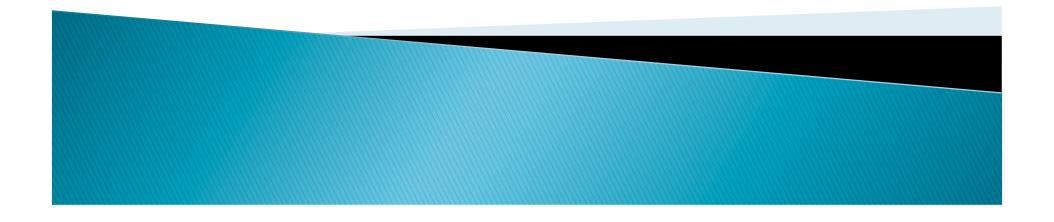
4.2. Properties of determinant



Determinant of A^T

- For 2x2 matrix det(A)=det(A^T).
- In general we have

Theorem 4.2.1 If A is a square matrix, then $det(A) = det(A^T)$.

- The simpliest way to prove this is to use the formula.
- The another method is to use the cofactor expansion along rows for A and that along columns for A^T. See p 190–191.



Effect of elementary operations on a determinant.

> The following will be important in computing:

Theorem 4.2.2 Let A be an $n \times n$ matrix.

- (a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det(B) = k det(A).
- (b) If B is the matrix that results when two rows or two columns of A are interchanged, then det(B) = -det(A).
- (c) If *B* is the matrix that results when a multiple of one row of *A* is added to another row or when a multiple of one column is added to another column, then det(B) = det(A).



- Proof (a): det(A)=a_i1C_i1+a_i2C_i2+... +a_inC_in.
 - If we multiply the ith-row by k, then each term in det(A) get multiplied by k.
- Proof (b): We can use formula.

- Suppose we exchanged two columns. Then in each elementary products in det(B).
- We can make a one-to-one correspondence between elementary products in det(A) to those of det(B) by identifying the same term up to signs.
- The sign in each term of B should be reversed from the corresponding one in A.
- To see in case we exchange two rows, we use A^{T} .

Proof (c): Add i-th row to j-th row. Cofactor expand det(A) along the j-th row. Then we have

$$det(A') = (a_{j1} + ka_{i1})C_{j1} + (a_{j2} + ka_{i2})C_{j2} + \dots + (a_{jn} + ka_{in})C_{jn}$$

= det(A) + k det(A'')

- Here A'' is a matrix obtained by replacing the j-th row of A by the i-th row of A.
- By Theorem 4.2.3 (a), det(A'')=0.
- For column case, we use A^{T} .
- See Example 1.



Theorem 4.2.3 Let A be an $n \times n$ matrix.

- (a) If A has two identical rows or columns, then det(A) = 0.
- (b) If A has two proportional rows or columns, then det(A) = 0.
- (c) $\det(kA) = k^n \det(A)$.
 - Proof (a): If A has two same rows, then after the exchange of the two rows, we still get A.
 By Theorem 4.2.2 (b), det(A)=-det(A). Thus det(A)=0.
 - Proof (b): If A has two proportional rows, then one row is a multiple of the other row, say by k. If we multiply the other row by 1/k, then the result has determinant 0. Thus det(A)=0 by Theorem 4.2.2 (a).

Proof (c): omit.

Simplifying cofactor expansion

- Given a matrix, we do row ad column operations of type Theorem 4.2.2 (c) to make many zeros.
- Example 4.



Determinats by Gaussian eliminations

- We can use Gaussian elimination to evaluate a determinant.
- Each multiplication by k of a row should be compensated by multiplying by1/k to the result.
- Each row exchange should be compensated by the multiplication by -1.
- For type (c), we do not need any compensations.
- See Example *.

- First we need. R ref of A. Then det(R)=0 iff det(A)=0. This follows since each elementary operation preserves det being 0 or nonzero.
- Proof: ->) If A is invertible, then ref of A is I. Thus, det(A) is nonzero.
- <-) If det(A) is not zero, then det(R) is not zero for the ref R of A. Thus R has no zero rows. Hence R is I. If ref of A is I, then A is invertible by Theorem 3.3.3.

Theorem 4.2.5 If A and B are square matrices of the same size, then det(AB) = det(A) det(B)

Proof: We need:

Lemma 4.2.8 Let *E* be an $n \times n$ elementary matrix and I_n the $n \times n$ identity matrix.

- (a) If E results by multiplying a row of I_n by k, then det(E) = k.
- (b) If E results by interchanging two rows of I_n , then det(E) = -1.
- (c) If E results by adding a multiple of one row of I_n to another, then det(E) = 1.

Lemma 4.2.9 If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

det(EB) = det(E) det(B)



- Proof of 4.2.8: Just computations
- Proof of 4.2.9. EB is just a result of row operation. det(EB) is just some number times det(B).

The number is det(E).

- Proof of 4.2.5: If A is singular (i.e. not invertible), then AB is singular (not invertible) also. By Theorem 4.2.4 both have determinant 0 and we are done.
- ▶ If A is invertible, then A=E_1E_2...E_k.
 - det(AB)=det(E_1E_2...E_kB)= det(E_1)det(E_2... E_k)det(B) =
 - $det(E_1)det(E_2)...det(E_k)det(B).$

 $det(\Delta) = det(E_1)det(E_2)...det(E_k).$

Thus the conclusion holds.

Computing determinants by LUdecompositions

- ► A=LU. det(A)=det(L)det(U).
- We just need to multiply the diagonals.
- Obtaining LU decompositions is around 2n³/3 which is much smaller than n!.



Determinant of an inverse matrix

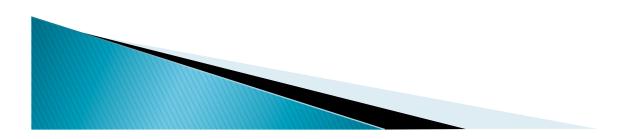
- Theorem 4.2.6. $det(A^{-1}) = 1/det(A)$.
- > Proof: $AA^{-1}=I$. $det(A)det(A^{-1})=det(I)=1$.
- Deteminant of A+B.
 - It is not true that det(A+B)=det(A)+det(B).
 - However, there are other invariants that we haven't learned that we compensate the difference.



A unifying theorem

Theorem 4.2.7 If A is an $n \times n$ matrix, then the following statements are equivalent.

- (a) The reduced row echelon form of A is I_n .
- (b) A is expressible as a product of elementary matrices.
- (c) A is invertible.
- (d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (e) $A\mathbf{x} = \mathbf{b}$ is consistent for every vector \mathbf{b} in \mathbb{R}^n .
- (f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector \mathbf{b} in \mathbb{R}^n .
- (g) The column vectors of A are linearly independent.
- (h) The row vectors of A are linearly independent.
- (i) $\det(A) \neq 0$.



Ex set 4.2.

- ▶ 1–10 Theory practise
- ▶ 11-18 Gaussian elimination
- ▶ 19–28 Theory

