3.4. Subspaces, Linear independence

Subspace

- A subspace is a set one can do scalar multiplication and addition and not leave the set.

Definition 3.4.1 A nonempty set of vectors in R^n is called a ***subspace*** of R^n if it is closed under scalar multiplication and addition.

1. A subspace is usually given by conditions.
2. We need to verify the conditions after scalar multiplications or additions.

- {O} is a susbspace
- Every subspace contains O. Why?
- W ={(x,y) in R^2|x> 0, y>0} is not a subspace. Why?
- W={(x,y,0) in R^3} is a subspace.
- W in R^n given by x_2=1,x_3=-1 a subspace?
- Let v_1, v_2,…,v_s is given in R^n.
  - Let W={c_1v_1+c_2v_2+…+c_sv_s| c_i in R}.
  - That is W is the set of all linear combinations of given vectors v_1, v_2,…, v_s.
  - Then W is a subspace.
- We write W=span{v_1,v_2,…,v_s}

Theorem 3.4.2 *If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are vectors in R^n, then the set of all linear combinations*

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s \tag{3}$$

is a subspace of R^n.

- Example 2: Span{O}={O}.
- Example 3: Span{(1,1,2,0)} is a line.
- Example 4.
  - A subspace in R^1: itself or {O}.
  - A subspace in R^2: itself, a line through O, {O}.
  - A subspace in R^3: itself, a plane through O, a line through O, {O}
  - A subspace in R^n: itself, a subspace $\approx R^i$, {O}.
  - Ax+By+C=D

Solution space of a linear system

Theorem 3.4.3 *If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system with n unknowns, then its solution set is a subspace of R^n.*

- Proof: W ={x|Ax=0}.
  - If x_0 is a solution, then kx_0 is a solution.
  - If x_1 and x_2 are solutions, then x_1+x_2 is a solution.
  - Thus W is closed under scalar multiplications and additions. Thus W is a subspace.
- If one has an inhomogeous system, then the solution space is not a subspace.
- See Example *.

Theorem 3.4.4

(*a*) *If A is a matrix with n columns, then the solution space of the homogeneous system* $A\mathbf{x} = \mathbf{0}$ *is all of* R^n *if and only if* $A = 0$.

(*b*) *If A and B are matrices with n columns, then* $A = B$ *if and only if* $A\mathbf{x} = B\mathbf{x}$ *for every* $\mathbf{x}$ *in* R^n.

- Philosophy: A is determined by Ax's.
- Proof:
  - (a) ->) A=0. Ax=0.
  - <-) Ax=0 for all x. Ae_1=0, Ae_2=0,…,Ae_n=O.
    - A=AI=A[e_1,e_2,..,e_n]=[Ae_1,Ae_2,…,Ae_n]=O.
    - Thus all columns of A are zero.
  - (b) Ax=Bx for all x. Ax-Bx=O. (A-B)x=O for all x.
    A-B=O.
    A=B.

Linear independence

- How can we find a good way to describe a subspaces…
  - Find equations… See as solutions spaces
  - Find parameters… Write a vector as a linear combination of vectors in unique way for a fixed set of vectors. These should be the least in number.
  - So we want to avoid "linearly dependent set of vectors":
    when some of the vectors in the set can be written as a linear combination of some others.
  - In such cases, the number can be reduced by eliminating these.

Definition 3.4.5 A nonempty set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ in R^n is said to be ***linearly independent*** if the only scalars $c_1, c_2, \ldots, c_s$ that satisfy the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0} \tag{9}$$

are $c_1 = 0, c_2 = 0, \ldots, c_s = 0$. If there are scalars, not all zero, that satisfy this equation, then the set is said to be ***linearly dependent***.

- {O} is linearly dependent. cO=O for all c.
- {v} v nonzero is linearly independent. cv=0 iff c=0.

Theorem 3.4.6 *A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ in R^n with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

- Proof: ->) 0=c_1v_1+c_2v_2+…+c_sv_s=0.
  - Not all c_is are zero. Say c_i is not.
  - Then c_iv_i= c_1v_1+…+c_(i-1)v_(i-1)+c_(i+1)v_(i+1)+…+c_sv_s.
  - v_i= = (c_1/c_i)v_1+…+(c_(i-1)/c_i)v_(i-1)+ (c_(i+1)/c_i)v_(i+1)+…+(c_s/c_i)v_s.
- <-) v_i= = d_1v_1+…+d_(i-1)v_(i-1)+ d_(i+1)v_(i+1)+…+d_sv_s.
  - Thus, d_1v_1+…+d_(i-1)v_(i-1)+(-1)v_1+ d_(i+1)v_(i+1)+…+d_sv_s=0

- Example 10. two vectors in R^n.
- Example 11. three vectors in R^n is dependent if one is a linear combination of the other two.
  - Thus, the three vectors lie in a common plane or a common plane or {O}.
  - Three vectors are linearly independent if there are no such planes, lines.

Linear independence and homogeneous linear systems

- Given v_1,v_2,…,v_s, write A=[v_1,v_2,…,v_s].
- We write c_1v_1+c_2v_2+…+c_nv_n=0 as

$$[v_1, v_2, \ldots, v_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Theorem 3.4.7 *A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the column vectors of A are linearly independent.*

- See Examples 12.

Theorem 3.4.8 *A set with more than n vectors in R^n is linearly dependent.*

Theorem 3.4.9 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* R^n.

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

Proof: (d)(g) equivalent by Th.3.4.7.
(g)->(h). (g)->(c). A^T is invertible. Use (g) for A^T. (h) follows
(h)->(g). (g) for A^T holds. A^T is invertible. -> A is invertibe -> (g).

Ex. Set. 3.4.

- 1-8 Span problem
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