

7_3 The fundamental spaces of a matrix.

Row space, column space, null space

- ✦ A $m \times n$ matrix
 - ✦ Row space of A : $\text{row}(A)$: span of row vectors in \mathbb{R}^n .
 - ✦ Column space of A : $\text{col}(A)$: span of column vectors in \mathbb{R}^m .
 - ✦ Null space of A : $\text{null}(A)$: the solution space of $Ax=0$.
- ✦ In addition: $\text{row}(A^T)$, $\text{col}(A^T)$, $\text{null}(A^T)$
- ✦ $\text{row}(A^T) = \text{col}(A)$, $\text{col}(A^T) = \text{row}(A)$.
- ✦ So, $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$, $\text{null}(A^T)$ are fundamental spaces of A .

Definition 7.3.1 The dimension of the row space of a matrix A is called the *rank* of A and is denoted by $\text{rank}(A)$; and the dimension of the null space of A is called the *nullity* of A and is denoted by $\text{nullity}(A)$.

Orthogonal complements

Definition 7.3.2 If S is a nonempty set in R^n , then the *orthogonal complement* of S , denoted by S^\perp , is defined to be the set of all vectors in R^n that are orthogonal to every vector in S .

- ✦ Example: A is $n \times n$ -matrix. The solution space of $Ax=0$ is exactly the orthogonal complement of row vectors of A .
- ✦ Example: two vectors in R^3 . The cross product solution.

Theorem 7.3.3 If S is a nonempty set in R^n , then S^\perp is a subspace of R^n .

Properties of the orthogonal complements

Theorem 7.3.4

- (a) *If W is a subspace of R^n , then $W^\perp \cap W = \{\mathbf{0}\}$.*
- (b) *If S is a nonempty subset of R^n , then $S^\perp = \text{span}(S)^\perp$.*
- (c) *If W is a subspace of R^n , then $(W^\perp)^\perp = W$.*

- ✦ Proof: (a) If v is in W^\perp and in W , then v is orthogonal to itself. $v \cdot v = \|v\|^2 = 0$. The length of v is zero and v is zero.
- ✦ (b) W^\perp is in $\text{span}(S)^\perp$ since any vector v orthogonal to S is orthogonal to every vector in $\text{span}(S)$.
 - ✦ $\text{span}(S)^\perp$ is in W^\perp . If v is orthogonal to $\text{span}(S)$, then v is orthogonal to S .
- ✦ (c) later.

$$\text{row}(A)^c = \text{null}(A)$$

Theorem 7.3.5 *If A is an $m \times n$ matrix, then the row space of A and the null space of A are orthogonal complements.*

Theorem 7.3.6 *If A is an $m \times n$ matrix, then the column space of A and the null space of A^T are orthogonal complements.*

✦ Proof: The solution space is a set of vectors orthogonal to the row vectors of A .

✦ $\text{row}(A)^c = \text{null}(A)$, $\text{null}(A)^c = \text{row}(A)$. (In \mathbb{R}^n)

✦ $\text{col}(A)^c = \text{null}(A^T)$, $\text{null}(A^T)^c = \text{col}(A)$. (In \mathbb{R}^m)

Theorem 7.3.7

- (a) *Elementary row operations do not change the row space of a matrix.*
- (b) *Elementary row operations do not change the null space of a matrix.*
- (c) *The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of the matrix.*

The row operations will change the column space.

Theorem 7.3.8 *If A and B are matrices with the same number of columns, then the following statements are equivalent.*

- (a) *A and B have the same row space.*
- (b) *A and B have the same null space.*
- (c) *The row vectors of A are linear combinations of the row vectors of B , and conversely.*

(a) \leftrightarrow (b). Null space is orthogonal complement of the row space.

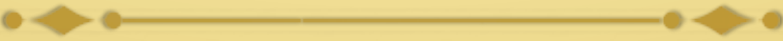
(c) \rightarrow (a). Clear. (a) \rightarrow (c). Row vectors of A span row space of B and conversely.

Finding rows by row reductions.

- ✦ $S = \{v_1, v_2, \dots, v_s\}$. Find a basis of $\text{Span } S$.
- ✦ 1. We form A where v_i s are rows. Apply Gauss–Jordan elimination. This does not change the span and finds the basis.
- ✦ 2. Find a basis in S . This is slightly different. We will do this later.
- ✦ Example 4. Given four vectors in \mathbb{R}^5 , we use Gauss–Jordan elimination to obtain the echelon form. The basis is the set of row vectors.

- ✦ Example 4(b). Find a basis of W^c .
 - ✦ Form 4×5 -matrix A . Obtain ref. Find the solution space and find its basis using the fundamental vectors.
- ✦ Example 5. Given v_1, v_2, v_3, v_4 in \mathbb{R}^5 , we find B such that the solution space of $Bx=0$ is $\text{span } W$.
 - ✦ Use the basis of W^c .

Determining whether a vector is in a given subspace.



- ✦ Problem 1. Given $S = \{v_1, v_2, \dots, v_s\}$ in \mathbb{R}^m , determine a condition on b_1, \dots, b_m so that $b = (b_1, \dots, b_m)$ will lie in $\text{span } S$.
- ✦ Problem 2. Given an $m \times n$ matrix A , find a condition on b_1, \dots, b_m so that b lies in $\text{col}(A)$.
- ✦ Problem 3. Given a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, determine a condition on b s.t. b is in $\text{ran } T$.
- ✦ Example 6.