to $\mathbb{CP}^3$ with some parts such as points and annuli "blown-up" to balls and solid tori. (Here the term "blown-up" is by no means precise and we invite the readers to make this precise.)

The arguments are basically the same as in the SO(3)-case in above Section 5. The only difference is that we do not need to consider the Klein four-group $\{I, I_A, I_B, I_C\}$-action although we need a Z_2-action on the pasting angle space T_2^3.

6.1. *The complex projective space*

Let CP^3 denote the complex projective space. According to the toric manifold theory, CP^3 admits a T^3-action given by

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0, z_1, z_2, z_3] = [e^{i\theta_1} z_0, e^{i\theta_2} z_1, e^{i\theta_3} z_2, z_3] \tag{6.1}$$

and the quotient map is given by

$$[z_0, z_1, z_2, z_3] \mapsto \pi(|z_0|^2, |z_1|^2, |z_2|^2) / \sum_{i=0}^{3} |z_i|^2, z_i \in \mathbb{C} \tag{6.2}$$

The image is a standard 3-simplex $\triangle^*$ in the positive quadrant of $\mathbb{R}^3$ given by $x_0 + x_1 + x_2 \leq \pi$ and the fibers are the orbits of T^3-action. The fibers are given by $\mathbb{R}^3$ quotient out by the standard lattice L^* with generators $(2\pi, 0, 0), (0, 2\pi, 0), (0, 0, 2\pi)$. (See Masuda [27] for the standard facts about toric manifolds.)

We can realize CP^3 as a quotient space of $\triangle^* \times \mathbb{R}^3$ under the equivalence relation given by L^*-action on $\mathbb{R}^3$ fibers and over the face given by $x_0 = 0$, $(\phi_0, \phi_1, \phi_2) \sim (\phi_0 + t, \phi_1, \phi_1)$ for any real t and over the face given by $x_1 = 0$, $(\phi_0, \phi_1, \phi_2) \sim (\phi_0, \phi_1 + t, \phi_1)$, $t \in \mathbb{R}$ and over the face $x_2 = 0$, $(\phi_0, \phi_1, \phi_2) \sim (\phi_0, \phi_1, \phi_1 + t)$, $t \in \mathbb{R}$ and over the face $x_0 + x_1 + x_2 = \pi$, $(\phi_0, \phi_1, \phi_2) \sim (\phi_0 + t, \phi_1 + t, \phi_1 + t)$, $t \in \mathbb{R}$. We prove the homeomorphism of CP^3 with the quotient space by using Hausdorff compactness property of both spaces.

We give another realization of CP^3 for our purposes.

Proposition 6.1. *By considering fibers of faces of Δ, we can realize CP^3 as the quotient space $\Delta \times T_2^3$ of under an equivalence relation given as follows:*

- *In the interior, the equivalence is trivially given.*
- *For the face a, the equivalence relation on $a \times T_2^3$ is given by*

$$(v, \phi_0, \phi_1, \phi_2) \sim (v', \phi_0', \phi_1', \phi_2')$$

  if and only if $v = v'$ and two vectors

$$(\phi_0, \phi_1, \phi_2) - (\phi_0', \phi_1', \phi_2') = s(2\pi, 2\pi, 2\pi) \text{ for } s \in \mathbb{R}.$$

  This is given by the $\mathbf{S}^1$-action generated by a vector normal to a.
- *For faces b, c, and d, the equivalence relation is defined again using the respective $\mathbf{S}_2^1$-action generated by vectors parallel to*

$$(-2\pi, 2\pi, 2\pi), (2\pi, -2\pi, 2\pi), (2\pi, 2\pi, -2\pi)$$

  perpendicular to b, c, d respectively.

- *In the edges and the vertices, the equivalence relation is induced from the facial ones.*
- *Finally an equivalence relation is given by the $\mathbb{Z}_2$-action sending (ϕ_0, ϕ_1, ϕ_2) to $(\phi_0 + 2\pi, \phi_1 + 2\pi, \phi_2 + 2\pi)$.*

There is a natural T_2^3-action of CP^3 making it a toric variety which is induced by the natural T_2^3-action on $\Delta \times T_2^3$ which behaves well-under the quotient map since the equivalence classes are mapped to equivalence classes under the action.

Proof. We basically induce a map from the first description of CP^3 to the description of the proposition and show the equivalence.

We first note that the lattice L in $\mathbb{R}^3$ generated by

$$(4\pi, 0, 0), (0, 4\pi, 0), (0, 0, 4\pi), (2\pi, 2\pi, 2\pi)$$

has a basis

$$(-2\pi, 2\pi, 2\pi), (2\pi, -2\pi, 2\pi), (2\pi, 2\pi, -2\pi).$$

Note that the fiber over the interior point in the above is $\mathbb{R}^3$ quotient out by this lattice.

We define a map $\mathcal{D} : \triangle^* \to \Delta$ given by the affine map

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix}$$

Then L^* is sent to L by 2 times the inverse transpose of the linear part of $\mathcal{D}$. We now verify that the quotient space of the toric bundle over Δ and the quotient space of the toric bundle over $\triangle^*$ are in one-to-one correspondence by using $\mathcal{D}$ and the inverse transpose of the linear part for the fibers. The compactness and Hausdorff property give us the homeomorphism property. $\qquad\square$

6.2. *The Klein four-group action*

There is a natural Klein four-group action generated by three transformations L sending a point $[z_0, z_1, z_2, z_3]$ to $[z_1, z_0, z_3, z_2]$ and M sending $[z_0, z_1, z_2, z_3]$ to $[z_2, z_3, z_0, z_2]$ and N sending $[z_0, z_1, z_2, z_3]$ to $[z_3, z_2, z_1, z_0]$ where z_i are complex numbers.

The above action of $\{I, L, M, N\}$ corresponds to our $\{I, I_A, I_B, I_C\}$-action exactly under this map as given in the following equations

$$\begin{aligned} I_A &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_A(x), \phi_0, 4\pi - \phi_1, 4\pi - \phi_2) \\ I_B &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_B(x), 4\pi - \theta_0, \phi_1, 4\pi - \phi_2) \\ I_C &: (x, \phi_0, \phi_1, \phi_2) \mapsto (I_C(x), 4\pi - \phi_0, 4\pi - \phi_1, \phi_2). \end{aligned} \qquad (6.3)$$

(This will be used in the next section.)

6.3. *The classical theorem recovered*

Theorem 6.1. $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ *is diffeomorphic to* CP^3 *considered as a* T^3-*fibration over* Δ *with the following properties:*

- *The interior of each edge of Δ corresponding to any region A, A', B, B', C, C' of $\tilde{\Delta}$ corresponds a solid torus fibration over it. The closure of the solid torus fibration is homeomorphic to the quotient space of the solid torus times an interval with a $\mathbb{Z}_2$-action at each boundary component giving an identification. Here, each solid torus at the end of the interval is identified to a 3-ball.*
- *Three of them meet in a 3-ball over each vertex of $\tilde{\Delta}$ according to the pattern of the edges of $\tilde{\Delta}$.*
- *The set of abelian characters $\chi_2(\Sigma)$ forms a subspace with an orbifold structure with 16 singularities. It consists of the two-torus fibrations over faces of Δ which meet at the boundary components of the above solid torus fibration.*
- *The boundary torus of the solid torus fibration corresponding to an endpoint of the intervals identifies to the boundary 2-spheres of vertex 3-balls by the $\mathbb{Z}_2$-action.*

We begin the proof divided into many subsections:

6.4. *Triangular characters*

A representation of $\pi_1(\Sigma)$ into $\mathrm{SU}(2)$ gives us representations of two pairs-of-pants which are identical by the reason similar to Proposition 4.2. By Proposition 3.2, the pair-of-pants representations correspond to generalized triangles. The triangles for the restricted representation for the two pairs-of-pants have the same angles in this case since we know that our space $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ is connected and it holds for at least one example. Therefore, we can form $\tilde{\Delta} \times T_2^3$ as in the $\mathrm{SO}(3)$-case and introduce equivalence relations, where the pasting maps are rotations fixing the vertices of the parametrizing triangles and they give us pasting angles in T_2^3. (Here, we are using triangles to denote a representation. The actual fixed point of c_2 is actually the opposite the third vertex. But since everywhere, it is reversed, there are no problems. We use triangles since they are concrete and easy to visualize.)

As an aside, we can consider triangles as a representation of punctured pair-of-pants with the puncture holonomy $-\mathrm{I}$. This approach can be followed but we will not pursue it here.

Here given three pasting angles in T_2^3, we see that the transformation

$$(\theta_0, \theta_1, \theta_2) \to (2\pi + \theta_0, 2\pi + \theta_1, 2\pi + \theta_2)$$

does not change characters as d_1 and d_2 do not change while the pasting maps P_0, P_1, P_2 are all changed by $-P_0, -P_1, -P_2$. Thus, we must use the generated $\mathbb{Z}_2$-action to $\tilde{\Delta} \times T_2^3$ first. Actually, we limit ourselves to $\tilde{\Delta} \times (T_2^3/\mathbb{Z}_2)$ from now on.

We find a description of $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ as a quotient space of $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2$: For the open domain of triangular characters, a representation of $\pi_1(\Sigma)$ gives us a unique triangle on $\mathbf{S}^2$ by Proposition 3.2 and hence a unique set of pasting maps up to $\mathbb{Z}_2$. Thus, the space of triangular characters is homeomorphic to $\tilde{\Delta}^o \times T_2^3/\mathbb{Z}_2$. By density, the map

$$\tilde{\Delta} \times T_2^3/\mathbb{Z}_2 \to \mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$$

is onto.

Here, the equivalence relation is simply defined as follows, two points of $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2$ is equivalent if they induce the same $\mathrm{SU}(2)$-characters. But we will go through each face of Δ below. Thus $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2/\sim$ is homeomorphic to $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$.

6.5. *The summary of equivalence relations*

Now we will summarize the equivalence relation over the each of the regions $a, b, c, dA, A', B, B', C, C', I, II, III$, and IV. (The details are in Appendix B.) Over each of the faces a, b, c, d, we have abelian representations. The space of characters over each face is the quotient of the face times T_2^3 by an action of $\mathbf{S}_2^1$. This gives a T_2^2-fibration over each face. We show that these tori over points of the edge of the face are identified with the tori over points of an edge of another face by crossing the ties in $\tilde{\Delta}$. Thus the total space is homeomorphic to $T_2^4/\mathbb{Z}_2$. (See Appendix B.1.)

Over each tie in any of the region A, A', B, B', C, C', the space of chacters is homeomorhic to $\mathbf{S}^1 \times B^2$. They form a fiber bundle with fibers homeomorphic to $\mathbf{S}^1 \times B^2$ over the interior of each edge of Δ corresponding to A, A', B, B', C, C'. These solid torus bundles are compactified by the $\mathbb{Z}_2$-actions on ends of the edges. (See Appendix B.2.)

Finally, we need (see Section 5.7 also):

Proposition 6.2. $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ *is homeomorphic to the quotient space of* $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2$ *by the above equivalence relations over the regions above faces* a, b, c, d, A, A', B, B', C, C', I, II, III, *and* IV.

Proof. Again there is a one-to-one map from $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2/\sim$ to $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ defined by sending the configuration to the character it produces from the generalized multiplication by geometry.

The map is clearly onto since any character is reached by a triangular path. Thus, every character can be achieved by a configuration in $\mathbf{S}^2$.

The one-to-one property can be achieved by first considering the interior of $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2$ which corresponds to triangular characters. This is achieved as in the SO(3)-case. For the boundary regions, the injectivity for each region times T_2^3 quotient by $\sim$ is clear. We note that the map restricted to $(a^o \cup b^o \cup c^o \cup d^o) \times T_2^3/\sim$ is injective since the angles of c_0, c_1, c_2 are all in different ranges in the boundary of Δ. The

restricted maps from

$$(A - U) \times T_2^3 / \sim, (A' - U) \times T_2^3 / \sim, (B - U) \times T_2^3 / \sim, (B' - U) \times T_2^3 / \sim,$$
$$(C - U) \times T_2^3 / \sim \text{ and } (C' - U) \times T_2^3 / \sim \quad (6.4)$$

have disjoint images since they are characterized by angles of c_0, c_1, c_2 are in disjoint ranges in $\partial \Delta$. The restricted maps from

$$I \times T_2^3 / \sim, II \times T_2^3 / \sim, III \times T_2^3 / \sim, \text{ and } IV \times T_2^3 / \sim$$

have mutually disjoint images since $h(c_i) = \pm I$ for every representation h in each of them, and the signs are different for these regions. Furthermore, we see that all these regions listed here map to disjoint subsets of the character space. By the injectivity over $\Delta^o \times T_2^3$, we obtain the injectivity of the map.

As in the SO(3)-case, $\tilde{\Delta} \times T_2^3 / \mathbb{Z}_2 / \sim$ is compact and $\mathrm{rep}(\pi_1(\Sigma), SU(2))$ is Hausdorff since $SU(2)$ is a compact Lie group. Thus, the map is a homeomorphism. $\quad\square$

6.6. *The proof of Theorem 6.1.*

We begin by noting that CP^3 is a toric variety of a tetrahedron Δ. We described it as a quotient space of $\Delta \times T_2^3 / \mathbb{Z}_2$ as in Section 6.1. The equivalence relation is the one given by the formula same as one on $\Delta \times T_2^3 / \mathbb{Z}_2$ restricted to the faces a, b, c, and d and there are no more. (See Proposition 6.1.) The interiors of the edges inherit two different equivalence relations and the fibers are homeomorphic to $\mathbf{S}^1$ over each point of the interiors of the edges and the vertex inherit three different equivalence relations and the fiber is a point over each vertex. We call these the *vertices* of CP^3.

We will now prove Theorem 6.1 by comparing it to CP^3: First, we will realize CP^3 as a $T_2^3 / \mathbb{Z}_2$-fibration over $\tilde{\Delta}$. Then we will remove the regular neighborhoods of the vertices. Compare the result to the SU(2)-character space with regular neighborhoods of vertex regions removed. This is accomplished by removing the regular neighborhoods of edge regions and comparing and comparing the topological types of the edge regions.

The quotient image of the four faces regions a, b, c, and d times $T_2^3 / \mathbb{Z}_2$ are four complex projective planes respectively since each face corresponds to a standard simplex. That is, we see that the standard 2-simplex in the standard 3-simplex $\triangle^*$ correspond to copies of CP^2 by the quotient relations (see equation (6.1)).

These faces homeomorphic to CP^2 meet transversally at six 2-spheres that fiber over the edges. Any three of the complex projective planes meet at a vertex which lie over a vertex of Δ. Any two of the six 2-spheres meet at a vertex also. Moreover, three of the six 2-spheres will meet at a vertex if their corresponding edges do. (These are standard facts from Fulton [21] and Masuda [27])

Let F be the set of four points over the vertex of Δ. Then we choose B to be an open regular T^3-invariant neighborhood of F homeomorphic to 6-dimensional balls as was chosen above. Thus B maps to a union B' of four three-balls that

are neighborhoods of the vertices of Δ. Remove from CP^3 the T_2^3-invariant regular neighborhood B.

Denote the resulting manifold by N where ∂N is a union of four 5-spheres. This corresponds to removing regular open neighborhoods of four corners of the tetrahedron, the result of which we denote by D. The edges of the tetrahedron correspond to six 2-spheres meeting at four vertices. If we remove points in B, the six subspaces are homeomorphic to $\mathbf{S}^1 \times I$. An open T^3-invariant regular neighborhood of each of the subspaces in N is homeomorphic to $\mathbf{S}^1 \times B^4 \times I$ where the triviality follows since the neighborhood is orientable. And hence their boundary is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^3 \times I$. This corresponds to removing regular neighborhood of the six edges in D.

The above can be seen by the theory of tubes under a compact group action (See Bredon [4].)

Denote by M the space $\tilde{\Delta} \times T^3/\mathbb{Z}_2/\sim$. Finally, each of the regions over I, II, III, and IV in M is homeomorphic to B^3. Let L be their union. Again, we find a neighborhood $N(L)$ of L in $\tilde{\Delta} \times T^3/\sim$ so that $N(L)$ with L removed coincides with $B - V$ over the points of Δ with edges removed. (This can be done by choosing a neighborhood B'' of the union of the vertex regions in $\tilde{\Delta}$ corresponding to B' under $\tilde{\Delta} \to \Delta$ and taking the inverse image.) Since L has contractible components and any map into $N(L)$ can be pushed into a contractible regular neighborhood of L, it follows that $N(L)$ is homotopy trivial. Therefore, $N(L)$ is a union of four six-dimensional balls.

Remove from M the union $N(L)$ and call the resulting space N'.

By Lemma B.1, each of the quotient spaces over the strips A, A', B, B', C, and C' in N' is homeomorphic to $\mathbf{S}^1 \times B^2 \times I$. Denote by J_A, $J_{A'}$, J_B, $J_{B'}$, J_C, and $J_{C'}$ the corresponding spaces. Let J be their union. By identifying Δ with edges and vertices removed with $\tilde{\Delta}$ with the strips A, A', B, B', C, and C' and regions I, II, III, and IV removed, we see that T_2^3-fibrations are the same ones where defined.

By Huebshmann [22], $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ is homeomorphic to a manifold, and so is our space $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2/\sim$ by Proposition 6.2. Let $N'(J)$ be the T_2^3-invariant set that is the neighborhood of J. Since we can homotopy any map into $N'(J)$ to a regular neighborhood of J homotopy-equivalent to J, $N'(J)$ is homotopy equivalent to J, and thus, $N'(J)$ is homeomorphic to the union of six copies of $\mathbf{S}^1 \times B^4 \times I$ as the ambient space is a topological manifold.

Let us denote the union $\mathcal{V}$ of the above six subspaces over the edges homeomorphic to $\mathbf{S}^1 \times I$ in N, and let $N(\mathcal{V})$ denote the regular T_2^3-invariant neighborhood of $\mathcal{V}$, which is homeomorphic to $\mathbf{S}^1 \times I \times B^4$. We can choose $N'(J) - J$ and $N(\mathcal{V}) - \mathcal{V}$ homeomorphic by choosing their basis images under the above T_2^3-actions to be the same. Moreover, $N(\mathcal{V})$ and $N'(J)$ are homeomorphic since they are six-dimensional compact orientable manifolds homotopy equivalent to circles.

Remove from N the regular T_2^3-invariant neighborhood $N(\mathcal{V})$ and remove from N' the regular T_2^3-invariant neighborhood $N'(J)$. Now consider the description of

$\Delta \times T_2^3/\sim$ and $N - N(\mathcal{V})$ as a torus fibration with equivalence relations that are same. (See Appendix B.1.) By our description, it follows that $N - N(\mathcal{V})$ and $N' - N'(J)$ are homeomorphic. We conclude that N' is homeomorphic to N. (We remark that the homeomorphism preserves the parameter I.)

Note that $\chi_2(\Sigma)$ in $N' - N'(J)$ correspond to $a'' \cup b'' \cup c'' \cup d'' \times T_2^3/\sim$ where a'', b'', c'', and d'' are disks in the faces a, b, c, and d respectively and hence it is homeomorphic to four 2-cells times 2-tori. They correspond to four copies of CP^2 in N.

Adding $N(L)$ to N', we recover M. Therefore M is homeomorphic to N with four six-balls reattached and hence M is homeomorphic to CP^3.

Let us discuss about the subspaces over the edge regions A, A', B, B', C, and C'. By Lemma B.1 each of the subspace is a union of solid torus time intervals over the 1-skeleton of a tetrahedron meeting each other at a 3-ball over a vertex, where the boundary of the torus fiber and 3-ball fiber here consists of abelian characters.

Let S be one of the 5-spheres that are a boundary component of N'. Assume without loss of generality it meets A, B, C. Then $S \cap (N' - N'(J))$ correspond to the circle k in $\partial\Delta$ with three closed arcs in A, B, C in the boundary removed, and $\chi_2(\Sigma) \cap S \cap (N' - N'(J))$ is a union of three embedded copies of $T^2 \times I$. The abelian part over the each tie in one of the edge regions is the boundary 2-torus of the solid torus. As we decrease $N'(J)$ to J by thinning the regular neighborhood, three embedded copies of $T^2 \times I$ will meet on these boundary 2-tori. The two copies get identified across ties in the edge regions by a reflection in one of the three gluing parameter coordinates. Hence, their composition equals $-\mathrm{I}$ in T_2^3 in gluing parameter coordinates before taking the quotients. The union is then homeomorphic to $T_2^2 \times I$ by identifying the top and the bottom where the identification homeomorphism is $-\mathrm{I}$. We call the result *a twisted* 3-*dimensional torus*. (See Appendix B.1 also.)

Now we can vary S. If we choose a parameter of boundary spheres S_t of N so that the corresponding image circle bounds disk D_t in $\tilde{\Delta}$ and D_t converges to one of the regions I, II, III, and IV as $t \to 0$. Then the parameter of 5-spheres S_t converges to the set above each of the regions, i.e., a three-ball representing the homomorphism factoring into a free group F_2-characters. The parameter of the twisted 2-dimensional tori in S_t will geometrically converge as $t \to 0$ to the boundary of the three-ball since the twisted torus corresponds to abelian characters which must accumulate to abelian characters in the subspace of free group characters, i.e., the boundary of the three-ball. Each of the parameters of the three solid tori in S_t geometrically converges to the three-ball as $t \to 0$ since the solid torus corresponds to the space of characters over the edges which must converge to a free group character modulo changing $h(c_i)$ to $\pm h(c_i)$. (See Lemma B.1 also for the clasping phenomena.) This identification of course extends to the boundary tori also, i.e., the abelian characters.

This completes the proof of Theorem 6.1.

Remark 6.1. The characters over A, A', B, B', C, and C' form a 4-dimensional

CW-complex meeting in a 3-dimensional cell over the regions I, II, III, and IV. They are twistedly embedded. We can picture them as a solid torus times interval "clasping" itself and two or three of them meet at the clasped 3-balls if their regions meet.

7. The topology of the quotient space $\tilde{\Delta} \times T_2^3 / \sim$ or $\mathcal{C}_0$

In this section, we will determine the topology of the quotient space $\tilde{\Delta} \times T_2^3 / \sim$. We will show that it is branch-covered by CP^3.

First, we discuss the group action by $\mathbb{Z}_2^4$ by considering the Klein four-group action on the fibers and the Klein group action on $\tilde{\Delta}$.

Clearly, there is a group V' of order 16 action on $\tilde{\Delta} \times T_2^3 / \mathbb{Z}_2 / \sim$ generated by the $\{I, I_A, I_B, I_C\}$-action as given by equations (6.3) and the Klein four-group acting on each of the fibers $\mathbf{S}_2^1 \times \mathbf{S}_2^1 \times \mathbf{S}_2^1 / \mathbb{Z}_2$ defined by

$$i_a : (\theta_0, \theta_1, \theta_2) \mapsto (\theta_0 + 2\pi, \theta_1 + 2\pi, \theta_2)$$
$$i_b : (\theta_0, \theta_1, \theta_2) \mapsto (\theta_0, \theta_1 + 2\pi, \theta_2 + 2\pi)$$
$$i_c : (\theta_0, \theta_1, \theta_2) \mapsto (\theta_0 + 2\pi, \theta_1, \theta_2 + 2\pi). \tag{7.1}$$

These transformations preserve equivalence classes: Each of I_A, I_B, I_C is described as a geometric transformation of taking two vertex points by their antipodes (see equation (6.3)) and each of i_a, i_b, and i_c amounts to multiplying two of the pasting maps by $-I$ is $SU(2)$. Hence, they are transformations on the set of the equivalence classes of $SU(2)$-representations whose quotient is the space of $SO(3)$-representations in $\mathcal{C}_0$.

The group V' is a Klein four-group extended by a Klein four-group: We show that $\{I, I_A i_a, I_B i_b, I_C i_c\}$ is a subgroup by considering the coordinate form of the action on $\Delta^o \times T_2^3 / \mathbb{Z}_2 / \sim$, we see that the group is isomorphic to a product group $V \times V$ and hence isomorphic to $\mathbb{Z}_2^4$.

7.1. *The fiberwise group action*

We discuss the fiberwise Klein four-group action over each of the regions. In the regions over the interiors of the faces of Δ, the action is free as we can see from the quotient formula.

Each region over I, II, III, or IV is a 3-ball and can be identified with a tetrahedron Δ since we can change $h(c_i) = \pm I$ to I and obtain a representation of F_2 to $SU(2)$ which gives a homeomorphism between these and $H(F_2)$. (See Section 3.2 and Proposition 3.2.)

Proposition 7.1. *The action of $\{I, i_a, i_b, i_c\}$ on the regions over each I, II, III, and IV, can be seen as the Klein group action on a tetrahedron if we identify it with a tetrahedron Δ. That is, $\{I, i_a, i_b, i_c\}$ that acts fiberwise on regions above I, II, III, and IV as involutory rotations about axes parallel to the coordinate direction through $(\pi/2, \pi/2, \pi/2)$.*

Proof. Consider a region I. The transformation i_a adds 2π to the pasting angle for c_1 and c_2. This means taking the antipodal fixed point in $\mathbf{S}^2$ and changing the respective rotation angles to be 2π minus the original ones. The third coordinate is the angle of rotation about a product, which does not change in this case. This is an involutary rotation about the one of the axis. For i_b and i_c, the arguments are similar. By an analogous consideration for each of the regions II, III, and IV, we are done. $\square$

Recall from Lemma B.1 that the subspace over a tie determined by the angles in each of A, B, C, A', B', and C' with $U = I \cup II \cup III \cup IV$ removed admits the quotient space under $\sim$ homeomorphic to $\mathbf{S}^1 \times B^2$.

By a *circle* we mean a subspace homeomorphic to the circle in a solid torus where the embedding is a homotopy equivalence. An *axis* circle is a subset of fixed points of an involution homeomorphic to a circle.

Lemma 7.1. *For the Klein four-group $\{I, i_a, i_b, i_c\}$ which acts on the solid torus over each tie in of A, B, C, A', B', and C' with $U = I \cup II \cup III \cup IV$ removed. The maps are involutions.*

- i_a *has the axis circle in the solid torus over each tie in $A - U$ and $A' - U$ where the other two i_b and i_c are without fixed points.*
- i_b *has the axis circle over the solid torus over each tie in $B - U$ and $B' - U$ where the other two i_a and i_c have no fixed points.*
- i_c *has the axis circle over the solid torus over each tie in $C - U$ and $C' - U$ where the other two i_a and i_b have no fixed points.*
- *The result of the quotient of the each region by the action of $\{I, i_a, i_b, i_c\}$ is homeomorphic to $\mathbf{S}^1 \times B^2$, i.e., a solid torus.*

Proof.

First, we discuss the region $B - U$. Let h be a representation over $B - U$. By the circle action described above, we can find a great circle in c_1^1 containing the two fixed points of $h(d_1)$ and $h(d_2)$ and a great circle c_2^1 containing the pair P of antipodal points fixed by $h(c_i)$ for $i = 1, 2$.

We recall Lemma B.1. Suppose first that $h(d_1)$ and $h(d_2)$ have the pairs of fixed points distinct from P. Then their respective fixed points f_1 and f_2 are on a great circle c_1^1. We suppose that the circle does not contain P as well. The other cases are in the closure of subspaces satisfying this condition as before and hence we can use the condition to parameterize our subspace of B.

Since we are on a particular segment in B determined by θ_1, we can choose a fixed point v_1 of $h(c_1)$ so that the angle of rotation at v_1 is the one given by θ_1.

We parameterize such configurations by (l, θ_1, θ_2) where l is the length of one of the shortest segment s between the fixed points of v_1 to the fixed point of f_1 and hence $l \leq \pi$ and $\theta_1/2$ is the angle between the great circle containing s the shortest segment to f_2 and θ_2 is the angle of rotation of $h(d_1)$. (See Figure B.1.) Recall that θ_1

is in $\mathbf{S}_2^1$ and θ_2 in $[0, 2\pi]$. Thus, our space is parameterized by $([0, \pi] \times \mathbf{S}_2^1 \times [0, 2\pi])/ \sim$ where $\sim$ is given by Lemma B.1 when $\theta_2 = 0, 2\pi$.

Now consider the action by $\{I, i_a, i_b, i_c\}$. This amounts to multiplying the pasting angles by $-I$. We start with i_b: Again i_b will transform by multiplying $h(d_1)$ and $h(d_2)$ by $-I$. This amounts to changing the fixed point to its antipode and changing the angles to 2π minus the original one by equation 3.1. This is identical with sending f_1 and f_2 to its antipodal points. Thus (l, θ_1, θ_2) is mapped to $(\pi - l, \theta_1, 2\pi - \theta_2)$. Thus an axis circle of fixed point of i_b is a circle given by $l = \pi/2$ and $\theta_2 = \pi$.

Next, i_a transforms by multiplying $h(d_2)$ by $-I$, and (l, θ_1, θ_2) is mapped to $(l, \theta_1 + 2\pi, \theta_2)$. Finally, i_c transforms by multiplying $h(d_1)$ by $-I$, and (l, θ_1, θ_2) is mapped to $(\pi - l, \theta_1 + 2\pi, 2\pi - \theta_2)$ and i_a and i_c do not have any fixed points.

The fundamental domain of the action is given by

$$[0, \pi/2] \times [0, 2\pi] \times [0, 2\pi]/ \sim,$$

and an equivalence

$$(\pi/2, \theta_1, \theta_2) \sim (\pi/2, \theta_1, 2\pi - \theta_2)$$
$$(l, 0, \theta_2) \sim (l, 2\pi, \theta_2) \tag{7.2}$$

Thus, the quotient space is homeomorphic to $\mathbf{S}^1 \times B^2$.

For region B', the reasoning is very similar, and we also have that i_b is a rotation about an axis circle of the solid torus of angle π and i_a s a translation by π-amount. i_c is a composition of the two.

For region A, we have the vertex denoted by v_0 is antipodal to v_1 and v_2 is on the segment between them. Let h be a representation over A. Again, we can assume that the fixed point f_1 of $h(d_1)$ and the one f_2 of $h(d_2)$ are well-defined on a dense set. Thus, we can choose the fixed point f_1 to lie on v_1 and the fixed point f_2 of angle in $[0, 2\pi]$ is on the sphere. We coordinatize by taking l to be the distance between v_0 and f_2 and θ_1 to be the rotation angle of $h(d_1)$ at v_1 and θ_2 to be the rotation angle of $h(d_2)$ at f_2. Here, since i_a sends $h(d_2)$ to $-h(d_2)$, we have that $i_a(l, \theta_1, \theta_2) = (\pi - l, \theta_1, 2\pi - \theta_2)$, and i_a has an axis circle of fixed points given by $l = \pi/2, \theta_2 = \pi$. Since i_c sends $h(d_1)$ to $-h(d_1)$, it follows that i_c is a translation $i_c(l, \theta_1, \theta_2) = (l, \theta_1 + 2\pi, \theta_2)$. Again i_b is a composition of the two and hence has no fixed point. We have a same result for the region A'.

For regions C and C', we have that i_c is a rotation about an axis circle of angle π and i_a is a translation in the axis direction and i_c is the composition of the two, similarly.

From the description of the actions, we see that each of the quotient spaces is homeomorphic to a solid torus. □

Proposition 7.2. *The fiberwise Klein four-group action on* $\tilde{\Delta} \times T_2^3/\mathbb{Z}_2/ \sim$ *is described as follows:*

- *The fiberwise Klein four-group action consists of order-two translations in*

the open domain above the interior of $\tilde{\Delta}$ and have no fixed points. So is the action on the fibers over the faces of $\tilde{\Delta}$.

- i_a is an involutionary rotation about the union of two disjoint 2-sphere $\mathbf{S}_A^2$ and $\mathbf{S}_{A'}^2$. Similarly i_b is a rotation about the union of two disjoint 2-spheres $\mathbf{S}_B^2$ and $\mathbf{S}_{B'}^2$, and i_c is one about the union of 2-spheres $\mathbf{S}_C^2$ and $\mathbf{S}_{C'}^2$.
- The spheres $\mathbf{S}_A^2$, $\mathbf{S}_{A'}^2$, $\mathbf{S}_B^2$, $\mathbf{S}_{B'}^2$, $\mathbf{S}_C^2$, and $\mathbf{S}_{C'}^2$ will be above each edge region with the same label.
- The spheres meet in the region above I, II, III, and IV in the axes of the corresponding action in the 3-ball.
- The intersection pattern is given by the intersection pattern of the axes of the action. That is, the two meet transversally at a point if and only if the corresponding edges in Δ do. The three meet transversally at a point if and only if the corresponding edges in Δ do.
- The quotient space has an orbifold structure with singular points the images of the above spheres which are again spheres with same incidence relations. The local group of the singular points is $\mathbb{Z}_2$ in the nonvertex point and is a Klein four-group at the vertices where any two of them meet. This can be made into a smooth orbifold structure.

Proof.

For the interior part, the first item is clear.

In the regions over the faces, the $\{I, i_a, i_b, i_c\}$-action is also free: consider these actions as translations in the tori $T_2^3/\sim$ where $\sim$ is generated by vectors parallel to the normal vectors of form $(\pm 2\pi, \pm 2\pi, \pm 2\pi)$ where the signs depend on which faces you are on:

Consider the face a of $\tilde{\Delta}$ for now. Then i_a, i_b, and i_c correspond to translations by $(0, 2\pi, 2\pi)$, $(2\pi, 0, 2\pi)$, and $(2\pi, 2\pi, 0)$ respectively. Then these vectors can be seen to be parallel to the plane of the quotient tori with generators $(4\pi, 0, -4\pi)$ and $(0, 4\pi, -4\pi)$ since we can change the signs of translation vectors mod 4π. We see easily that these translations are not trivial and the fiberwise Klein four-group acts freely here. (See Appendix B.1 for details.) Similar considerations show that the fiberwise Klein four-group action is free over the faces b, c and d.

Let us now go over to the regions A, B, C, A', B', and C': We construct the two sphere $\mathbf{S}_B^2$ by taking the axis circle of i_b over each solid tori in B removed with the vertex regions, as obtained from the ties. The parameter of axis circles in the parameter of the solid tori over the ties in $B - U$ will geometrically converge to an axis, a segment in the vertex 3-ball fixed by i_b, by approaching the vertex region and finally will double branch-cover the segment with two singular points (see Theorem 6.1). Thus, we obtain a sphere by compactifying the two ends of an annulus consisting of axis by two segments by Conner's work [9]. (i_a and i_c do not have fixed points here and they are not considered.)

The same constructions will give us all other spheres $\mathbf{S}_{B'}^2$, $\mathbf{S}_A^2$, $\mathbf{S}_{A'}^2$, $\mathbf{S}_C^2$, and $\mathbf{S}_{C'}^2$.

Note that the two spheres will meet if their compactifying edges will meet. Hence,

the result follows.

We can assume that they are smooth and have intersections so that their tangent planes meet only at the origin. (Thus, we can give an invariant Riemannian structure.)

Now we go over the action. For i_a will fix axes circles above A and A'. Since they fix the axis in the vertex 3-balls as well, $\mathbf{S}_A^2, \mathbf{S}_{A'}^2$ will be fixed. The action is a rotation of angle π in the solid tori and will be identity in the complementary 2-dimensions to the sphere and the solid tori. The similar reasoning also hold for i_b and i_c.

Since the quotient of the actions are always locally linear and the local groups are finite, we obtain that the quotient has an orbifold structure. Since these actions are finite, we can put the smooth orbifold structure on the quotient space.   □

7.2. *The topology*

Theorem 7.1. $\tilde{\Delta} \times T^3 / \sim$ *is homeomorphic to a quotient of* CP^3 *under the product of the two Klein four-group actions generated by fiberwise and axial action. The branch loci of* I_A, I_B, I_C *are given as follows: six 2-spheres corresponding to the axes of* I_A, I_B, *and* I_C. *There are two 2-spheres over each axis, and over each axis, the two 2-spheres are disjoint. Any two 2-spheres over respectively different axes meet at unique points over the center* $(\pi/2, \pi/2, \pi/2)$ *of* $\tilde{\Delta}$. *All three 2-spheres over different axis meet at the same point as above.*

Proof.

Now we consider the $\{I, I_A, I_B, I_C\}$-action on $\tilde{\Delta} \times T_2^3 / \mathbb{Z}_2 / \sim$.

Let us consider I_B first. Recall that a mid-tie is a tie in the middle of the strip, i.e., the half way locus along the angular parameterization. The solid tori in the edge regions are permuted or if the edge region is marked by B or B', then $\{I, I_B\}$ acts on it and in particular $\{I, I_B\}$ acts on the solid torus over the middle tie.

$\{I, I_B\}$ acts on the triangles corresponding to a representation above the middle tie by sending a triangle to another triangle by replacing v_1 by $-v_1$ and v_2 by $-v_2$ and fixing v_0. Also, the pasting angles behave $(\theta_0, \theta_1, \theta_2) \mapsto (\theta_0, 4\pi - \theta_1, 4\pi - \theta_2)$. Thus, the solid torus coordinate introduced above the mid-tie in B or B' behaves $(l, \theta_1, \theta_2) \to (l, 4\pi - \theta_1, 2\pi - \theta_2)$. Hence, the two fixed segments are given by equations $\theta_1 = 0, 2\pi$ and $\theta_2 = \pi$ (as $\theta_2 \in [0, 2\pi]$ an interval).

On the 3-torus over the fixed axis in the interior of $\tilde{\Delta}$, the pasting maps are also sent the same way. Thus $(\theta_0, \theta_1, \theta_2)$ is sent to $(\theta_0, 4\pi - \theta_1, 4\pi - \theta_2)$, and there are four fixed axes given by $\theta_1 = 0, 2\pi, \theta_2 = 0, 2\pi$, which are four circles. By the $\mathbb{Z}_2$-action, there are actually only two circles of fixed axes over these points. Thus, the union of entire fixed axes over the axis of I_B in $\tilde{\Delta}$ correspond to a union of two annuli compactified by segments to form two 2-spheres by Conner's work [9] as I_B is a smooth involution.

The action of $\{I, I_B\}$ in M permutes the four 6-dimensional balls which are

regular neighborhoods of the subspaces corresponding to the regions I, II, III, and IV. In $\mathbb{CP}^3$, this is true with the four 6-dimensional balls described in the above proof. Thus, the action of $\{\mathrm{I}, I_B\}$ on $\mathbb{CP}^3$ is conjugate to that on M by our identification above. Therefore, I_B is an involution with a fixed point a union of two disjoint 2-sphere.

Similar conclusions can be established about I_A and I_C. The corresponding spheres meet at a 3-torus over at the center $(\pi/2, \pi/2, \pi/2)$ of $\tilde{\Delta}$. Since they meet the 3-torus in orthogonal circles, we see that they meet at unique points: $\qquad\square$

Remark 7.1. Thus, we see that the final orbifold structure of $\mathcal{C}_0$ is simply given by $\mathbb{CP}^3$ with six 2-spheres denoted by $s_A, s'_A, s_B, s'_B, s_C$, and s'_C corresponding to involutions I_A, I_B, and I_C and the six 2-spheres $\mathbf{S}^2_B, \mathbf{S}^2_{B'}, \mathbf{S}^2_A, \mathbf{S}^2_{A'}, \mathbf{S}^2_C$, and $\mathbf{S}^2_{C'}$ in $\tilde{\Delta} \times T^3_2 / \sim$ which form the branch locus of the $\{I, i_a, i_b, i_c\}$-action. Denote the spheres by $S_A, S'_A, S_B, S'_B, S_C$, and S'_C respectively.

The three spheres $s_A, s'_A, s_B, s'_B, s_C$, and s'_C all meet at a point in the center of the tetrahedron. $S_A, S'_A, S_B, S'_B, S_C$, and S'_C also meet at a point in the "vertex" region. We see that s_A and S_A meet at a point in the middle of an "edge" region. and s_B and S_B do and so do s_C and S_C. Also, we can insert primes any where to get the incidences. There are no other incidence type.

We can now describe the orbifold structure on $\mathcal{C}_0$. The twelve spheres above have order two singularities everywhere. There are two points with local group $\mathbb{Z}^3_2$ and there are three with local group $\mathbb{Z}^2_2$.

Finally, the image of the first set of six 2-spheres map to three 2-spheres and and the second set of six 2-spheres map to another set of three 2-spheres.

8. The other component

In this section, we study the other component $\mathcal{C}_1$. We follow the basic strategy as in $\mathcal{C}_0$ case. First, we introduce an octahedron O and its blown-up compactification $\tilde{O}$ to parameterize the characters. This octahedron characterizes the characters of the two pairs-of-pants S_0 and S_1. Again, we compactify the octahedron by blowing up vertices into squares to prepare for the study of the characters of the fundamental group of the surface itself. Then we introduce equivalence relation so that $\tilde{O} \times T^3 / \sim$ becomes homeomorphic to $\mathcal{C}_1$. This will be done by considering the interior and each of the boundary regions as in the previous sections.

8.1. *The triangle region for the characters*

We will now discuss the other component, i.e., the component containing a representation h^* in Section 4.1 which correspond to one with an upper triangle and a lower triangle that are identical standard lunes with vertices $[1, 0, 0]$ and $[-1, 0, 0]$ and two edges passing through $[0, -1, 0]$ and $[0, 0, 1]$ where the vertices v_0, v_1, v_2 of the upper triangle and the vertices v'_0, v'_1, v'_2 of the lower triangle satisfy

$v_0 = v_0' = [1, 0, 0], v_1 = v_1' = [-1, 0, 0]$ and $v_2 = v_2' = [0, -1, 0]$ and pasting maps P_0 and P_2 identity and P_1 the rotation of angle π about $[0, 0, 1]$ sending v_1 to $-v_1'$.

A *triangular* representation is a representation such that the restriction to the fundamental groups of the pairs-of-pants S_0 and S_1 are triangular.

We can characterize the characters in triangular representations in $\mathcal{C}_1$ to be ones where each corresponding representation gives two triangles where each pasting map send a vertex v_i of the first triangle $\triangle_1$ to the corresponding vertex v_i' of the second triangle $\triangle_2$ or to the antipode $-v_i'$ of the vertex of the second triangle and at least one pasting map P_i sending v_i to $-v_i$ exists always.

A Klein group V acts on the upper triangle and the lower triangle by sending the vertices of the upper triangle v_0, v_1, v_2 to the vertices $\pm v_0, \pm v_1, \pm v_2$ for a triangle where there are two negatives or no negatives and sending the vertices of the lower triangle v_0', v_1', v_2' to $\pm v_0', \pm v_1', \pm v_2'$ with corresponding sign as the upper triangle case. In Δ^o, we have the formula as in the SO(3)-case.

By the Klein group action, we can make the triangular representation be so that the pasting map P_0 sends v_0 to v_0', P_1 sends v_1 to $-v_1'$ and P_2 sends v_2 to v_2' and hence satisfy the angle condition (8.1).

Lemma 8.1. *The subset of triangular characters in $\mathcal{C}_1$ is a dense open subset. Any character whose associated triangles is degenerate can be pushed to triangular ones by a path of deformations. Every character in the component $\mathcal{C}_1$ is associated with generalized triangles $(\triangle_0, \triangle_1)$ whose associated angles are*

$$(\theta_0, \theta_1, \theta_2) \ and \ (\theta_0, \pi - \theta_1, \theta_2). \tag{8.1}$$

Proof.

Choose a one near h^* that is triangular and hence it is represented by two generalized triangles $(\triangle_0, \triangle_1)$ with the angles $(\pi/2, \pi/2, \pi - \epsilon)$ and $(\pi/2, \pi/2, \pi - \epsilon)$ respectively and with vertices v_0, v_1, v_2 of $\triangle_0$ and v_0', v_1', v_2' of $\triangle_1$. Let us put them into standard positions. Thus, a pasting map must send the corresponding vertex to a vertex, say v, in the other triangle or to the antipodal point $-v$. Use the pasting maps P_0 and P_2 identity and P_1 a rotation at $[0, 0, 1]$ sending v_1 to $-v_1'$. The associated representation h_1 is clearly near h^*.

For some real numbers δ, δ', and δ'' with $|\delta|, |\delta'|, |\delta''| << \epsilon$, we can form two triangles with respective angles $(\pi/2 + \delta, \pi/2 - \delta', \pi - \epsilon + \delta'')$ and $(\pi/2 + \delta, \pi/2 + \delta', \pi - \epsilon + \delta'')$ with respective vertices (v_0, v_1, v_2) and (v_0', v_1', v_2'). Put these triangles in standard positions with the pasting map P_0 fixing v_0 and the corresponding P_1 sending v_1 to $-v_1'$ and P_2 sending v_2 to v_2'. If the pasting angles are different, the representation is different. One can form a near triangular representation h' to h_1 with triangles of angle $(\pi/2 + \delta, \pi/2 - \delta', \pi - \epsilon + \delta'')$ and $(\pi/2 + \delta, \pi + \delta', \pi - \epsilon + \delta'')$ respectively. Therefore, we obtain six parameters of characters near the character of h_1.

Therefore, the subset of triangular characters is not empty in $\mathcal{C}_1$. Moreover, the function $f_i : \mathcal{C}_1 \to R$ given by $[h] \mapsto \mathrm{tr} h(c_i)$ for each $i = 0, 1, 2$ is not con-

stant. Therefore, we see that the subset defined by $f_i = 3$ is an algebraic subset whose complement is dense and is locally path-connected. The subspace of the abelian characters are also 4-dimensional algebraic subset. Thus, any character can be pushed to a triangular character by a real analytic arc which is in triangular part except at the end. This proves the first part.

The set of triangular characters with the angle conditions forms an open subset of $\mathcal{C}_1$ since any nearby triangular character should have uniquely defined triangles up to isometry and uniquely pasting maps nearby.

Since all triangular characters satisfy the above angle condition, and these form a dense set, this completes the proof of the second part. $\square$

The set of possible nondegenerate triangles for $\triangle_0$ and $\triangle_1$ is then described as the intersection of $\tilde{\Delta}^o \cap \kappa(\tilde{\Delta}^o)$ where κ is the map sending $(\theta_0, \theta_1, \theta_2)$ to $(\theta_0, \pi - \theta_1, \theta_2)$. Since $\tilde{\Delta}^o$ is given by

$$\theta_0 + \theta_1 + \theta_2 > \pi$$
$$\theta_0 < \theta_1 + \theta_2 - \pi,$$
$$\theta_1 < \theta_2 + \theta_0 - \pi$$
$$\theta_2 < \theta_0 + \theta_1 - \pi, \tag{8.2}$$

it follows that our domain is an octahedron O given by eight equations

$$\theta_0 + \theta_1 + \theta_2 > \pi : (a)$$
$$\theta_0 + \theta_2 > \theta_1 : (a')$$
$$\theta_0 < \theta_1 + \theta_2 - \pi, : (c)$$
$$\theta_0 + \theta_1 < \theta_2 : (c')$$
$$\theta_1 < \theta_2 + \theta_0 - \pi : (b)$$
$$2\pi < \theta_0 + \theta_1 + \theta_2 : (b')$$
$$\theta_2 < \theta_0 + \theta_1 - \pi : (d)$$
$$\theta_1 + \theta_2 < \theta_0 : (d'). \tag{8.3}$$

The paranthesis indicates the name of the corresponding sides. See Section 8.2.

For a character h in $\mathcal{C}_1$, $\triangle_0$ and $\triangle_1$ are in the octahedron and they are related by κ.

8.2. *The compactification* $\tilde{O}$

We define $\tilde{O}$ as the subset of $\mathrm{Cl}(O)$ by additional requirements

$$\epsilon \leq \theta_0 \leq \pi - \epsilon, \epsilon \leq \theta_1 \leq \pi - \epsilon, \epsilon \leq \theta_2 \leq \pi - \epsilon.$$

We obtain six additonal faces corresponding to vertices

$$B_B : \theta_0 = \epsilon, B'_{B'} : \theta_0 = \pi - \epsilon, C_{C'} : \theta_1 = \epsilon, C'_C : \theta_1 = \pi - \epsilon,$$
$$A_A : \theta_2 = \epsilon, A'_{A'} : \theta_2 = \pi - \epsilon, \text{ for some } 0 < \epsilon < 1/10 \tag{8.4}$$

where respective faces are given by $\mathrm{Cl}(O)$ intersected with the planes of the respective equations. Actually, by $\tilde{O}$ we only mean the combinatorial facial structure of the above space.

We compactify O into $\tilde{O}$: We map O into $\tilde{\Delta} \times \tilde{\Delta}$ by sending $\triangle_0$ to $(\triangle_0, \triangle_1)$ by a map (I, κ) where $\triangle_1$ is the associated triangle with angles obtained by κ. Again the compactification is given by taking the closure of the image of O under the map $(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l'(0), l'(1), l'(2))$ where $v(i)$ and $l(i)$ denote the angle at v_i and the length of l_i of the upper triangle, and $v(i)'$ and $l(i)'$ denote the angle at v_i and the length of l_i of the lower triangle. We show that the result has the same combinatorial structures as $\tilde{O}$. (See Figure 9.)

Since geodesic rays in $\tilde{\Delta}^o$ correspond to points of the boundary of $\tilde{\Delta}$, and only the edge points of Δ or $\kappa(\Delta)$ are blown-up in $\tilde{\Delta}$ or $\kappa(\tilde{\Delta})$, we see that O compactifies in $\tilde{\Delta} \times \kappa(\tilde{\Delta})$ so that each vertex gets expanded to a quadrilateral and no other point of the boundary of O is changed: A ray in O gives us a ray in $\tilde{\Delta}$ and a ray in $\kappa(\tilde{\Delta})$. Therefore, a sequence of upper triangles and the corresponding sequence of the lower triangles in a given ray may converge to two different generalized triangles respectively. Recall that a ray a in $\tilde{\Delta}^o$ ending at the interior of an edge has a limit in a segment times the edge in $\tilde{\Delta}$. That is, a point is expanded to a segment. Considering a ray a and $\kappa(a)$, we see that at each vertex of O, a ray can converge to any point in an interval times an interval where the corresponding parameters of upper triangles and lower triangles converges respectively coordinate-wise. (See the proof of Theorem 1.1.) Thus O compactifies to a blown-up octahedron, i.e., an octahedron where each vertex is expanded to a square. The proof is similar to Theorem 1.1 and we omit it.

The Klein four-group $\{I, I_A, I_B, I_C\}$ acts on the resulting polyhedron $\tilde{O}$ isometrically. They are obtained by replacing v_i to $-v_i$ for some $i = 0, 1, 2$, and they have the same formula as in the $\mathcal{C}_0$ case. Of course, we need to verify these extensions but are analogously proven to the $\tilde{\Delta}$ case. See Figure 9 to see the topological actions.

They are as follows in terms of coordinates

$$I_A : (v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l'(0), l'(1), l'(2)) \mapsto$$
$$(\pi - v(0), \pi - v(1), v(2), \pi - l(0), \pi - l(1),$$
$$l(2), \pi - v(0)', \pi - v(1)', v(2)', \pi - l(0)', \pi - l(1)', l(2)') \quad (8.5)$$

where $v(i)$ and $l(i)$ denote the angle at v_i and the length of l_i of the upper triangle, and $v(i)'$ and $l(i)'$ denote the angle at v_i and the length of l_i of the lower triangle. The map I_B changes the triangle with vertices v_0, v_1, and v_2 to one with $v_0, -v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l(0)', l(1)', l(2)') \mapsto$$
$$(v(0), \pi - v(1), \pi - v(2), l(0), \pi - l(1), \pi - l(2),$$
$$v(0)', \pi - v(1)', \pi - v(2)', l(0)', \pi - l(1)', \pi - l(2)'). \quad (8.6)$$

$(\pi/2,\pi/2, \pi)$

$(0,\pi/2, \pi/2)$

$(\pi/2,0, \pi/2)$

$(\pi/2,\pi, \pi/2)$

$(\pi,\pi/2, \pi/2)$

$(\pi/2,\pi/2, 0)$

b

B $_B$

A $_A$

$(\pi/2, \pi/2, \pi)$

$(\pi, \pi/2, \pi/2)$

b'

$(\pi/2, \pi, \pi/2)$

c

c'

C $_{C'}$

d

d'

a'

A' $_{A'}$

B' $_{B'}$

$(\pi/2, \pi/2, 0)$
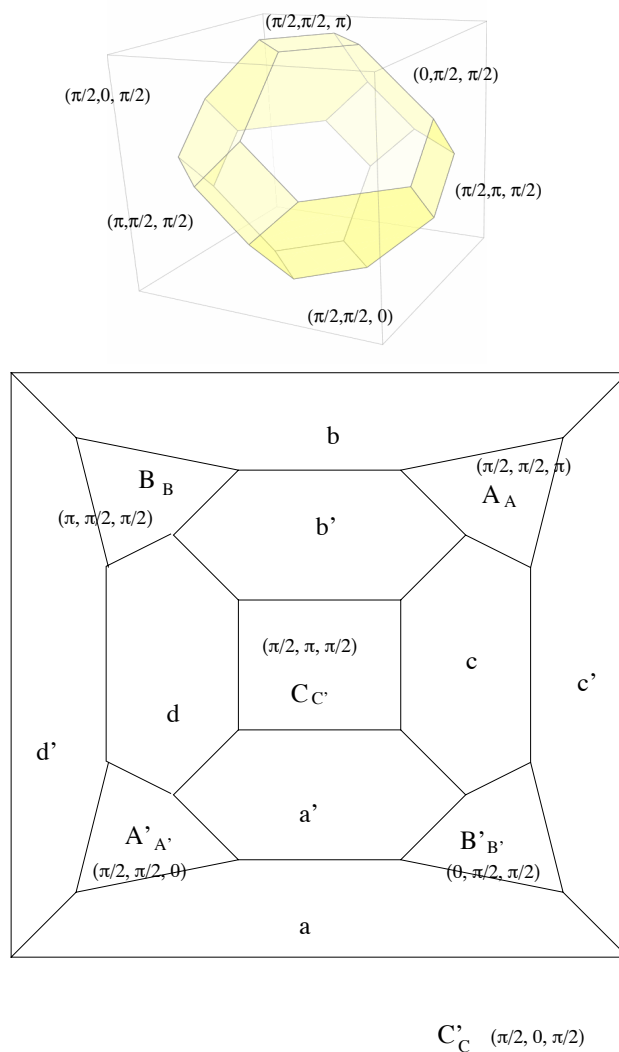
$(0, \pi/2, \pi/2)$

a

C' $_C$ $(\pi/2, 0, \pi/2)$

Fig. 9. $\tilde{O}$ and the faces of $\tilde{O}$ drawn topologically and on a plane using stereographic projection.

The map I_C changes the triangle with vertices v_0, v_1, and v_2 to one with $-v_0, v_1$, and $-v_2$:

$$(v(0), v(1), v(2), l(0), l(1), l(2), v(0)', v(1)', v(2)', l(0)', l(1)', l(2)') \mapsto$$
$$(\pi - v(0), v(1), \pi - v(2), \pi - l(0), l(1), \pi - l(2),$$
$$\pi - v(0)', v(1)', \pi - v(2)', \pi - l(0)', l(1)', \pi - l(2)'). \quad (8.7)$$

Since κ is an involution, we have $\kappa^{-1} = \kappa$. We label the regions by first what region in $\tilde{\Delta}$ it is from and second what region in $\kappa(\tilde{\Delta})$ it is from. For faces, from $\tilde{\Delta}$, we will simply denote it by a, b, c, or d according to where it came from. For

faces from $\kappa(\tilde{\Delta})$, we denote it by a', b', c', or d' according to where it came from among a, b, c, or d by κ respectively. Of course, here the orderings in labels are important. In the equation (8.3), the equations correspond to a, a', c, c', b, b', d, and d' respectively.

We describe the faces briefly. The parameters of each quadrilateral is given by tie parameters in $\tilde{\Delta}$ and $\iota(\tilde{\Delta})$ since the rectangles are really a product of the two middle ties.

- (A_A) This region is a quadrilateral. The top triangle of a point over this region is a lune with vertices v_0 and v_1, of angle $\pi/2$, and v_2 is on the segment between v_0 and v_1. The bottom triangle is a lune with vertices v_0 and $v_1 = -v_0$ and v_2 on the segment between v_0 and v_2. A point in corresponds to two generalized triangles $(\triangle_0, \kappa(\triangle_0))$. The function $l(1)$ measures the length of the edge between v_0 and v_2 and $l'(1)$ measures the length of the edge between v_0' and v_2' parameterize the square with values in $[0, \pi] \times [0, \pi]$. Here the lune and the map above for h^* correspond to the center of the square.

- ($A'_{A'}$) The upper triangle is a segment with one endpoint $v_0 = v_1$ both of angle $\pi/2$ and v_2 the other endpoint. The lower triangle is a segment with one endpoint $v_0 = v_1$ both of angle $\pi/2$ and v_2 the other endpoint.

- (B_B) This region is again a quadrilateral. The top triangle is a lune with vertices v_1 and v_2 antipodal to each other and v_0 is on an edge. The bottom one is a lune of same type.

- ($B'_{B'}$) This is again a quadrilateral and the top and the bottom triangles are segments with vertices $v_1 = v_2$ and v_0.

- ($C_{C'}$) This is again a quadrilateral and the top triangle is a lune with vertices v_0 and v_2 antipodal to each other. The bottom triangle is a segment with an endpoint $v_0 = v_1$ and the other endpoint v_1.

- (C'_C) This is again a quadrilateral and the top triangle and the bottom one are reversed ones for $C_{C'}$.

- (b) This region is a hexagon. A point correspond to a representation where the top triangle is a pointed lune and $v_0 = v_2$ and v_1 is antipodal to them. The bottom is a nondegenerate triangle except when we are at the boundary of b. The boundary consists of segments in A_A, B_B, $C_{C'}$, and in b', c', d'. This means that the bottom triangles are as in A, B, C', and b, c, d respectively.

- (c) The region is a hexagon. Again analogous statements to (b) hold.

- (d) The region is a hexagon. Again analogous statements to (b) hold.

- (a) The region is a hexagon. Analogous statements to (b) hold.

- ($a' - d'$) These are hexagons and the additional edges are among A_A, B'_B, and $C_{C'}$. The triangles of the points are easily learned from from a, b, c, and d respectively.

The regions from the faces of O are said to be *faces* still, and the regions from

the vertices of O are said to be *square regions*. Notice that edges are not blown-up here.

Note that $\{I, I_A\}$ acts on A_A and $A'_{A'}$ but does not act on other rectangular regions. Similarly $\{I, I_B\}$ acts on B_B and $B'_{B'}$ but does not act on other regions and $\{I, I_C\}$ acts on $C_{C'}$ and C'_C but does not act on other regions. Finally, I_A sends faces b to c and b' to c' and a to d and a' to d' and vice versa. I_B sends faces a to c and a' to c' and b to d and b' to d' and vice versa. I_C sends faces c to d and a' to b' and a to b and c' to d' and vice versa. The pasting angles are sent as in the $\mathcal{C}_0$ cases. One can verify this using a straightforward process.

8.3. *Equivalence relations on* $\tilde{O}$

We will now introduce equivalence relations on $\tilde{O} \times T^3$ so that the result is homeomorphic to the other component $\mathcal{C}_1$.

First we define a continuous map $\mathcal{T} : \tilde{O} \times T^3 \to \mathcal{C}_1$: We are given two triangles $(\triangle_0, \triangle_1)$ and three pasting angles, we define a representation. Put $\triangle_0$ and $\triangle_1$ into standard positions in the closed hemisphere H as defined above. That is, for vertices v_0, v_1, v_2, we have $v_0 = [1, 0, 0]$ and v_1 on the closure L of the component of $\partial H - \{[1, 0, 0], [-1, 0, 0]\}$ containing $[0, -1, 0]$ and $v_2 \in H$.

Let v_0, v_1, and v_2 be the vertices of $\triangle_0$ and v'_0, v'_1, and v'_2 be those of $\triangle_1$. Let $(\theta_0, \theta_1, \theta_2)$ denote the angles of $\triangle_0$.

We always have $v_0 = [1, 0, 0]$ and $v_1 \in L$. Thus, given a pasting angle (ϕ_0, ϕ_1, ϕ_2), we define the pasting map P_0 to be R_{v_0, ϕ_0}. Given ϕ_1, we define the pasting map P_1 sending v_1 to $-v'_1$ to be the

$$R_{-v'_1, \phi_1} R_{[0,0,1], d'(v_1, -v'_1)} = R_{[0,0,1], d'(v_1, -v'_1)} R_{v_1, \phi_1}$$

where $d(v_1, -v'_1)$ is the counter-clockwise arclength between v_1 and $-v'_1$.

Denote by L_0 the segment in H with endpoints v_0 and v_2 in all cases that has angle θ_0 with L at v_0. Let $\mathbf{S}^1_0$ be the great circle containing L_0. (Note that $\theta_0 = \pi/2$ when we are on $A_A, A'_{A'}, C_{C'}$, or C'_C and $\theta_0 = \pi$ when we are on B_B or $B'_{B'}$.)

Suppose that at least one of the triangles is nondegenerate. Then one of v_2 and v'_2 is in H^o and hence v_2 and v'_2 are not antipodal, and hence there exists a unique isometry R sending v_2 to v'_2 preserving the great circle through v_2 and v'_2 if $v_2 \neq v'_2$. If $v_2 = v'_2$ we let R be the identity. We define $P_2 = R_{v'_2, \phi_2} R = R R_{v_2, \phi_2}$. Since v_2 and v'_2 lie on L_0, R preserves $\mathbf{S}^1_0$.

Suppose that both triangles are degenerate, and they are on one of the interiors of the six squares corresponding to the six vertices of O.

Suppose that $(\triangle_0, \triangle_1)$ is in the square A_A. Put the two triangles into standard positions. They are both lunes with vertices on a segment L_0. Then R must preserve $\mathbf{S}^1_0$. Therefore, we define the pasting maps P_0, P_1, P_2 as in the two paragraphs ago. If $(\triangle_0, \triangle_1)$ is in the square $A'_{A'}$, we do similarly to A_A.

Suppose that $(\triangle_0, \triangle_1)$ is in the square B_B. In this case, v_0 has a π-angle and v_1 is in L and v_2 is in $-L$. The pasting maps P_0 and P_1 are defined as above. For

the pasting map P_2, we take as R the rotation preserving $\mathbf{S}_0^1 = \partial H$ as prescribed two paragraphs ago. For case $B'_{B'}$, the arguments are similar.

For case $C_{C'}$, $\triangle_0$ is a lune with vertices $v_0 = [1,0,0]$ and $v_2 = -v_0$ and v_1 is in L. We have that $\triangle_1$ is a segment with an endpoint $v'_0 = v'_2 = [1,0,0]$ and v'_1 is in L. Then the pasting maps P_0 and P_1 are defined as above. For the pasting map P_2, we use the composition of R preserving the great circle $\mathbf{S}_0^1$ with a rotation at $v_2 = [-1,0,0]$. For C'_C, the definitions are similar.

The continuous dependences of P_0, P_1, and P_2 on $\tilde{O} \times T^3$ follow from Proposition 1.4. Since h^* is represented by two triangles with vertices $[1,0,0], [0,1,0]$, and $[0,0,1]$ where ϕ_0, ϕ_1, and ϕ_2 are all zero, it follows that our map $\mathcal{T}$ is into $\mathcal{C}_1$. Since the subspace of triangular characters is dense in $\mathcal{C}_1$ and $\tilde{O}$ is compact, and our map is onto the triangular subspace, it follows that $\mathcal{T}$ is onto $\mathcal{C}_1$.

We now extend the action of $\{I, I_A, I_B, I_C\}$ on $\tilde{O} \times T^3$ from the action on $\tilde{O}$ in Section 8.2 as in equation (5.1). This is our first equivalence relation.

Proposition 8.1. *Using pasting map construction as above, we have a continuous onto map*

$$\mathcal{T} : \tilde{O} \times T^3 \to \mathcal{C}_1.$$

Furthermore, we have

$$\mathcal{T} \circ I_A = \mathcal{T} \circ I_B = \mathcal{T} \circ I_C = \mathcal{T}.$$

Proof. The continuity and the surjectivity were proved above. Consider the transformation I_A on $\tilde{O}$. It acts simultaneously on the upper and the lower triangles and on the pasting angles. By geometry, we see that this does not change the character constructed. The similar arguments hold for I_B and I_C.   □

We aim to introduce an equivalence relation on $\tilde{O} \times T^3$ so that the induced map is one-to-one onto.

8.3.1. *The equivalence relations on faces* a, b, c, d, a', b', c', *and* d'

There are equivalence relations on the faces a, b, c, d, a', b', c', and d'. We will consider a first. Then here the top triangle is a pointed point and the bottom triangle a nondegenerate triangle.

(a) We consider a first. The equivalence relation is given by the fact that a representation of a pair-of-pants corresponding to a pointed point has a stabilizer in SO(3) isomorphic to a circle. Thus the pasting maps P_0, P_1, P_2 are determined only up to a right multiplication by an isometry fixing the point. This changes the pasting angles to be added by the same angle. This changes P_i to $P_i R_{[1,0,0],\theta}$ as the pointed point is at $[1,0,0]$ in the standard position. We rewrite it as $P_i R_{[1,0,0],\theta} P_i^{-1} P_i$. The first three terms give us a rotation of angle θ at $P_i([1,0,0])$. This changes the rotation angles by θ, and gives us an $\mathbf{S}^1$-action given by sending

$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 + \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

This action gives us the equivalence relation on a. (This section is similar to Appendix A.1.) The quotient space is homeomorphic to a 2-torus, which we denote by $T^2_{a,2}$ as in the $\mathcal{C}_0$ case. Thus, the character space here is in one-to-one correspondence with $a \times T^2_{a,2}$. There are no more equivalences here since the lower triangle is nondegenerate.

Furthermore, the $\mathbf{S}^1$-action extends to $a \times T^3$ and still gives equivalent pastings.

For regions b, c, and d, these are all equivalent to a by the the action of the group generated by I_A, I_B, I_C. Thus, similar conclusions hold.

(b) The $\mathbf{S}^1$-action
$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 - \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $\phi_0 - \phi_1 + \phi_2 = 0$.

(c) The $\mathbf{S}^1$-action
$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 - \theta, \phi_1 + \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $-\phi_0 + \phi_1 + \phi_2 = 0$.

(d) The $\mathbf{S}^1$-action
$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 + \theta, \phi_2 - \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $\phi_0 + \phi_1 - \phi_2 = 0$.

There is again the order 3 translation group action giving us a 2-torus as a quotient in each case. Here the $\pm$ of $\pm\theta$ corresponds to the action of $\mathbf{S}^1$ on the pasting angles. (These $\mathbf{S}^1$-actions all extend to the boundary where they all give equivalent pastings.)

The upper and lower triangles are related by the relation in O:
$$(\theta_0, \theta_1, \theta_2) \leftrightarrow (\theta_0, \pi - \theta_1, \theta_2).$$

For regions a', b', c', and d', these are related to a, b, c, and d by switching the upper triangle with the lower ones and taking inverses of the pasting maps P_0, P_1, P_2.

(a') Here the upper triangle is nondegenerate in the interior of a' and the lower one is a pointed point, say v. Let v_0, v_1, v_2 denote the vertices of the upper triangle. The pasting map P_1 maps v_1 to the antipodal point of v and P_0, P_2 sends v_0, v_2 to v. By reasoning like the case a, we obtain that $\mathbf{S}^1$-action fixing v, gives us an $\mathbf{S}^1$-action
$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 - \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $\phi_0 - \phi_1 + \phi_2 = 0$

(b') By a similar reason to above, the $\mathbf{S}^1$-action is given by
$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 + \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $\phi_0 + \phi_1 + \phi_2 = 0$

(c') The $\mathbf{S}^1$-action is given by

$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 - \theta, \phi_1 - \theta, \phi_2 + \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $-\phi_0 - \phi_1 + \phi_2 = 0$

(d') The $\mathbf{S}^1$-action is given by

$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + \theta, \phi_1 - \theta, \phi_2 - \theta) \text{ for } \theta \in \mathbf{S}^1.$$

The equation for the equivalence class representatives is $\phi_0 - \phi_1 - \phi_2 = 0$.

Again, there is an order-3 translation group action in each case giving us a 2-torus as a quotient.

The $\mathbf{S}^1$-actions can be described as orthogonal directional one to the faces and the representative equations are tangent directions to the faces. Also, I_A, I_B, I_C permute $\mathbf{S}^1$-actions and the representative directions equations.

In the interior of a, b, c, a', b', c', the equivalence relation is precisely the ones given above since one of the upper and lower triangles is nondegenerate and the vertices are well-defined.

Up to the Klein group action, there are only two faces, say a' and c.

8.3.2. *Equivalence relations over the edges.*

When two faces meet, $\mathbf{S}^1$-actions induced from the two faces are different ones and they generate a T^2-action which gives equivalent points over the edges. In the interior of each edge, these two will be the only relations we need to consider: The upper triangle and the lower triangle are either a pointed-point or a pointed segment of length π with all the vertices equal to the endpoints. Hence there are two $\mathbf{S}^1$-actions corresponding to the upper triangle and the lower one. In the interior of the edge, these point or segments are well-defined up to isometries of $\mathbf{S}^2$.

Here we obtain as the quotient space a circle over each point of the interiors of edges as the quotient space, and the circles form annuli over the interiors of the edges. We obtain 12 open annuli.

Up to the Klein four group action, we only have three edges and three open annuli.

8.3.3. *The equivalence relations on faces* $A_A, A'_{A'}, B_B, B'_{B'}, C_{C'}, C'_C$.

We will introduce equivalences over the six regions, $A_A, A'_{A'}, B_B, B'_{B'}, C_{C'}, C'_C$. Each equivalence relation is simply the identification producing the same characters. We show that each of the quotient spaces is homeomorphic to $\mathbf{S}^3$ and its tubular neighborhood is homeomorphic to $\mathbf{S}^3 \times B^3$. Also, we show that there is a branch double-cover by $\mathbf{S}^3$ and by $\mathbf{S}^3 \times B^3$.

First, we go to the square A_A. Here the top triangle is a lune with vertices v_0 and v_1 and v_2 on an edge of a lune. The bottom triangle is a lune with vertices v'_0

and $v'_1 = -v'_0$ and v'_2 on an edge of the lune. The vertex angles are all $\pi/2$. These are put into standard positions.

We note that $d_1 = P_0^{-1}P_1$ and $d_2 = P_0^{-1}P_2$ for the pasting maps $P_0, P_1,$ and P_2. We will find preferred fixed points of d_1 and d_2 respectively on $\mathbf{S}^2$.

Fig. 10.   How to find a fixed point of $h(d_1)$ for region A_A.

First, the pasting map P_0 is a rotation $R([1, 0, 0], \theta_0)$.

And P_1 is a composition of rotation of angle θ_1 at $[-1, 0, 0]$ with $R([0, 0, 1], \pi)$. Thus, P_1 sends $[1, 0, 0]$ to $[-1, 0, 0]$ and has -1 as an eigenvalue. A fixed point is given as a vertex of a triangle with vertex $[0, 0, 1]$ and $[-1, 0, 0]$ with respective angles $\pi/2$ and $\theta_1/2$ respectively. We let u be this vertex which lies on the great circle $\mathbf{S}_o^1$ passing through $[\pm 1, 0, 0]$ and perpendicular to $\mathbf{S}_*^1$ passing through $[0, 1, 0]$ and $[0, 0, 1]$.

To obtain the fixed point of d_1, we draw a triangle with vertex u and $[1, 0, 0]$ with respective angles $\pi/2$ and $\pi - \theta_0/2$. The third vertex u_1 of this triangle is a fixed point of d_1. Again u_1 is on $\mathbf{S}_*^1$. The rotation angles is π clearly as the angle of the the vertex is $\pi/2$. Also, u_1 is the intersection point of $\mathbf{S}_*^1$ and the segment with vertices $[1, 0, 0]$ and $[-1, 0, 0]$ having angle $\pi - \theta_0/2 - \theta_1/2$ with the segment connecting $[1, 0, 0]$ with $[0, 0, 1]$. Thus, u_1 could be any point on $\mathbf{S}_*^1$. (See Figure 10.) From construction, the rotation angle of $h(d_1)$ is π always.

Now, $[1, 0, 0]$ and u_1 will give us a coordinate system κ_{u_1} on $\mathbf{S}^2$ so that $[1, 0, 0]$ has coordinates $[1, 0, 0]$ and u_1 the coordinates $[0, 0, 1]$.

Consider $\mathbf{S}^3$ as a double cover of SO(3), i.e., as a ball of radius 2π in $\mathbb{R}^3$ with the outer sphere identified to a point. The coordinate expression of d_2 with a choice of the set of fixed points and the angle correspond to a point of $\mathbf{S}^3$. The coordinates of

d_2 under the coordinate system κ_{u_1} give us a two-valued function on $A_A \times T^3 \to \mathbf{S}^3$. The function depends on the choice of the fixed point $u_1, -u_1$ for $h(d_1)$ and the choice among the pair in the set of fixed points of $h(d_2)$ as well. This generates a Klein four-group V action on $\mathbf{S}^3$. Therefore, we obtain a map $A_A \times T^3 \to \mathbf{S}^3/V$. (We remark that if we are on SU(2), we can find a canonical choice.)

By an easy topological argument, V acts by two transformations S and T: S on $\mathbf{S}^3$ is induced by a rotation of angle π at $[1,0,0]$; that is, sending an isometry to the isometry of same rotation angle but with the fixed point changed by the rotation of angle π at $[1,0,0]$ and by the transformation T equal to the antipodal map on $\mathbf{S}^3$. Then the set of fixed points of S is the circle through the identity, consisting of all rotations with fixed points $[1,0,0]$ and $[-1,0,0]$. The set of fixed points of T is empty. The set of fixed points of ST is given by isometries with fixed points at the circle though $[0,1,0]$ and $[0,0,1]$ of rotation angle π.

Remark 8.1. Here, it is useful to see $\mathbf{S}^3$ as a unit sphere in $\mathbb{R}^4$. Then S is represented as a diagonal matrix with entries $1, -1, -1, 1$ and T equals $-I$ and ST is a diagonal matrix with entries $-1, 1, 1, -1$. Thus, the set of fixed points of S is given by the xt-plane intersected with $\mathbf{S}^3$ and the set of fixed points of ST is given by the yz-plane intersected with $\mathbf{S}^3$. Hence, the branch locus is a link consisting of two trivial knots with one crossing point under a projection.

Finally, $\{I, I_A\}$ acts on A_A. This is the action sending v_0 to v_1 and v_1 to v_0 and fixing v_2 and v_0' to v_1' and v_1' to v_0' and fixing v_2'. The angles change $\theta_0 \to 2\pi - \theta_0$ and $\theta_1 \to 2\pi - \theta_1$ but θ_2 is not changed. (See Section 8.2 for equations of I_A.)

Lemma 8.2. I_A *acts on* $\mathbf{S}^3$ *as an involution with a circle as a fixed point set. The fixed points form a circle meeting each of the above two fixed circles of V twice. Let V' be the group generated by V and I_A, the resulting space $\mathbf{S}^3/V'$ is homeomorphic to* $\mathbf{S}^3$.

Proof. For the proof we work geometrically and the maps here are maps on $\mathbf{S}^2$ and hence in $\mathrm{SU}(2) = \mathbf{S}^3$ for the duration of the proof.

If we now construct the representation while the two triangles are changed as above. Then P_2 is not changed but P_0 is changed to be

$$R_{2\pi - \theta_0}(-v_0) = -R_{\theta_0}(v_0) = -P_0$$

by equation 3.1. and P_1 is changed to be

$$R_{2\pi - \theta_1}(v_1)R_\pi([0,0,1]) = -R_{\theta_1}(-v_1)R_\pi([0,0,1]) = -P_1$$

now for the new v_1 that is $-v_1$. Hence $h(d_1)$ is not changed and $h(d_2)$ changed to $-h(d_2)$. Define $\tilde{R} = R_\pi([0,0,1])$ with $\tilde{R}^{-1} = \tilde{R}$. Then we obtain $\tilde{R}h(d_1)\tilde{R}$ and $-\tilde{R}h(d_2)\tilde{R}$ to find coordinates. Our new u_1 is $\tilde{R}(u_1)$.

Suppose that we need an isometry L to move old u_1 to $[0,0,1]$ fixing $[1,0,0]$. To move $\tilde{R}(u_1)$ to $[0,0,1]$, we need L^{-1}. To find the coordinate of $-\tilde{R}h(d_2)\tilde{R}^{-1}$, we

conjugate by L^{-1} to obtain $-L^{-2}L\tilde{R}L^{-1}Lh(d_2)L^{-1}L\tilde{R}^{-1}L^{-1}L^2$. Thus, we act by $L^{-2}L\tilde{R}L^{-1}$ to obtain the fixed point of $-h(d_2)$ read from the new u_1 coordinates. Since $L\tilde{R}L^{-1}$ has a fixed point $L([0,0,1])$ and angle π and L^{-2} is a rotation at $[1,0,0]$ of an angel $2l$ for the distance l between old u_1 and $[0,1,0]$. We can use a triangle with angle $\pi/2$ at $L([0,0,1])$ and angle l at $[1,0,0]$ to show that $L^{-2}L\tilde{R}L^{-1} = \tilde{R}$ in fact. Thus the coordinate of new $h(d_2)$ is changed by $\tilde{R}$ from the old one. Thus, I_A is a left multiplication by $-\tilde{R}$ in this case. In the above coordinates, I_A corresponds to the diagonal matrix with entries $1, 1, -1, -1$ in the notation of Remark 8.1. $\square$

Hence, we obtain a map

$$A_A \times T^3 \to \mathbf{S}^3/V' \cong \mathbf{S}^3.$$

This map is seen to be surjective, which follows from the geometric constructions; essentially, any element of SO(3) can arise as $h(d_2)$. We show this by showing that the set of $h(d_2)$ with $h(d_1)$ fixed that arises is an open set by controlling P_2 in an open set fixing the angles θ_0, θ_1. The control of P_2 in an open set can be achieved by the multiplication by geometry construction. The set of $h(d_2)$ that arises with $h(d_1)$ fixed is a closed set since this is the image of a continuous map. (See the proof of Theorem 9.1 also.) Furthermore, under the equivalence relation we assigned,

$$A_A \times T^3/ \sim \to \mathbf{S}^3/V' \cong \mathbf{S}^3$$

is a homeomorphism.

For the region $A'_{A'} \times T^3$, we can do the similar argument: Hence, we obtain a map

$$A'_{A'} \times T^3 \to \mathbf{S}^3/V \cong \mathbf{S}^3.$$

This map is seen to be surjective, and

$$A'_{A'} \times T^3/ \sim \to \mathbf{S}^3/V' \cong \mathbf{S}^3$$

is a homeomorphism for our equivalence relation $\sim$.

We discuss B_B where I_B acts on. Here the top triangle is a lune with vertices v_1 and v_2 with v_0 on a segment. The vertex angle of the lune is $\pi/2$. The bottom triangle is a lune with an endpoint v'_1 and $v'_2 = -v'_1$ and v'_0 in the segment.

We change generators so that we use c_0 and c_2 for generators and $d'_1 = P_1^{-1}P_2$ and $d'_2 = P_1^{-1}P_0$. Then $d'_1 = d_2^{-1}$ and $d'_2 = d_1^{-1}d_2$. This is a change of index sending $(0,1,2)$ to $(2,0,1)$. Thus, the situation again reduces to the case $A_A \times T^3$ with I_B corresponding to I_A.

Let V'' be generated by V and I_B. We obtain a map

$$B_B \times T^3/ \sim \to \mathbf{S}^3/V'' \cong \mathbf{S}^3,$$

and

$$B_B \times T^3/ \sim \to \mathbf{S}^3/V'' \cong \mathbf{S}^3$$

is a homeomorphism.

For the region $B'_{B'} \times T^3$, we can do the similar argument: Hence, we obtain a map

$$B'_{B'} \times T^3/\mathbb{Z}_2 \to \mathbf{S}^3/V \cong \mathbf{S}^3.$$

This map is seen to be surjective, and

$$B'_{B'} \times T^3/\sim \to \mathbf{S}^3/V'' \cong \mathbf{S}^3.$$

Next, we go to the region $C_{C'}$. Here the vertex v_0 is antipodal to v_2 and v_0 and v_2 are the vertices of a lune, and v_1 is on the segment of the lune. The lower triangle is a segment with vertices $v'_0 = v'_2$ and v'_1.

In this case, the pasting map P_2 sends v_2 to $v'_2 = -v_2$ which is a composition of rotation of angle π at $[0, 1, 0]$ and a rotation at $[1, 0, 0] = v'_2$. Since P_0 is a rotation at $[1, 0, 0]$, it follows that $h(d_2)$ has a fixed point in the great circle $\mathbf{S}^1_*$ containing $[0, 1, 0]$ and $[0, 0, 1]$ by an argument similar to A_A case. $h(d_1)$ can be any isometry. Again I_C can be studied as in the I_A case. Let V''' denote the group generated by V and I_C.

$$C_{C'} \times T^3 \to \mathbf{S}^3/V''' \cong \mathbf{S}^3,$$

is surjective also, and

$$C_{C'} \times T^3/\sim \to \mathbf{S}^3/V''' \cong \mathbf{S}^3$$

is a homeomorphism.

Thus, we obtained the desired conclusions. The same can be said for the region C'_C.

We define the equivalence relation over the faces $A_A, A'_{A'}, B_B, B'_{B'}, C_{C'}, C'_C$ so that the above maps are injective. Here, we really have only three regions up to the V-action, say $A_A, B_B, C_{C'}$.

Also, we use $\{I, I_A, I_B, I_C\}$-action to identify characters.

Theorem 8.1. *The map* $\mathcal{T} : \tilde{O} \times T^3_2 \to \mathcal{C}_1$ *induces a homeomorphism* $\tilde{O} \times T^3_2/\sim \to$ $\mathcal{C}_1$ *where* $\sim$ *is given as above.*

Proof. We showed that $\mathcal{T}$ is onto by Proposition 8.1. Let us show that $\mathcal{T}$ is one-to-one. The interior $O \times T^3/\sim$ injects into triangular characters since triangular characters uniquely determine the triangles up to isometry and the choices of fixed points up to antipodal pairs and rotation angles are also determined by the standard positions.

The map from the boundary of O with vertex rectangles removed is also injective by Section 8.3.1. The maps from the regions above the square regions are injective up to the $\{I, I_A, I_B, I_C\}$-action by Section 8.3.3. The injectivity of the total map follows from these results and the fact that the images of $O \times T^3$ and $\partial \tilde{O} - (A_A \cup A'_{A'} \cup B_B \cup B'_{B'} \cup C_{C'} \cup C'_C) \times T^3$ and $A_A \times T^3, A'_{A'} \times T^3, B_B \times T^3, B'_{B'} \times T^3, C_{C'} \times T^3, C'_C) \times T^3$

are mutually disjoint up to action of $\{I, I_A, I_B, I_C\}$ from considering the angles of the triangles.

Since SO(3) is compact, the quotient space $\text{rep}(\pi_1(\Sigma_1), \text{SO}(3))$ is Hausdorff. Since $\tilde{O} \times T^3/\sim$ is compact and $\mathcal{C}_1$ is Hausdorff, we have a homeomorphism. $\qquad\square$

9. The SU(2)-pseudo-characters for the above component

We define the SU(2)-character space $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$ of a punctured genus 2 surface Σ_1 with the puncture holonomy $-I$ as the quotient space of the subspace of $\text{Hom}(\pi_1(\Sigma_1), \text{SU}(2))$ where $h(c) = -I$ by conjugations where c is a simple closed curve around the puncture.

A representation corresponding to a point of $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$ has a centralizer if and only if it is abelian. But no abelian character is in it. Thus, the SU(2)-character space is a manifold by Proposition 3.7 and the modified Proposition 1.2 of [16], where the relation is modified to equal $-I$.

Let T_2^3 be the product of three copies of $\mathbf{S}_2^1$. Let $\mathbb{Z}_2$ acts on it as before and we obtain $T_2^3/\mathbb{Z}_2$.

Theorem 9.1. $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$ *is homeomorphic to the filled octahedral manifold.*

Proof.

As before, we need to show that $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$ is homeomorphic to a quotient space of $\tilde{O} \times T_2^3/\mathbb{Z}_2$. Then we need to show that the quotient space is homeomorphic to the desired space.

Realize Σ_1 as a union of two pairs-of-pants S_0 and S_1 and a punctured annulus A inserted in place of c_1. Thus, the angles for c_1 in S_0 and c_1' in S_1 are related by θ to $2\pi - \theta$.

The triangular subset D_1 of $\mathcal{C}_1$ can be represented by $O^o \times T_2^3/\mathbb{Z}_2$. We define a triangular part D_2 of $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$ as the subset where the restrictions of each element to $\pi_1(S_0)$ and $\pi_1(S_1)$ are both triangular characters to SU(2), i.e., represented by a nondegenerate triangle. Using the trace functions for c_i, we show that D_2 is a dense open set with the complement equal to an algebraic subset. More precisely, there is a 16 to 1 map from $O^o \times T_2^3/\mathbb{Z}_2$ to D_1 and D_2 is an open connected subset. It follows that we can lift $O^o \times T_2^3/\mathbb{Z}_2$ into the dense open set D_2 in $\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$. The lift is injective since in O^o the two triangles are not degenerate and hence the isometry classes of the two triangles and the pasting angles are uniquely determined. (See Proposition 8.1 for details.)

By a geometric limit argument, we see that each character in

$$\text{rep}_{-I}(\pi_1(\Sigma_1), \text{SU}(2))$$

induces a representation where two pairs-of-pants are represented as two triangles (generalized) related by the angle conditions. Conversely, such a pair corresponds to a character of $\pi_1(\Sigma_1)$ to SU(2). Therefore, we have a surjective map from $\tilde{O} \times T^3/\mathbb{Z}_2$

to $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$. (Note that as before, we are using triangles, not the fixed point themselves, to represent these.)

Now we introduce equivalence relation $\sim$ on this space, which does not include ones given by I_A, I_B, I_C.

We will use the same equivalence relation on regions above a, b, c, d, a', b', c', and d' as in the SO(3)-case except that now the fibers are $T_2^3/\mathbb{Z}_2$. The equivalence relation is given by identifying all elements proportional to $(2\pi, 2\pi, 2\pi)$-vector in T_2^3 to O for face $a \times T_2^3$, by identifying all elements proportional to $(2\pi, -2\pi, 2\pi)$-vector in T_2^3 to O for face $a' \times T_2^3$, and so on. (See 8.3.1 for more details.) Thus, the character space over the interior of a face a^o is in one-to-one correspondence with $a^o \times T^2$. There are no more equivalences here since the lower triangle is nondegenerate. For all other faces, the similar consideration gives us that over the interior of a face, there is a 2-torus fibration. Over the interior of edges, the equivalence relation is one given by the two adjacent faces. Let $\tilde{O}'$ be the $\tilde{O}$ with six rectangular regions removed. Now, $\tilde{O}' \times T_2^3/\mathbb{Z}_2/\sim$ with the same equivalence relation as induced by faces extended to edges and vertices is the octahedral variety: This follows since this is $\tilde{O}' \times \mathbb{R}^3$ quotient by the lattice generated by $(4\pi, 0, 0), (0, 4\pi, 0), (0, 0, 4\pi), (2\pi, 2\pi, 2\pi)$, which has generators $(\pm 2\pi, \pm 2\pi, \pm 2\pi)$. By this description, we see that the quotient space is homeomorphic to the octahedral variety with six singular points removed. (See the page 50 in Fulton [21]).

This space is actually described as follows. CP^3 is a toric variety over a tetrahedron Δ. Then there are six spheres mapping to six edges of the tetrahedron. Over each vertex of Δ, the fiber is a singleton. We blow-up each of the four singleton mapping to a vertex of Δ to a two-dimensional sphere and now the six spheres do not meet. Now blow-down each of the spheres to a point. Then remove the six points, we obtain a manifold $\mathcal{O}$. (See page 50, Section 2.6 of Fulton [21].)

We see from this picture that if we remove the union of contractible $T_2^3/\mathbb{Z}_2$-invariant cone-neighborhoods of the six points, then the remaining is a manifold so that the closure in $\mathcal{O}$ is compact and has boundary a union of six 5-manifolds each homeomorphic to an $\mathbf{S}^3$-fibration over a 2-sphere.

For regions, A_A, B_B, $C_{C'}$, $A'_{A'}$, $B'_{B'}$, and C'_C, we will use a different but similar identification to the SO(3)-cases. Let us take the case A_A first. Here the pasting angles are in $\mathbf{S}_2^1$. By an extended multiplication by geometry for SU(2), we obtain the fixed point u_1 of $-h(d_1)$ and $-h(d_2)$ uniquely determined in this case. Therefore, by reading the coordinates of $-h(d_2)$ in terms of u_1, we obtain a map $A_A \times T_2^3 \to \mathcal{C}_1$ whose image is an embedded 3-sphere since given all points of $\mathbf{S}^2$ arise as a point and all rotation angles occur by our constructions, where again we used our control of P_2 with pasting angles fixed at v_0 and v_2. Moreover, $A_A \times T_2^3/\sim \to \mathcal{C}_1$ is an embedding since $\sim$ is chosen for this to hold.

Since $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$ is topologically a manifold, the tubular neighborhood of the region above A_A is homeomorphic to $\mathbf{S}^3 \times B^3$ and covers the tubular neighborhood of the 3-sphere in SO(3) case 4 to 1.

In all other cases B_B, $C_{C'}$, $A'_{A'}$, $B'_{B'}$, and C'_C, similar arguments show that the rectangle times T_2^3 maps to a 3-sphere in $\mathcal{C}_1$. and its tubular neighborhood homeomorphic to $\mathbf{S}^3 \times B^3$ branch covers the tubular neighborhood of the 3-sphere in $\mathcal{C}_1$ in 4 to 1 manner.

Finally, $\tilde{O} \times T_2^3/\sim$ is homeomorphic to $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$. Again, this follows from the injectivity and compactness and the Hausdorff property. (See Theorem 8.1 for the analogous proof.)

Now, we go over to the topology of $\tilde{O} \times T_2^3/\mathbb{Z}_2/\sim$: The space $\tilde{O}' \times T_2^3/\mathbb{Z}_2/\sim$ is a toric variety $\mathcal{O}'$ over the octahedron with vertices removed. By removing a thin neighborhood of square regions A_A,B_B,$C_{C'}$,$A'_{A'}$, $B'_{B'}$, and C'_C from $\mathcal{O}$, and considering the region M in our quotient space over it, we obtain a toric variety over an octahedron with neighborhoods of the ends removed.

Since the boundary of the closure M is a union of six five-dimensional manifolds homeomorphic to $\mathbf{S}^3 \times \mathbf{S}^2$. Thus, we can glue six neighborhoods of the three spheres over A_A,B_B,$C_{C'}$,$A'_{A'}$, $B'_{B'}$, and C'_C, homeomorphic to $\mathbf{S}^3 \times B^3$ to obtain the octahedral manifold. Hence, it follows that $\tilde{O} \times T_2^3/\mathbb{Z}_2/\sim$ is homeomorphic to the octahedral manifold. □

10. The topology of the other component $\mathcal{C}_1$

In this section, we will describe the topology of the other component $\mathcal{C}_1$.

We first discuss the fiberwise action and then the axial action induced from that of the octahedron O. We consider our space T_2^3-fibers over the octahedron with the equivalence relations $\tilde{O} \times T_2^3/\mathbb{Z}_2/\sim$ as determined in Section 8.

10.1. *The fiberwise action*

Again, we define i_a, i_b, and i_c on $T_2^3/\mathbb{Z}_2$:

$$i_a : (\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1 + 2\pi, \phi_2)$$
$$i_b : (\phi_0, \phi_1, \phi_2) \mapsto (\phi_0, \phi_1 + 2\pi, \phi_2 + 2\pi)$$
$$i_c : (\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 + 2\pi, \phi_1, \phi_2 + 2\pi). \tag{10.1}$$

Again, this amounts to changing some of the pasting maps by multiplications by $-\mathrm{I}$. The action sends equivalence classes to equivalence classes for $\mathrm{rep}_{-I}(\pi_1(\Sigma_1), \mathrm{SU}(2))$.

In the domain over the interior of $\tilde{O}$, the $\{I, i_a, i_b, i_c\}$-action is free and there is no branch locus.

In the regions over the interiors of faces, the $\{I, i_a, i_b, i_c\}$-action is also free: consider this action as translations in the tori $T_2^3/\sim$ where $\sim$ is generated by vectors parallel to the normal vectors of form $(\pm 2\pi, \pm 2\pi, \pm 2\pi)$ where the signs depend on which faces you are on.

Consider a^o for now. Then i_a, i_b, and i_c correspond to translations by $(2\pi, 2\pi, 0)$, $(0, 2\pi, 2\pi)$, and $(2\pi, 0, 2\pi)$ respectively. Then these vectors can be seen to be parallel to the plane of the quotient tori with generators $(4\pi, 0, -4\pi)$ and $(0, 4\pi, -4\pi)$

since we can change the signs of translation vectors mod 4π. We see easily that these translations are not trivial even up to the order 3 translation group action. Therefore, the fiberwise Klein four-group acts freely over each point of a^o.

Similar considerations shows that the fiberwise Klein four-group action is free over the interiors of the other faces b, c, d, a', b', c', d'.

The twelve edges of O form three classes given by three respective equations $\theta_0 = \pi/2, \theta_1 = \pi/2, \theta_2 = \pi/2$. That is, four edges satisfy $\theta_0 = \pi/2$, the next four edges satisfy $\theta_1 = \pi/2$ and the last four edges $\theta_2 = \pi/2$. Over the interiors of edges, the equivalence relation gives us circles. Thus, we have twelve annuli over the interiors of the edges.

Over the interiors of edges, by considering the equivalence classes, we deduce that if the edge is in the class given by $\theta_0 = \pi/2$, then the circle over the interior is fixed under i_b and each of i_a and i_c has no fixed points. This follows by looking at the equivalence classes.

Similarly, if the edge is in the class given by $\theta_1 = \pi/2$, then the circle over the interior of it is a fixed under i_c and each of i_a and i_b has no fixed points.

If the edge is in the class given by $\theta_2 = \pi/2$, then the circle over the interior of it is fixed under i_a and each of i_b and i_c has no fixed points.

For i_b, the four open annuli over the interiors of the edges given by $\theta_0 = \pi/2$, will extend to a circle in each of the four vertex 3-spheres where $\theta_0 = \pi/2$: $A_A, A'_{A'}, C_{C'}, C'_C$. (See Section 8.3.3 and the proof of Theorem 9.1.) Hence, we see that the four annuli patch up to a 2-torus since i_b is a smooth involution and the fixed points form a submanifold (See Conner [9]) as our manifolds and the maps are semialgebraic.

Similarly, i_a fixes a union of four annuli forming a torus over the edges given by $\theta_1 = \pi/2$. Finally, i_c fixes a union of four annuli forming a torus over the edges given by $\theta_2 = \pi/2$. Denote these torus by t_0, t_1, t_2 respectively.

Lemma 10.1. *Each of i_a, i_b, and i_c equals S, T, ST in some coordinates of each of the square regions $A_A, A'_{A'}, B_B, B'_{B'}, C_{C'}$, and $C_{C'}$.*

Proof. The action correspond to changing $h(d_i)$ to $\pm h(d_i)$ for $i = 1, 2$. See Remark 8.1. □

The tori are mutually disjoint in each of the 3-spheres over the vertex regions: First, the respective sets of fixed points of S, T, and ST are disjoint. i_a, i_b, and i_c are distinguished over each of the spheres and equal one of S, T, ST. Hence, the respective fixed sets of i_a, i_b, and i_c are mutually disjoint.

10.2. *The axes action*

Finally, let us study what are the branch loci of I_A, I_B, I_C. This is defined by equations (6.3). We note that these send equivalence classes to equivalence classes in $\tilde{O} \times T_2^3$.

The branch locus is as follows: I_A has a fixed axis in $\tilde{O}$. $\{I, I_A\}$ acts by taking the upper triangle with vertices v_0, v_1, v_2 and lower triangle with vertices v_0', v_1', v_2' so that v_0 and v_1 are changed to their antipodes and v_0' and v_1' are changed to their antipodes also and the corresponding pasting angles are changed by the map $\phi_i \to 4\pi - \phi_i$ for $i = 0, 2$. We aim to find an expression of I_A:

We need to consider the point over its axis in O^o and A_A and $A_{A'}'$ since other points on $\tilde{O}$ are sent to different points.

In a fiber in $O^o \times T_2^3/\mathbb{Z}_2$ over the axis part in O^o, the fixed points of I_A is a union of two circles given by $\phi_0 = 0, 2\pi$ and $\phi_2 = 0, 2\pi$ mod the $\mathbb{Z}_2$-action. The circle is parameterized by ϕ_2. In A_A, we show that I_A fixes a circle as we approach A_A, that is transversal to the fixed circles of S and ST and meeting each twice as in Section 8.3.3. This is also true in $A_{A'}'$. Thus, in $O^o \times T^3/\mathbb{Z}_2$, the set of fixed points is a union of two annuli and extends in A_A to the circle and similarly, we can show it extends in $A_{A'}'$ to a circle. Hence, the fixed subspace is homeomorphic to a 2-torus again using the smoothness of the involution and the manifold by Conner's work [9].

Similarly, the fixed set of I_B is a 2-torus meeting the fixed torus of I_A at the fiber over $(\pi/2, \pi/2, \pi/2)$ at two points $(0, 0, 0)$ and $(2\pi, 0, 2\pi)$ modulo 4π. The fixed set of I_C is a 2-torus meeting the two other tori at the fiber over $(\pi/2, \pi/2, \pi/2)$ at the two pairs of points. The three 2-tori meet in the fiber torus over $(\pi/2, \pi/2, \pi/2)$ in pairs.

Denote these tori by T_A, T_B, T_C respectively. Then T_A meets each of t_0 and t_1 at a pair of points respectively in regions A_A and $A_{A'}'$ by Lemmas 8.2 and 10.1. Similarly, T_B meets each of t_1 and t_2 at a pair of points respectively in regions B_B and $B_{B'}'$, and T_C meets each of t_2 and t_0 at a pair of points respectively in regions $C_{C'}$ and C_C'. (We now have the complete intersection pattern.)

Theorem 10.1. *The other component $\mathcal{C}_1$ is homeomorphic to the quotient of filled octahedral manifold with the product Klein four-group action given by the action of $I_A, I_B, I_C, i_a, i_b,$ and i_c.*

Appendix A. The equivalence relations for $\tilde{\Delta} \times T^3$.

The purpose of this appendix is to describe the equivalence class on $\tilde{\Delta} \times T^3$ so that the quotient space is homeomorphic to the identity component $\mathcal{C}_0$. We continue from Section 5 and we will use the notations from there.

A.1. *The equivalence over regions a, b, c, and d*

We can think of the region a as a subset of $[0, \pi] \times [0, \pi] \times [0, \pi]$ given by equation $\theta_0 + \theta_1 + \theta_2 = \pi$. a is to be the region given by $\theta_0, \theta_1, \theta_2 \geq 0$. Consider $a \times T^3$ and

the equivalence relation

$(x, \phi_0, \phi_1, \phi_2) \sim (y, \phi'_0, \phi'_1, \phi'_2)$ iff $x = y$ and
$$(\phi_0, \phi_1, \phi_2) - (\phi'_0, \phi'_1, \phi'_2) = s(2\pi, 2\pi, 2\pi), s \in \mathbb{R} \quad (A.1)$$

which was determined to have the same abelian characters. Thus, we consider this a circle action or translations generating the equivalence. For $b \times T^3$, the equivalence class is:

$(x, \phi_0, \phi_1, \phi_2) \sim (y, \phi'_0, \phi'_1, \phi'_2)$ iff $x = y$ and
$$(\phi_0, \phi_1, \phi_2) - (\phi'_0, \phi'_1, \phi'_2) = s(2\pi, -2\pi, 2\pi), s \in \mathbb{R} \quad (A.2)$$

For $c \times T^3$, it is generated by translations by vectors parallel to $(-2\pi, 2\pi, 2\pi)$ as above, and finally for $d \times T^3$, it is generated by translations by vectors parallel to $(2\pi, 2\pi, -2\pi)$ as above. Thus, there is an $\mathbf{S}^1$-action in each of the regions above these faces $a, b, c,$ and d.

Let us look at $a \times T^3 / \sim$ in more detail. So the equation determining the equivalence class representatives should be $\phi_0 + \phi_1 + \phi_2 = 0$ as $2\pi = 0$ in $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ with an order-three translation-group action. We take the subset of T^3 given by the equation. Looking at T^3 as a quotient space of the cube $[0, 2\pi]^3$, we see that the plane corresponds to the union of two triangles in T^3 with vertices $\triangle_1 = \triangle([2\pi, 0, 0], [0, 2\pi, 0], [0, 0, 2\pi])$ and $\triangle_2 = \triangle([2\pi, 2\pi, 0], [2\pi, 0, 2\pi], [0, 2\pi, 2\pi])$. Then two triangles form a 2-tori T' under the identifications of T^3. T' is isometric with a torus in the plane E^2 quotient out by a lattice generated by translation by $(2\pi\sqrt{2}, 0)$ and $(\pi\sqrt{2}, \pi\sqrt{6})$. There is another identification by the above $\mathbf{S}^1$-action for the union T'. The identification is given by a translation by $(\pi, \pi\sqrt{6}/3)$ in T', which is of order three. The quotient space T_a^2 is homeomorphic to a 2-torus. Thus, the quotient space here is in one-to-one correspondence with $a \times T_a^2$.

Similarly, for other faces, we obtain trivial 2-torus fibrations as well.

Proposition A.1.

(i) *The set of abelian characters is homeomorphic with* $T^4 / \sim$ *where* $\sim$ *is given by*

$$(\phi_0, \phi_1, \phi_2, \phi_3) \sim (2\pi - \phi_0, 2\pi - \phi_1, 2\pi - \phi_2, 2\pi - \phi_3)$$

where coordinates are rotation angles of $h(c_1), h(c_2), h(d_1), h(d_2)$ *respectively at a common fixed point determined up to the antipodal map.*

(ii) *Every abelian character* $\pi_1(\Sigma) \to SO(3)$ *is represented as an element over one of the regions* $a, b, c,$ *and* d *;*

(iii) *The subspace of abelian character is homeomorphic with* $a \times T_a^2 / \sim$ *where* $\sim$ *is given on* $(a \cap d) \times T_a^2$ *by* I_A *and on* $(a \cap c) \times T_a$ *by* I_B *and on* $(a \cap b) \times T_a$ *by* I_C *as in equation* (5.1).

(iv) *The above quotient space is homeomorphic to a generalized Seifert space where the base space is a sphere with four singular points of order two*

with fiber homeomorphic to T^2 and exceptional fibers are spheres with four singular points.

Proof. (i) For abelian representations, c_1, c_2, d_1, d_2 will map to an isometry fixing a common pair of points. If we choose one fixed point in the antipodal pair of fixed points, then the angles are given. If we choose the other fixed point, then the angles are given as the complementary one to the former ones.

The homeomorphism property follows easily.

(ii) If a representation h is abelian, then h has a common pair of antipodal points which are fixed points. The generalized triangle determined is then either a point or a segment of length π, and the gluing transformations have fixed points in the vertices. Thus, h corresponds to an element over one of the regions a, b, c, and d.

(iii) The angles θ_i of the triangle are half of the rotation angles. a is the fundamental region of $\partial\Delta \times T^3$ given by the above action of $\{I, I_A, I_B, I_C\}$. Thus, the quotient space is obtained from $a \times T_a^2$ by $\sim$.

(iv) When we take the quotient of $a \times T_a^2$, the four singular points come from the vertices of a which identifies to a point and the middle point of the edges of the triangle a. (See Lee-Raymond [26].) □

A.2. *Equivalence relations for regions I, II, III, IV*

The points over regions I, II, III, and IV are ones corresponding to the characters $\pi_1(\Sigma)$ that factor into F_2 sending c_0, c_1, and c_2 to identity since they have triangles with angles either equal to 0 or π. We choose a map from Σ to a bouquet of two circles such that d_1 and d_2 correspond to the generators of F_2. We denote those characters by $H(F_2)$ by an abuse of notation since $h : F_2 \to SO(3)$ corresponds to $h' : \pi_1(\Sigma) \to SO(3)$ so that $h' = h \circ f_*$.

The map $\mathcal{T}_i : i \times T^3 \to H(F_2)$ where $i = I, II, III$ or IV is defined by sending the configuration and assigned angles to the equivalence class of the associated representation. That is, $\mathcal{T}_i$ are restrictions of $\mathcal{T}$. We are interested in finding the equivalence relation to make these maps injective.

The map $\mathcal{T}_i$ can be constructed as follows for any region I, II, III, and IV: Let v_0, v_1, and v_2 be the three points on a great circle or a segment corresponding to the vertices of a triangle representing an element of I, II, III, or IV in the clockwise orientation and ϕ_0, ϕ_1, and ϕ_2 the associated pasting angles. To begin with, suppose that none of the angles are zero or π and that the vertices v_0, v_1, and v_2 in these cases are mutually distinct and do not coincide with an antipode of the other points. The vertices v_0, v_1, and v_2 give us the orientation of the boundary of the generalized triangle. For example, in case I, we take one triangle $\triangle_1$ in the outside of the great circle and another one $\triangle_2$ inside. In all cases, the two triangles are oriented in the clockwise direction. They have respective edges sharing a common great circle passing through v_0. We choose $\triangle_1$ and $\triangle_2$ in the following manner: $\triangle_1$ has two vertices v_0 and v_1 with angles $\pi - \phi_0/2$ and $\phi_1/2$ respectively. The remaining vertex

w_1 has an angle η_1, $0 \leq \eta_1 \leq \pi$. Since $\triangle_1$ is outside, the vertices v_1, v_0, and w_1 are in the clockwise direction in the boundary. $\triangle_2$ has two vertices v_0 and v_2 with angles $\pi - \phi_0/2$ and $\phi_2/2$ respectively. The remaining vertex w_2 has an angle η_2, $0 \leq \eta_2 \leq \pi$. Since $\triangle_2$ is inside, v_2, v_0, w_2 are in the clockwise direction in the boundary. (See Figure A.1 in the case of region I.)
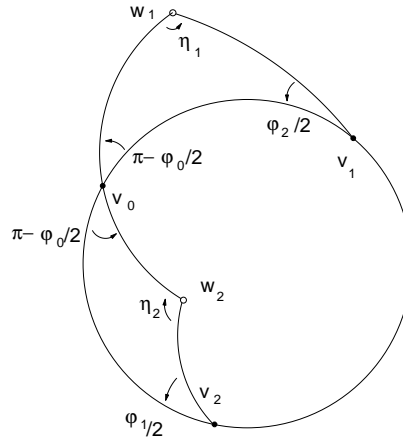

Fig. A.1.   Finding holonomy of d_1 and d_2 using triangles for region I.

Recall that $R_{x,\theta}$ denote an isometry with fixed points $x, -x$ which rotates about the x in the counter-clockwise direction of angle θ.

The isometry which is a composition of $R_{v_0,2\pi-\phi_0}$ and R_{v_1,ϕ_1} has a fixed point at w_1 and a rotation angle $2\pi - 2\eta_1$. Similarly, the isometry which is a composition of $R_{v_1,2\pi-\phi_0}$ and R_{v_2,ϕ_2} has a fixed point at w_2 and a rotation angle $2\pi - 2\eta_2$, which follows from the consideration as in Section 2. In this way, we obtain two isometries corresponding to d_1 and d_2.

Let us illustrate more details here for region I case only (The other regions are similarly treatable and the result follows from Proposition 2.1): We still suppose that any two of v_0, v_1, and v_2 are mutually distinct and do not coincide with an antipode of the other point: If ϕ_0 is zero, then we use w_1 to be $-v_1$ and d_1 is mapped to the rotation about $-v_1$ of angle $2\pi - \phi_1$ and w_2 is to be $-v_2$ and d_2 is a rotation about $-v_2$ of angle $2\pi - \phi_2$. (Here we are using lunes as degenerate triangles to compute these.)

If ϕ_0 is 2π and others are not zero or 2π, then we use w_1 to be v_1 and w_2 be v_2 and d_1 is a rotation about v_1 of angle ϕ_1 and d_2 is a rotation about v_2 of angle ϕ_2.

If ϕ_1 is zero and others are not zero or 2π, then we let w_1 be v_0 and d_1 is a rotation about v_0 of angle $2\pi - \phi_0$, and the other triangle is constructed as above by angles $2\pi - \phi_0$ and ϕ_2 and we obtain d_2 as in the above nondegenerate triangle T_2 case.

If ϕ_1 is 2π and others are not zero or 2π, then we let w_1 be $-v_0$ and d_1 be a rotation about $-v_0$ of angle ϕ_0 and we let w_2 and d_2 be determined as in the above nondegenerate T_2 case.

If ϕ_2 is zero and others are not zero or 2π, then we let w_2 be v_0 and d_2 is a rotation about v_0 of angle $2\pi - \phi_0$, and the other information is analogously determined to as above for the nondegenerate T_1 case.

If ϕ_2 is 2π and others are not zero or 2π, then we let w_2 be $-v_0$ and d_2 be a rotation about $-v_0$ of angle ϕ_0 and the other information is analogously determined as above for the nondegenerate T_1 case.

If $\phi_1 = \phi_2 = 0$ and $\phi_0 \neq 0, \pi$, then we let $w_1 = w_2 = v_0$ and d_1 and d_2 are respectively rotations of angle $2\pi - \phi_0$ at v_0.

If $\phi_1 = 0, \phi_2 = 2\pi$ and $\phi_0 \neq 0, \pi$, then we let $w_1 = v_0, w_2 = -v_0$ and d_1 is a rotation about v_0 of angle $2\pi - \phi_0$ and d_2 is a rotation of angle ϕ_0.

If $\phi_1 = 2\pi, \phi_2 = 0$, then we let $w_1 = -v_0, w_2 = v_0$ and d_1 is a rotation of angle ϕ_0 about $-v_0$ and d_2 is a rotation of angle $2\pi - \phi_0$ about v_0.

Finally, suppose that some of v_0, v_1, v_2 coincide or some of them coincide with an antipode of the other points. Then we have an abelian representation over one of the regions a, b, c, and d, which we studied above.

Remark A.1. Note that our construction of d_i and the fixed points w_i depends smoothly unless $\eta_i = 0, \pi$ on the data v_i, ϕ_i. Hence, our construction easily extends to a construction on SU(2) where the angles are restricted to be in $[0, 2\pi]$. However, to obtain d_1 and d_2 we have to multiply by $-I$ or to take the antipode to the result obtained from the triangle construction. (See Equation (3.2) and Section 2.2.)

Proposition A.2. *The map*

$$\mathcal{T} : \bigcup_{i=I, II, III, IV} \triangle_i \times T^3 \to H(F_2)$$

is onto. In fact, $\mathcal{T}_i$ is onto for each i.

Proof. Given a generalized triangle, we can control the distances between the vertices denoted by v_1 and v_2 and put any angles there. By putting the angle at the vertex denoted by v_0 be 2π, we can realize all segment with any given angles at vertices up to isometries. $\square$

Under the $\{I, I_A, I_B, I_C\}$-action as in Equation (5.1), the regions I, II, III, IV are actually identical. The maps $\mathcal{T}_i$s are also identical since we can easily show that $\mathcal{T}_i = \mathcal{T}_j \circ f$ holds for $f \in V$ sending i to j. (That is, we show that they correspond to the same characters.) Thus, we introduce the equivalence that element $x \in \triangle_i \times T^3$ is equivalent to $y \in \triangle_j \times T^3$ if $y = f(x)$ under $f \in V$ sending $\triangle_i$ to $\triangle_j$.

Finally, we introduce the equivalence relation $\sim$ so that two elements of $\triangle_i \times T^3$ are equivalent if it maps to the same point of $H(F_2)$. Since $\triangle_i$ is compact and $H(F_2)$ is a Hausdorff space, we obtain

Proposition A.3. $\mathcal{T}_i$ *induces a homeomorphism* $\triangle_i \times T^3/ \sim \to H(F_2)$ *for each* i, $i = I, II, III.IV$.

A.3. *The equivalence relation on* A, A', B, B', C, C'

Next, we wish to consider the representations for each of which exactly one of $h(c_i)$ is identity and does not correspond to representations over the interior of regions I, II, III, and IV. Define $U = I \cup II \cup III \cup IV$. Then these representations correspond to ones over the regions $A - U, A' - U, B - U$, $B' - U$, $C - U$, and $C' - U$.

Let us define $C^A, C^{A'}, C^B, C^{B'}, C^C$, and $C^{C'}$ as the subspace of $\mathcal{C}_0$ that can be described as having the restricted $\pi(S_0)$ characters as described by the generalized triangles in regions A, A', B, B', C, and C' respectively. In fact given a region R in any of A, B, C, A', B', C', we denote by C^R the subset of $\mathcal{C}_0$ coming from $R \times T^3$.

We call the subspace over some intersection region, i.e., of form $X \cap y$ where X is one of A, A', B, B', C, C' and y is one of a, b, c, d, the *abelian edges* of X. They correspond to abelian representations. In fact, every abelian representation in one of $C^A, C^{A'}, C^B, C^{B'}, C^C$, and $C^{C'}$ is from the abelian edges as we will see. We call them the *boundary abelian representations*.

A.3.1. *Region* $B - I - II$

We consider the region $B - I - II$ first and then we consider $B \cap I$ and $B \cap II$. We choose an element of $(B - I - II) \times T^3$ and analyze the triangles and pasting maps: Given a triangle here, we use the usual notations v_i, θ_i, l_is. Then $\theta_0 = \pi$ and $\theta_1 = \theta_2 \neq 0, \pi$ and v_1 and v_2 are antipodal and v_1 and v_2 are vertices of a lune L with angle $\theta_1 = \theta_2$ at v_1 or v_2.

This subsection is divided into two parts and the remaining subsections are similar: (i) We will first show how to construct the representations over $B - I - II$ by pasting angles using geometric constructions. (ii) Next, we will find the equivalence relation on $(B - I - II) \times T^3$ that gives precisely the space of characters over $B - I - II$.

(i) We start the construction process: We are given an element of $B - I - II$; hence, we choose a representative generalized triangle with vertices v_0, v_1, v_2. We now construct a representation h so that $h(c_1)$ fixes v_1 and $h(c_2)$ fixes v_2 and has the right pasting angles given by T^3. We use Proposition 2.1.

For this case, we will write the full details:

Suppose that v_0 is equal to v_1 or v_2. We choose v_1 and v_2 arbitrarily so that $v_2 = -v_1$. In the former case, $h(d_1)$ is realized as a rotation about v_1 of angle $\phi_1 - \phi_0$ and $h(d_2)$ as one about v_2 of angle $\phi_2 + \phi_0$. In the later case, $h(d_1)$ is realized as a rotation about v_1 of angle $\phi_1 + \phi_0$ and $h(d_2)$ as one about v_2 of angle $\phi_2 - \phi_0$. (These are abelian ones covered also in Appendix A.1.)

Now suppose that v_0 is distinct from v_1 and v_2. Suppose now that $0 < \phi_0, \phi_1, \phi_2 < 2\pi$. To compute the holonomy of $h(d_1)$ and $h(d_2)$, let l be the seg-

ment containing v_0, v_1, v_2 which is a side of the lune L. Then we draw a clockwise oriented triangle T_1 with vertices v_0, v_1 of angles $\phi_1/2$ and $\pi - \phi_0/2$ respectively. Then $h(d_1)$ is the rotation about the third vertex w_1 of T_1 of angle τ_1 equal to 2π minus twice the vertex angle η_1.

Similarly, we construct a clockwise oriented spherical triangle T_2 with vertices v_0, v_2 of angles $\pi - \phi_0/2$ and $\phi_2/2$ respectively. Then $h(d_2)$ is the rotation about the third vertex w_2 of T_2 of angle τ_2 equal to 2π minus twice the vertex angle η_2.

By allowing degenerate triangles, we can let $\phi_0 = 0$ or 2π and $0 < \phi_1, \phi_2 < 2\pi$ as well, in which case, w_1 becomes v_1 or v_2 respectively and T_1 a pointed segment or a pointed lune. Therefore, our representation is one which send c_1 and c_2 to rotations about v_1 and v_2 of respective rotation angles θ_1 and θ_2 and d_1 and d_2 to the rotations about v_1 and v_2. Thus, the characters are abelian ones and are over regions a, b, c or d. (Actually, these are only abelian ones if v_0 does not equal v_1 or v_2.) The equivalence over the region were studied above in Appendix A.1.

Now see Proposition 2.1 when ϕ_1 and ϕ_2 equal 0 or π.
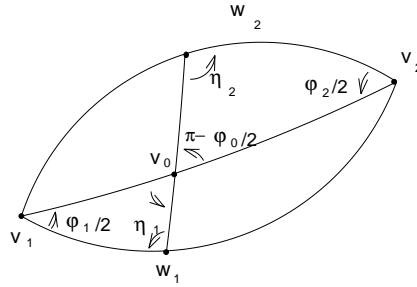
Fig. A.2.   Finding holonomy of d_1 and d_2 using triangles over $B - I - II$

Proposition A.4.

- *A representation h corresponding to an element of C^{B-I-II} is nonabelian if and only if both of $h(d_1)$ and $h(d_2)$ are not identity and moreover have no fixed points in v_1 and v_2 for any configuration corresponding to an element in $(B - I - II) \times T^3$ representing h.*
- *Suppose that h is nonabelian and is in C^{B-I-II}. Let $\mathbf{S}^1$ be the great circle passing through fixed points $\pm w_1$ of $h(d_1)$ and $\pm w_2$ of $h(d_2)$ or the great circle passing through w_1 at an angle $\eta_1 = \pi - \tau_1/2$ for the angle τ_1 of rotation of $h(d_1)$ when $\pm w_1 = \pm w_2$. Then $\mathbf{S}^1$ passes through none of v_1 and v_2.*

Proof. To prove the first statement, suppose that h represent an element of C^{B-I-II}. Since h comes from a triangle construction, we choose an element of $B - I - II$ and a representing generalized triangle with vertices v_0, v_1, and v_2. If

v_0 is not identical with v_1 and v_2 and $0 < \phi_0 < 2\pi$, then $h(d_1)$ and $h(d_2)$ are both not identity and the fixed points $\pm w_1$ and $\pm w_2$ are distinct from v_1, v_2. Thus, h is nonabelian.

If v_0 is not identical with v_1 and v_2 and $\phi_0 = 0, 2\pi$. Then if ϕ_1 is not zero or 2π, then w_1 equals one of v_1 and v_2. If ϕ_1 is zero or 2π, then $h(d_1) = $ I. Since the same consideration holds for $h(d_2)$ as well, h is abelian here.

If v_0 is identical with v_1 or v_2, then h is abelian and $h(d_1)$ and $h(d_2)$ have fixed points at v_1 or v_2.

This proves the first statement.

Suppose that $\mathbf{S}^1$ passes v_1 or v_2. Then it passes both. By construction, the intersection of $\mathbf{S}^1$ meets the great segment with endpoints v_1 and v_2 at v_0. This implies v_0 equals v_1 or v_2 or $\phi_0 = 0, \pi$ and hence, h is abelian. $\square$

Remark A.2. For purposes below, we look at the above construction from another angle: We suppose that $0 < \phi_0 < 2\pi$ and v_0 is distinct from v_1 and v_2 and $\pm w_1$ and $\pm w_2$ are distinct pairs. d_1 and d_2 correspond to two pairs of fixed points $(w_1, -w_1)$ and $(w_2, -w_2)$ in $\mathbf{S}^2$. We draw a great circle l through v_0, v_1, and $v_2 = -v_1$ separating w_1 and $-w_1$. The fact that $0 < \phi_0/2 < \pi$ implies $\{w_1, -w_1\}$ and $\{w_2, -w_2\}$ are distinct from $\{v_1, v_2\}$. Let s_0 denote the segment of length π with endpoints v_1 and v_2 through v_0 in l.

We change a definition here temporarily: the *lune* simply means a disk bounded by two segments of length π ending at a pair of antipodal points. If the vertex angle is $\leq \pi$, then the lune is convex.

We choose two segments s_1 and s_2 with vertices v_1 and v_2 through w_1 and w_2 respectively. Then we obtain the lune L bounded by two of these bounded by a union of s_1 and s_2. We constructed a lune L containing s_0. (See Figures A.2 and A.3.)

We see that the segment s' that is a union of two edges in T_1 and T_2 respectively through v_0 ends in w_1 and w_2 and is in $\mathbf{S}^1$. Let v_0 is the intersection point of s_0 and s'. The boundary of L and s_0 and s' bound four triangles two of which are T_1 with vertices v_1, v_0, and w_1 and T_2 with vertices v_2, v_0, and w_2.

Since $\phi_1/2$ and $\phi_2/2$ are angles at v_1 and v_2 of triangles T_1 and T_2 with vertices w_1 and w_2 respectively and $\phi_1/2 + \phi_2/2$ is the angle of the lune L containing both triangles T_1 and T_2 while agreeing with them in parts of edges.

Notice that we can change s_0 to another segment from v_1 and v_2 in L. We can even move s_0 outside L. Then we use a different choice of plus-minus signs among $\pm w_1$ and $\pm w_2$ so that the corresponding lune contains s_0 and we can construct the triangles as above to obtain the same isometries denoted by $h(d_1)$ and $h(d_2)$. (See Figure A.3.)

Now, we suppose that $0 < \phi_0 < 2\pi$ and v_0 is distinct from v_1 and v_2 and $\pm w_1 = \pm w_2$. In this case, we let the great circle $\mathbf{S}^1$ be the great circle through them that has angle η_1 of w_1 with the segment from w_1 to v_1. Then v_0 is the intersection $s_0 \cap \mathbf{S}^1$. We can choose two triangles T_1 and T_2 as above.

When s_0 is chosen to pass one of $\pm w_1$, we have $\phi_1 = \phi_2 = 0$ and the angles η_i of w_i satisfy $\eta_1 = \eta_2$ and $\eta_i = \pi - \tau_i/2$ for the rotation angles τ_i of $h(d_i)$ for each $i = 1, 2$. If s_0 does not pass one of $\pm w_1$, then $w_2 = -w_1$ and $\phi_1 = \pi - \phi_2$ and $\eta_1 = \pi - \eta_2$ also. Here, a hemisphere is divided into four triangles. Two of which are T_1 and T_2.

Again, we can move s_0 outside the hemisphere and we will be choosing a different pair from $\pm w_1, \pm w_2$.

We remark that the isometry equivalence class of the set of the six points $\pm w_1, \pm w_2, v_1$, and v_2 and the respective rotation angles of $h(d_1)$ and $h(d_2)$, τ_1 and τ_2, and $\theta_1 = \theta_2$ completely determines the conjugacy class of the representations over $B - I - II$ and vice versa. The isometry class space is homeomorphic to the conjugacy class of the characters over $B - I - II$. This will be used later in determining the topology of the space.

(ii) We introduce the equivalence relation on $(B - I - II) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ by demanding the $\mathcal{T}_{B-I-II}$ be injective.

To see more details, we will introduce it as equivalence relations in a geometric form. The equivalence relation is so that two are equivalent if they correspond to the same SO(3)-character of $\pi_1(\Sigma)$ with condition that the configurations are on $B - I - II$.

By Proposition A.4, the subspace of $B \times T^3$ giving us abelian characters is precisely

$$(v(1), 0, \phi_0, \phi_1, \phi_2), (v(1), \pi, \phi_0, \phi_1, \phi_2),$$
$$(v(1), l(1), 0, \phi_1, \phi_2),$$
$$(0, l(1), \phi_0, \phi_1, 2\pi - \phi_1), (\pi, l(1), \phi_0, \phi_1, 2\pi - \phi_1)$$
$$\text{for } v(1), l(1) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi]. \qquad \text{(A.3)}$$

For the region $B \times T^3$, we first introduce an obvious equivalence relation given by:

$(v(1), 0, \phi_0, \phi_1, \phi_2) \sim (v(1), 0, \phi_0 - t, \phi_1 + t, \phi_2 - t)$, for every $t \in \mathbf{S}^1$

$(v(1), \pi, \phi_0, \phi_1, \phi_2) \sim (v(1), \pi, \phi_0 - t, \phi_1 - t, \phi_2 + t)$ for every $t \in \mathbf{S}^1$

$(v(1), 0, \phi_0, \phi_1, \phi_2) \sim (v(1), \pi, 2\pi - \phi_0, \phi_1, \phi_2)$

$(v(1), l(1), 0, \phi_1, \phi_2) \sim (v(1), l(1)', 0, \phi_1, \phi_2)$

$\qquad\qquad\qquad\qquad$ for an arbitrary pair of $l(1)$ and $l(1)'$ and

$(0, l(1), \phi_0, \phi_1, 2\pi - \phi_1) \sim (0, l(1)', \phi_0', \phi_1', 2\pi - \phi_1')$

$\qquad\qquad$ for an arbitrary pair of $l(1)$ and $l(1)'$ and the same third angles

$(\pi, l(1), \phi_0, \phi_1, 2\pi - \phi_1) \sim (\pi, l(1)', \phi_0', \phi_1', 2\pi - \phi_1')$

$\quad$ for an arbitrary pair of $l(1)$ and $l(1)'$ and the same third angles.

$$\text{(A.4)}$$

Here by the same third angle, we mean that the triangle with a side $l(1)$ and

angles $\pi - \phi_0/2, \pi - \phi_1/2$ at the two vertices and the triangle with a side $l(1)'$ and angles $\pi - \phi_0'/2, \pi - \phi_1'/2$ has the same third angle. This case of course includes the generalized triangle cases, which might be ambiguous but not the third angle itself. Or we can use the equation from spherical trigonometry:

$$
\begin{aligned}
& -\cos(\pi - \phi_0/2)\cos(\pi - \phi_1/2) + \sin(\pi - \phi_0/2)\sin(\pi - \phi_1/2)\cos l(1) = \\
& \quad -\cos(\pi - \phi_0'/2)\cos(\pi - \phi_1'/2) + \sin(\pi - \phi_0'/2)\sin(\pi - \phi_1'/2)\cos l'(1). \quad \text{(A.5)}
\end{aligned}
$$

Note that we include the degenerate triangles and in that case the equation is not a sufficient condition and the condition is a geometric one.

Note that these elements are all of the ones in $B \times T^3$ corresponding to abelian characters and the complete equivalence relations for abelian characters on B. The first two are really restrictions of equivalence relations on $b \times T^3$ and $d \times T^3$, i.e., the $\mathbf{S}^1$-action there. (Finally, abelian characters are always equivalent to abelian edge ones by the final three items above.)

There is also an $\mathbf{S}^1$-action giving us further equivalences on the subset of $B \times T^3$ corresponding to nonabelian characters: Let v_0, v_1, v_2, w_1 and w_2 be obtained for a nonabelian representation h coming from $B \times T^3$. By Proposition A.4, w_1 and w_2 do not coincide with v_1 or v_2 and there exists a great circle $\mathbf{S}^1$ containing w_1 and w_2 determined in Remark A.2 and $\mathbf{S}^1$ contains none of v_1, v_2. Then there exists $v_0, v_1,$ and v_2 where v_0 is not identical with v_1 or v_2. For any choice of v_0 in a great circle $\mathbf{S}^1$ containing w_1 and w_2, we choose a corresponding s_0 between v_1 and v_2 in the collection of all great segments between v_1 and v_2. This gives us a generalized triangle with vertices $v_0, v_1,$ and w_1 and possibly $\geq \pi$ angle at v_1. (See equation (3.2). The generalized triangle with one vertex of angle in $[2\pi, 4\pi]$ and the other two angles in $[0, 2\pi]$ also needs to be considered here in multiplications by geometry.)

If the angle at v_2 is $\geq \pi$, we subtract π from it. This amounts to replacing w_2 with its antipode and the angle η_2 to be $\pi - \eta_2$ and thus $h(d_2)$ is unchanged. (See Figure A.3. The generalized triangle $v_1 v_0 w_2$ is replaced by $v_1 v_0 (-w_2)$.) Now the angle of the triangle at v_2 is in $[0, \pi)$. The angle of the triangle at v_0 is always in $[0, \pi)$, and our operation does not change the rotation angle of the pasting map P_0 at v_0. (This is described also in the last part of Remark A.2.)

Similarly, we can repeat the constructions for v_0, v_2, w_2. Therefore, there is an $\mathbf{S}^1$-action on $(B - I - II) \times T^3$ by choosing one among the space of great segments, denoted by s_0, between v_1 and v_2 forming $\mathbf{S}^1$.

(Actually the choice of fixed point for $h(d_i), i = 1, 2$ among an antipodal pair gives possibilities for s_0 and conversely the choice of s_0 forces choices of fixed points of $h(d_i), i = 1, 2$.)

Suppose now that we are over the abelian edges $(B - I - II) \cap b$ or $(B - I - II) \cap d$. For boundary abelian characters, the angles at points denoted by w_1 and w_2 matter only and the equivalence relations are clear. This can be seen as an $\mathbf{S}^1$-action on
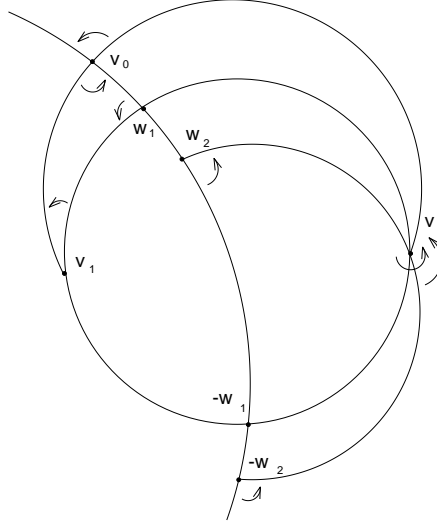
Fig. A.3.   The circle action.

the abelian edges given by changing pasting angles

$$(\phi_0, \phi_1, \phi_2) \mapsto (\phi_0 - t, \phi_1 + t, \phi_2 - t) \text{ if } l(1) = 0 \qquad (A.6)$$

$$\mapsto (\phi_0 - t, \phi_1 + t, \phi_2 + t) \text{ if } l(1) = \pi \text{ for } t \in [0, 2\pi). \qquad (A.7)$$

This is a continuous extension of the above $\mathbf{S}^1$-action and is a restriction of the $\mathbf{S}^1$-action on the subspace of abelian characters in Appendix A.1.

By the circle action, we can always make our situation so that $\phi_1 = 0$ or $\phi_2 = 0$. We call these the *canonical representatives*, and every representation or character over B has canonical representatives. Then the map

$$\mathcal{T}^1_{B-I-II} : (B - I - II) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \to C^{B-I-II} \text{ and}$$

$$\mathcal{T}^2_{B-I-II} : (B - I - II) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to C^{B-I-II}$$

are easily seen to be surjective.

The two maps are generically 4 to 1 since there is a Klein four-group action on $(B - I - II) \times T^3$ given as follows and which acts on the domains of the above two maps:

(a) There is a $\mathbb{Z}_2$-action on the initial spaces given by changing s_0 to $-s_0$ which goes to the trivial action in the second space. (This is a transformation in the $\mathbf{S}^1$-action.)

(b) This is a transformation I_B on $(B - I - II) \times T^3$ induced by sending the generalized triangle with vertices v_0, v_1, v_2 to another generalized triangle by sending v_1 and v_2 to their antipodal points and v_0 to itself. This sends parameters in the following way:

$$(\theta_1, l(1), \phi_0, \phi_1, \phi_2) \mapsto (\pi - \theta_1, \pi - l(1), \phi_0, 2\pi - \phi_1, 2\pi - \phi_2) \qquad (A.8)$$

Note that $\{\mathrm{I}, I_B\}$ acts on the union of fibers above mid-tie but sends other ties in $B-I-II$ to different ones. I_A and I_C do not preserve B and hence do not appear in equivalences for B only. For end ties, we have further equivalences introduced by mapping into the regions above I, II, III, or IV.

Proposition A.5. *The equivalence relation $\sim$ generated by all of above, i.e, by equations* (A.4) *and equation* (A.8) *and the* $\mathbf{S}^1$*-action, gives us a homeomorphism:*

$$\mathcal{T}_{B-I-II} : (B - I - II) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \; \to C^{B-I-II}.$$

Furthermore,

$$\mathcal{T}^1_{B-I-II} : (B - I - II) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \; \to C^{B-I-II} \;\; and$$

$$\mathcal{T}^2_{B-I-II} : (B - I - II) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \; \to C^{B-I-II}$$

are homeomorphisms where $\sim$ is given by the equation (A.4) *and the Klein four-group action above denoted (a) and (b).*

Proof. The surjectivity follows by definition of C^{B-I-II}.

The injectivity will follow if we show that two elements of $(B - I - II) \times T^3$ map to a same element, then they are equivalent. For abelian representations, this is easy to show from equation (A.4) and the $\mathbf{S}^1$-action. For a nonabelian representation h, we have a great circle containing fixed points of $h(d_1)$ and $h(d_2)$ with nonzero angles by Proposition A.4. We make both of the elements have angles 0 at v_1 by the $\mathbf{S}^1$-action. Then $w_1 = v_0$ in both cases. Then we see that the two give the same isometry class of fixed points and rotation angles if and only if they are the same or differ by taking two of w_i to be their antipodes respectively, which amounts to changing s_0 to $-s_0$ and the move described by equation (A.8).

We prove that the map is a homeomorphism: Elements of C^{B-I-II} can be grouped into a family of ties C_θ given by $\theta_1 = \theta_2 = \theta$ where $0 < \theta < \pi$, i.e., images of the ties. Note that $C_\theta = C_{\pi-\theta}$. For each tie in $B - I - II$ where $\theta_1 = \theta_2 = \theta'$ for a constant θ', we obtain a subset $C'_{\theta'}$ in $(B - I - II) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim$ where $C'_{\theta'} = C'_{\pi-\theta'}$. We have a map from the compact set $C'_{\theta'}$ to $C_{\theta'}$ which is injective by the above paragraph and is surjective by definition. The map $\mathcal{T}_{B-I-II}$ is a proper map by the above property of preserving θ'. By the above injectivity and the Hausdorff property of C^{B-I-II}, we have a homeomorphism.

The proofs of the last two statements are from the fact that the $\mathbf{S}^1$-action and the $\mathbb{Z}_2$-action restrict to an action of Klein four-group action in the two spaces. □

A.3.2. *Regions B_π and B_0*

Let B_π be the subset of B where $\theta(1) = \theta(2) = \pi$, i.e., $B_\pi = B \cap I$, and B_0 the subset of B where $\theta(1) = \theta(2) = 0$, i.e., $B_0 = B \cap II$.

We consider the restriction $\mathcal{T}_{B_\pi}|B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. Here the characters are in $H(F_2)$ since $h(c_i)$ are all identity.

By considering the choice of s_0, we have an $\mathbf{S}^1$-action, which was described above for the region B above. A representation corresponds to the two fixed points of $h(d_1)$ and $h(d_2)$ and their rotation angles.

There is again a Klein four-group action: (a) For each map, there is a $\mathbb{Z}_2$-action on $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ given by changing s_0 to $-s_0$ which goes to trivial action in the SO(3)-character space. (This is a transformation in the $\mathbf{S}^1$-action.)

(b) Again there is a $\mathbb{Z}_2$-action on $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ by taking the fixed points v_1 and v_2 to the antipodal ones respectively.

$$(\pi, l(1), \phi_0, \phi_1, \phi_2) \mapsto (\pi, \pi - l(1), \phi_0, 2\pi - \phi_1, 2\pi - \phi_2) \qquad \text{(A.9)}$$

We call the map an *augmented I_B-action* and the two generate a Klein four group action. (Compare to equation (A.8).)

As above, $\sim$ now defines a *restricted equivalence relation* on $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ given by equation (A.4) and the $\mathbf{S}^1$-action and the above $\mathbb{Z}_2$-action.

We define similar equivalence relations for B_0 as well, also called the *restricted equivalence relation* with a $\mathbb{Z}_2$-action including the *augmented I_B-action*:

$$(0, l(1), \phi_0, \phi_1, \phi_2) \mapsto (0, \pi - l(1), \phi_0, 2\pi - \phi_1, 2\pi - \phi_2) \qquad \text{(A.10)}$$

Recall that B_π and B_0 are intervals homeomorphic to $I = [0, \pi]$.

Proposition A.6. *We can identify $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim$ with $H(F_2)$ by sending elements to the fixed points of $h(d_1)$ and $h(d_2)$ respectively and rotation angles and their distances and $\sim$ is given by equations (A.4) and (A.9) and the $\mathbf{S}^1$-action. Furthermore, we can identify $B_\pi \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim$ with $H(F_2)$ and similarly $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim$ with $H(F_2)$ where $\sim$ is given by equation (A.4) and the Klein four-group action. The same statements hold with B_π replaced with B_0.*

Proof. The existence and the continuity and the surjectivity of the map from $B_\pi \times T^3$ can be proved by showing every elements of $H(F_2)$ can be constructed by triangle constructions.

The injectivity is proved as follows: Suppose first that we have a representation h for an element of $H(F_2)$ and h is not abelian so that $h(d_1)$ and $h(d_2)$ are both not I by Proposition A.4. Suppose that some element of $B_\pi \times T^3$ corresponds to h. Let v_0, v_1, v_2, w_1, w_2 denote the vertices of the corresponding lune and the respective fixed points of $h(d_1)$ and $h(d_2)$. Moreover $\pm w_1 \neq \pm w_2$ since otherwise h is abelian. The fixed points $\pm w_1$ of $h(d_1)$ and $\pm w_2$ of $h(d_2)$ are well-defined, we have four choices by choosing one from each. Then the rotation angle at w_1 determines a great circle where v_1 lies and the rotation angle at w_2 determine a great circle where $v_2 = -v_1$ lies. This determines v_1 and v_2. Once v_1 and v_2 are determined, then v_0 is determined to be w_1 if we choose a configuration where $\phi_1 = 0$. This implies that the Klein four-group action on the spaces of quadruples and their angles give the only ambiguities. These are covered by our equivalences based on the $\mathbf{S}^1$-action and the $\mathbb{Z}_2$-action.

For the space of abelian characters, the injectivity is followed by equation (A.4). Since $B_\pi \times T^3/ \sim$ is compact, it follows that our map is a homeomorphism.

The final statements on $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\}/ \sim$ and $B_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\}/ \sim$ are proved using the fact that the $\mathbf{S}^1$-action restricts to a Klein four-group action.   □

A.4. *Region* $B' - III - IV$

First, we describe how to obtain the corresponding character from an element of $(B' - III - IV) \times T^3$. Over the region $B' - III - IV$, the triangle is a segment s with a vertex v_0 and another vertex $v_1 = v_2$. Drawing segments s_1 and s_2 of length π from v_1 to $-v_1$ of angles $\phi_1/2$ and $\phi_2/2$ respectively and drawing a segment s_0 from v_0 of angle $\pi - \phi_0/2$ to the segment s, we obtain two triangles T_1 and T_2 sharing an edge s. The respective vertices w_1 and w_2 of T_1 and T_2 are fixed points of $h(d_1)$ and $h(d_2)$ and the respective vertex angles are π minus the half of those of $h(d_1)$ and $h(d_2)$.

The configuration space that we have to consider is given by the point $v_1 = v_2$ with the rotation angles for $h(c_1)$ and $h(c_2)$ and the fixed points w_1 and w_2 and their rotation angles for $h(d_1)$ and $h(d_2)$ respectively. Again, we will find the topology of the configuration space and hence $C^{B'-III-IV}$ in Section 7.

Let us discuss the equivalence relation to make $(B' - III - IV) \times T^3$ equal to $C^{B'-III-IV}$:

By a similar reason to Proposition A.4, the subspace of $B' \times T^3$ giving us abelian characters is precisely

$$(v(1), 0, \phi_0, \phi_1, \phi_2), (v(1), \pi, \phi_0, \phi_1, \phi_2), (v(1), l(1), 0, \phi_1, \phi_2),$$
$$(0, l(1), \phi_0, \phi_1, \phi_1), (\pi, l(1), \phi_0, \phi_1, \phi_1)$$
$$\text{for } v(1), l(1) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi] \qquad \text{(A.11)}$$

For the region $B' \times T^3$, we introduce the equivalence relation given by

$(v(1), 0, \phi_0, \phi_1, \phi_2) \sim (v(1), 0, \phi_0 + t, \phi_1 + t, \phi_2 + t)$, for every $t \in \mathbf{S}^1$

$(v(1), \pi, \phi_0, \phi_1, \phi_2) \sim (v(1), \pi, \phi_0 + t, \phi_1 - t, \phi_2 - t)$ for every $t \in \mathbf{S}^1$

$(v(1), 0, \phi_0, \phi_1, \phi_2) \sim (v(1), \pi, 2\pi - \phi_0, \phi_1, \phi_2).$

$(v(1), l(1), 0, \phi_1, \phi_2) \sim (v(1), l(1)', 0, \phi_1, \phi_2)$

$\qquad\qquad\qquad$ for an arbitrary pair of $l(1)$ and $l(1)'$ and

$(0, l(1), \phi_0, \phi_1, \phi_1) \sim (0, l(1)', \phi_0', \phi_1', \phi_1')$

$\qquad\qquad$ for an arbitrary pair of l_1 and l_1' and the same third angles

$(\pi, l(1), \phi_0, \phi_1, \phi_1) \sim (\pi, l(1)', \phi_0, \phi_1, \phi_1)$

$\qquad$ for an arbitrary pair of $l(1)$ and $l(1)'$ and the same third angles.

$$\text{(A.12)}$$

They correspond to abelian characters. The same third angle condition means the triangle angle condition as indicated by the equation (A.5) with $\pi - \phi_1/2$ replaced by $\phi_1/2$ and $\pi - \phi_1'/2$ replaced by $\phi_1'/2$

From now on, given an element of $(B' - III - IV) \times T^3$ and the corresponding representation h, we assume that the fixed points of $h(d_i)$ are well-defined and the angles are nonzero for $i = 1, 2$ since otherwise the above equivalence relation is enough. As above, we can choose $s_1 = s_2$ differently not changing $v_1 = v_2$ but changing v_0 on the great circle $\mathbf{S}^1$ containing w_1 and w_2 and this changes ϕ_is but the representation itself does not change. We may have to consider a generalized triangle with one angle $\geq \pi$, where we have to subtract by π. Here $\mathbf{S}^1$ is determined as in the case B. This gives us an $\mathbf{S}^1$-action, which we will not describe in detail. For each abelian edge region, the $\mathbf{S}^1$-action is defined similarly to equation (A.6) extending the $\mathbf{S}^1$-action continuously and is a restriction of the $\mathbf{S}^1$-action in Appendix A.1.

We also have a notion of *canonical representatives* for a representation or character over B' by making $\phi_1 = 0$ or $\phi_2 = 0$ using the $\mathbf{S}^1$-action so that the restriction to $(B' - III - IV) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1$ or $(B' - III - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\}$ are still onto.

We also describe a Klein four-group action: The first one is given by choosing $s_1 = s_2$ to be $-s_1 = -s_2$ or equivalently letting v_0 be its antipode $-v_0$.

Choosing $v_1 = v_2$ to be its antipode gives us I_B-transformation as above, which is a $\mathbb{Z}_2$-action as well. The formulas are exactly the same. (See equations (A.8).) The above equation (A.12) and this $\mathbf{S}^1$-action and the I_B-transformation gives us the needed equivalence relation $\sim$.

Thus, there is an onto map $\mathcal{T}_{B'-III-IV} : (B' - III - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \to C^{B'-III-IV}$. (Actually, $B' - III - IV$ is equivalent to $B - I - II$ under I_A but we will not need this fact explicitly for the equivalences on $B' - III - IV$ itself.)

Proposition A.7. *There is a well-defined induced homeomorphism*

$$\mathcal{T}_{B'-III-IV} : (B' - III - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{B'-III-IV}.$$

Furthermore, the restricted maps

$$F^1_{B-III-IV} : (B' - III - IV) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \to C^{B'-III-IV} \ and$$

$$F^2_{B-III-IV} : (B' - III - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to C^{B'-III-IV}$$

are homeomorphisms where $\sim$ is given by the Klein four-group action above.

A.4.1. *Regions B'_π and B'_0*

Let B'_π be the subset of B where $\theta(1) = \pi$, i.e., $B'_\pi = B \cap IV$, and B'_0 the subset of B where $\theta(1) = 0$, i.e., $B'_0 = B \cap III$. The restricted equivalence relation is given in $B'_\pi \times T^3$ by the ones given by equation (A.12) and the circle action and the $\mathbb{Z}_2$-action of form (A.9) and (A.10). We can define a restricted equivalence relation on $B'_0 \times T^3$ the same way.

Similarly, we have:

Proposition A.8.

$$\mathcal{T}_{B'_\pi} : B'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \rightarrow H(F_2)$$

is a homeomorphism where $\sim$ *denotes the restricted equivalence relation given in* $B' \times T^3$.

$$\mathcal{T}_{B'_0} : B'_0 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \rightarrow H(F_2)$$

is a homeomorphism where $\sim$ *denotes the restricted equivalence relation given in* $B' \times T^3$. *Furthermore, we can identify* $B'_\pi \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim$ *with* $H(F_2)$ *and similarly* $B'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim$ *with* $H(F_2)$ *where* $\sim$ *is given by equation* (A.12) *and the Klein four-group action. The same statements hold with* B_π *replaced with* B_0.

A.5. *Region* $A - I - IV$

Let us now consider C^{A-I-IV} and $C^{A'-II-III}$. (Again, they are equivalent under I_B but we will not need this fact.)

We consider $A - I - IV$ first. Here the corresponding generalized triangle is a lune with vertices v_0 and $v_1 = -v_0$ and the vertex v_2 is on an edge.

Let v_0 and v_1 be an arbitrary pair of antipodal points. First, we discuss how to construct a representation h so that $h(c_i)$ fix v_i for $i = 0, 1$ and has the right pasting angles for an element of C^{A-I-IV}.

To obtain the transformations $h(d_1)$ and $h(d_2)$, we consider a triangle T_2 with vertices v_2, v_0 and w_2 and with angles $\phi_2/2$ and $\pi - \phi_0/2$ at v_2 and v_0 respectively. The vertex w_2 is the fixed point of $h(d_2)$ and the vertex angle η_2 is the half of 2π minus the rotation angle τ_2 of $h(d_2)$, and $h(d_1)$ is a rotation about v_1 with angle $\phi_1 + \phi_0$ and $w_1 = v_1$. Since we defined $h(c_i)$ and $h(d_j)$ for $i = 1, 2$ and $j = 1, 2$, we are done.

Let us discuss the equivalence relation:

First, the following is obvious:

Proposition A.9. *A representation* h *corresponding to an element of* C^{A-I-IV} *is nonabelian if and only if* $h(d_2)$ *is not identity and does not fix* v_0 *and* v_1 *for any configuration corresponding to an element of* $(A - I - IV) \times T^3$ *representing* h.

By Proposition A.9, the subset of $A \times T^3$ corresponding to the abelian representations is

$$(v(0), 0, \phi_0, \phi_1, \phi_2), (v(0), \pi, \phi_0, \phi_1, \phi_2), (v(0), l(0), \phi_0, \phi_1, 0),$$
$$(0, l(0), \phi_0, 2\pi - \phi_0, \phi_2), (\pi, l(0), \phi_0, 2\pi - \phi_0, \phi_2)$$
$$\text{for } v(0), l(0) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi] \qquad (A.13)$$

For the region $A \times T^3$, we first introduce the equivalence relation given by

$$(v(0), 0, \phi_0, \phi_1, \phi_2) \sim (v(0), 0, \phi_0 + t, \phi_1 - t, \phi_2 - t)$$
$$(v(0), \pi, \phi_0, \phi_1, \phi_2) \sim (v(0), \pi, \phi_0 + t, \phi_1 - t, \phi_2 + t)$$
$$(v(0), 0, \phi_0, \phi_1, \phi_2) \sim (v(0), \pi, \phi_0, \phi_1, 2\pi - \phi_2)$$
$$(v(0), l(0), \phi_0, \phi_1, 0) \sim (v(0), l(0)', \phi_0, \phi_1, 0)$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)' \text{ and}$$
$$(0, l(0), \phi_0, 2\pi - \phi_0, \phi_2) \sim (0, l(0)', \phi_0', 2\pi - \phi_0', \phi_2')$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)' \text{ and the same third angles}$$
$$(\pi, l(0), \phi_0, 2\pi - \phi_0, \phi_2) \sim (\pi, l(0)', \phi_0', 2\pi - \phi_0', \phi_2')$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)' \text{ and the same third angles.}$$

(A.14)

By the same third angles, we mean that the third angle of the triangle with a side of length $l(1) = \pi - l(0)$ and angles $\pi - \phi_0/2$ and $\phi_2/2$ at the end is same as the third angle of the triangle with a side of length $l(1)' = \pi - l(0)'$ and the angles $\pi - \phi_0'/2$ and $\phi_2'/2$ at the end.

$$-\cos(\pi - \phi_0/2)\cos\phi_2/2 + \sin(\pi - \phi_0/2)\sin\phi_2/2\cos(\pi - l(0)) =$$
$$-\cos(\pi - \phi_0'/2)\cos\phi_2'/2 + \sin(\pi - \phi_0'/2)\sin\phi_2'/2\cos(\pi - l(0)'). \quad \text{(A.15)}$$

Note again that we include the degenerate triangles and in that case the equation is not an enough condition and the condition is a geometric one.

From now on, we may assume that the fixed points $\pm w_2$ of $h(d_2)$ are well defined and the angle τ_2 at w_2 is not 0 or 2π since otherwise the above equivalences are enough. Also, each of $\pm w_2$ does not coincide with v_0, v_1.

An $\mathbf{S}^1$-action is given by taking a different segment s_2 with endpoints v_0 and $v_1 = -v_0$. Then v_2 is obtained as an intersection of s_2 with the great circle through w_2 that has angle $\eta_2 = \pi - \tau_2/2$ with respect to the shortest segment to v_0 from w_2. This again involves generalized triangles and changing the angle $\theta_1/2$ to $\theta_1/2 - \pi$ if $\theta_1/2 \geq \pi$. Again there is an $\mathbf{S}^1$-action on the abelian edge regions given by restricting the $\mathbf{S}^1$-action in Appendix A.1, which extends these ones continuously.

If we choose a segment s_2 with endpoints v_1 and v_0 passing through w_2, then $v_2 = w_2$, $h(d_2)$ has the rotation angle ϕ_2, $\phi_0 = 0$, and $h(d_1)$ is now a rotation about v_1 with angle ϕ_1. We can also choose so that $\phi_1 = 0$ by making ϕ_0 be equal to 2π minus the rotation angle of $h(d_1)$. We call these the *canonical* representatives for a representation or character. Thus, every representation over A has *canonical* representatives.

Thus, the configurations that we need to classify are ones where each is given by an antipodal pair (v_0, v_1) and a choice of fixed point $w_2, -w_2$ and the rotation angles of $h(d_2)$ and $h(d_1)$.

Also, there is a Klein four-group action again: First, there is a $\mathbb{Z}_2$-action given by choosing $-s_2$ instead of s_2. This is in the $\mathbf{S}^1$-action.

Second, the different antipodal choices of v_0, v_1 give us an I_A-action:

$$(v(0), l(0), \phi_0, \phi_1, \phi_2) \mapsto (\pi - v(0), \pi - l(0), 2\pi - \phi_0, 2\pi - \phi_1, \phi_2). \quad \text{(A.16)}$$

We define the equivalence relation $\sim$ on $(A - I - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ to be one generated by the equivalence relations given by equation (A.14) and the $\mathbf{S}^1$-action and equation (A.16).
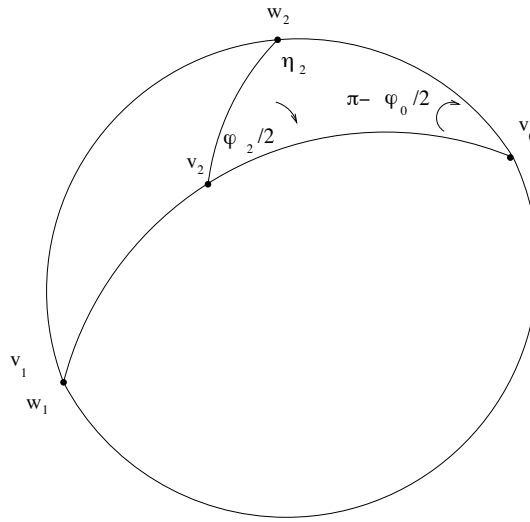

Fig. A.4.   Finding holonomy of d_1 and d_2 using triangles for case $A - I - IV$.

Therefore we define a map

$$\mathcal{T}_{A-I-IV} : (A - I - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \to C^{A-I-IV},$$

which is onto by definition. We note that $(A - I - IV) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 \to C^{A-I-IV}$ and $A - I - IV \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 \to C^{A-I-IV}$ are onto since we can let $\phi_0 = 0$ or let $\phi_1 = 0$ by the $\mathbf{S}^1$-action.

Proposition A.10. *The induced map*

$$\mathcal{T}_{A-I-IV} : (A - I - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{A-I-IV}$$

is a homeomorphism. Furthermore, the restrictions

$$\mathcal{T}^0_{A-I-IV} : (A - I - IV) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{A-I-IV}$$

$$\mathcal{T}^1_{A-I-IV} : (A - I - IV) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \to C^{A-I-IV}$$

are homeomorphisms where $\sim$ is given by the equation (A.14) *and the Klein four-group action.*

Proof. The proof is similar to Proposition A.5. That is, the $\mathbf{S}^1$-action gives the equivalences for the nonabelian parts and the abelian parts are studied by equation (A.14). □

A.5.1. *Regions A_π and A_0*

Again let A_π and A_0 denote the subsets of A defined by $\theta(0) = \pi$ and $\theta(0) = 0$ respectively.

Also, on $A_\pi \times T^3$ and $A_0 \times T^3$, there is a $\mathbb{Z}_2$-action sending v_0, v_1 to their antipodes respectively and not changing v_2.

$$(\pi, l(0), \phi_0, \phi_1, \phi_2) \mapsto (\pi, \pi - l(0), 2\pi - \phi_0, 2\pi - \phi_1, \phi_2).$$
$$(0, l(0), \phi_0, \phi_1, \phi_2) \mapsto (0, \pi - l(0), 2\pi - \phi_0, 2\pi - \phi_1, \phi_2). \tag{A.17}$$

(These can be considered an augmented I_A-transformation.) $A_\pi \times T^3$ has an equivalence relation given by equation (A.14) and the $\mathbf{S}^1$-action and the $\mathbb{Z}_2$-group action given by equation (A.17). Similar statements hold for $A_0 \times T^3$ as well.

Proposition A.11.

$$\mathcal{T}_{A_\pi} : A_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \rightarrow H(F_2)$$

is a homeomorphism. Furthermore, so are the following restrictions

$$\mathcal{T}_{A_\pi}^0 : A_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \rightarrow H(F_2) \ and$$

$$\mathcal{T}_{A_\pi}^1 : A_\pi \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \rightarrow H(F_2).$$

We can replace A_π with A_0 and the above statements hold again.

Proof. This can be proved as in Proposition A.6.

However, the second statement can be shown by the direct expression of coordinates and the fact that the Klein four-group acts on both coordinate spaces can be used: The map

$$\mathcal{T}_{A_\pi} : A_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 \rightarrow H(F_2)$$

is given by

$$(\pi, l(0), 0, \phi_1, \phi_2) \rightarrow (l(0), \phi_1, \phi_2),$$

which is clearly onto. (Recall from Section 3.2 for the second coordinates.) □

A.6. *Region $A' - II - III$*

Now, consider the region A' where a corresponding triangle is a segment with endpoints $v_0 = v_1$ and v_2.

First, we find the representation h corresponding to an element of $(A' - II - III) \times T^3$: To obtain $h(d_1)$ and $h(d_2)$, we note first that $h(d_1)$ is a rotation about v_1

with angle $\phi_1 - \phi_0$. We next consider a triangle T_2 with vertices v_2, v_0 and w_2 and with angles $\phi_2/2$ and $\pi - \phi_0/2$ at v_2 and v_0 respectively. The vertex w_2 is the fixed point of $h(d_2)$ and the vertex angle is π minus the one half of the rotation angle of $h(d_2)$.

Similarly to Proposition A.9, we show that the subset of $A' \times T^3$ corresponding to the abelian characters is

$$(v(0), 0, \phi_0, \phi_1, \phi_2), (v(0), \pi, \phi_0, \phi_1, \phi_2), (v(0), l(0), \phi_0, \phi_1, 0),$$
$$(0, l(0), \phi_0, \phi_0, \phi_2), (\pi, l(0), \phi_0, \phi_0, \phi_2)$$
$$\text{for } v(0), l(0) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi] \qquad \text{(A.18)}$$

For the region $A' \times T^3$, we first introduce the equivalence relation given by

$$(v(0), 0, \phi_0, \phi_1, \phi_2) \sim (v(0), 0, \phi_0 + t, \phi_1 + t, \phi_2 + t)$$
$$(v(0), \pi, \phi_0, \phi_1, \phi_2) \sim (v(0), \pi, \phi_0 + t, \phi_1 + t, \phi_2 - t)$$
$$(v(0), 0, \phi_0, \phi_1, \phi_2) \sim (v(0), \pi, \phi_0, \phi_1, 2\pi - \phi_2)$$
$$(v(0), l(0), \phi_0, \phi_1, 0) \sim (v(0), l(0)', \phi_0, \phi_1, 0)$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)',$$
$$(0, l(0), \phi_0, \phi_0, \phi_2) \sim (0, l(0)', \phi_0', \phi_0', \phi_2')$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)' \text{ and the same third angles}$$
$$(\pi, l(0), \phi_0, \phi_0, \phi_2) \sim (\pi, l(0)', \phi_0', \phi_0', \phi_2')$$
$$\text{for an arbitrary pair of } l(0) \text{ and } l(0)' \text{ and the same third angles.}$$

$$\text{(A.19)}$$

The same third angle condition means the triangle angle condition as indicated by equation (A.15).

From now on, we may assume that the fixed points of $h(d_i)$ are well-defined and the angles are nonzero for $i = 1, 2$ since otherwise the above equivalences are enough.

There is an $\mathbf{S}^1$-action also given by choosing a different segment from v_0 to $-v_0$ and taking v_2 as intersection of a great circle $\mathbf{S}^1$ through w_2 having π minus the $1/2$ of the angle of $h(d_2)$ with the segment from v_0 to w_2.

If we choose a segment s with endpoints v_1 and v_0 passing through w_2, then $v_2 = w_2$ and $h(d_2)$ has the rotation angle ϕ_2. Now, we obtain $\phi_0 = 0$. We can also make $\phi_1 = 0$. We call these the *canonical* representatives. Thus, every representation over A' has canonical representatives. For each abelian edge region, the $\mathbf{S}^1$-action is given by restricting one in Appendix A.1.

Therefore we define a map $\mathcal{T}_{A'} : A' \times \mathbf{S}^I \times \mathbf{S}^1 \times \mathbf{S}^1 \to C^{A'}$. The map is onto by same reasoning as above.

Again, we define $\sim$ on $(A' - II - III) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ by the above equivalence relations, i.e., equation (A.19) and the $\mathbf{S}^1$-action and the $\{I, I_A\}$-action, given by same formula as equation (A.16). The $\{I, I_A\}$-action and the $\mathbf{S}^1$-action restrict to the respective Klein four-group actions on $(A' - II - III) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$ and

$(A' - II - III) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1$. We define equivalence relations on these by equation (A.19) and the Klein four-group action.

We obmit the proof of the following:

Proposition A.12. *The induced map*

$$\mathcal{T}_{A'-II-III} : (A' - II - III) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{A'-II-III}$$

is a homeomorphism. The restrictions

$$\mathcal{T}_{A'-II-III} : (A' - II - III) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{A'-II-III} \ and$$

$$\mathcal{T}_{A'-II-III} : (A' - II - III) \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \to C^{A'-II-III}$$

are homeomorphisms.

A.6.1. *Regions A'_π and A'_0*

Again let A'_π and A'_0 denote the subsets of A' defined by $\theta_0 = \pi$ and $\theta_0 = 0$ respectively. For these, I_A has an *augmented action* as given by (A.17). The equivalence relation $\sim$ on $A'_\pi \times T^3$ is given by equation (A.19) and the $\mathbf{S}^1$-action and the above augmented I_A-action. The actions restrict to a Klein four-group action on $A'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$ and $A'_\pi \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1$. We define equivalence relations on these by equation (A.19) and the Klein four-group action. Similarly, we define equivalence relations on $A'_0 \times T^3$ and $A'_0 \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$ and $A'_0 \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$.

By introducing appropriate $\sim$ from equation (A.19) and the $\mathbf{S}^1$-action and the augmented $\{I, I_A\}$-action, we obtain:

Proposition A.13.

$$\mathcal{T}_{A'_\pi} : A'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2)$$

is a homeomorphism. Furthermore, the restriction are homeomorphisms

$$\mathcal{T}^0_{A'_\pi} : A'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{A'_\pi}$$

$$\mathcal{T}^1_{A'_\pi} : A'_\pi \times \mathbf{S}^1 \times \{0\} \times \mathbf{S}^1 / \sim \to C^{A'_\pi}.$$

Also, the same statements are true with A'_0 replacing A'_π.

Proof. The proof is similar to the case B'_π.

The second statement can be also proved by the map

$$\mathcal{T}_{A'_\pi} : A'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2)$$

is given by

$$(\pi, l(0), 0, \phi_1, \phi_2) \mapsto (l(0), \phi_1, \phi_2) \qquad \qquad \square$$

A.7. *Region* $C - I - III$

Finally, we go to $C^{C-I-III}$ and $C^{C'-II-IV}$. Suppose that we are over C. Then an element of C corresponds to a lune with vertices v_0 and $v_2 = -v_0$ and v_1 is on an edge.

First, we obtain a map $(C - I - III) \times T^3 \to C^{C-I-III}$: Given an element of $(C - I - III) \times T^3$, to construct $h(d_1)$ and $h(d_2)$, we consider a triangle T_1 with vertices v_1, v_0 and w_1 and with angles $\phi_1/2$ and $\pi - \phi_0/2$ at v_1 and v_0 respectively for the lune with vertices v_0 and v_2 and v_1 on the edge. The vertex w_1 is the fixed point of $h(d_1)$ and the vertex angle η_1 is π minus the half of the rotation angle of $h(d_1)$ and $h(d_2)$ is a rotation about v_2 with angle $\phi_2 - \phi_0$ and $w_2 = v_2$.
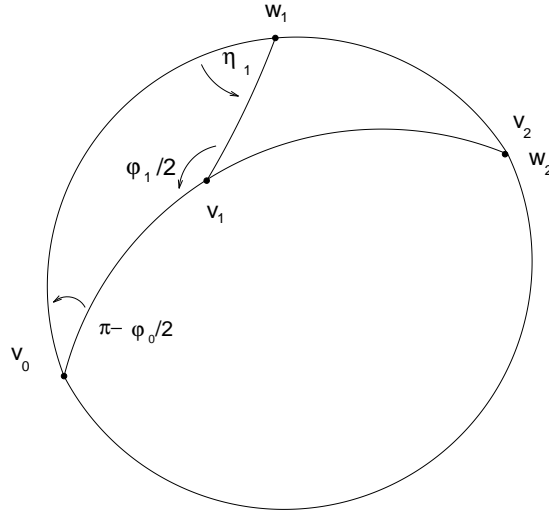
Fig. A.5.   Finding holonomy of d_1 and d_2 using triangles.

We now discuss the equivalence relation on $C \times T^3$:

Proposition A.14. *A representation h corresponding to an element of $C \times T^3$ is nonabelian if and only if $h(d_1)$ are not identity and does not fix v_0 and v_2 for any configuration corresponding to an element of $C \times T^3$ representing h.*

The subset of $C \times T^3$ corresponding to the abelian representations is

$$(v(2), 0, \phi_0, \phi_1, \phi_2), (v(2), \pi, \phi_0, \phi_1, \phi_2), (v(2), l(2), \phi_0, 0, \phi_2),$$
$$(0, l(2), \phi_0, \phi_1, 2\pi - \phi_0), (\pi, l(2), \phi_0, \phi_1, 2\pi - \phi_0)$$
$$\text{for } v(2), l(2) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi]. \qquad \text{(A.20)}$$

For the region $C \times T^3$, we introduce the equivalence relation given by

$$(v(2), 0, \phi_0, \phi_1, \phi_2) \sim (v(2), 0, \phi_0 + t, \phi_1 + t, \phi_2 - t)$$
$$(v(2), \pi, \phi_0, \phi_1, \phi_2) \sim (v(2), \pi, \phi_0 + t, \phi_1 - t, \phi_2 - t)$$
$$(v(2), 0, \phi_0, \phi_1, \phi_2) \sim (v(2), \pi, \phi_0, 2\pi - \phi_1, \phi_2).$$
$$(v(2), l(2), \phi_0, 0, \phi_2) \sim (v(2), l(2)', \phi_0, 0, \phi_2)$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and

$$(0, l(2), \phi_0, \phi_1, 2\pi - \phi_0) \sim (0, l(2)', \phi_0', \phi_1', 2\pi - \phi_0')$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and the same third angles

$$(\pi, l(2), \phi_0, \phi_1, 2\pi - \phi_0) \sim (\pi, l(2)', \phi_0', \phi_1', 2\pi - \phi_0')$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and the same third angles.

(A.21)

The same third angle means the triangle angle condition as indicated by

$$-\cos(\pi - \phi_0/2)\cos\phi_1/2 + \sin(\pi - \phi_0/2)\sin\phi_1/2\cos l(2) =$$
$$-\cos(\pi - \phi_0'/2)\cos\phi_1'/2 + \sin(\pi - \phi_0'/2)\sin\phi_1'/2\cos l(2)'. \quad \text{(A.22)}$$

Choose an element of $C \times T^3$. We have a lune with vertices v_0 and $v_2 = -v_0$ and v_1 is on an edge. Again there is an $\mathbf{S}^1$-action given by choosing a different segment s with endpoints v_0 and v_2 and taking v_1 as an intersection point of s with a great circle $\mathbf{S}^1$ through w_1 which has π minus the angle $1/2$ of the rotation angle of $h(d_1)$ with respect to the shortest segment from w_1 to v_0. For each abelian edge region, the $\mathbf{S}^1$-action is given by the restriction of ones in Appendix A.1 and continuously extends the above.

If we choose a segment s with endpoints v_2 and v_0 passing through w_1, then $v_1 = w_1$ and $h(d_1)$ has the rotation angle ϕ_1. Here we obtain $\phi_0 = 0$.

We can also make $\phi_2 = 0$ by making ϕ_0 have angle 2π minus the angle of $h(d_2)$. We call these the *canonical* representatives. Thus, every representation over C has canonical representatives.

Again $\{I, I_C\}$ acts on these spaces. We define the equivalence relation $\sim$ using the above ones by equation (A.21) and the $\mathbf{S}^1$-action and the $\{I, I_C\}$-action.

We obmit the proof of the following:

Proposition A.15. *The induced map*

$$\mathcal{T}_{C-I-III} : (C - I - III) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{C-I-III}$$

is a homeomorphism. Furthermore, the restrictions

$$\mathcal{T}_{C-I-III}^0 : (C - I - III) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{C-I-III} \text{ and}$$

$$\mathcal{T}_{C-I-III}^2 : (C - I - III) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to C^{C-I-III}$$

where $\sim$ *is given by equation* (A.21) *and the Klein four-group given by restricting the* $\mathbf{S}^1$*-action and* $\{I, I_C\}$*-action.*

A.7.1. *Regions* C_π *and* C_0

Again let C_π and C_0 denote the subsets of C defined by $\theta_0 = \pi$ and $\theta_0 = 0$ respectively. The equivalence relations on $C_\pi \times T^3$ and $C_0 \times T^3$ are defined by equation (A.21) and by the $\mathbf{S}^1$-action and the *augmented* $\{\mathrm{I}, I_C\}$-action:

$$(0, l(2), \phi_0, \phi_1, \phi_2) \mapsto (0, \pi - l(2), 2\pi - \phi_0, \phi_1, 2\pi - \phi_2)$$

$$(\pi, l(2), \phi_0, \phi_1, \phi_2) \mapsto (\pi, \pi - l(2), 2\pi - \phi_0, \phi_1, 2\pi - \phi_2). \qquad (\mathrm{A.23})$$

Define $\sim$ on $C_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$ by equation (A.21) and an induced Klein four-group action on. Similarly we define $\sim$ on $C_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\}$. We define respective equivalence relations on C_0 cases.

Proposition A.16.

$$\mathcal{T}_{C_\pi} : C_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2)$$

is a homeomorphism. Similarly,

$$\mathcal{T}_{C_\pi} : C_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2) \ and$$

$$\mathcal{T}_{C_\pi} : C_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to H(F_2)$$

are homeomorphisms. The same statements hold with C_π *replaced by* C_0.

The proof is similar to the above B_π case. Again, the second statement can be proved using the onto map

$$\mathcal{T}_{C_\pi} : C_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 \to H(F_2)$$

given by

$$(\pi, l(2), 0, \phi_1, \phi_2) \to (\pi - l(2), \phi_1, \phi_2).$$

A.8. *Region* $C' - II - III$

We are over C' finally. Here, for our configuration, a triangle is a segment with endpoints v_1 and $v_0 = v_2$.

First, we construct a map $C' \times T^3 \to C^{C'}$: To construct $h(d_1)$ and $h(d_2)$ for the representation corresponding to a given configuration, we consider a triangle T_1 with vertices v_1, v_0 and w_1 and with angles $\phi_1/2$ and $\pi - \phi_0/2$ at v_2 and v_0 respectively for a pointed segment with vertices v_1 and $v_0 = v_2$. The vertex w_1 is the fixed point of $h(d_1)$ and the vertex angle is π minus the half of the rotation angle of $h(d_2)$. $h(d_2)$ is a rotation about v_2 with angle $\phi_2 - \phi_0$.

Similarly to Proposition A.14, the subset of $C' \times T^3$ corresponding to the abelian characters is

$$(v(2), 0, \phi_0, \phi_1, \phi_2), (v(2), \pi, \phi_0, \phi_1, \phi_2),$$

$$(v(2), l(2), \phi_0, 0, \phi_2), v(2), l(2), \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi].$$

$$(0, l(2), \phi_0, \phi_1, \phi_0), (\pi, l(2), \phi_0, \phi_1, \phi_0)$$

$$\text{for } v(2), l(2) \in [0, \pi], \phi_0, \phi_1, \phi_2 \in [0, 2\pi]. \qquad (\mathrm{A.24})$$

For the region $C' \times T^3$, we introduce the equivalence relation given by

$$(v(2), 0, \phi_0, \phi_1, \phi_2) \sim (v(2), 0, \phi_0 + t, \phi_1 + t, \phi_2 + t)$$
$$(v(2), \pi, \phi_0, \phi_1, \phi_2) \sim (v(2), \pi, \phi_0 + t, \phi_1 - t, \phi_2 + t)$$
$$(v(2), 0, \phi_0, \phi_1, \phi_2) \sim (v(2), \pi, \phi_0, 2\pi - \phi_1, \phi_2)$$
$$(v(2), l(2), \phi_0, 0, \phi_2) \sim (v(2), l(2)', \phi_0, 0, \phi_2)$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and

$$(0, l(2), \phi_0, \phi_1, \phi_0) \sim (0, l(2)', \phi_0', \phi_1', \phi_0')$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and the same third angles

$$(\pi, l(2), \phi_0, \phi_1, \phi_0) \sim (\pi, l(2)', \phi_0', \phi_1', \phi_0')$$

for an arbitrary pair of $l(2)$ and $l(2)'$ and the same third angles.

(A.25)

The same third angle condition means the triangle angle condition as indicated by

$$- \cos(\pi - \phi_0/2) \cos \phi_1/2 + \sin(\pi - \phi_0/2) \sin \phi_1/2 \cos l(2) =$$
$$- \cos(\pi - \phi_0'/2) \cos \phi_1'/2 + \sin(\pi - \phi_0'/2) \sin \phi_1'/2 \cos l(2)'. \quad \text{(A.26)}$$

We may assume without loss of generality that fixed points of $h(d_1)$ and $h(d_2)$ are well-defined. Again, there is an $\mathbf{S}^1$-action. The equivalence relation is given on $C' \times T^3$ by the above equation (A.25) and the $\mathbf{S}^1$-action and the $\{\mathrm{I}, I_C\}$-action.

If we choose a segment s with endpoints v_2 and v_0 passing through w_1, then $v_1 = w_1$ and $h(d_1)$ has the rotation angle ϕ_1. Here $\phi_0 = 0$. We can also make $\phi_2 = 0$. We call these the *canonical* representatives. Thus, every representation over C' has *canonical* representatives.

Proposition A.17. *The induced map*

$$\mathcal{T}_{C'-II-IV} : (C' - II - IV)\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times / \sim \to C^{C'-II-IV}$$

is a homeomorphism. Furthermore, the restrictions

$$\mathcal{T}^0_{C'-II-IV} : (C' - II - IV) \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to C^{C'-II-IV} \text{ and}$$

$$\mathcal{T}^2_{C'-II-IV} : (C' - II - IV) \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to C^{C'-II-IV}$$

where $\sim$ *is given by equation* (A.25) *and the Klein four-group given by restricting the* $\mathbf{S}^1$-*action and the* $\{\mathrm{I}, I_C\}$-*action.*

A.8.1. C'_π *and* C'_0

Again let C'_π and C'_0 denote the subsets of C defined by $\theta_0 = \pi$ and $\theta_0 = 0$ respectively. Again equivalence relations on $C'_\pi \times T^3$ and $C'_0 \times T^3$ are given by equation (A.25) and the $\mathbf{S}^1$-action and augmented $\{\mathrm{I}, I_C\}$-actions. as given by equation (A.23).

The equivalence relations on $C'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1$ and $C'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\}$ are given by equation (A.25) and the Klein four-group action given by restricting the $\mathbf{S}^1$-action and the $\{I, I_C\}$-action.

Proposition A.18.

$$\mathcal{T}_{C'_\pi} : C'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2)$$

is a homeomorphism. Similarly,

$$\mathcal{T}_{C'_\pi} : C'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2) \ and$$

$$\mathcal{T}_{C'_\pi} : C'_\pi \times \mathbf{S}^1 \times \mathbf{S}^1 \times \{0\} / \sim \to H(F_2)$$

are homeomorphisms. The same statements hold with C'_π replaced by C'_0.

The proof is similar to the above C_π case. Again, the second statement can be proved using the map

$$\mathcal{T}_{C'_\pi} : C'_\pi \times \{0\} \times \mathbf{S}^1 \times \mathbf{S}^1 / \sim \to H(F_2)$$

given by

$$(\pi, l(2), 0, \phi_1, \phi_2) \to (l(2), \phi_1, \phi_2).$$

Appendix B. The equivalence relations for $\mathbf{Hom}(\pi_1(\Sigma), \mathbf{SU(2)})/\mathbf{SU(2)}.$

In this section, we will describe the equivalence relation on $\tilde{\Delta} \times T_2^3$ so that the quotient space is homeomorphic to $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(2))/\mathrm{SU}(2)$. We continue from Section 6.5 and use the notations from the section.

B.1. *The space of abelian* SU(2)*-characters*

We study the faces a, b, c, and d of Δ, which correspond to abelian characters. We denote by $\chi_2(\Sigma)$ the subspace of abelian SU(2)-characters in $\mathrm{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$. Given an abelian SU(2)-representation h, $h(c_1), h(c_2), -h(d_1)$, and $-h(d_2)$ can be represented as a point in $\mathbf{S}^2$ with rotation angles in T_2^4. Since we can choose the fixed point to be a common one or its antipode, there is a $\mathbb{Z}_2$-action on T_2^3 given by $(\theta_1, \theta_2, \phi_1, \phi_2)$ to $(4\pi - \theta_1, 4\pi - \theta_2, 4\pi - \phi_2, 4\pi - \phi_2)$. Thus, the space $\chi_2(\Sigma)$ is homeomorphic to $T_2^4/\mathbb{Z}_2$, where the 16 singular points are given by $(\pm 2\pi, \pm 2\pi, \pm 2\pi, \pm 2\pi)$.

Reinterpreting in terms of pairs-of-pants decomposition, we see that each abelian character corresponds to a point of $\partial\tilde{\Delta} \times T_2^4/\mathbb{Z}_2/\sim$ by choosing $h(c_i)$ to have a fixed point with angles in $[0, 2\pi]$.

For regions, a, b, c, and d, the circle action is a simply lifted one from the circle action above for the SO(3)-case.

- For the face a, the equivalence relation on $a \times T_2^3$ is given by $(v, \phi_0, \phi_1, \phi_2) \sim (v', \phi_0', \phi_1', \phi_2')$ if and only if $v = v'$ and two vectors

$$(\phi_0, \phi_1, \phi_2) - (\phi_0', \phi_1', \phi_2') = s(2\pi, 2\pi, 2\pi) \text{ for } s \in \mathbb{R}.$$

- For faces b, c, and d, the equivalence relation is defined again using the respective $\mathbf{S}^1$-action generated by vectors parallel to $(-2\pi, 2\pi, 2\pi), (2\pi, -2\pi, -2\pi), (2\pi, 2\pi, -2\pi)$ perpendicular to b, c, d respectively.

The interiors of the quotients of $a \times T_2^3, b \times T_2^3, c \times T_2^3$, and $d \times T_2^3$ are mutually disjoint.

By an analogous reasoning to Appendix A.1, the fibers over the points of faces are 2-tori. (The only we have to double the size of every vector in the argument here): Looking at $a \times T_2^3 / \sim$ in more detail, the equation determining the equivalence class representatives should be $\phi_0 + \phi_1 + \phi_2 = 0$ as $4\pi = 0$ in $\mathbf{S}_2^1 \times \mathbf{S}_2^1 \times \mathbf{S}_2^1$ with an order-three translation action. We take the subset of T_2^3 given by this equation. Contemplating T_2^3 as a quotient space of the cube $[0, 4\pi]^3$, we see that the plane corresponds to the union of two triangles with vertices $\triangle_1 = \triangle([4\pi, 0, 0], [0, 4\pi, 0], [0, 0, 4\pi])$ and $\triangle_2 = \triangle([4\pi, 4\pi, 0], [4\pi, 0, 4\pi], [0, 4\pi, 4\pi])$. Then two triangles form a 2-tori T' under the identifications of T^3. There is an identification by the above $\mathbf{S}^1$-action which gives us an order-three translation action. T' is isometric with a torus in the plane quotient out by a lattice generated by translation by $(4\pi\sqrt{2}, 0)$ and $(2\pi\sqrt{2}, 2\pi\sqrt{6})$. Then the order-three identification group is generated by a translation by $(2\pi, 2\pi\sqrt{6}/3)$. The quotient space $T_{2,a}^2$ is homeomorphic to a 2-torus. Thus, the character space here is in one-to-one correspondence with $a \times T_{2,a}^2$. Similarly, we obtain $T_{2,b}^2, T_{2,c}^2$, and $T_{2,d}^2$ for respective faces b, c, and d.

Recall that for regions, A, B, C, A', B', and C', note that fixing all three angles determines a segment, a tie, which joins a point of an edge of a face to a point in the edge of another face.

We take a union of $a \times T_{2,a}^2$, $b \times T_{2,b}^2$, $c \times T_{2,c}^2$, and $d \times T_{2,d}^2$. Note that as we cross an edge through a tie from a face to another face, we change one of the vertex of a lune triangle to its antipode. We have four faces of $\tilde{\Delta} \times T_2^3 / \mathbb{Z}_2$ glued over the edges by isomorphisms with mappings such as $(\phi_0, \phi_1, \phi_2) \to (4\pi - \phi_0, \phi_1, \phi_2)$, $(\phi_0, \phi_1, \phi_2) \to (\phi_0, 4\pi - \phi_1, \phi_2)$, and $(\phi_0, \phi_1, \phi_2) \to (\phi_0, \phi_1, 4\pi - \phi_2)$, which occurs if we change one of the vertex to its antipode as we cross an edge. (See third item in each set of equations (A.4), (A.12), (A.14), (A.19), (A.21) and (A.25).) Therefore, the 2-tori get identified across the edges by homeomorphisms. (Over the interiors of edges the fibers are still homeomorphic to a 2-torus.) Around each vertex the gluing gives $-I$ as the identification holonomy around the vertex, we have a 2-sphere with four singularities which is obtained as $T_2^3 / \sim$ by the $\mathbb{Z}_2$-action generated by $-I$. (Here by, $-I$, we mean sending (x, y) to $(4\pi - x, 4\pi - y)$, that is $-I \mod 4\pi$.)

Hence, we can consider the union as a fibration over $\partial \tilde{\Delta}$ with fibers homeomorphic to T^2 except at vertices where the fibers are homeomorphic to a 2-sphere

which is the boundary of a three-ball parameterizing SU(2)-characters factored by the free group F_2 of rank 2 up to changing some of $h(c_i) = \pm I$ to I. (See Proposition 3.4 and Appendix B.2) Here the sphere can be considered a quotient of a torus by $\mathbb{Z}_2$-action generated by $-I$ as in the above paragraph.

Actually, we take a quotient of the first two factors T_2^2 of T_2^4 first, we obtain a sphere with four singularities, corresponding to the vertices of Δ, which is identical with $\partial \Delta$ in a natural way. In this way, we see naturally that $\chi_2(\Sigma)$ is a fibered space as described above.

B.2. *Regions above* A, A', B, B', C, C'.

Now, we go over the regions A, A', B, B', C, and C'. Recall $U = I \cup II \cup III \cup IV$.

Lemma B.1. *The subspace over a tie in one of the regions A, B, C, A', B', and C' but not in U is homeomorphic to $\mathbf{S}^1 \times B^2$. Thus, over the interior of each edge corresponding to A, A', B, B', C, C', there is a bundle over an open interval with fibers homeomorphic to the solid tori. If a tie is in U, the subspace over it is identical with the subspace over I, II, III, or IV respectively and hence is homeomorphic to a 3-ball and can be considered as having been obtained from a $\mathbb{Z}_2$-action on the solid torus. Hence, the region above each of A, A', B, B', C, C' is homeomorphic to the quotient space of a solid torus times an interval with the solid torus over each end identified with a 3-ball.*

Proof.

We work similarly to Appendix A.3. We form $h(d_1)$ and $h(d_2)$ for a representation h using triangle constructions from a configuration where we use $-P_0^{-1}$ instead of P_0 so that the construction is the same. Sometimes, we need to use a generalized triangle since we have angles in $[0, 4\pi)$ (See equation (3.2). The generalized triangle with one vertex of angle in $[2\pi, 4\pi]$ and the other two angles in $[0, 2\pi]$ also need to be considered here in multiplications by geometry.) Here, their fixed points and angles are determined from the geometric constructions. The circle action exists and here we do not need to change the fixed points of $h(d_1)$ and $h(d_2)$ up to antipodes.

First, suppose that we are on $B - U$. Let h be a representation over a tie in $B - U$. Then $h(c_1)$ and $h(c_2)$ have a pair of respective fixed points antipodal to each other. Suppose that h is nonabelian. Then $h(d_1)$ and $h(d_2)$ respectively have well-defined pairs of fixed points $\pm w_1$ and $\pm w_2$ distinct from this pair. Let w_1 and w_2 be ones with respective rotation angles for $h(d_1)$ and $h(d_2)$ in $[0, 2\pi]$. (See Figure B.1.)

The subspace of such characters is dense. We can parameterize such configurations by (l, θ_1, θ_2) where l is the length of the shortest segment s between the fixed point v_1 of $h(c_1)$ to w_1 and hence $l \leq \pi$ and $\theta_1/2$ is the counter-clockwise angle at v_1 between s and the shortest segment to the fixed point w_2 of $h(d_2)$ for $\theta_1 \in [0, 4\pi]$ mod 4π. And θ_2, $0 \leq \theta_2 \leq 2\pi$, is the angle of rotation of $h(d_1)$ and equals 2π minus the twice of the angle at w_1 between s and one of the geodesic segment t between
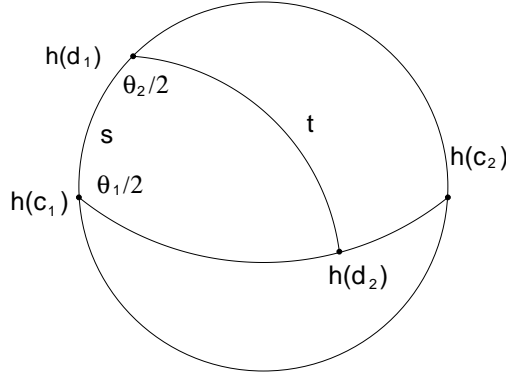
Fig. B.1.   Finding topology of space over regions $A - U$.

the fixed points w_1 and w_2 corresponding to the angle of $\theta_1/2$ between l and s. (The segment t is unique if w_1 and w_2 are not antipodal.)

Since the segments l, s, and t bound a generalized triangle with angles $\theta_1/2$ at a vertex v_1 and $\theta_2/2$ at w_1 and $\theta_3/2$ at w_2, it follows that $2\pi - \theta_2$ is the angle of rotation of $h(d_1)$: To see this, recall our holonomy constructions for B and we are letting the segment s_0 between v_1 and v_2 pass through w_2, i.e., we are in a canonical position. See Section A.3.1. (See Figure B.1.)

Since the fixed point of $h(d_2)$ may lie anywhere in $\mathbf{S}^2$, it follows that θ_1 can vary within $\mathbf{S}_2^1$.

The abelian characters over the tie in $B - U$ are in the limit set of the nonabelian characters. Thus, our space is $[0, \pi] \times \mathbf{S}_2^1 \times [0, 2\pi]$ quotient by some equivalence relation. Here the equivalence relation, we have to consider, is when $\theta_2 = 0, 2\pi$, i.e., when $h(d_1)$ is I or $-I$ and thus the length l is not defined well.

$$(l, \theta_1, 0) \sim (l', \theta_1, 0) \text{ and } (l, \theta_1, 2\pi) \sim (l', \theta_1, 2\pi).$$

The identification takes place over the abelian subspace and we verify easily that there is no other identification.

The quotient space is homeomorphic to a disk times a circle. The circle corresponds to the second factor and the disk the other two factors. The boundary region corresponds to the subspace of abelian characters, forming a two-torus.

For the region $B' - U$, the reasoning is identical and the corresponding results hold.

For the region $A - U$, let h be a representation over it. The fixed point v_1 of $h(c_1)$ is antipodal to the fixed point v_0 of $h(c_0)$. Thus, we choose a fixed point of $h(d_1)$ to be v_1 and the rotation angle to vary in $\mathbf{S}_2^1$. Then the distance from v_1 to the fixed point of $h(d_2)$ and the rotation angle of $h(d_2)$ gives us a parameterization $[0, \pi] \times \mathbf{S}_2^1 \times [0, 2\pi]$. Again the equivalence relations occur when the third coordinate is zero or 2π. The quotient space is homeomorphic to a solid torus with boundary containing all of the abelian representations in C^A.

For the region $A' - U$, the reasonings are similar and the corresponding results hold.

For the region $C - U$, let h be a representation over it. The fixed point v_2 of $h(c_2)$ is antipodal to the fixed point v_0 of $h(c_0)$ for a triangle with vertices v_0, v_1, v_2 representing an element of $C - U$. We choose a fixed point of $h(d_2)$ be v_2 and the rotation angle to vary in $\mathbf{S}_2$. The distance from v_0 to the fixed point of $h(d_1)$ and the rotation angle of $h(d_1)$ and the rotation angle of $h(d_2)$ give us a parameterization $[0, \pi] \times \mathbf{S}_2^1 \times [0, 2\pi]$. The equivalence relation occurs when the third factor is equal to 0 or 2π.

For region $C' - U$, the reasoning is similar.

For each of regions I, II, III, and IV, we obtain 3-dimensional balls by Section 3.2. For regions I, II, III, and IV, the subspace of $\operatorname{rep}(\pi_1(\Sigma), \mathrm{SU}(2))$ over each corresponds to the set of representations factoring to homomorphisms from F_2 to $\mathrm{SU}(2)$ by changing each representation h by changing $h(c_i)$ to $-h(c_i)$ for $i = 0, 1, 2$ to make it I. Therefore, each region above $I, II, III,$ or IV is homeomorphic to a 3-ball.

They are mutually disjoint and their boundary sets are the subspaces of abelian characters. Moreover, each of this is parameterized by (l, θ_1, θ_2) where l is the distance between the fixed points of $h(d_1)$ and $h(d_2)$ and θ_i is the angle of rotation of $h(d_i)$ for $i = 1, 2$. See Section 3.3 for details.

Thus, we showed that over each of the regions $A - U, A' - U, B - U, B' - U, C - U, C' - U$, we have a bundle over an open interval with solid torus fibers.

If a tie is in an edge region intersected with one of a vertex region I, II, III, or IV, then the subspace over the tie is actually onto the subspace over the vertex region. This follows since given a representation h over one of the ties, $h(c_i)$ are $\pm \mathrm{I}$ and just $h(d_1)$ and $h(d_2)$ are important and we obtain all possibilities of $h(d_1)$ and $h(d_2)$ on the tie by construction.

For region B intersected with region I, from the information $(l, \theta_1, \theta_2) \in [0, \pi] \times [0, 4\pi] \times [0, 2\pi]$, we can construct a pair of fixed points of $h(c_1)$ and $h(c_2)$ provided that we are given an orientation on the great circle containing the fixed point of $h(d_1)$ and $h(d_2)$. Changing the orientation of the great circle has the effect $(l, \theta_1, \theta_2) \rightarrow (\pi - l, 4\pi - \theta_1, \theta_2)$. Thus, the axis of this transformation is a union of two segments and the quotient space is a 3-ball. (This is related to Proposition A.6 and the analogous propositions.) Over the region B, the solid torus bundle over the interval gets identified with a 3-ball by an involution over the end. (We call the phenomenon the *clasping*.)

For region A intersected with the vertex region I, we also have a transformation $(l, \theta_1, \theta_2) \rightarrow (\pi - l, 4\pi - \theta_1, \theta_2)$. This gives a quotient space homeomorphic to B^3. For region A', the reasoning is similar.

For regions above ties in regions C, C' intersected with the vertex region I, we can again switch v_0 and v_2. The transformation can be written in the same way as above.

For intersections with vertex regions II, III, or IV, we omit the obvious generalizations of the proofs.   □

Acknowledgments

References

[1] M. Atiyah and R. Bott, The Yang Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London* Ser. A **308** (1983) 523–615.

[2] T. Baird, The moduli space of flat G-bundles over a nonorientable surface, Ph.D. Thesis, University of Toronto, Toronto (2008).

[3] M. Berger, *Geometry* I, (Springer-Verlag, Berlin, 1994).

[4] G. Bredon *Introduction to compact transformation groups*, (Acadmic Press, New York, 1972).

[5] S. Bradlow, O. Garcia-Prada, and P. Gothen, Surface group representations and U(p, q)-Higgs bundles, *J. Differential Geom.* **64** (2003) 111–170.

[6] S. Bradlow, O. Garcia-Prada, and P. Gothen, Representations of surface groups in the general linear group, in *Proceedings of the XII Fall Workshop on Geometry and Physics*, *Publ. R. Soc. Mat. Esp. 7* (R. Soc. Mat. Esp., Madrid, 2004), pp. 83–94.

[7] S. Choi, Geometric structures on orbifolds and the varieties of holonomy representations, *Geometriae Dedicata* **104** (2004) 161–199.

[8] S. Choi and W. M. Goldman, The deformation spaces of projectively flat structures on 2-orbifold, *American Journal of Mathematics* **127** (2005) 1019–1102.

[9] P. Conner, *Differentiable periodic maps,* Second edition, Lecture Notes in Mathematics, Vol. 738, Springer, Berlin, 1979.

[10] K. Corlette, Flat G-bundles with canonical metrics, *J. Differential Geom.* **28** (1988) 361–382.

[11] U. V. Desale and S. Ramanan, Classification of vector bundles of rank 2 on hyperelliptic curves, *Invent. Math.* **38** (1976/77) 161–185.

[12] S.K. Donaldson, Twisted harmonic maps and the self-duality equations, *Proc. London Math. Soc.* (3) **55** (1987) 127–131.

[13] S. K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, *J. Differential Geom.* **18** (1983) 269–277.

[14] N.J. Hitchin, Lie groups and Teichmüller space, *Topology* **31** (1992) 449–473.

[15] W. Goldman, Geometric structures on manifolds and varieties of representations, *Contemp. Math.* **74** (1988) 169–198.

[16] W. Goldman, The Symplectic nature of fundamental groups of surfaces, *Advances in Mathematics* **54** (1984) 200–225.

[17] W. Goldman, Representations of fundamental groups of surfaces, in *Lecture Notes in Mathematics*, vol 1167, (Springer, Berlin, 1985), pp. 95–117 .

[18] W. Goldman, Convex real projective structures on compact surfaces, *J. Differential Geometry* **31** (1990) 791–845.

[19] T. Goodwillie, A response to a question about Lie groups, http://www.lehigh.edu/ dmd1/tg33.

[20] C. Florontino and S. Lawton, Singularities of free group character varieties, archive:0907.4720v2

[21] W. Fulton. *Introduction to toric varieties,* Annals of Mathematics Studies, No. 131. (Princeton University Press, Princeton, 1993)

[22] J. Huebschmann. Smooth structures on certain moduli spaces for bundles on a surface, *Journal of Pure and Applied Algebra* **126** (1998) 183–221.

[23] J. Huebschmann, Poisson geometry of flat connections for $SU(2)$-bundles on surfaces, *Mathematische Zeitschrift* **221** (1996) 243–259.

[24] L.C. Jeffrey, J. Weitsman, Toric structures on the moduli space of flat connections on a Riemann surface: volumes and the moment map, *Adv. Math.* **106** (1994) 151–168.

[25] L.C. Jeffrey and J. Weitsman, Toric structures on the moduli space of flat connections on a Riemann surface. II. Inductive decomposition of the moduli space, *Math. Ann.* **307** (1997) 93–108.

[26] K.B. Lee and Frank Raymond, Seifert manifolds, in *Handbook of geometric topology* (North-Holland, Amsterdam, 2002), pp. 635–705,

[27] M. Masuda, Equivariant cohomology distinguishes toric manifolds, *Advances in Mathematics*, 218 (2008) 205–212,

[28] M.S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.* (2) **89** (1969) 14–51.

[29] M.S. Narasimhan and C.S., Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* (2) **82** (1965) 540–567.

[30] P.E. Newstead, Topological properties of some spaces of stable bundles, *Topology* **6** (1967) 241–262.

[31] P.E. Newstead, Stable bundles of rank and odd degree over a curve of genus 2, *Topology* **7** (1968) 205–215

[32] C.T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992) 5–95.

[33] S. Ramanan, Orthogonal and spin bundles over hyperelliptic curves, *Proc. Indian Acad. Sci. Math. Sci.* **90** (1981) 151–166.

[34] M. Reid, *The complete intersection of two or more quadrics,* Ph.D. Thesis, Trinity College, Cambridge (1972).

[35] W. Thurston. *Three-Dimensional Geometry and Topology*: vol 1, (Princeton University Press,1997).