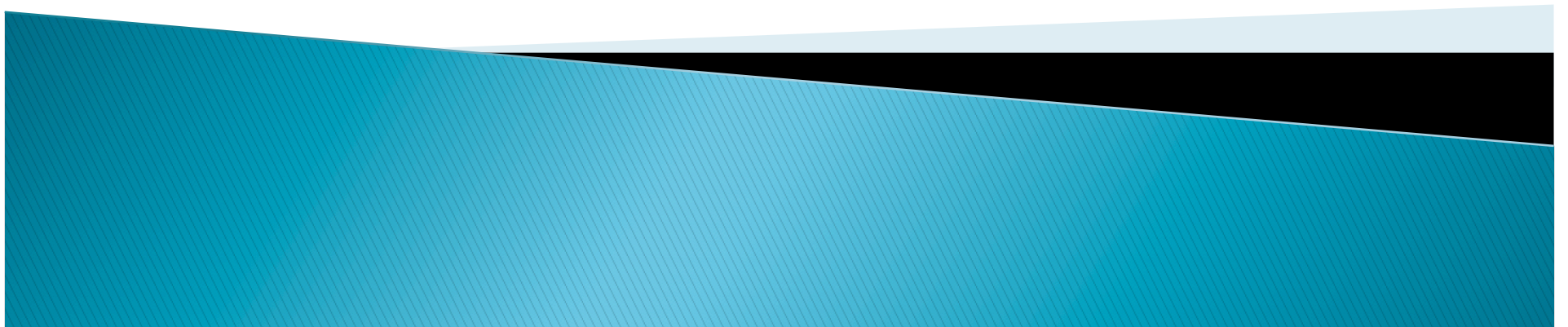


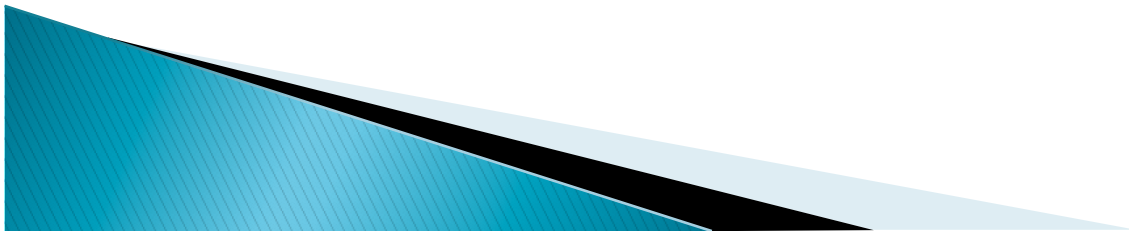
2.1. Introduction to Systems of linear equations



Linear Systems

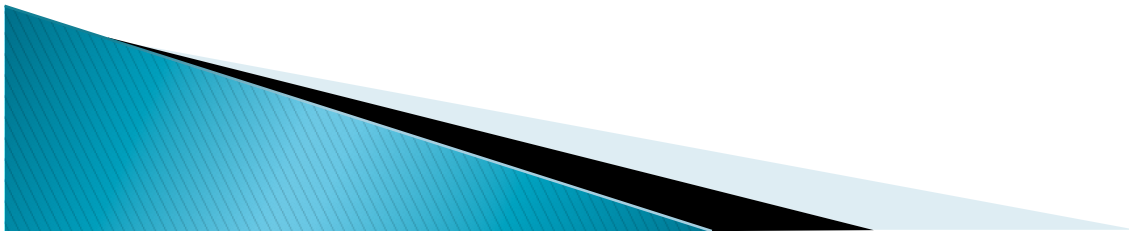
- ▶ Recall lines and planes
- ▶ $x+y=4$, $x-5y=1$. Solve this system.
- ▶ $x+y-z=0$, $x+y+z=1$, $x-y=1$. Solve this system.

- ▶ System of linear equations:
 - ▶ $a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$
 - ▶ $a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$
 - ▶
 - ▶ $a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$
- ▶ The system is homogeneous if all b_i s are zero.
- ▶

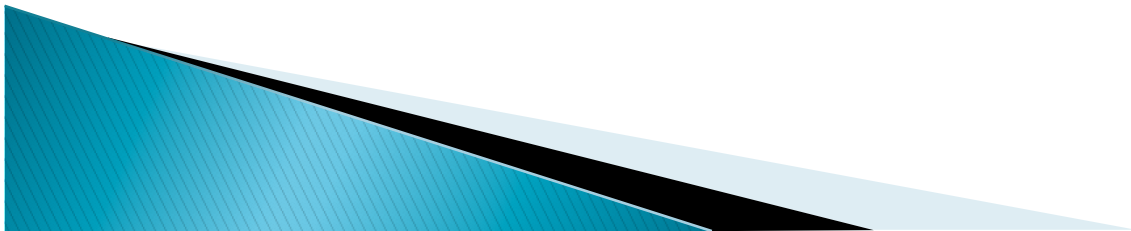


Theoretical: Do solutions exist and if so, how many and in what form?

- ▶ To see this, we play with lines first.
 - Two lines are parallel and are disjoint. \rightarrow no solution
 - Two lines are parallel and coincide. \rightarrow infinitely many solutions
 - Two lines are not parallel and intersect at a unique point. \rightarrow unique solution
- ▶ In the 3-space, see Figure 2.1.2.



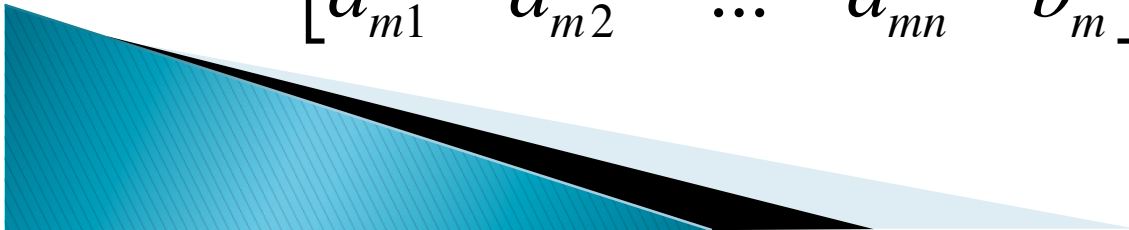
- ▶ Theorem 2.1.1. Every system of linear equations have zero, one, or infinitely many solutions.
- ▶ Proof: We will see this through Gaussian eliminations.
- ▶ Examples:
 - ▶ $x-y=0, x+y=1 \rightarrow x-y=0, 2y=1 \rightarrow y=1/2, x=-1/2$
 - ▶ $x-y=2, 2x-2y=0 \rightarrow x-y=0, 0=-4 \rightarrow$ no solution
 - ▶ $x-y=-1, 3x-3y=-3 \rightarrow x-y=-1, 0=0 \rightarrow y=t, x=t-1.$



Augmented matrices and elementary row operations

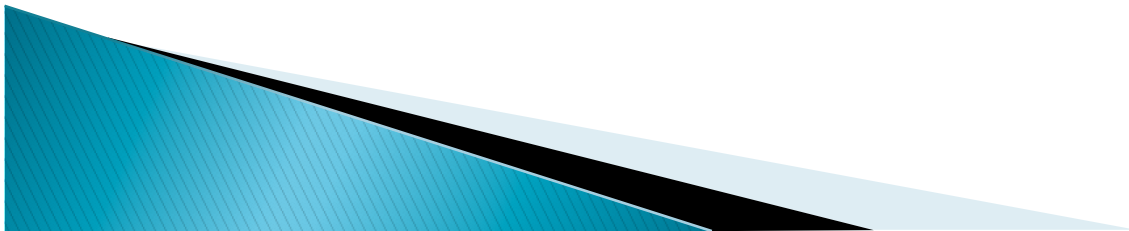
$$\begin{array}{cccccc} a_{11}x_1 + & a_{12}x_2 + & \dots & a_{1n}x_n & = & b_1 \\ a_{21}x_1 + & a_{22}x_2 + & \dots & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + & \dots & a_{mn}x_n & = & b_m \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right]$$



Elementary operations

- ▶ These operations can be used to solve a system of equations
 - 1. Multiply an equation through by a nonzero constant
 - 2. Interchange to equations
 - 3. Add a multiple of one equation to another.
- ▶ For a matrix (an augmented one)
 - Multiply a row through a nonzero constant
 - Interchange two rows
 - Add a multiple of one row to another.



- ▶ Example
- ▶ See Example 6
- ▶ Example 7. Write a vector $w=(4,1,3)$ as a linear combination of vectors $a=(1,1,1)$, $b=(2,-1,2)$, $c=(0,0,1)$.
 - Solve $W=c_1a+c_2b+c_3c$.
 - Solve $(4,1,3)=c_1(1,1,1)+c_2(2,-1,2)+c_3(0,0,1)$
 - $(4,1,3)=(c_1+2c_2,c_1-c_2,c_1+2c_2+c_3)$

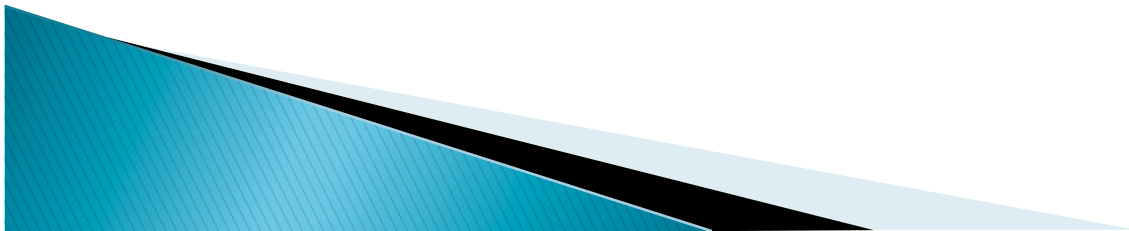
$$c_1 + 2c_2 = 4$$

$$c_1 - c_2 = 1$$

$$c_1 + 2c_2 + c_3 = 3$$

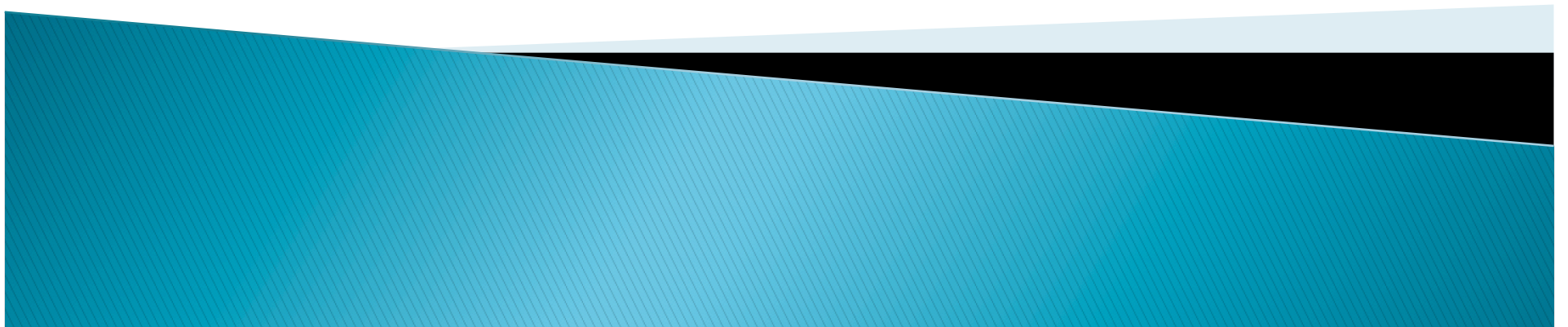
Comments Ex.2.1

- ▶ 1–4 Linearity of equations
- ▶ 5–6 solution checking
- ▶ 7–8 Graphing problems
- ▶ 9,10 solving
- ▶ 11,12 Finding equations (Eliminate t)
- ▶ 15,16 number of solutions
- ▶ 17–24 System \leftrightarrow augmented matrix
- ▶ 25,26 Elementary row operations



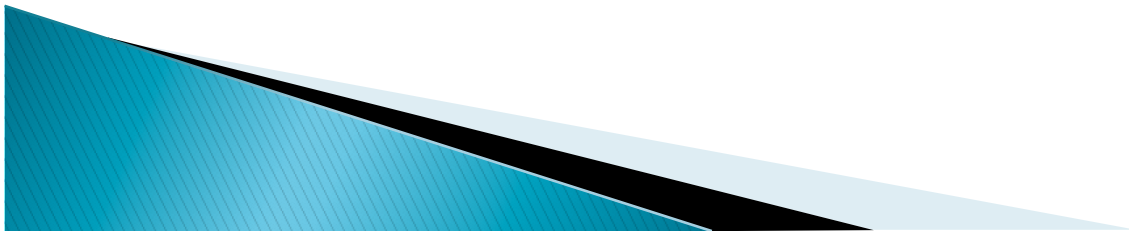
2.2. Solving equations by row reduction

Row reduced echelon form



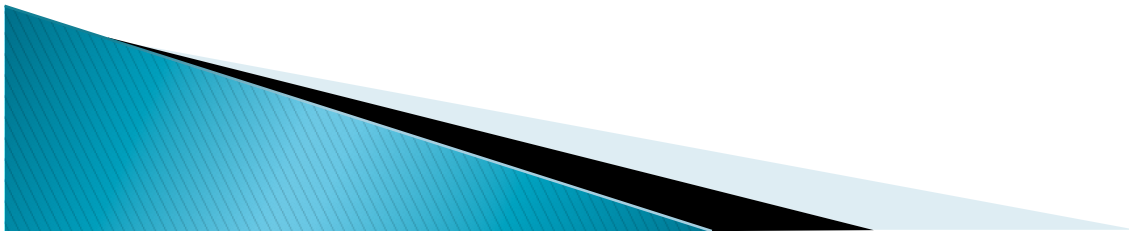
Reduced row echelon form

- ▶ Recall Ex. 6.
 - A) If a row is not zero, the first nonzero element is 1. (Leading 1)
 - B) A zero row lie below all nonzero rows.
 - C) The leading 1 of the lower row starts later than the leading 1 of the upper row (up to here row echelon form)
 - D) Each column containing a leading 1 is zero elsewhere.



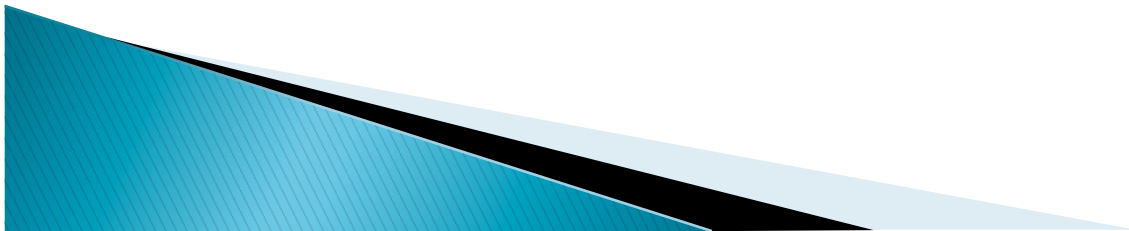
Examples

- ▶ See Examples 1, 2.
- ▶ Once one obtains a reduced row echelon matrix, then what do we do?
- ▶ Example 4 (a) no solutions (contradiction always arises)
- ▶ Example 4 (b) OK.
 - $x + 3z = -1, y - 4z = 2$
 - z is not associated with any leading one.
 - We call such z a free variable.
 - Let assign t to z . Then $x = -1 - 3t, y = 2 + 4t, z = t$
 - Is the solution space. (infinite case)
 - In the unique case there is no free variable.



- ▶ It is often desirable to write solution spaces as linear combinations of column vectors.
- ▶ For Example 4 (b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 - 3t \\ 2 + 4t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$



Gauss Jordan Eliminations

- ▶ Using elementary row operations, one can make any matrix into a reduced row echelon form.
- ▶ Thus any system of equations can be solved.
- ▶ The basic step.
 - 1. Find a first column with a nonzero entry. (The column is usually the first one.)
 - 2. Interchange so that the nonzero entry is at the top row
 - 3. Multiply to make it to be 1.
 - 4. Use 1, to make the entries below it to become 0.

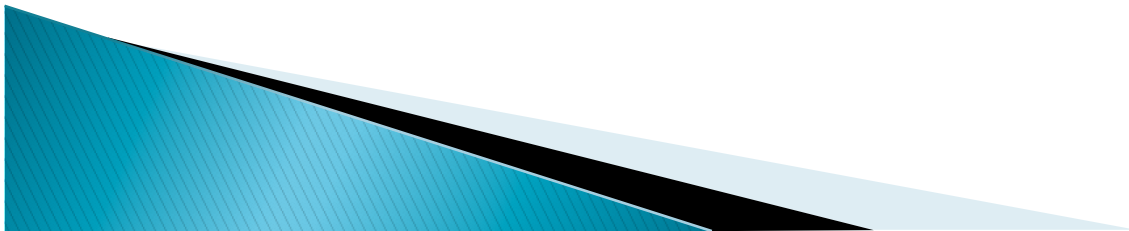


- Now take the first row out of considerations and work in $(m-1) \times n$ matrix A' below.
- We work the same way:
 - Find the first column A' with nonzero entry.
 - Interchange to move it to the first row of A' .
 - Scalar-multiply to make it 1.
 - Make the entries below 1 to be zero.
- We obtain row-echelon form.
- Now use each leading 1 to make the other entries in the same column become zero.
- This is called the Gauss-Jordan elimination.
- Gaussian elimination: not do the reduction.
- Pivot positions: positions of leading 1s.
- Pivot columns: columns that containing leading 1s.



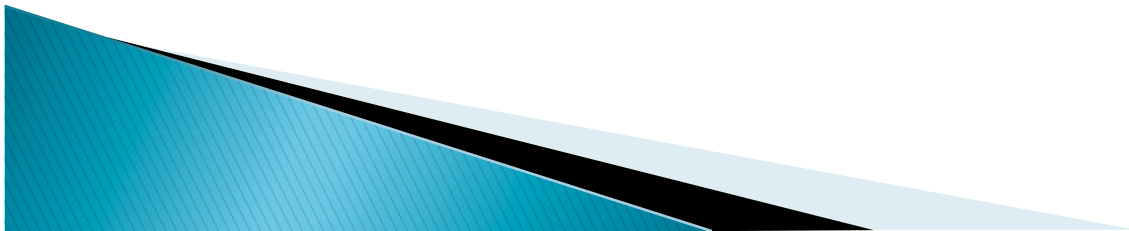
Solving equations

- ▶ A system of linear equations
- ▶ Take an augmented matrix
- ▶ Gauss–Jordan elimination
- ▶ Make it back into a system of linear equations.
- ▶ Take free variables: variables not associate with leading 1s.
- ▶ Solve
- ▶ Finally make it into a linear combinations of column vectors.
- ▶ See Board for an example.



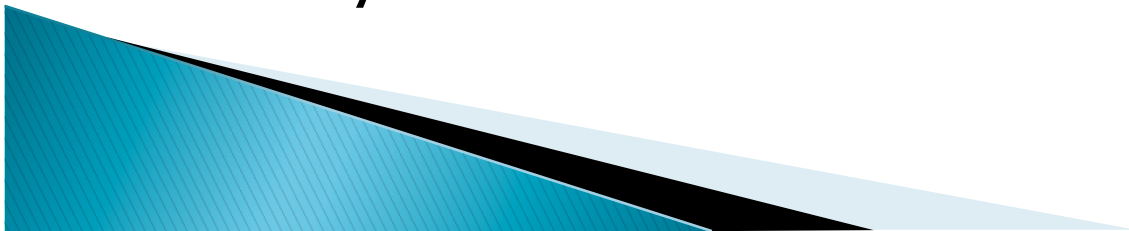
Back substitution

- ▶ One can use Gaussian elimination only.
- ▶ Make it back into a system of linear equations.
- ▶ In this case, we solve from the lowest equation.
- ▶ See the black board.
- ▶ Concluding Theorem: A system of linear equations have either no solution or a unique solution or infinitely many solutions.



Homogeneous linear system

- ▶ If the system of linear equations consists of only homogeneous linear equations, we call the system homogeneous.
- ▶ This is equivalent to all constant terms being 0.
- ▶ A homogeneous linear system always have at least one solution $(0,0,\dots,0)$ a trivial solution.
- ▶ Theorem 2.2.1. A homogeneous system has either a unique trivial solution or infinitely many solutions.



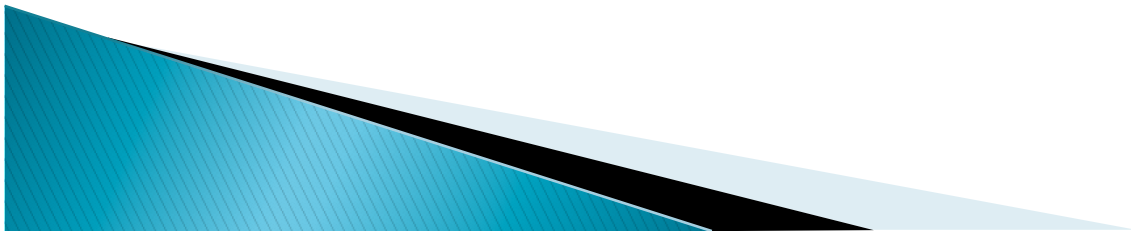
Consequences

- ▶ Theorem 2.2.2. homogeneous linear system with n unknowns. Its reduced echelon matrix has r nonzero rows. Then the system has $n-r$ free variables.
- ▶ This is actually the dimension of the solution space....



Computer implementations

- ▶ We always use largest nonzero element to put in the top rows.
- ▶



Comments on Ex. 2.2.

- ▶ 1–8 recognizing forms (reduced row echelon forms, row echelon forms...)
- ▶ 9–14. Solving given reduced echelon forms.
- ▶ 17–20 Back substitution
- ▶ 23–42 Solving. Gauss–Jordan
- ▶ 43–46 A bit tricky....
- ▶ 52...

