# 7_3 The fundamental spaces of a matrix.

# Row space, column space, null space

- A mxn matrix
  - Row space of A: row(A): span of row vectors in $R^n$.
  - Column space of A: col(A): span of column vectors in $R^m$.
  - Null space of A: null(A): the solution space of Ax=O.
- In addition: row($A^T$), col($A^T$), null($A^T$)
- row($A^T$) =col(A), col($A^T$)=row(A).
- So, row(A), col(A), null(A), null($A^T$) are fundamental spaces of A.

**Definition 7.3.1** The dimension of the row space of a matrix $A$ is called the ***rank*** of $A$ and is denoted by $\text{rank}(A)$; and the dimension of the null space of $A$ is called the ***nullity*** of $A$ and is denoted by $\text{nullity}(A)$.

# Orthogonal complements

**Definition 7.3.2** If $S$ is a nonempty set in $R^n$, then the ***orthogonal complement*** of $S$, denoted by $S^\perp$, is defined to be the set of all vectors in $R^n$ that are orthogonal to every vector in $S$.

- Example: A is nxn-matrix. The solution space of Ax=0 is exactly the orthogonal complement of row vectors of A.
- Example: two vectors in $R^3$. The cross product solution.

**Theorem 7.3.3** *If $S$ is a nonempty set in $R^n$, then $S^\perp$ is a subspace of $R^n$.*

# Properties of the orthogonal complements

**Theorem 7.3.4**

(*a*) *If* $W$ *is a subspace of* $R^n$, *then* $W^{\perp} \cap W = \{\mathbf{0}\}$.

(*b*) *If* $S$ *is a nonempty subset of* $R^n$, *then* $S^{\perp} = \operatorname{span}(S)^{\perp}$.

(*c*) *If* $W$ *is a subspace of* $R^n$, *then* $(W^{\perp})^{\perp} = W$.

- Proof: (a) If v is in $W^c$ and in W, then v is orthogonal to itself. v.v.=$||v||^2$=0. The length of v is zero and v is zero.
- (b) $S^c$ is in span$(S)^c$ since any vector v orthogonal to S is orthogonal to every vector in span(S).
  - Span$(S)^c$ is in $S^c$. If v is orthogonal to Span(S), then v is orthogonal to S.
- (c) later.

# $\text{row}(A)^c = \text{null}(A)$

**Theorem 7.3.5** *If $A$ is an $m \times n$ matrix, then the row space of $A$ and the null space of $A$ are orthogonal complements.*

**Theorem 7.3.6** *If $A$ is an $m \times n$ matrix, then the column space of $A$ and the null space of $A^T$ are orthogonal complements.*

- Proof: The solution space is a set of vectors orthogonal to the row vectors of A.
- $\text{row}(A)^c = \text{null}(A)$, $\text{null}(A)^c = \text{row}(A)$. (In $R^n$ )
- $\text{col}(A)^c = \text{null}(A^T)$, $\text{null}(A^T)^c = \text{col}(A)$. (In $R^m$)

**Theorem 7.3.7**

(*a*) *Elementary row operations do not change the row space of a matrix.*

(*b*) *Elementary row operations do not change the null space of a matrix.*

(*c*) *The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of the matrix.*

The row operations will change the column space.

**Theorem 7.3.8** *If A and B are matrices with the same number of columns, then the following statements are equivalent.*

(*a*) *A and B have the same row space.*

(*b*) *A and B have the same null space.*

(*c*) *The row vectors of A are linear combinations of the row vectors of B, and conversely.*

(a)<->(b). Null space is orthogonal complement of the row space.

(c)->(a). Clear. (a)->(c). Row vectors of A span row space of B and conversely.

# Finding rows by row reductions.

- $S = \{v\_1, v\_2, \cdots, v\_s\}$. Find a basis of Span S.
- 1. We form A where v_is are rows. Apply Gauss–Jordan elimination. This does not change the span and finds the basis.
- 2. Find a basis in S. This is slightly different. We will do this later.
- Example 4. Given four vectors in $R^5$, we use Gauss-Jordan elimination to obtain the echelon form. The basis is the set of row vectors.

- Example 4(b). Find a basis of $W^c$.
  - Form 4x5-matrix A. Obtain ref. Find the solution space and find its basis using the fundamental vectors.
- Example 5. Given v_1,v_2,v_3,v_4 in $R^5$, we find B such that the solution space of Bx=0 is span W.
  - Use the basis of $W^c$.

# Determining whether a vector is in a given subspace.

- Problem 1. Given S={v_1,v_2,..,v_s} in $R^m$, determine a condition on b_1,⋯,b_m so that b=(b_1,⋯,b_m) will lie in span S.
- Problem 2. Given an mxn matrix A, find a condition on b_1,..,b_m so that b lies in col(A).
- Problem 3. Given a linear transformation T:$R^n$->$R^m$, determine a condition on b s.t. b is in ranT.
- Example 6.