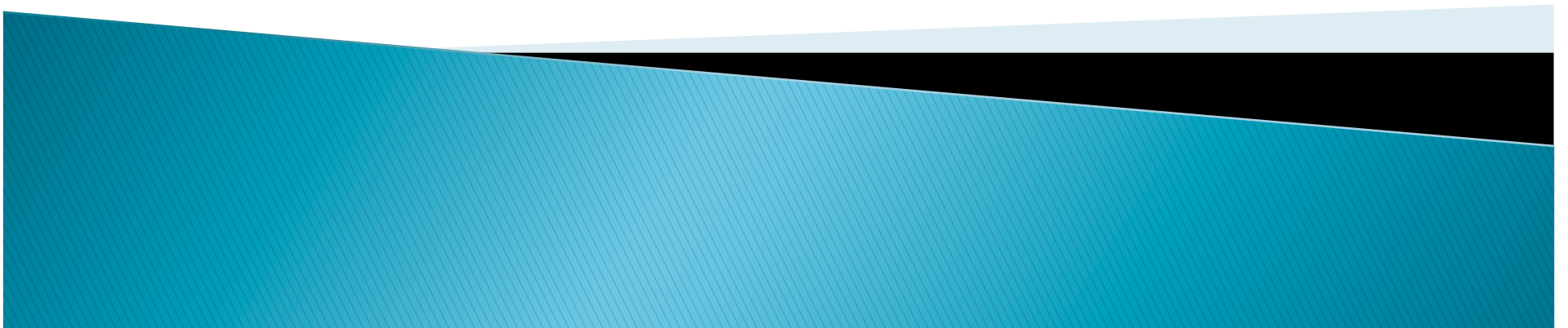


4.4. A first look at Eigenvalues and eigenvectors



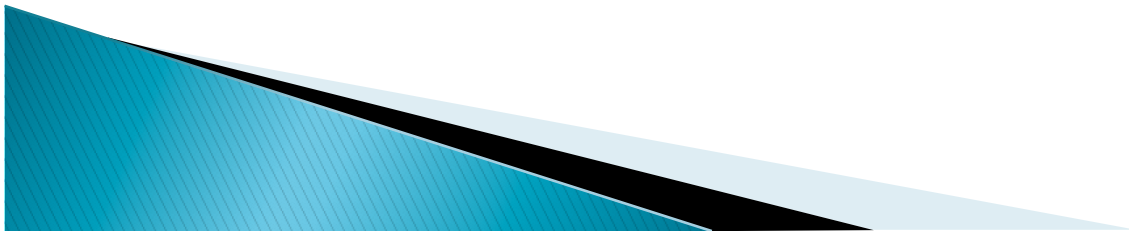
Fixed points

- ▶ $Ax=x$. $(I-A)x=0$.
- ▶ If $I-A$ is invertible, then there are only trivial solutions.
- ▶ Thus we have:

Theorem 4.4.1 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

- (a) *A has nontrivial fixed points.*
- (b) *$I - A$ is singular.*
- (c) $\det(I - A) = 0$.

- ▶ Example 1.

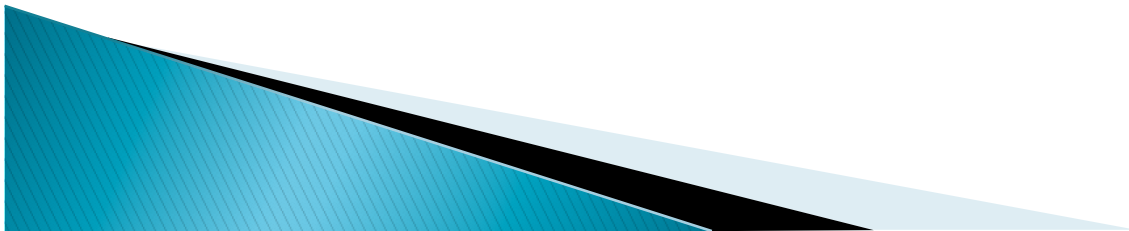


Eigenvalue and Eigenvectors

- ▶ $A\mathbf{x} = L\mathbf{x}$ for a real number L (could be zero)

Definition 4.4.3 If A is an $n \times n$ matrix, then a scalar λ is called an *eigenvalue* of A if there is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$. If λ is an eigenvalue of A , then every nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$ is called an *eigenvector* of A corresponding to λ .

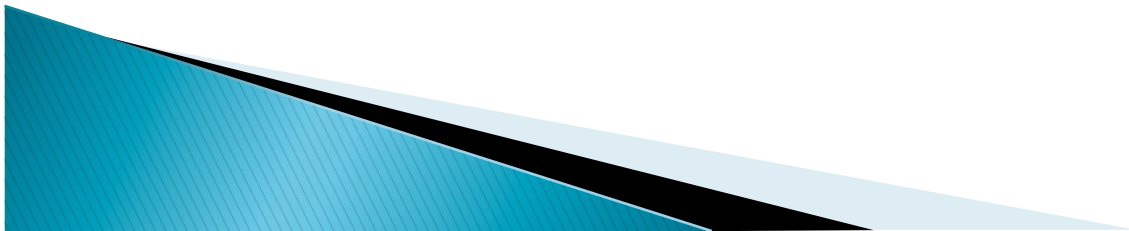
- ▶ $(L I - A)\mathbf{x} = 0$. This has a nontrivial solution if and only if $L I - A$ is singular if and only if $\det(L I - A) = 0$.
- ▶ This is called the characteristic equation.



Theorem 4.4.4 *If A is an $n \times n$ matrix and λ is a scalar, then the following statements are equivalent.*

- (a) λ is an eigenvalue of A .
- (b) λ is a solution of the equation $\det(\lambda I - A) = 0$.
- (c) The linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

▶ **Example 2**



Eigenvalues of triangular matrices

- ▶ A $n \times n$ triangular matrix with diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$.
- ▶ Then $\det(LI - A) = (L - a_{11})(L - a_{22}) \dots (L - a_{nn})$.
- ▶ Thus the eigenvalues are $a_{11}, a_{22}, \dots, a_{nn}$.

Theorem 4.4.5 *If A is a triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A .*



Theorem 4.4.6 *If λ is an eigenvalue of a matrix A and \mathbf{x} is a corresponding eigenvector, and if k is any positive integer, then λ^k is an eigenvalue of A^k and \mathbf{x} is a corresponding eigenvector.*

Theorem 4.4.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

- (a) *The reduced row echelon form of A is I_n .*
- (b) *A is expressible as a product of elementary matrices.*
- (c) *A is invertible.*
- (d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- (e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector \mathbf{b} in R^n .*
- (f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector \mathbf{b} in R^n .*
- (g) *The column vectors of A are linearly independent.*
- (h) *The row vectors of A are linearly independent.*
- (i) *$\det(A) \neq 0$.*
- (j) *$\lambda = 0$ is not an eigenvalue of A .*

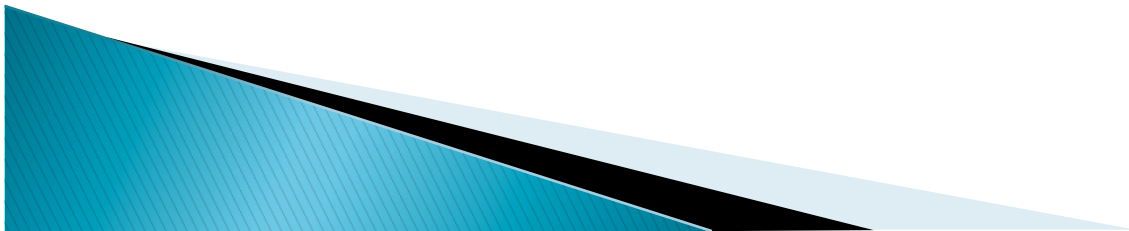
Complex eigenvalues

- ▶ There might be complex roots of characteristic polynomial, with only real coefficients.
- ▶ Thus when a complex root appears its complex conjugate appears as a root also.
- ▶ Thus eigenvalues appear as real numbers or as complex numbers in conjugate pairs.



Multiplicity of eigenvalues

- ▶ When you factor the characteristic polynomial, one of the following happens:
 1. Factor completely into distinct real linear factors.
 2. Some real linear factors may be repeated.
 3. There might be quadratic factors, which may be repeated.
- ▶ If we allow complex numbers, then a characteristic polynomial factors completely into linear factors which may be repeated.
- ▶ The multiplicity of an eigenvalue L_i is the number of times $(L - L_i)$ appears in the factorization.

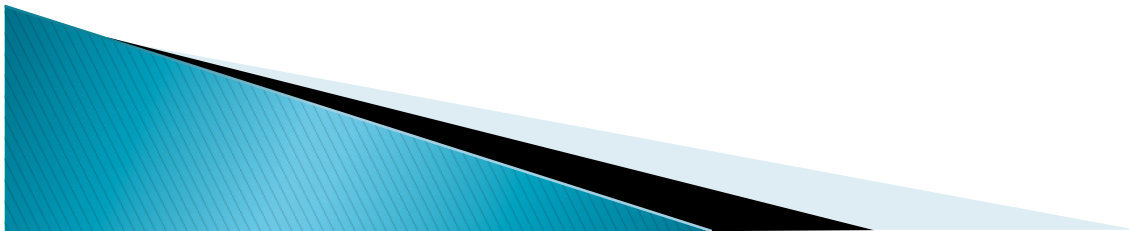


Theorem 4.4.8 *If A is an $n \times n$ matrix, then the characteristic polynomial of A can be expressed as*

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of A and $m_1 + m_2 + \cdots + m_k = n$.

▶ **Example ****



Eigenvalue analysis of 2x2-matrices

- ▶ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- ▶ $\text{Det}(L I - A) = (L - a)(L - d) - bc = L^2 - (a + d)L + (ad - bc) = L^2 - \text{tr}(A)L + \det(A)$.
- ▶ Discriminants

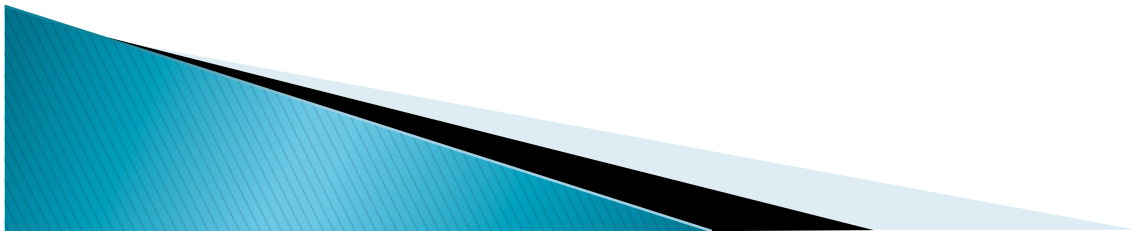
Theorem 4.4.9 *If A is a 2×2 matrix with real entries, then the characteristic equation of A is*

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

and

- (a) *A has two distinct real eigenvalues if $\text{tr}(A)^2 - 4 \det(A) > 0$;*
- (b) *A has one repeated real eigenvalue if $\text{tr}(A)^2 - 4 \det(A) = 0$;*
- (c) *A has two conjugate imaginary eigenvalues if $\text{tr}(A)^2 - 4 \det(A) < 0$.*

- ▶ In case (a), A has two eigenvectors not parallel to each other.
- ▶ In case (b), A may have only one eigenvector.
eg. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.
- ▶ In case (c), A have two complex eigenvectors not parallel to each other.
- ▶ See Example 5.



Eigenvalues of Symmetric 2x2 matrices

When given 2x2 **symmetric** matrix, we see that

$$\text{tr}(A)^2 - 4\det(A) =$$

$$(a+d)^2 - 4(ad-b^2) = (a-d)^2 + 4b^2 \geq 0.$$

Thus, A has only real eigenvalues.

If A has a repeated eigenvalue, then $(a-d)=b=0$.

Thus $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

Theorem 4.4.10 *A symmetric 2×2 matrix with real entries has real eigenvalues. Moreover, if A is of the form*

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{23}$$

then A has one repeated eigenvalue, namely $\lambda = a$; otherwise it has two distinct eigenvalues.

Theorem 4.4.11

- (a) *If a 2×2 symmetric matrix with real entries has one repeated eigenvalue, then the eigenspace corresponding to that eigenvalue is \mathbb{R}^2 .*
- (b) *If a 2×2 symmetric matrix with real entries has two distinct eigenvalues, then the eigenspaces corresponding to those eigenvalues are perpendicular lines through the origin of \mathbb{R}^2 .*

▶ Example 6: (Ex 5(a))



Expressions for determinants and traces in terms of eigenvalues.

Theorem 4.4.12 *If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (repeated according to multiplicity), then:*

(a) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

(b) $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

- ▶ Proof a) $\det(LI - A) = (L - \lambda_1) \cdots (L - \lambda_n)$.
 - Let $L=0$. Then $\det(-A) = (-1)^n \lambda_1 \cdots \lambda_n$.
 - Since $\det(-A) = (-1)^n \det(A)$, we have the result.
- ▶ Proof b) In $\det(LI - A)$, the L^{n-1} terms come from the diagonal product $(L - a_{11})(L - a_{22}) \cdots (L - a_{nn})$: Why?
 - The coefficient is $-(a_{11} + a_{22} + \cdots + a_{nn}) = -\text{tr}(A)$
 - In $(L - \lambda_1) \cdots (L - \lambda_n)$, the L^{n-1} term has a coefficient $-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$



Eigenvalues by numerical methods.

- ▶ For $n < 5$, there is an exact algebraic method since we can solve such polynomials.
- ▶ For $n \geq 5$, there are no algebraic method.
- ▶ But there are numerical approximations to eigenvalues and eigenvectors.

