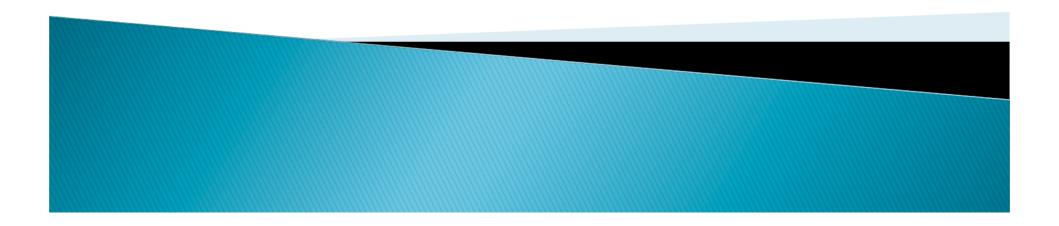
# 3.4. Subspaces, Linear independence



## Subspace

A subspace is a set one can do scalar multiplication and addition and not leave the set.

**Definition 3.4.1** A nonempty set of vectors in  $\mathbb{R}^n$  is called a *subspace* of  $\mathbb{R}^n$  if it is closed under scalar multiplication and addition.

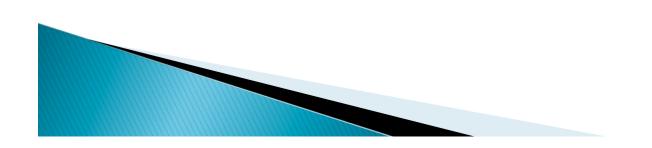
 A subspace is usually given by conditions.
 We need to verify the conditions after scalar multiplications or additions.



- {O} is a susbspace
- Every subspace contains O. Why?
- W ={(x,y) in  $R^2|x>0$ , y>0} is not a subspace. Why?
- ▶ W={(x,y,0) in R<sup>3</sup>} is a subspace.
- W in R<sup>n</sup> given by x\_2=1,x\_3=-1 a subspace?
- ▶ Let v\_1, v\_2,...,v\_s is given in R<sup>n</sup>.
  - $\circ$  Let W={c\_1v\_1+c\_2v\_2+...+c\_sv\_s| c\_i in R}.
  - That is W is the set of all linear combinations of given vectors v\_1, v\_2,..., v\_s.

 $\circ$  Then W is a subspace.

We write W=span{v\_1,v\_2,...,v\_s}



**Theorem 3.4.2** If  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$  are vectors in  $\mathbb{R}^n$ , then the set of all linear combinations

(3)

 $\mathbf{x} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \dots + t_s \mathbf{v}_s$ 

is a subspace of  $\mathbb{R}^n$ .

- Example 2: Span{O}={O}.
- Example 3: Span{(1,1,2,0)} is a line.
- Example 4.
  - $\circ$  A subspace in R<sup>1</sup>: itself or {O}.
  - $\circ$  A subspace in R<sup>2</sup>: itself, a line through O, {O}.
  - $\circ$  A subspace in R³: itself, a plane through O
    - (Ax+By+C=O), a line through O, {O}
  - $\circ$  A subspace in R<sup>n</sup>: itself, a subspace ≈R<sup>i</sup>, {O}.



## Solution space of a linear system

**Theorem 3.4.3** If  $A\mathbf{x} = \mathbf{0}$  is a homogeneous linear system with *n* unknowns, then its solution set is a subspace of  $\mathbb{R}^n$ .

### • Proof: W = $\{x|Ax=0\}$ .

- $\circ$  If x\_0 is a solution, then kx\_0 is a solution.
- $\circ$  If x\_1 and x\_2 are solutions, then x\_1+x\_2 is a solution.
- Thus W is closed under scalar multiplications and additions.
  Thus W is a subspace.
- If one has an inhomogeous system, then the solution space is not a subspace.
- See Example \*.

#### Theorem 3.4.4

- (a) If A is a matrix with n columns, then the solution space of the homogeneous system  $A\mathbf{x} = \mathbf{0}$  is all of  $R^n$  if and only if A = 0.
- (b) If A and B are matrices with n columns, then A = B if and only if  $A\mathbf{x} = B\mathbf{x}$  for every  $\mathbf{x}$  in  $\mathbb{R}^n$ .
  - Philosophy: A is determined by Ax's.
  - Proof:
    - (a) ->) A=0. Ax=0.
    - o <-) Ax=0 for all x. Ae\_1=0, Ae\_2=0,...,Ae\_n=0.</p>
      - A=AI=A[e\_1,e\_2,..,e\_n]=[Ae\_1,Ae\_2,...,Ae\_n]=O.
      - Thus all columns of A are zero.
    - (b) Ax=Bx for all x. Ax-Bx=O. (A-B)x=O for all x. A-B=O.
      A=B.



## Linear independence

- How can we find a good way to describe a subspaces...
  - $\circ$  Find equations... See as solutions spaces
  - Find parameters... Write a vector as a linear combination of vectors in unique way for a fixed set of vectors. These should be the least in number.
  - So we want to avoid "linearly dependent set of vectors": when some of the vectors in the set can be written as a linear combination of some others.
  - In such cases, the number can be reduced by eliminating these.



**Definition 3.4.5** A nonempty set of vectors  $S = {\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s}$  in  $\mathbb{R}^n$  is said to be *linearly independent* if the only scalars  $c_1, c_2, \dots, c_s$  that satisfy the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_s\mathbf{v}_s = \mathbf{0} \tag{9}$$

are  $c_1 = 0, c_2 = 0, ..., c_s = 0$ . If there are scalars, not all zero, that satisfy this equation, then the set is said to be *linearly dependent*.

- ▶ {O} is linearly dependent. cO=O for all c.
- {v} v nonzero is linearly independent. cv=0 iff c=0.



**Theorem 3.4.6** A set  $S = {\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_s}$  in  $\mathbb{R}^n$  with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

- ▶ Example 10. two vectors in R<sup>n</sup>.
- Example 11. three vectors in R<sup>n</sup> is dependent if one is a linear combination of the other two.
  - Thus, the three vectors lie in a common plane or a common plane or {O}.
  - Three vectors are linearly independent if there are no such planes, lines.

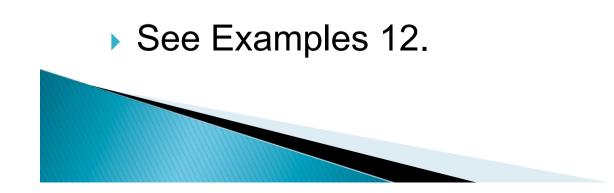


## Linear independence and homogeneous linear systems

- Given v\_1,v\_2,...,v\_s, write A=[v\_1,v\_2,...,v\_s].
- We write c\_1v\_1+c\_2v\_2+...+c\_sv\_s=0 as

$$\begin{bmatrix} v_1, v_2, \dots, v_s \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

**Theorem 3.4.7** A homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution if and only if the column vectors of A are linearly independent.



**Theorem 3.4.8** A set with more than n vectors in  $\mathbb{R}^n$  is linearly dependent.

**Theorem 3.4.9** If A is an  $n \times n$  matrix, then the following statements are equivalent.

- (a) The reduced row echelon form of A is  $I_n$ .
- (b) A is expressible as a product of elementary matrices.
- (c) A is invertible.
- (d)  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- (e)  $A\mathbf{x} = \mathbf{b}$  is consistent for every vector  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- (f)  $A\mathbf{x} = \mathbf{b}$  has exactly one solution for every vector  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- (g) The column vectors of A are linearly independent.
- (*h*) The row vectors of A are linearly independent.

Proof: (d)(g) equivalent by Th.3.4.7. (g)->(h): (g)->(c).  $A^{T}$  is invertible. Use (g) for  $A^{T}$ . (h) follows (h)->(g): (g) for  $A^{T}$  holds.  $A^{T}$  is invertible. -> A is invertible -> (g).

## Ex. Set. 3.4.

- 1-8 Span problem
- 9,10 independence
- 13-16 span problem
- 17-22 linear independence
- > 23-26 Subspaces

