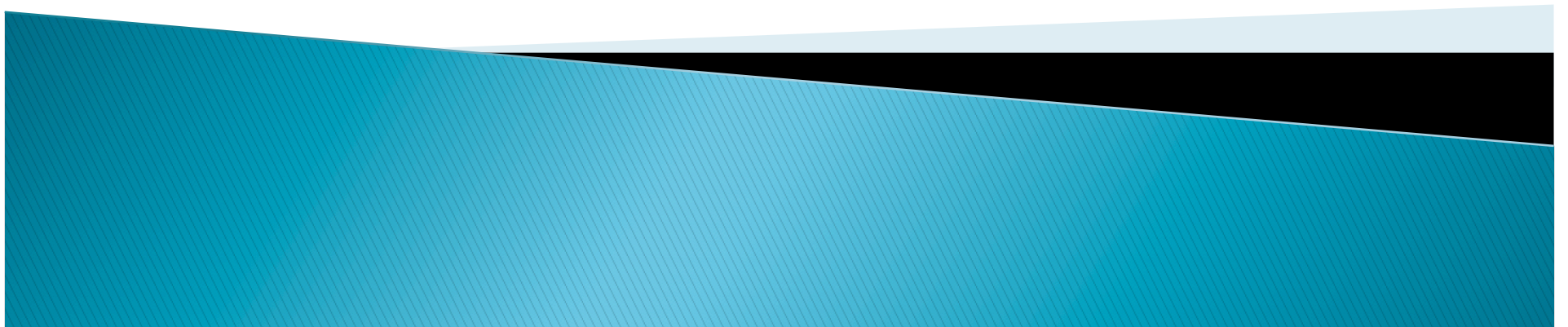


Introduction to Linear algebra

Summer 2008

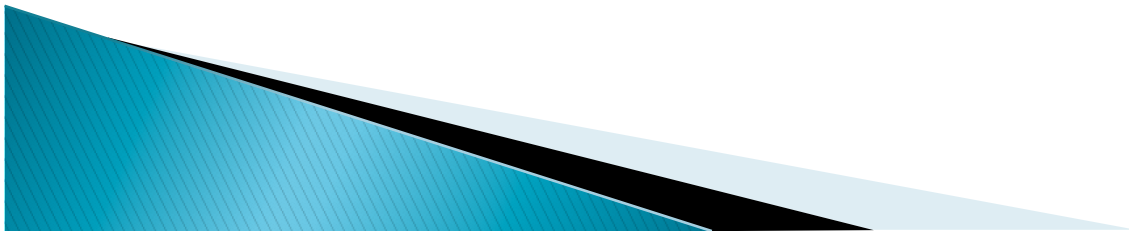


Outline of the course

- ▶ The informations are all in math.kaist.ac.kr/~schoi/sumlin2008.htm.
- ▶ Basic purpose is for you to understand how to compute in linear algebra.
- ▶ One should progress acquiring many abstract notions through concrete computations.
- ▶ We can also see many applications.



- ▶ Ch 1: vectors
- Ch 2: System of linear equations.
- ▶ Ch 3: Matrices
 - Operations on matrices
 - Find inverses
 - Factorizations LU-decompositions
- ▶ Ch4: Determinants
 - Cofactor expansion- how to compute
 - Properties – how to use
 - Cramer's rule



- ▶ Ch 5.3: Gauss–Seidel and Jacobi iterations.
(mid term)
- ▶ Ch 6: Linear transformations (abstract)
 - Matrices as linear transformations
 - Geometry
 - Kernel, range
 - Composition, invertibility
- ▶ Ch 7: Dimension and structure (abstract)
 - Basis and dimension
 - Dimension theorem, rank
 - Best approximation, QR–decomposition



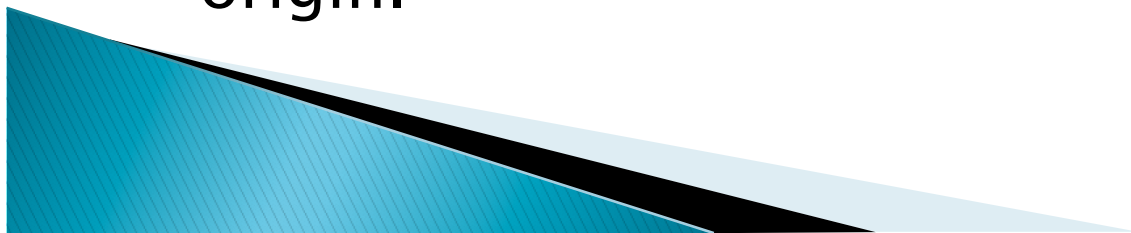
▶ Ch 8: Diagonalization

- Matrix representation of linear transformations
- Similarity, diagonalizability
- Orthogonal diagonalizability
- Quadratic forms
- Singular value decompositions
- The pseudo-inverse



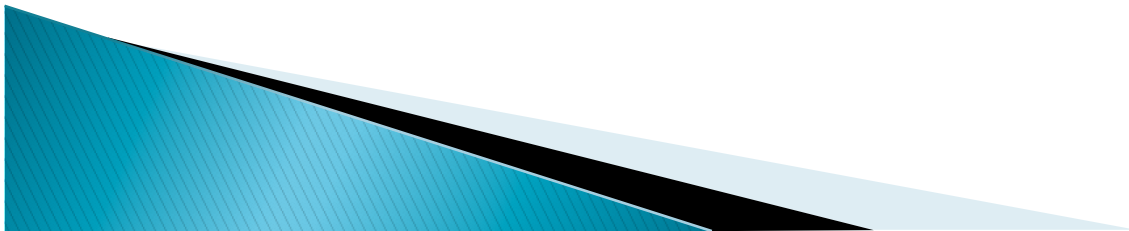
Vectors

- ▶ Vectors are abstract notions: standing for direction and the size in the Euclidean space:
- ▶ A vector have an initial point and the terminal point.
- ▶ displacement of position, velocity(speed +direction) =change of displacement per unit time,
forces vector (amount of force+direction)= change of velocity per unit time.
- ▶ Free vectors: like forces without origin
- ▶ Bound vectors: like displacement with a given origin.

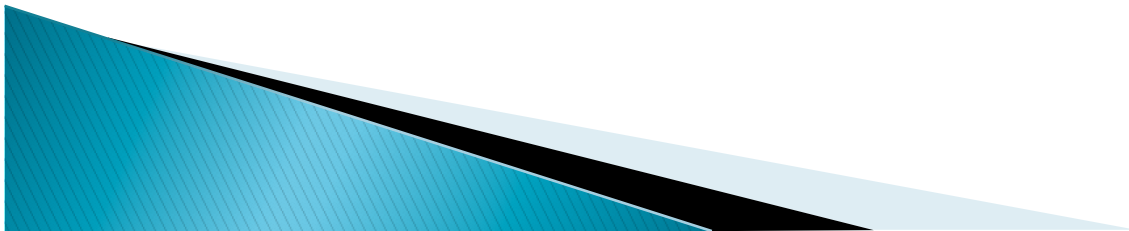


Vector additions

- ▶ Parallelogram rule: position the initial points of two vectors at a given point. Then form the two vectors to be sides of a parallelogram. Take the diagonal vector.
- ▶ Triangle rule: The second vector is at the final point of a first vector. Take the displacement of the terminal point of the second vector from the initial point of the first vector.

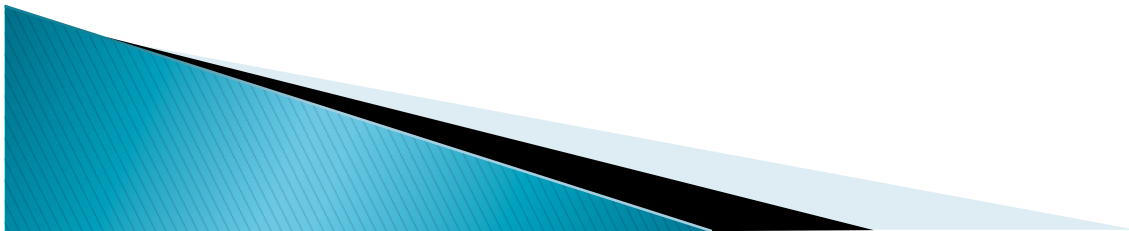


- ▶ Vector addition viewed as translations or as displacements.
- ▶ Example:
 - displacement: Daejeon is at northwest of Pusan by 300 km. Seoul is at north of Daejeon by 200km. Seoul is 500km from Pusan in north north west direction.
 - Velocity: A ship going in 5 km per hour to east (as seen by a shipmate) meeting a southerly wind of 5 km per hour.
 - Force: An Egyptian slave pulling a cart driven by an ox.



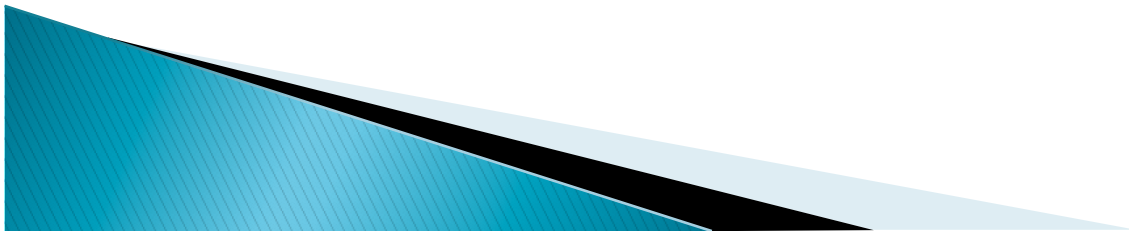
Scalar multiplications

- ▶ $-V$ is the vector in the opposite direction to V of the same length.
- ▶ $V - W = V + (-W)$.
- ▶ Scalar multiplications:
 - A real number k (called scalar).
 - For $k > 0$, kV is the vector in the same direction of the length k times that of V .
 - For $k < 0$, kV is the vector in the opposite direction of length $-k$ times that of V .
 - $-V = (-1)V$.

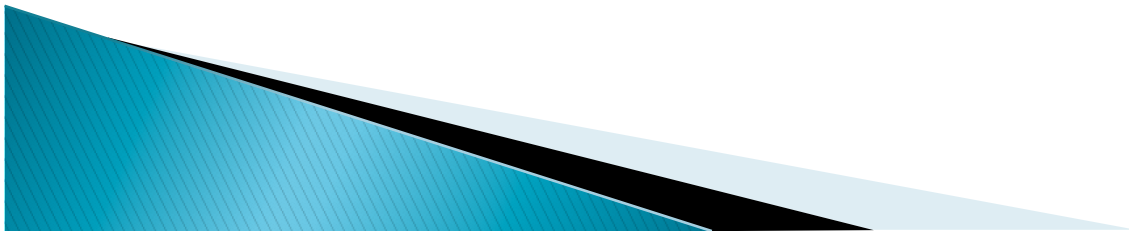


Vectors in coordinates

- ▶ The notion of vectors exists without coordinates. But the computations of vector addition is hard.
- ▶ A coordinate system on (Euclidean) 2-space is a two perpendicular axes: i.e. a line with directions. The point is given a coordinate by perpendicularly projecting to the axis.
- ▶ This introduced by Descartes, a revolutionary idea at the time. Turning geometry into algebra. Hence the name Cartesian plan. This was made use by Newton in solving the tangent problem...

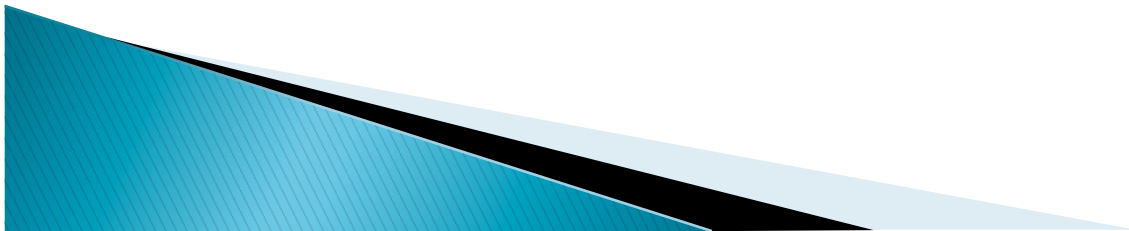


- ▶ A vector in 2-space is described by two ordered number (a,b) .
 - a is obtained by vertical projection to the x -axis.
 - b is obtained by horizontal projection to the y -axis.
- ▶ A point $P \leftrightarrow (a,b)$
- ▶ $P(a,b)$ $O(0,0)$
- ▶ x -axis $(x,0)$, y -axis $(0,y)$

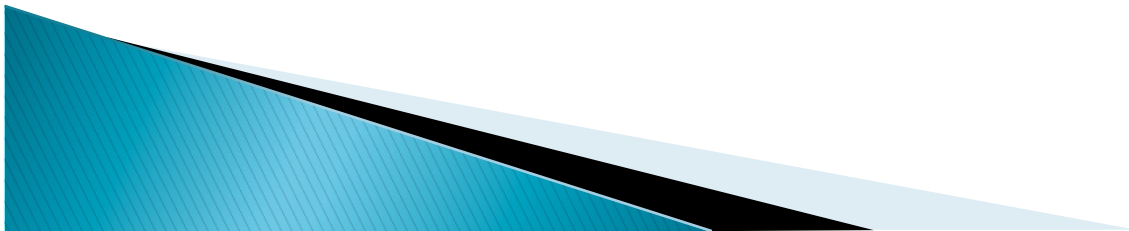


A rectilinear coordinate system in 3-space

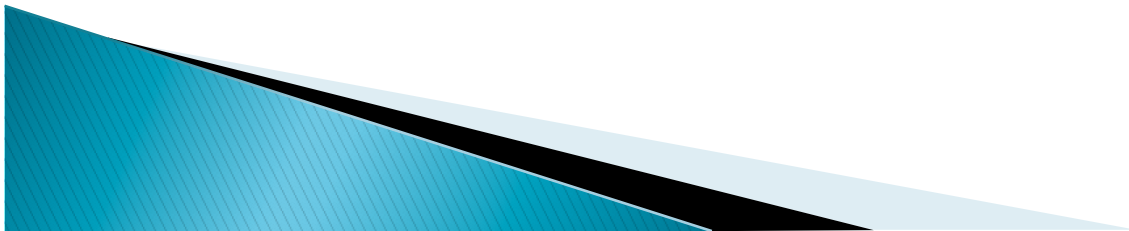
- ▶ In 3-dimensional Euclidean space, we have mutually perpendicular 3-axis: x-axis, y-axis, z-axis.
- ▶ The three axis meet at the origin O.
- ▶ The right handed system, z-axis: head, x-axis: the right arm, y-axis: the left arm.
- ▶ The left handed system, z-axis: head, x-axis: the left arm, y-axis: the right arm.



- ▶ Or you can use the right hand. Z-axis: the thumb, finger start: x-axis, finger end: y-axis.
- ▶ A point $P \leftrightarrow (a,b,c)$, $P(a,b,c)$
- ▶ a is obtained by the projection to the x-axis, b is obtained by the projection to the y-axis, and c by the projection to the z-axis.



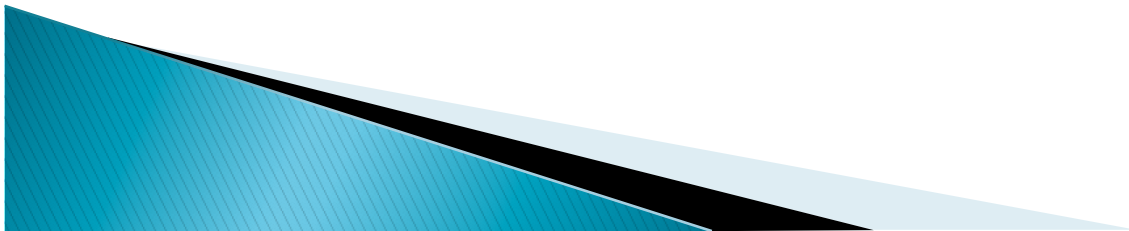
- ▶ Now, vector additions, scalar multiplications are easy:
 - $(a,b) + (a', b') = (a+a', b+b')$.
 - $k(a,b) = (ka, kb)$.
 - $(a,b,c) + (a',b',c') = (a+a', b+b', c+c')$
 - $k(a,b,c) = (ka, kb, kc)$.
- ▶ These are verified by addition rules (triangle rules in particular.) Such a verification process can be a hard one for one to grasp.



- ▶ The displacement between P,Q is

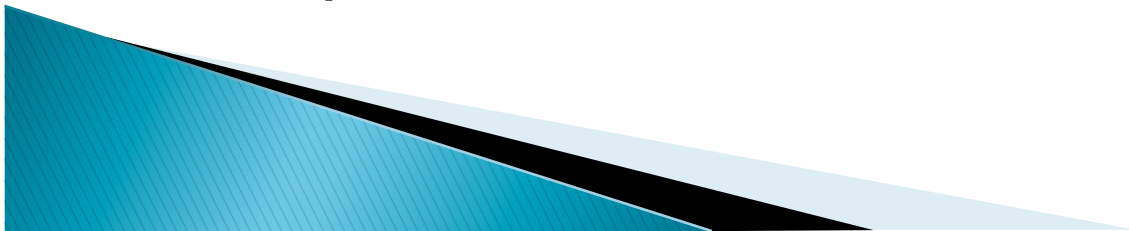
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

- ▶ In coordinates $(x,y)-(x',y')=(x-x',y-y')$
and $(x,y,z) - (x',y',z')=(x-x',y-y',z-z')$
- ▶ Lesson: Any vector operations can be done better in coordinate-wise addition, multiplications.



Higher-dimensional spaces

- ▶ We saw that 2-space can be coordinatized to be a set of pairs or ordered sets of two real numbers.
- ▶ A 3-space correspond to a set of triples of real numbers.
- ▶ We define the n -space as something that can be coordinatized by a set of n -tuples of real numbers, i.e., an ordered sequece (x_1, x_2, \dots, x_n) .
- ▶ The set is denoted by R^n . This is said to be an n -space.



- ▶ R^1 , R^2 , R^3 , visible spaces
- ▶ R^4 , R^5 , Higher-dimensional spaces.
- ▶ Actually, higher-dimensional spaces are useful.
- ▶ Graphics (x,y,h,s,b) (x,y) coordinates, h hue, s saturation, b brightness
- ▶ 4-dimensional space can be drawn in 3-space by coloring differently.
- ▶ Economic analysis: economic indicators (GDP, KOSPI, KOSDAK, Export, Import, retail, inflation rate, oil price). All these coordinates are relevant to economic analysis to predict something else.



n-vector addition and scalar multiplications

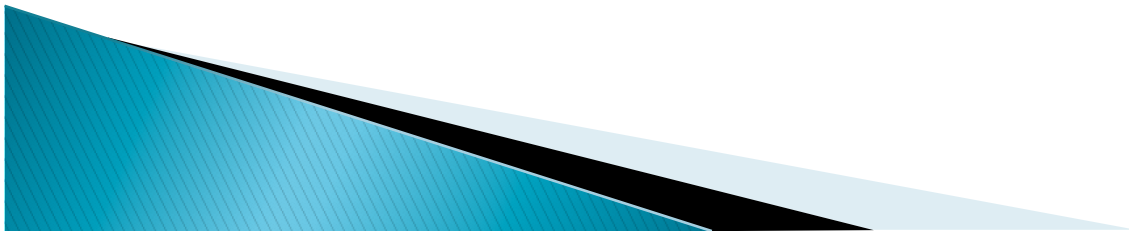
▶ Definition:

- $v+w = (v_1+w_1, v_2+w_2, \dots, v_n+w_n)$
- $kv = (kv_1, kv_2, \dots, kv_n)$
- $-v = (-v_1, -v_2, \dots, -v_n)$
- $w-v = w+(-v) = (v_1-w_1, v_2-w_2, \dots, v_n-w_n)$

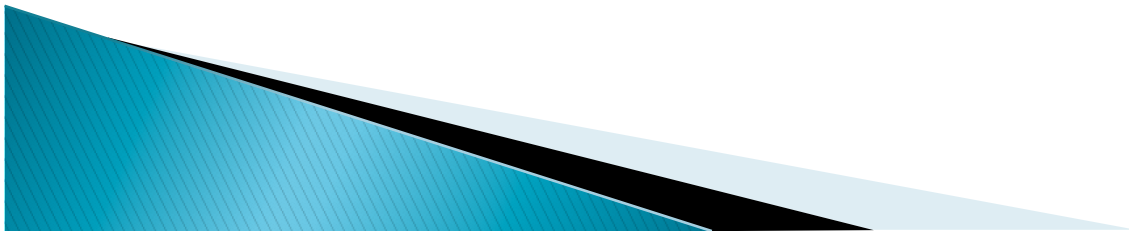
▶ Theorem: Laws

- $u+v=v+u$, $(u+v)+w=u+(v+w)$, $u+0=0+u=u$,
- $u+(-u)=0$, $(k+l)u=ku+lu$, $k(u+v)=ku+kv$,
- $k(lu)=(kl)u=l(ku)$, $1u=u$, $0v=0$, $k0=0$, $(-1)v=-v$.

▶ This needs to be verified.



- ▶ Two vectors are *parallel or collinear* if one vector is a scalar multiple of the other vector.
- ▶ Two vectors are in the *same direction* if one is a positive scalar multiple of the other vector.
 - $(5,5,10), (1,1,2)$
- ▶ Two vectors are in the opposite direction if one is a negative scalar multiple of the other vector.
 - $(2,2,2), (-1,-1,-1)$



Linear combinations

- ▶ A vector w in \mathbb{R}^n is a linear combination of the vectors v_1, v_2, \dots, v_n if
 - $w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$
 - c_1, c_2, \dots, c_n are coefficients (may not be unique).
- ▶ $(1, 1, 2, 1) = 1(1, 0, 0, 0) + 1(0, 1, 0, 0) + 2(0, 0, 1, 0) + 1(0, 0, 0, 1)$
- ▶ $(3, 4) = 2(1, 1) + 2(0, 1) + 1(1, 0) = (1, 1) + 3(0, 1) + 2(1, 0)$



RGB color model

- ▶ $r=(1,0,0)$ red, $b=(0,1,0)$ blue, $g=(0,0,1)$ green.
- ▶ Each point of the screen has three points to be lit by an electron.
- ▶ The RGB-space is all the linear combinations of all these vectors. That is, some of these are lit at the same time. (RGB-color cube)
- ▶ $c=cr+db+eg$, c,d,b in $\{0,1\}$ or in $[0,1]$.
- ▶ These can create most colors.
- ▶ See Figure 1.1.19.



matrix notation for vectors

- ▶ (x_1, x_2, \dots, x_n) as a column vector
 - i.e., $n \times 1$ -vector
 - This is more standard

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

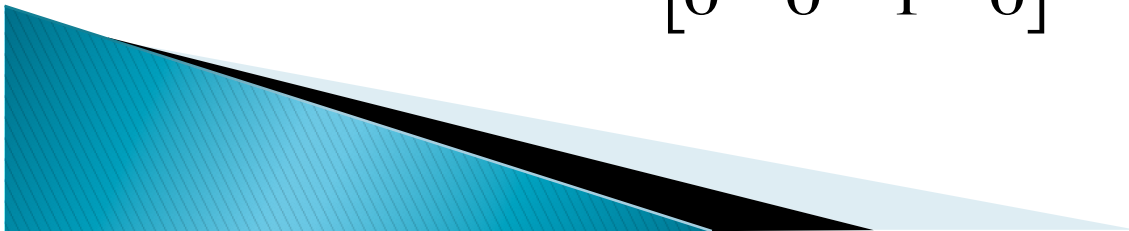
- ▶ Sometimes as a row vector
 $[x_1, x_2, \dots, x_n]$, i.e., $1 \times n$ -vector



matrices

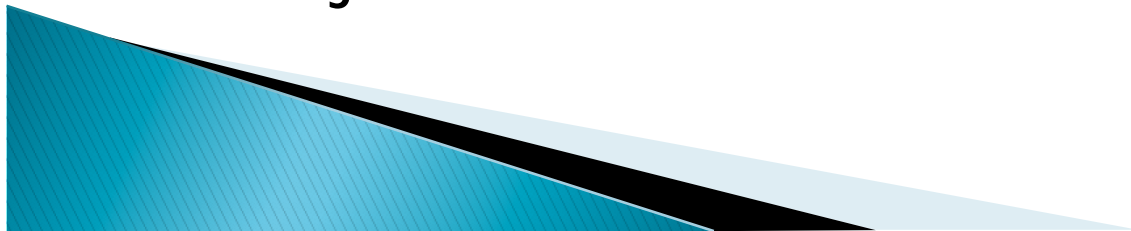
- ▶ A matrix has m-rows and n-columns.
- ▶ Each position is meaningful.
- ▶ A *row vector* is a one row → make it into a 1xn-matrix.
- ▶ A *column vector* is a one column → make it into a nx1-matrix.

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



matrices and graphs

- ▶ A graph is a set of vertices and segments connecting two.
- ▶ Directed graph is a graph with directions on segments.
- ▶ A segment may have two directions: two-way connection. Otherwise it is a one-way connection.
- ▶ An *adjacency matrix* is given by letting (i,j) -position be 1 if there is an edge directed from i to j .



- ▶ There can be no 2s...
- ▶ See Figure 1.1.20.
- ▶ Conversely, given a matrix with 0 and 1s only, we can find a graph. (perhaps not on euclidean plane.)

