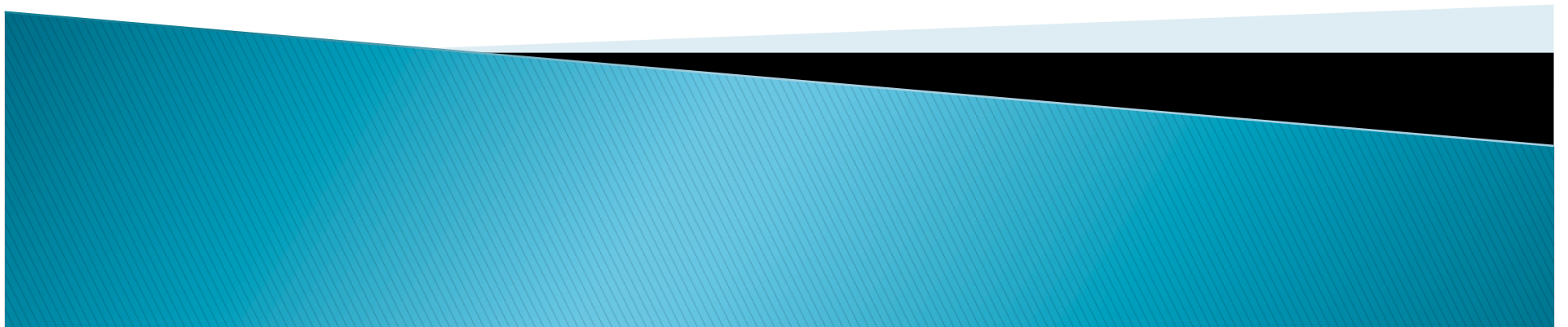


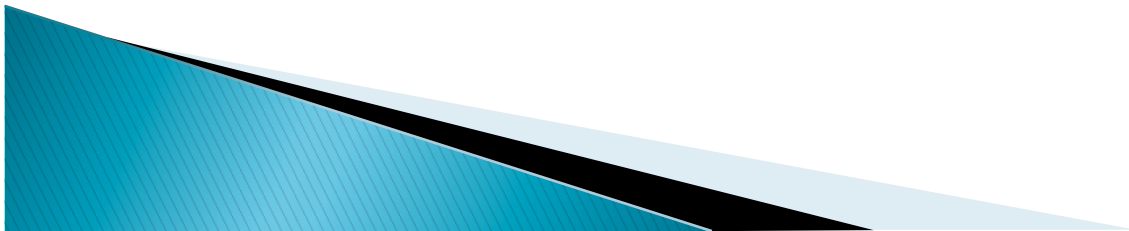
3.5. The geometry of linear systems

Solutions for inhomogeneous systems.
Consistency
Geometric interpretations



Translated subspaces

- ▶ W is a subspace.
- ▶ $x_0 + W = \{v = x_0 + w \mid w \text{ is in } W\}$
- ▶ This is not a subspace in general but is called an affine subspace (linear manifold, flat).
- ▶ For example $x_0 + \text{span}\{v_0, v_1, \dots, v_s\} = \{v = x_0 + c_0v_1 + \dots + c_s v_s\}$
- ▶ $y = 1$ in \mathbb{R}^2 . $\{(x, 1) \mid x \text{ in } \mathbb{R}\} = (0, 1) + \{(x, 0) \mid x \text{ in } \mathbb{R}\}$
- ▶ $Ax + By + Cz = D$ in \mathbb{R}^3 translated from $Ax + By + Cz = 0$ since they are parallel.



The solution space of $Ax=b$ and that of $Ax=0$

- ▶ $W=\{x|Ax=b\}$, $W_0=\{x|Ax=0\}$
- ▶ Let x be in W . Take one x_0 in W . Then $x-x_0$ is in W_0 .
 - $A(x-x_0)=Ax-Ax_0=b-b=0$.
- ▶ Given an element x in W_0 . $x+x_0$ is in W .
- ▶ $A(x+x_0)=Ax+Ax_0=0+b=b$.
- ▶ Thus, $W=x_0+W_0$.

Theorem 3.5.1 *If $Ax = b$ is a consistent nonhomogeneous linear system, and if W is the solution space of the associated homogeneous system $Ax = 0$, then the solution set of $Ax = b$ is the translated subspace $x_0 + W$, where x_0 is any solution of the nonhomogeneous system $Ax = b$ (Figure 3.5.1).*

- ▶ $W = \{(x,y) \mid x+y=1\}$ is obtained from
- ▶ $W_0 = \{(x,y) \mid x+y=0\}$ adding $(1,0)$ in W .
- ▶ $W = \{(x,y,z) \mid Ax+By+Cz=D\}$ is obtained from $W_0 = \{(x,y,z) \mid Ax+By+Cz=0\}$ by a translation by (x_0,y_0,z_0) for any point of W .
- ▶ $W = \{(x,y,z) \mid x+y+z=1, x-y=0\}$
 - $= \{(s+1/2, 1/2, s) \mid s \in \mathbb{R}\}$
 - $= \begin{bmatrix} s + 1/2 \\ 1/2 \\ s \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
 - Here $(1/2, 1/2, 0)$ is in W and $\{s(1,0,1)\}$ are solutions of the homogeneous system.



- ▶ Solution to $A\mathbf{x}=\mathbf{b}$ can be written as $\mathbf{x}=\mathbf{x}_h+\mathbf{x}_0$ where \mathbf{x}_0 is a particular solution and \mathbf{x}_h is a homogeneous solution.

Theorem 3.5.2 *A general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding a particular solution of $A\mathbf{x} = \mathbf{b}$ to a general solution of $A\mathbf{x} = \mathbf{0}$.*

Theorem 3.5.3 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

- (a) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (b) $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b} in R^m (i.e., is inconsistent or has a unique solution).

Theorem 3.5.4 *A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.*

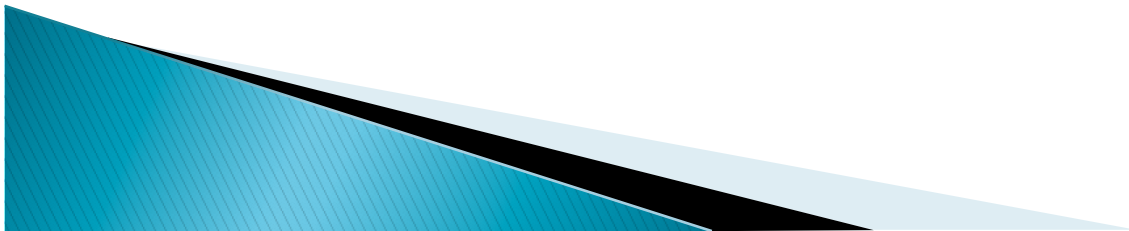


Consistency of a linear equation.

- ▶ $A\mathbf{x}=\mathbf{b}$ can be written as $x_1\mathbf{v}_1+x_2\mathbf{v}_2+\dots+x_n\mathbf{v}_n=\mathbf{b}$.

Theorem 3.5.5 *A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is in the column space of A .*

- ▶ This can be used to tell whether a certain vector can be written as a linear combination of some other vectors
- ▶ Example 2.



Hyperplanes

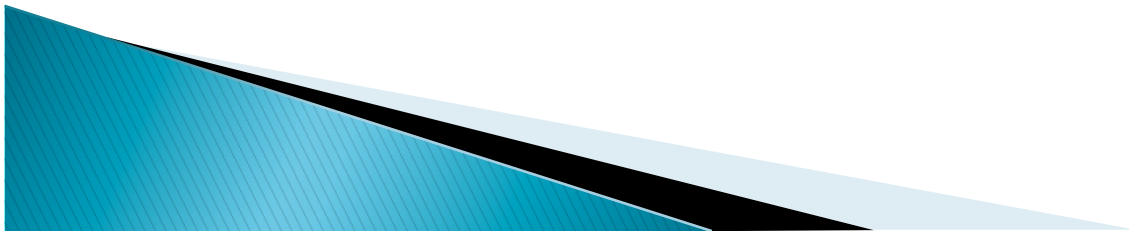
- ▶ $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ in \mathbb{R}^n . (a_i not all zero)
- ▶ The set of points (x_1, x_2, \dots, x_n) satisfying the equation is said to be a hyperplane.
- ▶ $b=0$ if and only if the hyperplane passes O .
- ▶ We can rewrite $a \cdot x = b$ where $a = (a_1, \dots, a_n)$ and $x = (x_1, \dots, x_n)$.
- ▶ A hyperplane with normal a .
- ▶ $a \cdot x = 0$. An orthogonal complement of a .
- ▶ Example 3.



Geometric interpretations of solution spaces.

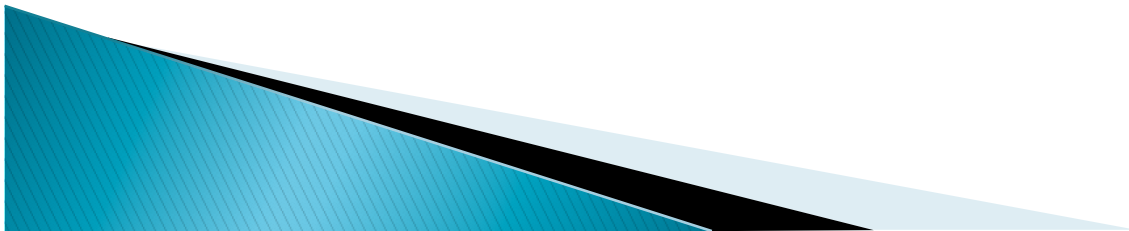
- ▶ $a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$
- ▶ $a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$
- ▶
- ▶ $a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m$

- ▶ This can be written: $a_1 \cdot x = 0$, $a_2 \cdot x = 0$,
..., $a_m \cdot x = 0$.



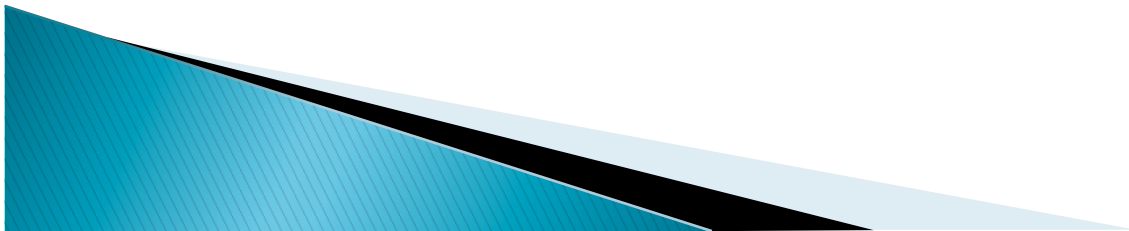
Theorem 3.5.6 *If A is an $m \times n$ matrix, then the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A .*

▶ See Example 4



Look ahead

- ▶ The set of solutions of a system of linear equation can be solved by Gauss–Jordan method.
- ▶ The result is the set W of vectors of form $x_0 + t_1 v_1 + \dots + t_s v_s$ where t_i are free variables.
- ▶ We show that $\{v_1, v_2, \dots, v_n\}$ is linearly independent later.
- ▶ Thus $W = x_0 + W_0$. W is an affine subspace of dimension s .



Ex. Set 3.5.

- ▶ 1–4 solving
- ▶ 5–8 linear combinations
- ▶ 7–10 span
- ▶ 11–20 orthogonality

