Topology: Midterm Exam (Spring 2006)

Justify your answers fully.

1. (30 pts.) Let I be the interval $[0,1]$ in $\mathbf{R}$ with the standard subspace topology. Compare the following three topologies on $I \times I$. That is, list all pairs where one is finer than the other, and list all pairs which are not comparable among the all six pairs.

   (a) The product topology.

   (b) The dictionary order topology.

   (c) The subspace topology as a subspace of $\mathbf{R} \times \mathbf{R}$ given the dictionary order topology.

2. Let the set $\mathbf{Z}^+$ of positive integers be given a discrete topology. Let $\mathbf{Z}^+ \times I$ be given a product topology. Define X as the quotient space of $\mathbf{Z}^+ \times I$ by the equivalence relation

$$(x,t) \sim (y,s) \text{ iff (i) } t = 0 \text{ and } s = 0; \text{ or (ii) } x = y \text{ and } t = s$$

   (a) (10 pts.) Let $\mathbf{R}^2$ be given the standard metric and the standard topology. Let Y be the union of length 1 segments with vertices at $(0,0)$, in the positive quadrant of $\mathbf{R}^2$, and of slope $1/i$ for $i = 1, 2, \ldots$. Let Y be given a subspace topology from $\mathbf{R}^2$. Prove or disprove that X is homeomorphic to Y.

   (b) (10 pts.) Let f be a function $X \to \mathbf{R}$ induced from the map $F : \mathbf{Z}^+ \times [0,1] \to \mathbf{R}$ defined by setting $F(x,t) = xt$ for $x \in \mathbf{Z}^+, t \in [0,1]$. Prove or disprove that f is a continuous function on X.

   (c) (10 pts.) Let g be a function defined on Y defined as follows: g restricted to a segment of slope $1/i$ is defined to be i times the distance function on $\mathbf{R}^2$ restricted to the segment. Prove or dispove that g is a continuous function on Y.

3. Let $\mathbf{R}^3 - \{(0,0,0)\}$ be given the subspace topology from the standard topology of $\mathbf{R}^3$. Let $\mathbf{R}P^2$ be the quotient space of $\mathbf{R}^3 - \{(0,0,0)\}$ with the equivalence relation $v \sim sw$ iff v, w are non-zero vectors and s is a nonzero real number.

   (a) (10 pts). Prove or disprove that $\mathbf{R}P^2$ is a Hausdorff space.

   (b) (10 pts.) Let S^3 be the unit sphere with the subspace topology. Define an equivalence relation $\sim$ where $v \sim w$ iff $v = \pm w$ for two unit vectors v, w. Let $S^3/\sim$ be given a quotient topology. Prove or disprove that it is a compact space.

   (c) (10pts.) Define a homeomorphism from $S^3/\sim$ to $\mathbf{R}P^2$. Prove that it is a homeomorphism.

Problems 4 and 5 in the next page.

4. (30 pts.) Let $\bar{d}(x,y) = \min\{|x-y|, 1\}$ be the standard bounded metric on $\mathbf{R}$. Define a metric on $\mathbf{R}^\omega$ by

$$D(x,y) = \sup\{\bar{d}(x_i,y_i)/i\} \text{ for } x=(x_i), y=(y_i) \in \mathbf{R}^\omega.$$

Prove that the metric topology induced by D is the product topology on $\mathbf{R}^\omega$.

5. Let $I \times I$ be given the dictionary order and the dictionary order topolgy.

(a) (10 pts.) Does $I \times I$ have the least upper bound property?

(b) (10 pts.) Prove or disprove that $I \times I$ is compact.

(c) (10 pts.) Prove or disprove that $I \times I$ is connected.