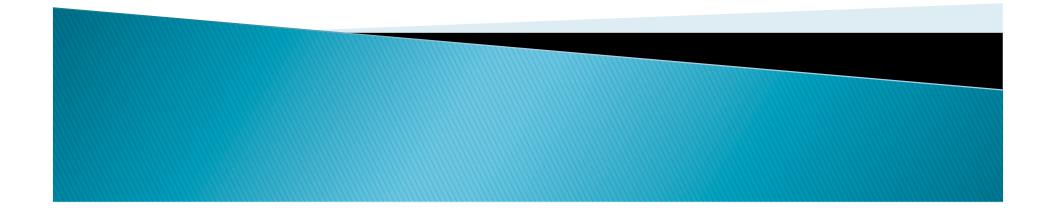
1.2.Dot Products, Orthogonality



lengths

• Length, norm, magnitude of a vector $v = (v_{1,...,v_n})$ is $||v|| = (v_1^2 + v_2^2 + ... + v_n^2)^{1/2}$.

- Examples v=(1,1,...,1) ||v||=n^{1/2}.
- Unit vectors u=v/||v|| corresponds to directions.
- Standard unit vectors
 - i=(1,0), j=(0,1) in R²
 - i=(1,0,0), j=(0,1,0), k=(0,0,1) in R^3
 - $e_1 = (1,0,...,0), e_2 = (0,1,...,0), ..., e_n = (0,0,...,1)$ in R^n .



- v=(v₁,v₂, ...,v_n)= v₁e₁ + v₂e₂ + ... + v_n e_n.
 This is a unique expression.
- Distances when given position vectors:
- $P(x_1,x_2) = ||P_1P_2|| = ((x_2-x_1)^2 + (y_2 y_1)^2)^{1/2}$ in 2-space.
- In 3-space $d(P_1,P_2) = ||P_2-P_1|| = ((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2)^{1/2}$ in 2-space.
- In n-space u=(u_1,u_2,...,u_n), v=(v_1,v_2, ...,v_n),
- Then $d(u,v) = ((u_1-v_1)^2+(u_2-v_2)^2+...+(u_n-v_n)^2)^{1/2}$.



- Theorem. Two position vectors u,v in Rⁿ.
 o d(u,v)≥0, d(u,v)=d(v,u), d(u,v)=0 if and only if u=v.
- Proof: Use the formula.
- We now introduce dot product. Given two vectors, the dot product gives you a real number.
- The dot product generalizes length and angle and is useful to compute many quantities. In fact, it is more fundamental than angles. Given
 - $u = (u_1, u_2, ..., u_n), v = (v_1, v_2, ..., v_n),$
 - $u \cdot v = u_1 v_1 + u_2 v_2 + ... + u_n v_n$.

Properties of dot products

- $||v|| = (v \cdot v)^{1/2}.$
- Theorem 1.2.7
 - u•v=v•u, Symmetry
 - u•(v+w)= u•v+u•w. distributivity
 - k(u•v)=(ku)•v homogenous
 - ∘ $v \bullet v \ge 0$, and $v \bullet v = 0$ if and only if v = 0. positivity
- Theorem 1.2.8.
 - 0•v=v•0=0
 - (u+v)•w=u•w+v•w
 - u•(v-w)=u•v-u•w, (u-v)•w=u•w-v•w
 - $k(u \bullet v) = u \bullet (kv)$



- Theorems 1.2.6,1.2.7 gives us a means to compute as one does with real numbers. (See board.)
- Theorem 1.2.8:u, v nonzero vectors in R², R³.
 If θ is an angle between u and v, then
 cosθ = u•v/||u•v|| or θ=cos⁻¹(u•v/||u•v||).
- Proof: Use cosine law
 - $||v-u||^2 = ||u||^2 + ||v||^2 2||u||||v||\cos\theta$.
 - Now $||v-u||^{2=} (v-u) \cdot (v-u) = (v-u) \cdot v (v-u) \cdot u = v \cdot v u \cdot v v \cdot u + u \cdot u = ||v||^2 2u \cdot v + ||u||^2.$
 - $\circ ||v||^2 2u \bullet v + ||u||^2 = ||u||^2 + ||v||^2 2||u||||v||\cos\theta.$
 - We simplify to get above.



- We consider θ to be in [0, π] interval.
- Orthogonality.
 - $u \bullet v = 0$ iff $\cos \theta = 0$ iff $\theta = \pi/2$.
 - Two nonzero vectors in 2- or 3-spaces are perpendicular if and only if their dot product is zero.
- See Example 5,6.(See board)

- Definition. We extend he above formula to hold for n-space as well.
- Thus two vectors in n-spaces are *orthogonal* if their dot product is zero. A nonempty set of vectors is said to be an orthogonal set if each pair of distinct vectors are orthogonal.
 Use perpendicular for nonzero-vectors.

- Zero vector 0 is orthorgonal to every vector in Rⁿ. Actually, it is the only such vector in Rⁿ.
- {(1,0,0),(0,1,0),(0,0,1)}
- Orthonormal set. Two vectors are orthonormal if they are orthogonal and have length 1. A set of vectors is *orthonormal* if every vector in the set has length 1 and each pair of vectors is orthogonal.

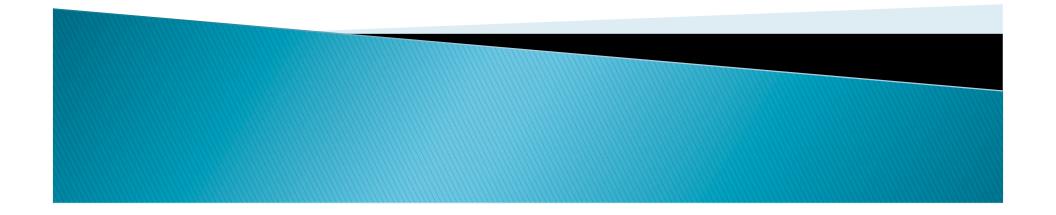


- Phythagoras theorem: If u and v are orthogonal vectors, then ||u+v||²=||u||²+||v||².
- Proof: $||u+v||^2 = (u+v) \cdot (u+v) = ||u||^2 + ||v|| + 2u \cdot v = ||u||^2 + ||v||^2$.
- Cauchy-Swartz inequality
 - $(u.v)^2 \leq ||u||^2 ||v||^2$ or $|(u.v)| \leq ||u||||v||^2$
- Proof: If u=0 or v=0, then true.
 - (See board.)
- Triangle inequality: u,v,w vectors.
 - $\circ ||u+v|| \le ||u||+||v||.$
- Proof: $||u+v||^2 = (u+v).(u+v) =$
- $||u||^{2}+2(u.v)+||v||^{2} \leq ||u||^{2}+2|u.v|+||v||^{2} \leq$
- $||u||^2+2||u||||v||+||v||^2$

- Theorem 1.2.14. Parallelogram equation for vectors. $||u+v||^2+||u-v||^2 = 2(||u||^2+||v||^2)$.
- Proof: see board
- Triangle inequality: u,v,w vectors d(u,v)≤d(u,w)+d(w,v).
- Proof: see board.



1.3. Vector equations of lines and planes



Lines

- General equation for lines in 2-space: Ax+By= C. (A,B not both zero)
- Ax+By=0 (passes origin)
- Another method (parametric equation): Let a line pass through x_0.
 If x is a point on the line, then x-x_0 is always parallel to a fixed vector say v.
- Thus x-x_0=tv for some real number t.
- $x = x_0 + tv$. (t is called a parameter)

- $(x,y)=(x_0,y_0)+t(a,b)$.
- ▶ x=x_0+ta, y=y_0+tb.
- In 3-space, (x,y,z)=(x_0,y_0,z_0)+t(a,b,c). Thus, x=x_0+ta, y=y_0+tb, z=z_0+tc.
- Given two points x_1, x_0 in R² or R³, we try to find a line through them.
 - The line is parallel to $x_1 x_0$.
 - Thus $x = x_0 + t(x_1 x_0)$ or $x = (1-t)x_0 + tx_1$.
 - This is a *two-point vector equation*.
 - If t is in [0,1], then the point is in the segment with endpoints x_0, x_1.



- Actually, one can turn the general equation to parametric equation in R² and conversely.
- General to parametric: Find two points in it and use the two-point vector equation.
 - 7x+5y=35. (5,0) and (7,0).

- X=(1-t)(0,7)+t(5,0). x=5t, y=7-7t.
- Parametric to general: Eliminate t from the equation:
 - x=5t, y=7-7t. Then 7x+5y = 35. This is the general equation.
- Final comment: to give general equations for lines in 3-space, we need two equations.

A plane in R³

- From a plane S in R³, we can obtain a point x_0 and a perpendicular vector n.
- From x_0, and n, we can obtain a pointnormal equation of S:
 - n.(x-x_0)=0.
- Conversely, any x satisfying the equation lies in S.
- $(A,B,C).(x-x_0,y-y_0,z-z_0)=0.$
 - $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.
 - Ax+By+Cz=D. (general equation of S.)
 - Rmk:The coefficients give us the normal vector.

- Actually S passes 0 if and only if D=0.
- There is also a parametric equation of a plane:
 - Given a plane W, let x_0 be a point and let v_1 and v_2 be two vectors parallel to W.
 - Then t_1v_1+t_2v_2 is also parallel to W for any real numbers t_1 and t_2 by parallelogram laws. Thus x_0+t_1v_1+t_2v_2 lies in W.
 - Conversely, given any point x in W, x- x_0 is parallel to W and hence equals t_1v_1+t_2v_2 for some real numbers t_1 and t_2.
 - Thus x=x_0+t_1v_1+t_2v_2 is the equation of points of W.

- Examples: Given a point, and two vectors, find parametric equations.
- Given three points x_0,x_1,x_2 on W, find a parametric equation

 $x = x_0+t_1(x_1-x_0)+t_2(x_2-x_0).$

- From general equation to a parametric equation. (Example 7)
 - Solution: is to find three distinct point and use the above.
- From parametric equation to a general equation.

(not yet studied.)

- In general Rⁿ:
- A line through x_0 parallel to v:
 - $X=x_0+tv$.
- A plane through x_0 parallel to v_1, v_2.
 - $X = x_0 + t_1 v_1 + t_2 v_2$
- Actually, we can do s-dimensional subspace with s parallel vectors. But we stop here.
- See Example 8 (page 34)



Comments on homework

- Ex set 1.2. Mostly computations.
- ▶ 1.2:13–16 use the definition
- 1.2: 32–35 Sigma notations (expect to know)
- 1.3: Two planes are parallel if their normal vectors are parallel. (perpendicular: the same)
- Finding normal vectors to the plane: Take the coefficients. (1.3:26-35)
- 1.3:37–38. Finding intersection line: Find two points in the intersections.
- 1.3:39–40. Use substitutions.

