Topology: Final Exam (Spring 2006)

Justify your answers fully.

1. Let X be a space. (Y, d) a metric space. Prove or disprove:

   (a) (10pts.) The product topology on Y^X is strictly finer than the topology of pointwise convergence.

   (b) (10pts.) If X is compact, then the uniform topology on Y^X is the same as the compact convergence topology.

   (c) (10pts.) The compact convergence topology is finer than the pointwise convergence topology.

   (d) (10pts.) Let X be a compact Hausdorff space and an infinite set. If X has a topology strictly weaker than the discrete topology, then the compact convergence topology is strictly finer than the pointwise convergence topology.

2. Let $H = \{(x, y) \in \mathbf{R}^2 | y > 0\}$ and $L = \{(x, 0) | x \in \mathbf{R}\}$. Let $H \cup L$ be given a topology generated by the subbasis consisting of all open balls in H and the set of form $x \cup D$ for $x \in L$ and D an open ball in H tangent to L at x. This space is called the Niemytski space. Prove the following statements:

   (a) (5 pts.) The space is Hausdorff.

   (b) (10 pts.) The space is regular.

   (c) (10 pts.) The subspace topology of L is discrete.

   (d) (15 pts.) The space is not normal.

3. (30 pts.) Show that a compact Hausdorff space is normal.

4. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(y)$ is compact for each $y \in Y$. Prove the following statements.

   (a) (5 pts.) If U is an open set containing $p^{-1}(y)$, then there is a neighborhood W of y such that $p^{-1}(W) \subset U$.

   (b) (5pts.) If X is Hausdorff, then so is Y.

   (b) (10pts.) If X is regular, then so is Y.

   (c) (10pts.) If X is locally compact, then so is Y.

   (d) (10pts.) If X is second-countable, then so is Y.