7_5. The rank theorem

Column rank = row rank.

A deep thought indeed!

The rank theorem

Theorem 7.5.1 (***The Rank Theorem***) *The row space and column space of a matrix have the same dimension.*

- Proof: Let $T:R^n \to R^m$ be defined by T(x)=Ax. Then
  - dim ranT=dim column space A.
  - Ker T= null A.
  - dim ran T + dim ker T = n.
    - Choose a basis a_1,..,a_k in ker T. dim kerT =n.
    - Expand a_k+1,…,a_n in R^n to a basis.
    - T(a_k+1)…,T(a_n) is independent. They span ranT.
    - Thus n-k = dim ran T.
  - dim column space A + nullity A =n.
  - rank A+ nullity A=n. The Proof is done.
- Example 1:

Theorem 7.5.2 *If A is an $m \times n$ matrix, then*

$$\operatorname{rank}(A) = \operatorname{rank}(A^T) \quad (3)$$

- $\operatorname{rank}(A^T)+\operatorname{nullity}(A^T)=m$. ($A^T$ is nxm matrix)
- $\operatorname{rank}(A)+\operatorname{nullity}(A^T)=m$.
- Thus the dimension of four fundamental space is determined from a single number rank A.
- dim row A = k, dim null A=n-k, dim colA=k, dim nullA^T=m-k.
- See Example 2.

The relationship between consistency and rank.

Theorem 7.5.3 (***The Consistency Theorem***) *If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the following statements are equivalent.*

(*a*) $A\mathbf{x} = \mathbf{b}$ *is consistent.*

(*b*) $\mathbf{b}$ *is in the column space of A.*

(*c*) *The coefficient matrix A and the augmented matrix $[A \mid \mathbf{b}]$ have the same rank.*

- Proof: (a) <->(b) by Theorem 3.5.5.
  (a)<->(c). Put both into ref. Then the number of the nonzero rows are the same for consistency.
- Example 3:

Definition 7.5.4 An $m \times n$ matrix A is said to have ***full column rank*** if its column vectors are linearly independent, and it is said to have ***full row rank*** if its row vectors are linearly independent.

Theorem 7.5.5 *Let A be an $m \times n$ matrix.*

(*a*) *A has full column rank if and only if the column vectors of A form a basis for the column space, that is, if and only if* $\operatorname{rank}(A) = n$.

(*b*) *A has full row rank if and only if the row vectors of A form a basis for the row space, that is, if and only if* $\operatorname{rank}(A) = m$.

- Proof: clear

Theorem 7.5.6 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* R^m.

(c) *A has full column rank.*

- Proof: (a)<->(b) Theorem 3.5.3.
- (a)<->(c). Ax=0 can be written x_1a_1+…+x_na_n=0. The trival solution <-> a_i independent. <-> A has full column rank.
- Example 5.

Overdetermined and underdetermined

- A mxn-matrix.
  - If m>n, then overdetermined.
  - If m <n, then underdetermined.

Theorem 7.5.7 *Let A be an $m \times n$ matrix.*

(a) (***Overdetermined Case***) *If $m > n$, then the system $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$ in R^m.*

(b) (***Underdetermined Case***) *If $m < n$, then for every vector $\mathbf{b}$ in R^m the system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.*

- Proof(a): m>n. The column vectors of A cannot span R^m.
- (b): m<n. The column vectors of A is linearly dependent. Ax=0 has infinitely many solutions. Use Theorem 3.5.2.

Matrices of form A^TA and AA^T.

- AA^T. The ij-th entry is a_i.a_j. a_i column vector
- A^TA. The ij-th entry is r_i.r_j. r_i row vector

Theorem 7.5.8 *If A is an $m \times n$ matrix, then:*

(*a*) *A and A^TA have the same null space.*

(*b*) *A and A^TA have the same row space.*

(*c*) *A^T and A^TA have the same column space.*

(*d*) *A and A^TA have the same rank.*

- Proof (a). null A is a subset of null A^TA. (if Ax=0, then $A^TAx=0$).
  - null A^TA is a subset of null A. (If $A^TAv=0$, then v is orthogonal to every row vector of A^TA. Since A^TA is symmeric, v is orthogonal to every column vectors of A^TA. Thus, $v^TA^TAv=0$. $(Av)^TAv=0$. Thus Av.Av=0 and Av=0.
  - (b) By Theorem 7.3.5. The complements are the same.
  - (c). The column space of A^T is the row space of A.
  - (d). From (b).

Theorem 7.5.9 *If A is an $m \times n$ matrix, then:*

(a) *A^T and AA^T have the same null space.*

(b) *A^T and AA^T have the same row space.*

(c) *A and AA^T have the same column space.*

(d) *A and AA^T have the same rank.*

Unifying theorem.

Theorem 7.5.10 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* R^m.

(c) *A has full column rank.*

(d) A^TA *is invertible.*

Proof) (a)<->(b)<->(c). Done before.
(c)<->(d). A^TA is an nxn matrix. A^TA is invertible if and only if A^TA is of full rank. By Theorem 7.5.8(d), this is if and only if A is full rank.

Theorem 7.5.11 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A^T\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A^T\mathbf{x} = \mathbf{b}$ *has at most one solution for every vector* $\mathbf{b}$ *in* R^n.

(c) A *has full row rank.*

(d) AA^T *is invertible.*

- Example 7: