3.3. Elementary matrices

A method to find A^{-1}

Elementary matrices

- Elementary operations:
  - 1. Interchange two rows.
  - 2. Multiply a row by a nonzero constant
  - 3. Add a multiple of one row to another.
- Elementary matrix is a matrix that results from a single elementary row operation to I_n.

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 3.3.1 *If A is an $m \times n$ matrix, and if the elementary matrix E results by performing a certain row operation on the $m \times m$ identity matrix, then the product EA is the matrix that results when the same row operation is performed on A.*

- See Example 1.
- Each elementary operation has an elementary operation that reverses it.
  - 1) multiply a row by c (nonzero)
  - 1') multiply the same row by 1/c
  - 2) interchange row i and row j. 2')=2)
  - 3) Add c times a row i to row j.
  - 3') Add –c times a row i to row j.

- An elementary matrix E=e(I) for an operation e. Then E'=e'(I) is the inverse of E.
- Proof: EE'=e(E')=e(e'(I)) = I.
  E'E=e'(E)=e'(e(I))=I.

Theorem 3.3.2 *An elementary matrix is invertible, and the inverse is also an elementary matrix.*

Theorem 3.3.3 *If A is an $n \times n$ matrix, then the following statements are equivalent; that is, they are all true or all false.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

- Proof: (a)->(b)->(c)->(a)
- (a)->(b).
  - Ref of A is I_n. There is a sequence of elementary moves making A into I_n. Each operation is a multiplication by an elementary matrix.
  - E_kE_{k-1}…E_2E_1A=I_n.
  - A=E_1^{-1}E_2^{-1}…E_{k-1}$^{-1}$E_k^{-1}I_n

- (b)->(c)
  - If A is a product of elementary matrices and each elementary matrix is invertible. Thus, so is A.
- (c)->(a) Suppose A is invertible.
  - Let R be ref of A.
  - Then $E_k \ldots E_2 E_1 A = R$.
  - Then R is invertible. By Theorem 3.2.4, either R has zero rows or $R = I_n$. The former implies R is not invertible.
  - Thus $R = I_n$.

Row equivalence

- If a matrix B is obtained from A by applying a sequence of row operations. Then B is row equivalent to A. A≅B.
- A ≅B iff B≅A, A≅B, B≅C-> A≅C, A≅A.

A square matrix A is invertible if and only if it is row equivalent to the identity matrix of the same size.

Theorem 3.3.4 *If A and B are square matrices of the same size, then the following are equivalent:*

(a) *A and B are row equivalent.*

(b) *There is an invertible matrix E such that* $B = EA$.

(c) *There is an invertible matrix F such that* $A = FB$.

Inversion algorithm

The Inversion Algorithm *To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to I, and then perform the same sequence of operations on I to obtain A^{-1}.*

- Proof: E_k…E_2E_1A=I. E_k…E_2E_1I=A^{-1}
- Example 3.

Solving linear equations by matrix inversions

- If A is invertible, then Ax=b can be solved by x =A^{-1}b.

Theorem 3.3.5 *If $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n unknowns, and if the coefficient matrix A is invertible, then the system has a unique solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.*

Theorem 3.3.6 *If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system of n equations in n unknowns, then the system has only the trivial solution if and only if the coefficient matrix A is invertible.*

- Proof) <-) Ax=0. x=A^{-1}0=0.
- ->) Let A' be A augmented with 0 column.
- Then the ref for the augmented A' is I augmented with 0s since 0s are the only solutions.
- augmented. Thus ref of A=I.

Theorem 3.3.8

(*a*) *If A and B are square matrices such that $AB = I$ or $BA = I$, then A and B are both invertible, and each is the inverse of the other.*

(*b*) *If A and B are square matrices whose product AB is invertible, then A and B are invertible.*

- Proof: (a) Suppose AB=I.
  - Then show Bx=0 has a unique solution 0. Use Thm 3.3.6.
  - ABx=0-> Ix=0. Thus x=0. Done. B is invertible.
  - $ABB^{-1}=B^{-1}$. Thus $A=B^{-1}$ and A is invertible.
- (b) AB invertible. $I=(AB)(AB)^{-1}=A(B(AB)^{-1})$. $I=(AB)^{-1}(AB)=((AB)^{-1}A)B$.

Theorem 3.3.9 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector $\mathbf{b}$ in R^n.*

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector $\mathbf{b}$ in R^n.*

- Proof: (a)(b)(c)(d) equivalent.
- We show (c)(e)(f) equivalent. (f)->(e)->(c)->(f).
- (f)->(e). Has a solution -> consistent.
- (e)->(c). Ax=e_1,Ax=e_2,…,Ax=e_n are consistent.
  - Let c_1,c_2,…,c_n be the respective solutions.
  - Then A[c_1,c_2,…,c_n]=I_n. C=[c_1,c_2,…,c_n] is an nxn-matrix. By Theorem 3.3.8, A is invertible.
- (c)->(f) Theorem 3.3.5.

- Solving multiple linear system with common coefficients.
  - We can simplify by stacking bs.
  - See Example 7.
- Consistency of linear systems

3.3.10 The Consistency Problem For a given matrix A, find all vectors $\mathbf{b}$ for which the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

- Example 8.

Ex. Set. 3.3.

- 1-6 Recognizing elementary matrices, finding inverse. 7,8.
- 9-12 Find inverse
- 13-14,15-22 Inverse finding
- 29-34 Find consistency conditions.
- D5-D7… Interesting theoretical sides…