2.1. Introduction to Systems of linear equations

Linear Systems

- Recall lines and planes
- $x+y=4$, $x-5y=1$. Solve this system.
- $x+y-z=0$, $x+y+z=1$, $x-y=1$. Solve this system.

- System of linear equations:
- a_11 x_1+a_12 x_2+…+a_1n x_n=b_1
- a_21 x_1+a_22 x_2+…+a_2n x_n=b_2
- …….
- a_m1 x_1+a_m2 x_2+…+a_mn x_n=b_m
- The system is homogeneous if all b_i s are zero.
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Theoretical: Do solutions exists and if so, how many and in what form?

- To see this, we play with lines first.
  - Two lines are parallel and are disjoint. -> no solution
  - Two lines are parallel and coincide. -> infinitely many solutions
  - Two lines are not parallel and intersect at unique point.-> unique solution
- In the 3-space, see Figure 2.1.2.

- Theorem 2.1.1. Every system of linear equations have zero, one, or infinitely many solutions.
- Proof: We will see this through Gaussian eliminations.
- Examples:
- $x-y=0$, $x+y=1$ -> $x-y=0, 2y=1$ -> $y=1/2$, $x=-1/2$
- $x-y=2$, $2x-2y=0$ -> $x-y=0$, $0=-4$ -> no solution
- $x-y=-1$, $3x-3y=-3$ -> $x-y=-1$, $0=0$ -> $y=t$, $x=t-1$.

Augmented matrices and elementary row operations

$$
\begin{array}{cccccc}
a_{11}x_1 + & a_{12}x_2 + & \ldots & a_{1n}x_n & = & b_1 \\
a_{21}x_1 + & a_{22}x_2 + & \ldots & a_{2n}x_n & = & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{m1}x_1 + & a_{m2}x_1 + & \ldots & a_{mn}x_n & = & b_m
\end{array}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn} & b_m
\end{bmatrix}
$$

Elementary operations

- These operations can be used to solve a system of equations
  - 1. Multiply an equation through by a nonzero constant
  - 2. Interchange to equations
  - 3. Add a multiple of one equation to another.
- For a matrix (an augmented one)
  - Multiply a row through a nonzero constant
  - Interchange two rows
  - Add a multiple of one row to another.

- Example
- See Example 6
- Example 7. Write a vector w=(4,1,3) as a linear combination of vectors a=(1,1,1), b=(2,−1,2), c=(0,0,1).
  - Solve W=c_1a+c_2b+c_3c.
  - Solve (4,1,3)=c_1(1,1,1)+c_2(2,−1,2)+c_3(0,0,1)
  - (4,1,3)=(c_1+2c_2,c_1−c_2,c_1+2c_2+c_3)

$$
\begin{aligned}
c_1 + 2c_2 \quad\quad &= 4 \\
c_1 - c_2 \quad\quad &= 1 \\
c_1 + 2c_2 + c_3 &= 3
\end{aligned}
$$

Comments Ex.2.1

- 1-4 Linearity of equations
- 5-6 solution checking
- 7-8 Graphing problems
- 9,10 solving
- 11,12 Finding equations (Eliminate t)
- 15,16 number of solutions
- 17-24 System <-> augmented matrix
- 25,26 Elementary row operations

2.2. Solving equations by row reduction

Row reduced echelon form

Reduced row echelon form

- Recall Ex. 6.
  - A) If a row is not zero, the first nonzero element is 1. (Leading 1)
  - B) A zero row lie below all nonzero rows.
  - C) The leading 1 of the lower row starts later than the leading 1 of the upper row (up to here row echelon form)
  - D) Each column containing a leading 1 is zero elsewhere.

Examples

- See Examples 1, 2.
- Once one obtains a reduced row echelon matrix, then what do we do?
- Example 4 (a) no solutions (contradiction always arises)
- Exampe 4 (b) OK.
  - $x + 3z = -1$, $y - 4z = 2$
  - z is not associated with any leading one.
  - We call such z a free variable.
  - Let assign t to z. Then $x = -1 - 3t$, $y = 2 + 4t$, $z = t$
  - Is the solution space. (infinite case)
  - In the unique case there is no free variable.

- It is often desirable to write solution spaces as linear combinations of column vectors.
- For Example 4 (b)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1-3t \\ 2+4t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

Gauss Jordan Eliminations

- Using elementary row operations, one can make any matrix into a reduced row echelon form.
- Thus any system of equations can be solved.
- The basic step.
  - 1. Find a first column with a nonzero entry. (The column is usually the first one.)
  - 2. Interchange so that the nonzero entry is at the top row
  - 3. Multiply to make it to be 1.
  - 4. Use 1, to make the entries below it to become 0.

- Now take the first row out of considerations and work in (m-1)xn matrix A’ below.
- We work the same way:
  - Find the first column A’ with nonzero entry.
  - Interchange to move it to the first row of A’.
  - Scalar-multiply to make it 1.
  - Make the entries below 1 to be zero.
- We obtain row-echelon form.
- Now use each leading 1 to make the other entries in the same column become zero.
- This is called the Gauss-Jordan elimination.
- Gaussian elimination: not do the reduction.
- Pivot positions: positions of leading 1s.
- Pivot columns: columns that containing leading 1s.

Solving equations

- A system of linear equations
- Take an augmented matrix
- Gauss–Jordan elimination
- Make it back into a system of linear equations.
- Take free variables: variables not associate with leading 1s.
- Solve
- Finally make it into a linear combinations of column vectors.
- See Board for an example.

Back substitution

- One can use Gaussian elimination only.
- Make it back into a system of linear equations.
- In this case, we solve from the lowest equation.
- See the black board.
- Concluding Theorem: A system of linear equations have either no solution or a unique solution or infinitely many solutions.

Homogeneous linear system

- If the system of linear equations consists of only homogeneous linear equations, we call the system homogeneous.
- This is equivalent to all constant terms being 0.
- A homogeneous linear system always have at least one solution (0,0,…,0) a trivial solution.
- Theorem 2.2.1. A homogeneous system has either a unique trivial solution or infinitely many solutions.

Consequences

- Theorem 2.2.2. homogeneous linear system with n unknowns. Its reduced echelon matrix has r nonzero rows. Then the system has n-r free variables.
- This is actually the dimension of the solution space….

Computer implementations

- We always use largest nonzero element to put in the top rows.
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Comments on Ex. 2.2.

- 1-8 recognizing forms (reduced row echelon forms, row echelon forms…)
- 9-14. Solving given reduced echelon forms.
- 17-20 Back substitution
- 23-42 Solving. Gauss-Jordan
- 43-46 A bit tricky….
- 52…