Introduction to Linear algebra

Summer 2008

Outline of the course

- The informations are all in math.kaist.ac.kr/~schoi/sumlin2008.htm.
- Basic purpose is for you to understand how to compute in linear algebra.
- One should progress aquiring many abstract notions through concrete computations.
- We can also see many applications.

- Ch 1: vectors
- Ch 2: System of linear equations.
- Ch 3: Matrices
  - Operations on matrices
  - Find inverses
  - Factorizations LU-decompositions
- Ch4: Determinants
  - Cofactor expansion- how to compute
  - Properties – how to use
  - Cramer’s rule

- Ch 5.3: Gauss-Seidel and Jacobi iterations. (mid term)
- Ch 6: Linear transformations (abstract)
  - Matrices as linear transformations
  - Geometry
  - Kernel, range
  - Composition, invertibility
- Ch 7: Dimension and structure (abstract)
  - Basis and dimension
  - Dimension theorem, rank
  - Best approximation, QR-decomposition

- Ch 8: Diagonalization
  - Matrix representation of linear transformations
  - Similarity, diagonalizability
  - Orthogonal diagonalizability
  - Quadratic forms
  - Singular value decompositions
  - The pseudo-inverse

Vectors

- Vectors are abstract notions: standing for direction and the size in the Euclidean space:
- A vector have an initial point and the terminal point.
- displacement of position, velocity(speed +direction) =change of displacement per unit time,
  forces vector (amount of force+direction)= change of velocity per unit time.
- Free vectors: like forces without origin
- Bound vectors: like displacement with a given origin.

Vector additions

- Parallelogram rule: position the initial points of two vectors at a given point. Then form the two vectors to be sides of a parallelogram. Take the diagonal vector.
- Triangle rule: The second vector is at the final point of a first vector. Take the displacement of the terminal point of the second vector from the initial point of the first vector.

- Vector addition viewed as translations or as displacements.
- Example:
  - displacement: Daejeon is at northwest of Pusan by 300 km. Seoul is at north of Daejeon by 200km. Seoul is 500km from Pusan in north north west direction.
  - Velocity: A ship going in 5 km per hour to east (as seen by a shipmate) meeting a southerly wind of 5 km per hour.
  - Force: An Egyptian slave pulling a cart driven by an ox.

Scalar multiplications

- $-V$ is the vector in the opposite direction to V of the same length.
- $V - W = V + (-W)$.
- Scalar multiplications:
  - A real number k (called scalar).
  - For $k > 0$, kV is the vector in the same direction of the length k times that of V.
  - For $k < 0$, kV is the vector in the opposite direction of length $-k$ times that of V.
  - $-V = (-1)V$.

Vectors in coordinates

- The notion of vectors exists without coordinates. But the computations of vector addition is hard.
- A coordinate system on (Euclidean) 2-space is a two perpendicular axes: i.e. a line with directions. The point is given a coordinate by perpendicularly projecting to the axis.
- This introduced by Descartes, a revolutionary idea at the time. Turning geometry into algebra. Hence the name Cartesian plan. This was made use by Newton in solving the tangent problem…

- A vector in 2-space is described by two ordered number (a,b).
  - a is obtained by vertical projection to the x-axis.
  - b is obtained by horizontal projection to the y-axis.
- A point P <-> (a,b)
- P(a,b) O(0,0)
- x-axis (x,0), y-axis (0,y)

A rectilinear coordinate system in 3-space

- In 3-dimensional Euclidean space, we have mutually perpendicular 3-axis: x-axis, y-axis, z-axis.
- The three axis meet at the origin O.
- The right handed system, z-axis: head, x-axis: the right arm, y-axis: the left arm.
- The left handed system, z-axis: head, x-axis: the left arm, y-axis: the right arm.

- Or you can use the right hand. Z-axis: the thumb, finger start: x-axis, finger end: y-axis.
- A point P <-> (a,b,c), P(a,b,c)
- a is obtained by the projection to the x-axis, b is obtained by the projection to the y-axis, and c by the projection to the z-axis.

- Now, vector additions, scalar multiplications are easy:
  - $(a,b) + (a', b') = (a+a', b+b')$.
  - $k(a,b) = (ka, kb)$.
  - $(a,b,c) + (a',b',c') = (a+a,b+b',c+c')$
  - $k(a,b,c) = (ka, kb, kc)$.
- These are verified by addition rules (triangle rules in particular.) Such a verification process can be a hard one for one to grasp.

- The displacement between P,Q is

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

- In coordinates (x,y)-(x',y')=(x-x',y-y') and (x,y,z) - (x',y',z')=(x-x',y-y',z-z')

- Lesson: Any vector operations can be done better in coordinate-wise addition, multiplications.

Higher-dimensional spaces

- We saw that 2-space can be coordinatized to be a set of pairs or ordered sets of two real numbers.
- A 3-space correspond to a set of triples of real numbers.
- We define the n-space as something that can be coordinatized by a set of n-tuples of real numbers, i.e., an ordered sequece $(x_1, x_2, \ldots, x_n)$.
- The set is denoted by R^n. This is said to be an n-space.

- R^1, R^2, R^3, visible spaces
- R^4, R^5, …. Higher-dimensional spaces.
- Actually, higher-dimensional spaces are useful.
- Graphics (x,y,h,s,b) (x,y) coordinates, h hue, s saturation, b brightness
- 4-dimensional space can be drawn in 3-space by coloring differently.
- Economic analysis: economic indicators (GDP, KOSPI, KOSDAK, Export, Import, retail, inflation rate, oil price). All these coordinates are relevant to economic analysis to predict something else.

n-vector addition and scalar multiplications

- Definition:
  - $v+w = (v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$
  - $kv = (kv_1, kv_2, \ldots, kv_n)$
  - $-v = (-v_1, k-v_2, \ldots, -v_n)$
  - $w-v = w+(-v) = (v_1-w_1, v_2-w_2, \ldots, v_n-w_n)$
- Theorem: Laws
  - $u+v=v+u$, $(u+v)+w=u+(v+w)$, $u+0=0+u=u$,
  - $u+(-u)=0$, $(k+l)u= ku+lu$, $k(u+v)= ku+kv$,
  - $k(lu)=(kl)u=l(ku)$, $1u=u$, $0v=O$, $kO=O$, $(-1)v=-v$.
- This needs to be verified.

- Two vectors are *parallel or collinear* if one vector is a scalar multiple of the other vector.
- Two vectors are in the *same direction* if one is a positive scalar multiple of the other vector.
  - (5,5,10), (1,1,2)
- Two vectors are in the opposite direction if one is a negative scalar multiple of the other vector.
  - (2,2,2), (−1,−1,−1)

Linear combinations

- A vector w in R^n is a linear combination of the vectors v_1, v_2, …,v_n if
  - $w = c_1 v_1 + c_2 v_2 + \ldots c_n v_n$
  - $c_1, c_2, \ldots c_n$ are coefficients (may not be unique).
- (1,1,2,1)=1(1,0,0,0)+1(0,1,0,0)+2(0,0,1,0)+ 1(0,0,0,1)
- (3,4) = 2(1,1)+2(0,1)+1(1,0) = (1,1)+3(0,1)+2(1,0)

RGB color model

- r=(1,0,0) red, b=(0,1,0) blue, g=(0,0,1) green.
- Each point of the screen has three points to be lit by an electron.
- The RGB-space is all the linear combinations of all these vectors. That is, some of these are lit at the same time. (RGB-color cube)
- c=cr+db+eg, c,d,b in {0,1} or in [0,1].
- These can create most colors.
- See Figure 1.1.19.

matrix notation for vectors

- $(x_1, x_2, \ldots, x_n)$ as a column vector
  - i.e., nx1-vector
  - This is more standard

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Sometimes as a row vector

  $[x_1, x_2, \ldots, x_n]$, i.e., 1xn-vector

matrices

- A matrix has m-rows and n-columns.
- Each position is meaningful.
- *A row vector* is a one row -> make it into a 1xn-matrix.
- *A column vector* is a one column -> make it into a nx1-matrix.

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

matrices and graphs

- A graph is a set of vertices and segments connecting two.
- Directed graph is a graph with directions on segments.
- A segment may have two directions: two-way connection. Otherwise it is a one-way connection.
- An *adjacency matrix* is given by letting (i,j)-position be 1 if there is an edge directed from i to j.

- There can be no 2s…
- See Figure 1.1.20.
- Conversely, given a matrix with 0 and 1s only, we can find a graph. (perhaps not on euclidean plane.)