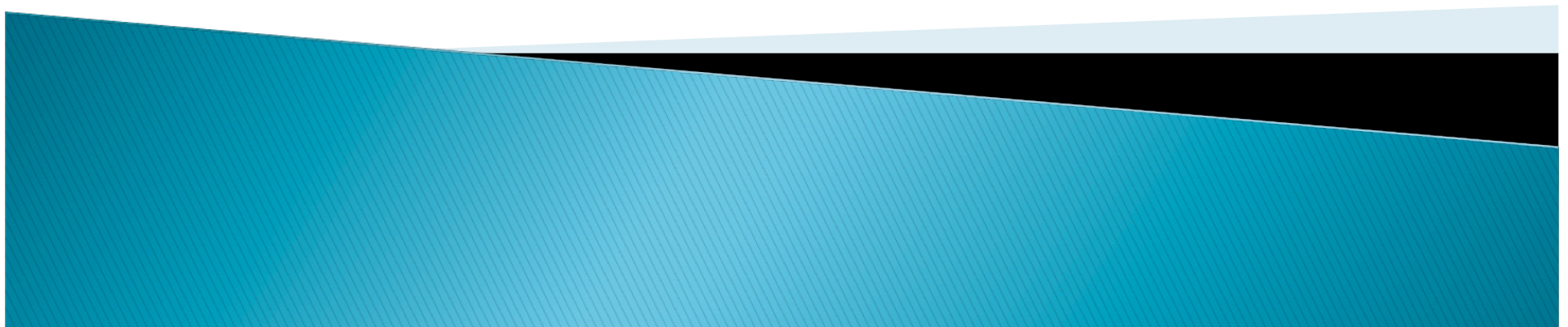


# 1.2.Dot Products, Orthogonality

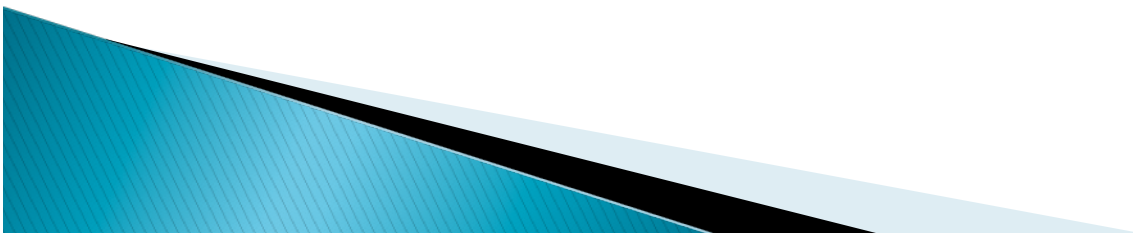


# lengths

- ▶ Length, norm, magnitude of a vector  $v=(v_1, \dots, v_n)$  is  $\|v\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$ .
- ▶ Examples  $v=(1, 1, \dots, 1)$   $\|v\|=n^{1/2}$ .
- ▶ Unit vectors  $u=v/\|v\|$  corresponds to directions.
- ▶ Standard unit vectors
  - $i=(1,0)$ ,  $j=(0,1)$  in  $R^2$
  - $i=(1,0,0)$ ,  $j=(0,1,0)$ ,  $k=(0,0,1)$  in  $R^3$
  - $e_1=(1,0,\dots,0)$ ,  $e_2=(0,1,\dots,0)$ ,  $\dots$ ,  $e_n=(0,0,\dots,1)$  in  $R^n$ .



- ▶  $v=(v_1,v_2, \dots,v_n)= v_1e_1 + v_2e_2 + \dots +v_n e_n$ .
  - This is a unique expression.
- ▶ Distances when given position vectors:
- ▶  $D(P_1,P_2) = \|P_1P_2\|$   
 $=((x_2-x_1)^2+(y_2-y_1)^2)^{1/2}$  in 2-space.
  
- ▶ In 3-space  
 $d(P_1,P_2) = \|P_2-P_1\|$   
 $=((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2)^{1/2}$  in 3-space.
  
- ▶ In n-space  $u=(u_1,u_2,\dots,u_n)$ ,  $v=(v_1,v_2,\dots,v_n)$ ,  
 Then  $d(u,v) = ((u_1-v_1)^2+(u_2-v_2)^2+\dots+(u_n-v_n)^2)^{1/2}$ .



- ▶ Theorem. Two position vectors  $u, v$  in  $\mathbb{R}^n$ .
  - $d(u, v) \geq 0$ ,  $d(u, v) = d(v, u)$ ,  $d(u, v) = 0$  if and only if  $u = v$ .
- ▶ Proof: Use the formula.
  
- ▶ We now introduce dot product. Given two vectors, the dot product gives you a real number.
- ▶ The dot product generalizes length and angle and is useful to compute many quantities. In fact, it is more fundamental than angles. Given  $u = (u_1, u_2, \dots, u_n)$ ,  $v = (v_1, v_2, \dots, v_n)$ ,  
 $u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$ .



# Properties of dot products

- ▶  $\|v\| = (v \cdot v)^{1/2}$ .
- ▶ Theorem 1.2.7
  - $u \cdot v = v \cdot u$ , Symmetry
  - $u \cdot (v+w) = u \cdot v + u \cdot w$ . distributivity
  - $k(u \cdot v) = (ku) \cdot v$  homogenous
  - $v \cdot v \geq 0$ , and  $v \cdot v = 0$  if and only if  $v=0$ . positivity
- ▶ Theorem 1.2.8.
  - $0 \cdot v = v \cdot 0 = 0$
  - $(u+v) \cdot w = u \cdot w + v \cdot w$
  - $u \cdot (v-w) = u \cdot v - u \cdot w$ ,  $(u-v) \cdot w = u \cdot w - v \cdot w$
  - $k(u \cdot v) = u \cdot (kv)$



- ▶ Theorems 1.2.6, 1.2.7 gives us a means to compute as one does with real numbers. (See board.)
- ▶ Theorem 1.2.8:  $u, v$  nonzero vectors in  $\mathbb{R}^2, \mathbb{R}^3$ . If  $\theta$  is an angle between  $u$  and  $v$ , then
 
$$\cos\theta = u \cdot v / (\|u\| \|v\|) \text{ or}$$

$$\theta = \cos^{-1}(u \cdot v / (\|u\| \|v\|)).$$
- ▶ Proof: Use cosine law
  - $\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$
  - Now  $\|v-u\|^2 = (v-u) \cdot (v-u) = (v-u) \cdot v - (v-u) \cdot u$   
 $= v \cdot v - u \cdot v - v \cdot u + u \cdot u = \|v\|^2 - 2u \cdot v + \|u\|^2.$
  - $\|v\|^2 - 2u \cdot v + \|u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$
  - We simplify to get above.



- ▶ We consider  $\theta$  to be in  $[0, \pi]$  interval.
- ▶ Orthogonality.
  - $u \cdot v = 0$  iff  $\cos \theta = 0$  iff  $\theta = \pi/2$ .
  - Two nonzero vectors in 2- or 3-spaces are perpendicular if and only if their dot product is zero.
- ▶ See Example 5,6.(See board)
- ▶ Definition. We extend the above formula to hold for n-space as well.
- ▶ Thus two vectors in n-spaces are *orthogonal* if their dot product is zero. A nonempty set of vectors is said to be an orthogonal set if each pair of distinct vectors are orthogonal.
- ▶ Use “perpendicular” for nonzero-vectors.

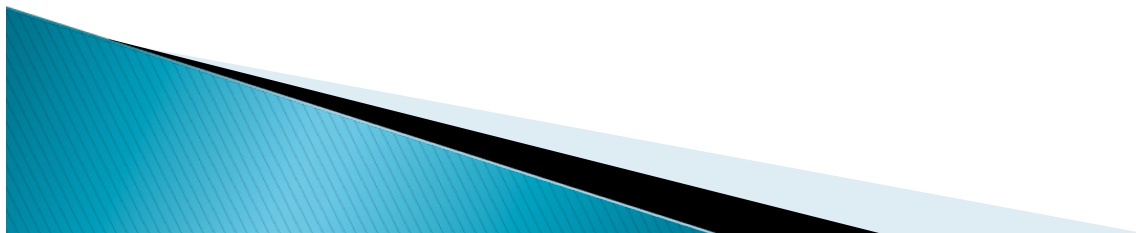


- ▶ Zero vector  $0$  is orthogonal to every vector in  $\mathbb{R}^n$ . Actually, it is the only such vector in  $\mathbb{R}^n$ .
- ▶  $\{(1,0,0),(0,1,0),(0,0,1)\}$
- ▶ Orthonormal set. Two vectors are orthonormal if they are orthogonal and have length 1. A set of vectors is *orthonormal* if every vector in the set has length 1 and each pair of vectors is orthogonal.





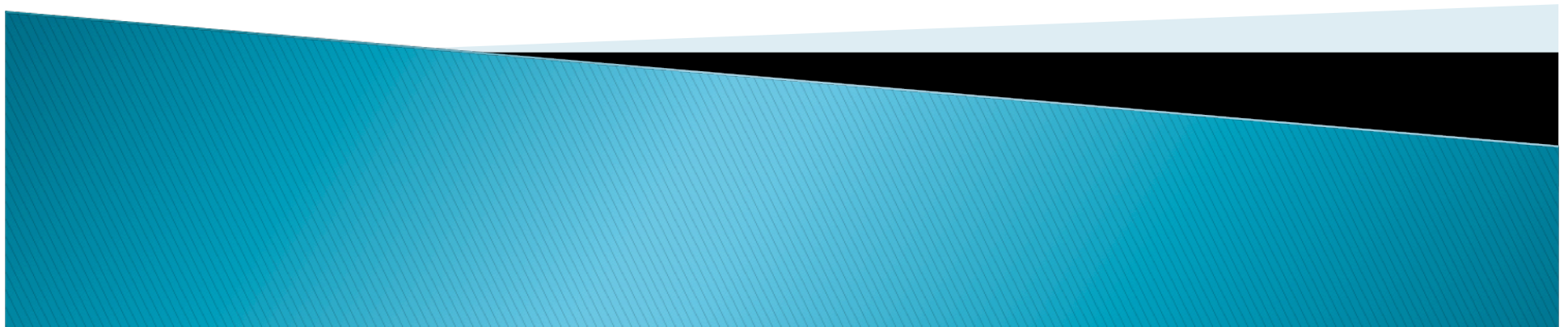
- ▶ Pythagoras theorem: If  $u$  and  $v$  are orthogonal vectors, then  $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ .
- ▶ Proof:  $\|u+v\|^2 = (u+v) \cdot (u+v) = \|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2$ .
- ▶ Cauchy-Swartz inequality
  - $(u \cdot v)^2 \leq \|u\|^2 \|v\|^2$  or  $|u \cdot v| \leq \|u\| \|v\|$
- ▶ Proof: If  $u=0$  or  $v=0$ , then true.
  - (See board.)
- ▶ Triangle inequality:  $u, v, w$  vectors.
  - $\|u+v\| \leq \|u\| + \|v\|$ .
- ▶ Proof:  $\|u+v\|^2 = (u+v) \cdot (u+v) =$
- ▶  $\|u\|^2 + 2(u \cdot v) + \|v\|^2 \leq \|u\|^2 + 2|u \cdot v| + \|v\|^2 \leq$
- ▶  $\|u\|^2 + 2\|u\| \|v\| + \|v\|^2$



- ▶ Theorem 1.2.14. Parallelogram equation for vectors.  $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$ .
- ▶ Proof: see board
- ▶ Triangle inequality:  $u, v, w$  vectors  
 $d(u, v) \leq d(u, w) + d(w, v)$ .
- ▶ Proof: see board.



# 1.3. Vector equations of lines and planes



# Lines

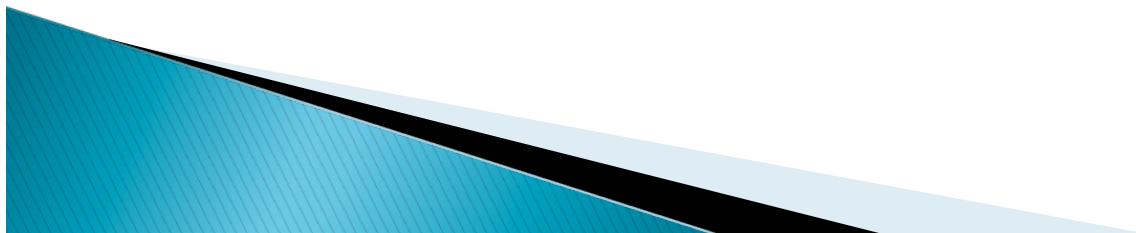
- ▶ General equation for lines in 2-space:  
 $Ax+By= C$ . (A,B not both zero)
- ▶  $Ax+By=0$  (passes origin)
- ▶ Another method (parametric equation): Let a line pass through  $x_0$ .  
If  $x$  is a point on the line, then  $x-x_0$  is always parallel to a fixed vector say  $v$ .
- ▶ Thus  $x-x_0=tv$  for some real number  $t$ .
- ▶  $x = x_0+tv$ . ( $t$  is called a parameter)



- ▶  $(x,y)=(x_0,y_0)+t(a,b)$ .
- ▶  $x=x_0+ta, y=y_0+tb$ .
  
- ▶ In 3-space,  $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$ .  
Thus,  $x=x_0+ta, y=y_0+tb, z=z_0+tc$ .
  
- ▶ Given two points  $x_1, x_0$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , we try to find a line through them.
  - The line is parallel to  $x_1 - x_0$ .
  - Thus  $x= x_0+t(x_1-x_0)$  or  $x= (1-t)x_0+tx_1$ .
  - This is a *two-point vector equation*.
  - If  $t$  is in  $[0,1]$ , then the point is in the segment with endpoints  $x_0, x_1$ .



- ▶ Actually, one can turn the general equation to parametric equation in  $\mathbb{R}^2$  and conversely.
- ▶ General to parametric: Find two points in it and use the two-point vector equation.
  - $7x+5y=35$ .  $(5,0)$  and  $(7,0)$ .
  - $X=(1-t)(0,7)+t(5,0)$ .  $x=5t$ ,  $y=7-7t$ .
- ▶ Parametric to general: Eliminate  $t$  from the equation:
  - $x=5t$ ,  $y=7-7t$ . Then  $7x+5y = 35$ . This is the general equation.
- ▶ Final comment: to give general equations for lines in 3-space, we need two equations.



# A plane in $\mathbb{R}^3$

- ▶ From a plane  $S$  in  $\mathbb{R}^3$ , we can obtain a point  $x_0$  and a perpendicular vector  $n$ .
- ▶ From  $x_0$ , and  $n$ , we can obtain a *point-normal equation* of  $S$ :
  - $n \cdot (x - x_0) = 0$ .
- ▶ Conversely, any  $x$  satisfying the equation lies in  $S$ .
- ▶  $(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$ .
  - $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ .
  - $Ax + By + Cz = D$ . (general equation of  $S$ .)
  - Rmk: The coefficients give us the normal vector.



- ▶ Actually  $S$  passes  $0$  if and only if  $D=0$ .
- ▶ There is also a parametric equation of a plane:
  - Given a plane  $W$ , let  $x_0$  be a point and let  $v_1$  and  $v_2$  be two vectors parallel to  $W$ .
  - Then  $t_1v_1+t_2v_2$  is also parallel to  $W$  for any real numbers  $t_1$  and  $t_2$  by parallelogram laws.  
Thus  $x_0+t_1v_1+t_2v_2$  lies in  $W$ .
  - Conversely, given any point  $x$  in  $W$ ,  $x-x_0$  is parallel to  $W$  and hence equals  $t_1v_1+t_2v_2$  for some real numbers  $t_1$  and  $t_2$ .
  - Thus  $x=x_0+t_1v_1+t_2v_2$  is the equation of points of  $W$ .





- ▶ Examples: Given a point, and two vectors, find parametric equations.
- ▶ Given three points  $x_0, x_1, x_2$  on  $W$ , find a parametric equation
$$x = x_0 + t_1(x_1 - x_0) + t_2(x_2 - x_0).$$
- ▶ From general equation to a parametric equation.  
(Example 7)
  - Solution: is to find three distinct point and use the above.
- ▶ From parametric equation to a general equation.  
(not yet studied.)



- ▶ In general  $\mathbb{R}^n$ :
- ▶ A line through  $x_0$  parallel to  $v$ :
  - $X=x_0+tv$ .
- ▶ A plane through  $x_0$  parallel to  $v_1, v_2$ .
  - $X=x_0+t_1v_1+t_2v_2$
- ▶ Actually, we can do  $s$ -dimensional subspace with  $s$  parallel vectors. But we stop here.
- ▶ See Example 8 (page 34)



# Comments on homework

- ▶ Ex set 1.2. Mostly computations.
  - ▶ 1.2:13-16 use the definition
  - ▶ 1.2: 32-35 Sigma notations (expect to know)
  - ▶ 1.3: Two planes are parallel if their normal vectors are parallel. (perpendicular: the same)
  - ▶ Finding normal vectors to the plane: Take the coefficients. (1.3:26-35)
  - ▶ 1.3:37-38. Finding intersection line: Find two points in the intersections.
  - ▶ 1.3:39-40. Use substitutions.
- 