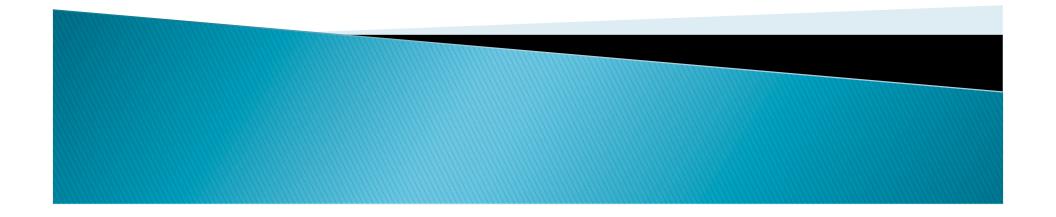
8.1. Matrix representations of linear transformations Matrix of a linear operator with respect to a basis.



Matrix of linear operators w.r.t. a basis

- One can use different representation of a transformation using basis.
- If one uses a right basis, the representation get simpler and easier to understand.
- ▶ x->Tx.
- [x]_B -> [Tx]_B = A[x]_B for some matrix A depending on B.
- How does one find A_B?

This amounts to change of coordinates.
 (Coordinates are usually not canonical.)

Theorem 8.1.1 Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for \mathbb{R}^n , and let

$$A = \left[[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B \mid \dots \mid [T(\mathbf{v}_n)]_B \right]$$
(4)

Then

$$[T(\mathbf{x})]_B = A[\mathbf{x}]_B \tag{5}$$

for every vector \mathbf{x} in \mathbb{R}^n . Moreover, the matrix A given by Formula (4) is the only matrix with property (5).

- A is called the matrix of t w.r.t. the basis B.
- $[T]_B = A = [[T(v_1)]_B, [T(v_2)]_B, ..., [T(v_n)]_B].$
- $[T(x)]_B = [T]_B[x]_B$.
- If S is the standard basis, [T]_S is the standard matrix for T.
- Example 1.
- Example 2. A matrix realized as a rotation....



Changing basis

- What is the relationship between [T]_B and [T]_B' for two basis B and B'.
- $[T]_B[x]_B = [T(x)]_B$. $[T]_B'[x]_B' = [T(x)]_B'$.
- $P_{(B->B')}[T(x)]_B = [T(x)]_B'$
- $P_(B -> B')[x]_B = [x]_B'$
- $[T]_B'[x]_B' = [T(x)]_B'$
- $T]_B'P[x]_B=P[T(x)]_B.$
- $(P^{-1}[T]_B'P)[x]_B = [T(x)]_B.$
- Compare to $[T]_B[x]_B=[T(x)]_B$.
- Thus $P^{-1}[T]_B'P=[T]_B$.

Theorem 8.1.2 If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_n\}$ are bases for \mathbb{R}^n , then $[T]_B$ and $[T]_{B'}$ are related by the equation $[T]_{B'} = P[T]_B P^{-1}$ (12)

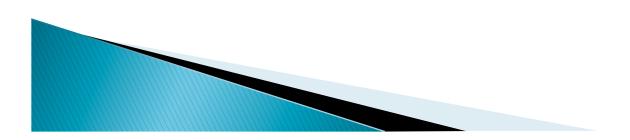
in which

$$P = P_{B \to B'} = \left[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \dots \mid [\mathbf{v}_n]_{B'} \right]$$
(13)

is the transition matrix from B to B'. In the special case where B and B' are orthonormal bases the matrix P is orthogonal, so (12) is of the form

$$[T]_{B'} = P[T]_B P^T \tag{14}$$

[T]_B=P⁻¹[T]_B'P. [T]_B=P^T[T]P if B, B' orthonormal basis.



S (standard basis)->B.

Theorem 8.1.3 If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, and if $B = {\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n}$ is a basis for \mathbb{R}^n , then [T] and $[T]_B$ are related by the equation

$$[T] = P[T]_B P^{-1}$$
(17)

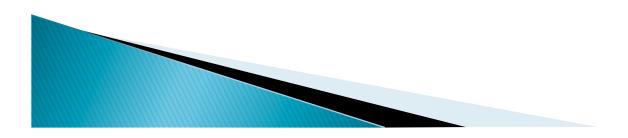
in which

$$P = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \dots \mid \mathbf{v}_n] \tag{18}$$

is the transition matrix from B to the standard basis. In the special case where B is an orthonormal basis the matrix P is orthogonal, so (17) is of the form

$$[T] = P[T]_B P^T \tag{19}$$

- Example 3.
- Example 4. Any reflection can be made into a reflection on x-axis by changing basis or changing coordinates



Base changes for transformations T:Rⁿ->R^m

- Suppose that we choose basis B for Rⁿ and B' for R^m.
- ▶ x->T(x).
- $[x]_B -> [T(x)]_B'$
- $A[x]_B = [T(x)]_B'$. What is A?

Theorem 8.1.4 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, let $B = {\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n}$ and $B' = {\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m}$ be bases for \mathbb{R}^n and \mathbb{R}^m , respectively, and let

$$A = \left[[T(\mathbf{v}_1)]_{B'} \mid [T(\mathbf{v}_2)]_{B'} \mid \dots \mid [T(\mathbf{v}_n)]_{B'} \right]$$
(23)

Then

$$[T(\mathbf{x})]_{B'} = A[\mathbf{x}]_B \tag{24}$$

for every vector \mathbf{x} in \mathbb{R}^n . Moreover, the matrix A given by Formula (23) is the only matrix with property (24).

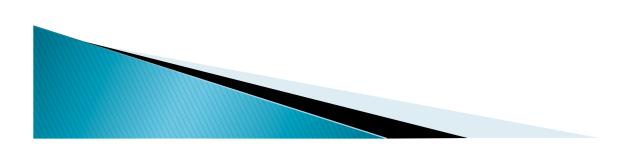
Some formula [T]_B',B -> [[T(v_1)]_B',[T(v_2)]_B',..., [T(v_n)]_B'] and

- [T(x)]_B' = [T]_B',B[x]_B
- Example 6.
- Remark: For operators T:Rⁿ->Rⁿ, [T]_B=[T]_B,B.



Effect of changing basis

- ▶ B_1,B_2 for Rⁿ, B'_1,B'_2 for R^m.
- U transition matrix from B_2->B_1
- V transition matrix from B'_2->B'_1
- $T]_B'_1,B_1 = V[T]_B'_2,B_2U^{-1} (*)$
- Proof: $[T(x)]_B'_1 = [T]_B'_1, B_1[x]_B_1$.
 - $V[T(x)]_B'_2 = [T]_B'_1, B_1U[x]_B_2$
 - $[T(x)]_B'_2 = (V^{-1}[T]_B'_1, B_1U)[x]_B_2$
 - % Use [w]_B'=P_{B->B'}[w]_B.



Representing Linear operators with two basis.

- Actually, we can use two basis for Rⁿ as well.
- ▶ [T]_B',B.
- What we used was [T]_B=[T]_B,B. B'=B.
- So change of basis formula: [T]_B_1=P[T]_BP⁻¹ for P=P_B->B_1.
- V,U=P in this case.
- Thus this follows from (*)

