4.4. A first look at Eigenvalues and eigenvectors

Fixed points

- $Ax=x$. $(I-A)x=0$.
- If I–A is invertible, then there are only trivial solutions.
- Thus we have:

Theorem 4.4.1 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A has nontrivial fixed points.*

(b) *$I - A$ is singular.*

(c) $\det(I - A) = 0$.

- Example 1.

Eigenvalue and Eigenvectors

- Ax = Lx for a real number L (could be zero)

Definition 4.4.3 If A is an $n \times n$ matrix, then a scalar λ is called an ***eigenvalue*** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. If λ is an eigenvalue of A, then every nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is called an ***eigenvector*** of A corresponding to λ.

- (LI -A)x=0. This has a nontrivial solution if and only if LI–A is singular if and only if det(LI–A)=0.
- This is called the characteristic equation.

Theorem 4.4.4 *If A is an $n \times n$ matrix and λ is a scalar, then the following statements are equivalent.*

(a) λ *is an eigenvalue of* A.

(b) λ *is a solution of the equation* $\det(\lambda I - A) = 0$.

(c) *The linear system* $(\lambda I - A)\mathbf{x} = \mathbf{0}$ *has nontrivial solutions.*

- Example 2

Eigenvalues of triangular matrices

- A nxn triangular matrix with diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$.
- Then $\det(LI-A)=(L-a_{11})(L-a_{22})\ldots(L-a_{nn})$.
- Thus the eigenvalues are a_{11}, a_{22}, ..., a_{nn}.

Theorem 4.4.5 *If A is a triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.*

Theorem 4.4.6 *If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, and if k is any positive integer, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.*

Theorem 4.4.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.*

(g) *The column vectors of A are linearly independent.*

(h) *The row vectors of A are linearly independent.*

(i) $\det(A) \neq 0$.

(j) *$\lambda = 0$ is not an eigenvalue of A.*

Complex eigenvalues

- There might be complex roots of characteristic polynomial, with only real coefficients.
- Thus when a complex root appears its complex conjugate appears as a root also.
- Thus eigenvalues appear as real numbers or as complex numbers in conjugate pairs.

Multiplicity of eigenvalues

- When you factor the characteristic polynomial, one of the following happens:
  1. Factor completely into distinct real linear factors.
  2. Some real linear factors may be repeated.
  3. There might be quadratic factors, which may be repeated.
- If we allow complex numbers, then a characteristic polynomial factors completely into linear factors which may be repeated.
- The multiplicity of an eigenvalue L_i is the number of times (L-L_i) appears in the factorization.

Theorem 4.4.8 *If A is an $n \times n$ matrix, then the characteristic polynomial of A can be expressed as*

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A and $m_1 + m_2 + \cdots + m_k = n$.

- Example **

Eigenvalue analysis of 2x2-matrices

- A=[(a,b),(c,d)].
- Det(LI−A)=(L−a)(L−d)−bc=L^2−(a+d)L+(ad−bc) = L^2−tr(A)L+det(L).
- Discriminants

Theorem 4.4.9 *If A is a 2×2 matrix with real entries, then the characteristic equation of A is*

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

and

(*a*) *A has two distinct real eigenvalues if* $\operatorname{tr}(A)^2 - 4\det(A) > 0$*;*

(*b*) *A has one repeated real eigenvalue if* $\operatorname{tr}(A)^2 - 4\det(A) = 0$*;*

(*c*) *A has two conjugate imaginary eigenvalues if* $\operatorname{tr}(A)^2 - 4\det(A) < 0$.

- In case (a), A has two eigenvectors not parallel to each other.
- In case (b), A may have only one eigenvector. eg. A=[[2,1],[0,2]].
- In case (c), A have two complex eigenvectors not parallel to each other.
- See Example 5.

Eigenvalues of Symmetric 2x2 matrices

When given 2x2 symmetric matrix, we see that
$tr(A)^2-4det(A)=$
$(a+d)^2-4(ad-b^2) = (a-d)^2+4b^2 \geq 0.$
Thus, A has only real eigenvalues.
If A has a repeated eigenvalue, then (a-d)=b=0.
Thus A=[[a,0],[a,0]].

Theorem 4.4.10 *A symmetric 2×2 matrix with real entries has real eigenvalues. Moreover, if A is of the form*

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{23}$$

then A has one repeated eigenvalue, namely $\lambda = a$; otherwise it has two distinct eigenvalues.

Theorem 4.4.11

(*a*) *If a* 2×2 *symmetric matrix with real entries has one repeated eigenvalue, then the eigenspace corresponding to that eigenvalue is* R^2.

(*b*) *If a* 2×2 *symmetric matrix with real entries has two distinct eigenvalues, then the eigenspaces corresponding to those eigenvalues are perpendicular lines through the origin of* R^2.

- Example 6: (Ex 5(a))

Expressions for determinants and traces in terms of eigenvalues.

Theorem 4.4.12 *If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (repeated according to multiplicity), then:*

(*a*) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

(*b*) $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

- Proof a) det(LI–A)=(L–L_1)…(L–L_n).
  - Let L=0. Then det(–A)=(-1)^{n}L_1…L_n.
  - Since det(–A)=(–1)^{n}det(A), we have the result.
- Proof b) In det(LI–A), the L^{n-1} terms come from the diagonal product (L–a_11)(L–a_22)…(L–a_nn): Why?
  - The coefficent is –(a_11+a_22+…+a_nn)=tr(A)
  - In (L–L_1)…(L–L_n), the L^{n-1} term has a coefficient –(L_1+L_2+…+L_n)

Eigenvalues by numerical methods.

- For n < 5, there is an exact algbraic method since we can solve such polynomials.
- For n≥ 5, there are no algebraic method.
- But there are numerical approximations to eigenvalues and eigenvectors.