3.1. Operations on matrices

Matrix notation, operations, row and column vectors, product AB(important), transpose

Matrix notation

- Matrix: a rectangular array of real numbers.
- mxn-matrix: m rows and n columns
- square matrix: nxn-matrix
- Notation A=[a_ij]_mxn, A=[a_ij], (A)_ij=a_ij

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Operations on matrices

- $A=B$ if and only if they have same size and the same entries: $a_{ij}=b_{ij}$ for all i,j,
- $A+B$: $(A+B)_{ij}=A_{ij}+B_{ij}$
- $A-B$: $(A-B)_{ij}=A_{ij}-B_{ij}$
- cA: $(cA)_{ij}=c(A)_{ij}=ca_{ij}$, $-A= (-1)A$.
- See Ex 1,2,3.

Row and column vectors

- Row vectors: r=[r_1, r_2,…,r_n]; i.e., 1xn matrix
- Column vectors: c=

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

- Think of mxn matrix as m row n-vectors in a column.
- Think of it as n column m-vectors in a row.

- [c_1,c_2,…,c_n]=

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

- r_i(A)=[a_i1,a_i2,..,a_in]
- c_j(A)= $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Define Ax where A: mxn-matrix, x n-vector

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + & \ldots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \ldots & +a_{2n}x_n & = b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots & a_{mn}x_n & = b_m \end{array}$$

$$\begin{bmatrix} a_{11}x_1 + & a_{12}x_2 + & \ldots & +a_{1n}x_n \\ a_{21}x_1 + & a_{22}x_2 + & \ldots & +a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \ldots & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- We define Ax to be:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- Ax=[a_1,a_2,…,a_n]=x_1a_1+….+x_na_n where a_i are column vectors.

Thus, we can write a system of linear equations as Ax=b for b mx1 column vector.

- See examples top. Page 83 and Example 4.
- Linearity Property: Theorem 3.1.5:
  A mxn matrix, u,v column n-vector. Then
  - (a) A(cu) = c(Au), c a real number
  - (b) A(u+v)=Au+Av.
  - Equivalently A(cu+dv)=c(Au)+d(Av), c,d reals
- Proof: Simply use definitions and follow…

Product AB: Natural Definition

- A mxn-matrix, B nxr matrix. Define AB mxr-matrix so that $(AB)x=A(Bx)$ for any r-vector x.
- This is the associativity which is needed….
  - $B=[b_1,b_2,\ldots,b_r]$
  - $Bx=x_1b_1+x_2b_2+\ldots+x_rb_r$
  - $A(Bx) = A(x_1b_1+x_2b_2+\ldots+x_rb_r)$
    $=x_1Ab_1+x_2Ab_2+\ldots+x_rAb_r$
  - $(AB)x$ must equal $x_1(Ab_1)+x_2(Ab_2)+\ldots+x_r(Ab_r)$
- Definition $AB=[Ab_1,Ab_2,\ldots,Ab_r]$.
- See Example 5.

Specific entry of AB

- A $m \times s$ matrix, B $s \times n$ matrix -> AB $m \times n$ matrix
- $(AB)_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{is}b_{sj}$
  $= r_i(A)c_j(B)$ or $r_i(A).c_j(B)$ (second as vectors)
- Proof: $AB=[Ab_1,Ab_2,\ldots,Ab_n]$
  =

- Remove brackets to get a familiar formula

$$\begin{bmatrix} \begin{bmatrix} a_{11}b_{11}+ & a_{12}b_{21}+ & \cdots & +a_{1s}b_{s1} \\ a_{21}b_{11}+ & a_{22}b_{21}+ & \cdots & +a_{2s}b_{s1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11}+ & a_{m2}b_{21}+ & \cdots & a_{mn}b_{s1} \end{bmatrix}, \begin{bmatrix} a_{11}b_{12}+ & a_{12}b_{22}+ & \cdots & +a_{1s}b_{s2} \\ a_{21}b_{12}+ & a_{22}b_{22}+ & \cdots & +a_{2s}b_{s2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{12}+ & a_{m2}b_{22}+ & \cdots & a_{mn}b_{s2} \end{bmatrix}, \cdot\cdot \\ \cdot\cdot, \begin{bmatrix} a_{11}b_{1n}+ & a_{12}b_{2n}+ & \cdots & +a_{1s}b_{sn} \\ a_{21}b_{1n}+ & a_{22}b_{2n}+ & \cdots & +a_{2s}b_{sn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{1n}+ & a_{m2}b_{2n}+ & \cdots & a_{mn}b_{sn} \end{bmatrix} \end{bmatrix}$$

- Theorem, The ij entry of AB is the product of i-th row vector of A times the j-th colmn vector of B.
- See Example 7.

Finding rows and columns of AB.

- AB=[Ab_1,…,Ab_n]. Thus c_j(AB)=Ac_j(B). This is the column rule
  - Ax=x_1c_1(A)+…+x_sc_s(A) (A mxs x sx1)
- r_i(AB)= r_i(A)B. Row rule. (Use Dot. Product rule t see this.)

$$AB = \begin{bmatrix} r_1(A)B \\ r_2(A)B \\ \vdots \\ r_m(A)B \end{bmatrix}$$

  - yB=y_1r_1(B)+…+y_sr_s(B) (B sxn, y 1xs) (To see, this, yB=[yb_1,…,yb_n]=

$$\left[\begin{bmatrix} y_1b_{11}+ \\ y_2b_{21}+ \\ \vdots \\ y_sb_{s1} \end{bmatrix}, \begin{bmatrix} y_1b_{12}+ \\ y_2b_{22}+ \\ \vdots \\ y_sb_{s2} \end{bmatrix}, \ldots, \begin{bmatrix} y_1b_{1n}+ \\ y_2b_{2n}+ \\ \vdots \\ y_sb_{sn} \end{bmatrix}\right] = \begin{bmatrix} y_1b_{11}, y_1b_{12}, \ldots, y_1b_{1n} \\ +y_2b_{21}, +y_2b_{22}, \ldots, +y_2b_{2n} \\ \vdots \\ +y_sb_{s1}, +y_sb_{s2}, \ldots, +y_sb_{sn} \end{bmatrix} = \begin{bmatrix} y_1r_1(B) \\ +y_2r_2(B) \\ \vdots \\ +y_sr_s(B) \end{bmatrix}$$

- Theorem 3.1.8.
  - (a) the j-th column of AB is a linear combination of columns of A with coefficients from j-th column of B.
  - (b) the i-th row of AB is a linear combinations of rows of B with coefficients from the i-th row of A.

Transpose

- A mxn… A^T mxn rows become columns and vice versa.
- (A^T)_ij=(A)_ji.
- See Example 10.

Trace

- Given nxn-matrix A (1x1 also), tr(A)= sum of the diagonal entries A_11,A_22,…,A_nn.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & -1 & 4 \\ 1 & 1 & 1 \end{bmatrix}, trA = 1$$

- $trA = trA^T$.

Inner and outer matrix product

- Same size (nx1) column matrix u,v
- $u^T v$ is 1x1 matrix or a number: matrix inner product.
  - $u^T v = u.v = v.u = v^T u$
- uv^T is nxn matrix: matrix outer product
- See Example 11
- $tr(uv^T)=tr(vu^T)=u.v$.

Ex set 3.1.

- 1-14 recognition, computations.
- 16-22 computations, rules,,, traces

3.2. Algebraic properties

Properties, inverses

Addition, scalar multiplication rules

- Theorem 3.2.1. a,b scalars, A,B,C same size
  - (a) A+B=B+A commutativity
  - (b) A+(B+C)=(A+B)+C
  - (c) (ab)A=a(bA)
  - (d) (a+b)A=aA+bA
  - (f) a(A+B)=aA+aB
- Proof: simple calculations…

Multiplication rules

- AB is not necessarily equal BA. (not commutative) See Example 1.
- Some times AB=BA. Then A and B commute.
  - See Example *
- Theorem 3.2.2. a, A mxn B pxq C rxs
  - (a) A(BC)=(AB)C (n=p, q=r)
  - (b)A(B+C)=AB+AC. (n=p=r, q=s)
  - (c) (B+C)A=BA+CA. (q=s=m, p=r)
  - (d) A(B-C)=AB-AC (e) (B-C)A=BA-CA
  - (f) a(BC)=(aB)C=B(aC) (q=r)

- Proof (a) The rest is omitted.
  - Let c_j be the j-th column of C.
  - Question: What is j-th column of DC for some D?
  - j-th column of (AB)C is (AB)c_j.
  - j-th column of A(BC) is A(BC)_j. (BC)_j = Bc_j. Thus, A(Bc_j).
  - We showed A(Bx)=(AB)x for any vector x.
- Zero matrix O: all the entries are 0.
- A+O=O+A=A; must be of same size
- A-A=A+(-A)=O
- 0A=O
- OA=O (may be different size zero matrices)
- If cA=O, then either c=0 or A=O.
- See Example 3. AB=O but BA is not O.

Identity matrix.

- [1], $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- I_m denotes mxm identity matrix
- A mxn matrix AI_n= A, I_mA= A
- Note that I_m is already in reduced row echelon form.
- Conversely, a row reduced mxm-matrix either has zero rows or equals I_m (Theorem 3.2.4)

Inverses

- A nxn matrix. If B is nxn and satisfy $AB=BA=I_n$, then A is invertible and B is an inverse of A.
- See Example 4.
- A matrix may not have an inverse. (See Example 5)
  - When a row or a column of it is zero.
  - When two rows (or two columns) are the same…
  - But there are more than these…
  - We will figure out precisely when…

Properties of Inverse

- Theorem 3.2.6. A invertible. B,C inverses. Then B=C.
- Thus, we denote the inverse of A as A^{-1}.
- The inverse of 2x2-matrix is easy to obtain:

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad-bc\neq 0, A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- This is a computational result
- Example **, Example 8.

- $(AB)^{-1} = B^{-1}A^{-1}$, A,B nxn matrix
- Proof: $(AB)(B^{-1}A^{-1})=A(BB^{-1})A^{-1}=AA^{-1}=I$ and $(B^{-1}A^{-1})AB=B^{-1}(A^{-1}A)B=B^{-1}B=I$. Now use uniqueness

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Powers of matrix

- $A^0=I$, $A^n=AA\ldots A$ n-times
- $A^{-n}=(A^{-1})^n=A^{-1}A^{-1}\ldots A^{-n}$ n-times
- $A^{r+s}=A^rA^s$, $(A^r)^s=A^rA^r\ldots A^r = A^{rs}$

- Theorem 3.2.9. A invertible
  - (a) A^{-1} is invertible and $(A^{-1})^{-1}=A$.
  - (b) A^n is invertible and $(A^n)^{-1}=A^{-n}=(A^{-1})^n$.
  - (c) kA is invertible and $(kA)^{-1}=k^{-1}A^{-1}$

- Proof (b): $A^{-n}A^n=(A^{-1})^nA^n$. $A^nA^{-n}=A^n (A^{-1})^n =I$ by cancelation from the middle.
- Thus, this operation is very similar to taking powers in real numbers….

Matrix polynomials.

- $p(x)=a_0+a_1x+\ldots+a_mx^m$
- Matrix polynomial in A (nxn-matrix):
  - $p(A) = a_0I+ a_1A+\ldots+a_mA^m$.
- See Example 12.
- $p_1(A)p_2(A)=(p_1p_2)(A)=(p_2p_1)(A)$ $=p_2(A)p_1(A)$

Transpose again

Theorem 3.2.10 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

- Proof (e): A mxn, B nxs. AB mxs $(AB)^T$ sxm
  - B^TA^T sxm also.
  - ji-th entry of $(AB)^T$ is (AB)_ij = r_i(A)c_j(B)
  - = r_j(B^T)c_i(A^T) is the ji-th entry of B^TA^T.
  - We need to show
    r_i(A)c_j(B) = r_j(B^T)c_i(A^T)

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Properties of traces

Theorem 3.2.11 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem 3.2.12 *If A and B are square matrices with the same size, then:*

(*a*) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$

(*b*) $\operatorname{tr}(cA) = c\operatorname{tr}(A)$

(*c*) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

(*d*) $\operatorname{tr}(A - B) = \operatorname{tr}(A) - \operatorname{tr}(B)$

(*e*) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

- 3.2.12 (e) state tr(AB)=tr(BA) for square matrices. See Example.
- Proof: A nxm, B mxn AB nxn…

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} B_{ji} = \sum_{j=1}^{m} \sum_{i=1}^{n} B_{ji} A_{ij}$$

$$= \sum_{j=1}^{m} (BA)_{jj} = tr(BA)$$

- Product of row and column vector to be useful later

Theorem 3.2.13 *If* $\mathbf{r}$ *is a* $1 \times n$ *row vector and* $\mathbf{c}$ *is an* $n \times 1$ *column vector, then*

$$\mathbf{rc} = \operatorname{tr}(\mathbf{cr}) \qquad (11)$$

- Proof: Let $u=r^T$, $v=c$. $u^Tv=tr(uv^T)$. ((25) sec3.1)
  - Thus $rc=tr(r^Tc^T)=tr((cr)^T)=tr(cr)$

Transpose and dot product.

- $Au.v=u.A^Tv$ and $u.Av=A^Tu.v$
- Proof: Use $u.v= v^Tu$. –(*) Why true?
  - $Au.v=v^T(Au)=(v^TA)u=(Av)^Tu=u.A^Tv$.
  - $u.Av=(Av)^Tu=(v^TA^T)u=v^T(A^Tu)=A^Tu.v$
- In the dot product, A moves across the dot by transposing….

Ex. Set 3.2.

- 1-6 confirmation by direct computations for specific matrices
- 7, 8 find unknown
- 9-12 confirmation
- 13-18 computations
- 32-37 a bit harder