1 Introduction

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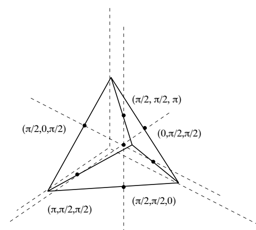
Some helpful references

- W. Thurston, Lecture notes...: A chapter on orbifolds, 1977. (This is the principal source)
- W. Thurston, Three-dimensional geometry and topolgy, PUP, 1997
- M. Berger, Geometry I, Springer
- J. Ratcliffe, Foundations of hyperbolic manifolds, Springer
- M. Kapovich, Hyperbolic Manifolds and Discrete Groups, Birkhauser.
- My talk `http://math.kaist.ac.kr/~schoi/Titechtalk.pdf`



2 Geometries

2.1 Euclidean geometry

Euclidean geometry

- The Euclidean space is $\mathbb{R}^n$ and the group $Isom(\mathbb{R}^n)$ of rigid motions is generated by $O(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.
- A system of linear equations gives us a subspace (affine or linear)
- This gives us the model for Euclidean axioms....

2.2 Spherical geometry

Spherical geometry

- Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$.
- Many great sphere exists and they are subspaces... (They are given by homogeneous system of linear equations in $\mathbb{R}^{n+1}$.)
- The lines are replaced by great circles and lengths and angles are also replaced.
- The transformation group is $O(n+1)$.

Spherical trigonometry

- Many spherical triangle theorems exist... `http://mathworld.wolfram.com/SphericalTrigonometry.html`
- Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \quad (1)$$
$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \quad (2)$$

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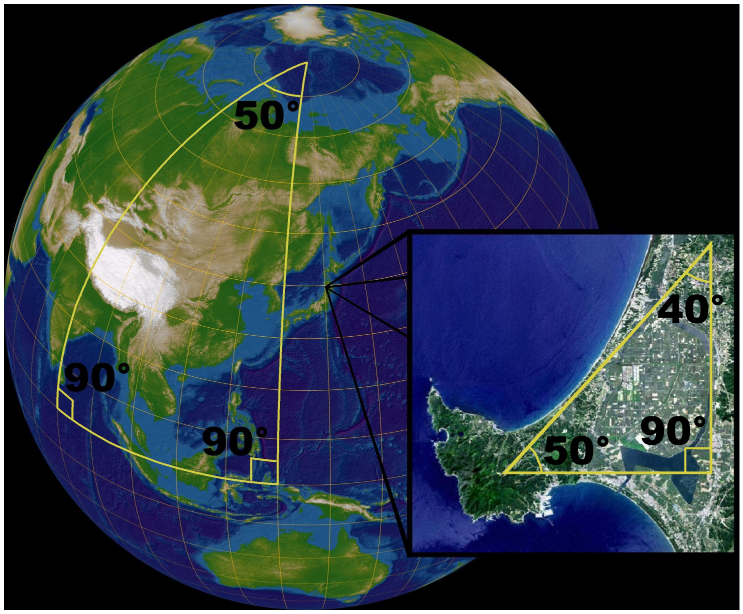
- 

2.3 Affine geometry

Affine geometry

- A vector space $\mathbb{R}^n$ becomes an affine space by forgetting the origin.
- An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.
- The Euclidean space $\mathbb{R}^n$ with the group $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations form the affine geometry.
- Of course, angles and lengths do not make sense. But the notion of lines exists.
- The set of three points in a line has an invariant based on ratios of lengths.

2.4 Projective geometry

Projective geometry

- Projective geometry was first considered from fine art.
- Desargues (and Kepler) first considered points at infinity.
- Poncelet first added infinite points to the euclidean plane.
- Projective transformations are compositions of perspectivities. Often, they send finite points to infinite points and vice versa. (i.e., two planes that are not parallel).

- The added points are same as ordinary points up to projective transformations.
- Lines have well defined infinite points and are really circles topologically.
- Some notions lose meanings. However, many interesting theorems can be proved. Duality of theorems plays an interesting role.
- See for an interactive course: `http://www.math.poly.edu/courses/projective_geometry/`
- and `http://demonstrations.wolfram.com/TheoremeDePappusFrench/`, `http://demonstrations.wolfram.com/TheoremeDePascalFrench/`, `http://www.math.umd.edu/~wphooper/pappus9/pappus.html`, `http://www.math.umd.edu/~wphooper/pappus/`

- Formal definition with topology is given by Felix Klein using homogeneous coordinates.
- The projective space $\mathbb{R}P^n$ is $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R}$.
- Each point is given a homogeneous coordinates: $[v] = [x_0, x_1, ..., x_n]$.
- The projective transformation group $\mathrm{PGL}(n+1, \mathbb{R}) = GL(n+1, \mathbb{R})/\sim$ acts on $\mathbb{R}P^n$ by each element sending each ray to a ray using the corresponding general linear maps.
- Here, each element of g of $\mathrm{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $GL(n+1, \mathbb{R})$ of g. Also any coordinate change can be viewed this way.

- The affine geometry can be "imbedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{R}P^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, ..., x_n]\}$. This is called an affine patch. The subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$.
- The subspace of points $\{[0, x_1, x_2, ..., x_n]\}$ is the complement homeomorphic to $\mathbb{R}P^{n-1}$. This is the set of infinities, i.e., directions in $\mathbb{R}P^n$.
- From affine geometry, one can construct a unique projective geometry and conversely using this idea. (See Berger for the complete abstract approach.)

- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponding to a projective subspace in $\mathbb{R}P^n$ in a one-to-one manner while the dimension drops by 1.
- The independence of points are defined. The dimension of a subspace is the maximal number of independent set minus 1.
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Acutally a line is $\mathbb{R}P^1$ or a line $\mathbb{R}^1$ with a unique infinity.

- Cross ratios of four points on a line (x, y, z, t). There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. Thus $b = b(x, y, z, t)$ is the cross-ratio.
- If the four points are on $\mathbb{R}^1$, the cross ratio is given as

$$(x, y; z, t) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

  if we can write

$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$

- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2.
- For us $n = 2$ is important. Here we have a familiar projective plane as topological type of $\mathbb{R}P^2$, which is a Mobius band with a disk filled in at the boundary. `http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/`

2.5 Conformal geometry

Conformal geometry

- Reflections of $\mathbb{R}^n$. The hyperplane $P(a, t)$ given by $a \cot x = b$. Then $\rho(x) = x + 2(t - a \cdot x)a$.
- Inversions. The hypersphere $S(a, r)$ given by $|x - a| = r$. Then $\sigma(x) = a + (\frac{r}{|x-a|})^2 (x - a)$.

- We can compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This can be accomplished by a stereographic projection from the unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ from the northpole $(0, 0, ..., 1)$. Then these reflections and inversions induce conformal homeomorphisms.

- The group of transformations generated by these homeomorphisms is called the Mobius transformation group.

- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.

- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobius transformation is a either a fractional linear transformation of form

$$z \mapsto \frac{az + b}{cz + d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

  or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.

- In higher-dimensions, a description as a sphere of null-lines and the special Lorentzian group exists.

Poincare extensions

- We can identify E^{n-1} with $E^{n-1} \times \{O\}$ in E^n.

- We can extend each Mobius transformation of $\hat{E}^{n-1}$ to $\hat{E}^n$ that preserves the upper half space U: We extend reflections and inversions in the obvious way.

- The Mobius transformation of $\hat{E}^n$ that preserves the open upper half spaces are exactly the extensions of the Mobius transformations of $\hat{E}^{n-1}$.

- $M(U^n) = M(\hat{E}^{n-1})$.

- We can put the pair $(U^n, \hat{E}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobius transformation.

- Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere.

2.6 Hyperbolic geometry

Lorentzian geometry

- A hyperbolic space H^n is defined as a complex Riemannian manifold of constant curvature equal to -1.

- Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions.

- But it is realized as a "sphere" in a Lorentzian space.

- A Lorentzian space is $\mathbb{R}^{1,n}$ with an inner product

$$x \cdot y = -x_0y_0 + x_1y_1 + \cdots + x_{n-1}y_{n-1} + x_ny_n.$$

- A Lorentzian norm $||x|| = (x \cdot y)^{1/2}$, a positive, zero, or positive imaginary number.
- One can define Lorentzian angles.
- The null vectors form a light cone divide into positive, negative cone, and 0.
- Space like vectors and time like vectors and null vectors.
- Ordinary notions such as orthogonality, orthonormality,...

Lorentz group

- A Lorentzian transformation is a linear map preserving the inner-product.
- For J the diagonal matrix with entries $-1, 1, ..., 1$, $A^tJA = J$ iff A is a Lorentzian matrix.
- A Lorentzian transformation sends time-like vectors to time-like vectors. It is either positive or negative.
- The set of Lorentzian transformations form a Lie group $O(1, n)$.
- The set of positive Lorentzian transformations form a Lie subgroup $PO(1, n)$.

Hyperbolic space

- Given two positive time-like vectors, there is a time-like angle

$$x \cdot y = ||x||||y||cosh\eta(x, y)$$

- A hyperbolic space is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone.
- Topologically, it is homeomorphic to $\mathbb{R}^n$. Minkowsky model
- One induces a metric from the Lorentzian space which is positive definite.
- This gives us a Riemannian metric of constant curvature -1. (The computation is very similar to the computations for the sphere.)
- $PO(1, n)$ is the isometry group of H^n which is homogeneous and directionless.
- A hyperbolic line is an intersection of H^n with a time-like two-dimensional vector subspace.
- The hyperbolic sine law, The first law of cosines, The second law of cosines...

- One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π.
- One can assign any lengths to a hyperbolic triangle.
- The triangle formula can be generalized to formula for quadrilateral, pentagon, hexagon.
- Basic philosophy here is that one can push the vertex outside and the angle becomes distances between lines. (See Ratcliffe, `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)
- hyperbolic law of sines:

$$\sin A/\sinh a = \sin B/\sinh b = \sin C/\sinh c$$

- hyperbolic law of cosines:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$$

$$\cosh c = (\cosh A \cosh B + \cos C)/\sinh A \sinh B$$

Beltrami-Klein models of hyperbolic geometry

- Beltrami-Klein model is directly obtained from the hyperboloid model.
- $d_k(P,Q) = 1/2 \log |(AB, PQ)|$ where A, P, Q, B are on a segment with endpoints A, B and

$$(AB, PQ) = \left| \frac{AP}{BP} \frac{BQ}{AQ} \right|.$$

- There is an imbedding from H^n onto an open ball B in the affine patch $\mathbb{R}^n$ of $\mathbb{R}P^n$. This is standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$.
- B can be described as a ball of radius 1 with center at O.
- The isometry group $PO(1,n)$ also maps injectively to a subgroup of $PGL(n+1,\mathbb{R})$ that preserves B.
- The projective automorphism group of B is precisely this group.
- The metric is induced on B. This is precisely the metric given by the log of the cross ratio. Note that $\lambda(t) = (\cosh t, \sinh t, 0, ..., 0)$ define a unit speed geodesic in H^n. Under the Riemannian metric, we have $d(e_1, (\cosh t, \sinh t, 0, ..., 0)) = t$ for t positive.
- Under d_k, we obtain the same. Since any geodesic segment of same length is congruent under the isometry, we see that the two metrics coincide. Betrami-Klein model

- Beltrami-Klein model is nice because you can see outside. The outside is the anti-de Sitter space http://en.wikipedia.org/wiki/Anti_de_Sitter_space
- Also, we can treat points outside and inside together.
- Each line (hyperplane) in the model is dual to a point outside. (i.e., orthogonal by the Lorentzian inner-product) A point in the model is dual to a hyperplane outside. Infact any subspace of dimenstion i is dual to a subspace of dimension $n-i-1$ by orthogonality.
- For $n = 2$, the duality of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
- The distance between two hyperplanes can be obtained by two dual points. The two dual points span an orthogonal plane to the both hyperperplanes and hence provide a shortest geodesic.

The conformal ball model (Poincare ball model)

- The stereo-graphic projection H^n to the plane P given by $x_0 = 0$ from the point $(-1, 0, ..., 0)$.
- The formula for the map $\kappa : H^n \to P$ is given by
$$\kappa(x) = \left(\frac{y_1}{1+y_0}, ..., \frac{y_n}{1+y_0}\right),$$
where the image lies in an open ball of radius 1 with center O in P. The inverse is given by
$$\zeta(x) = \left(\frac{1+|x|^2}{1-|x|^2}, \frac{2x_1}{1-|x|^2}, ..., \frac{2x_n}{1-|x|^2},\right).$$
- Since this is a diffeomorphism, B has an induced Riemannian metric of constant curvature -1.

- We show
$$\cosh d_B(x, y) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)},$$
and inversions acting on B preserves the metric. Thus, the group of Mobius transformations of B preserve metric.
- The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1-|x|^2)^2$.
- It follows that the group of Mobius transformations acting on B is precisely the isometry group of B. Thus, $Isom(B) = M(\mathbf{S}^{n-1})$.
- Geodesics would be lines through O and arcs on circles perpendicular to the sphere of radius 1.

The upper-half space model.

- Now we put B to U by a Mobius transformation. This gives a Riemannian metric constant curvature -1.
- We have by computations $\cosh d_U(x, y) = 1 + |x - y|^2/2x_n y_n$ and the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(E^{n-1})$.
- Geodesics would be arcs on lines or circles perpendicular to E^{n-1}.
- Since $\hat{E}^1$ is a circle and $\hat{E}^2$ is the complex sphere, we obtain $Isom^+(B^2) = PSL(2, \mathbb{R})$ and $Isom^+(B^3) = PSL(2, \mathbb{C})$.
- Orientation-preserving isometries of hyperbolic plane can have at most one fixed point. elliptic, hyperbolic, parabolic.

$$z \mapsto e^{i\theta}, z \mapsto az, a \neq 1, a \in \mathbb{R}^+, z \mapsto z + 1$$

- Isometries of a hyperbolic space: loxodromic, hyperbolic, elliptic, parabolic.
- Up to conjugations, they are represented as Mobius transformations which has forms
  - $z \mapsto \alpha z, Im\alpha \neq 0, |\alpha| \neq 1.$
  - $z \mapsto az, a \neq 1, a \in \mathbb{R}^+.$
  - $z \mapsto e^{i\theta} z, \theta \neq 0.$
  - $z \mapsto z + 1.$

3 Discrete group actions

Discrete groups and discrete group actions

- A discrete group is a group with a discrete topology. (Usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group.
- We have many notions of a group action $\Gamma \times X \to X$:
  - The action is effective, is free
  - The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with compact open topology.
  - The action has *discrete orbits* if every x has a neighborhood U so that the orbit points in U is finite.
  - The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
  - The action is *properly discontinuous* if for every compact subset K the set of γ such that $K \cap \gamma(K) \neq \emptyset$ is finite.

- discrete action < discrete orbit < wandering < properly discontinuous. This is a strict relation (Assuming X is a manifold.)
- The action is wandering and free and gives manifold quotient (possibly non-Hausdorff)
- The action of Γ is free and properly discontinuous if and only if X/Γ is a manifold quotient (Hausdorff) and $X \to X/\Gamma$ is a covering map.
- Γ a discrete subgroup of a Lie group G acting on X with compact stabilizer. Then Γ acts properly discontinuously on X.
- A complete (X, G) manifold is one isomorphic to X/Γ.
- Suppose X is simply-connected. Given a manifold M the set of complete (X, G)-structures on M up to (X, G)-isotopies are in one-to-one correspondence with the discrete representations of $\pi(M) \to G$ up to conjugations.

Examples

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. (free, properly discontinuous.)
- The modular group $PSL(2, \mathbb{Z})$ the group of Mobius transformations or isometries of hyperbolic plane given by $z \mapsto \frac{az+b}{cz+d}$ for integer a, b, c, d and $ad - bc = 1$. `http://en.wikipedia.org/wiki/Modular_group`. This is not a free action.

Convex polyhedrons

- A *convex subset* of H^n is a subset such that for any pair of points, the geodesic segment between them is in the subset.
- Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.
- Some facts about convex sets:
  - The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in H^n.
  - The interior C^o, the boundary ∂C are defined in $\hat{C}$.
  - The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.

Convex polytopes

- A side C is a nonempty maximal convex subset of ∂C.
- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in H^n.
- A polytope is a convex polyhedron with finitely many vertices and is the convex hull of its vertices inH^n.
- A polyhedron P in H^n is a generalized polytope if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices.

Examples of Convex polytopes

- A compact simplex: convex hull of $n+1$ points in H^n.
- Start from the origin expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x \to sx$ for $s > 0$ and x is the coordinate vector of an affine patch using in fact any vector coordinates. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.
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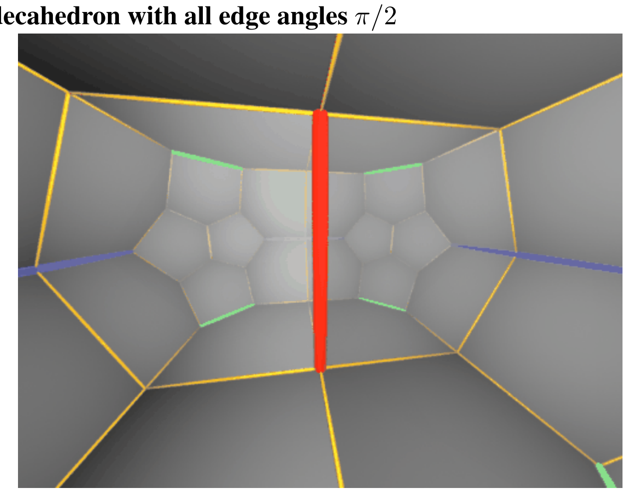
Regular dodecahedron with all edge angles $\pi/2$



Fundamental domain of discrete group action

- Let Γ be a group acting on X.
- A *fundamental domain* for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X.
- The fundamental domain is locally finite if the above closures are locally finite.

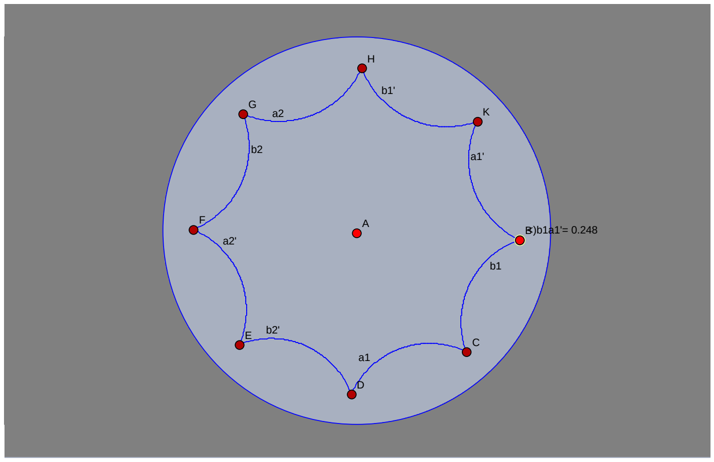
- The *Dirichlet domain* for $u \in X$ is the intersection of all $H_g(u) = \{x \in X | d(x,u) < d(x,gu)\}$. Under nice conditions, $D(u)$ is a convex fundamental polyhedron.
- The regular octahedron example of hyperbolic surface of genus 2 is an example of a Dirichlet domain with the origin as u.

Tessellations

- A tessellation of X is a locally-finite collection of polyhedra covering X with mutually disjoint interiors.
- Convex fundamental polyhedron provides examples of exact tessellations.
- If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by $\Phi = \{g \in \Gamma | P \cap g(P)$ is a side of $P\}$.

Side pairings and Poincare fundamental polyhedron theorem

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}$.
- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.
- Cycle relations (This should be cyclic):
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.
- Then
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$.
  - $g_{S_1} g_{S_2} .... g_{S_l}$ has order k.
- Example: the octahedron in the hyperbolic plane giving genus 2-surface.
- The period is the number of sides coming into a given side R of codimension two.



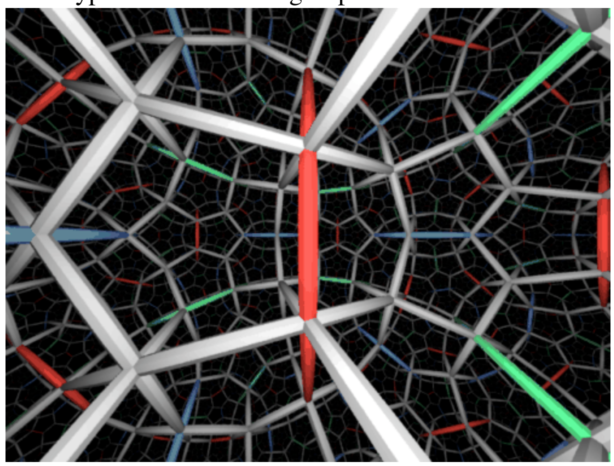
- $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C),$
- $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G),$
- $(b2, G), (b2', D), (a1, D), (a1', K), \ldots$
- Poincare fundamental polyhedron theorem is the converse. (See Kapovich P. 80–84):
- Given a convex polyhedron P in X with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete group generated by the side-pairing isometries.
- If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds.
- The results are always complete.
- See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html`

Reflection groups

- A discrete reflection group is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π.
- Then the side pairing such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.
- The reflection group has presentation $\{S_i : (S_i S_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$.
- These are examples of Coxeter groups. `http://en.wikipedia.org/wiki/Coxeter_group`

Icosahedral reflection group

One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group.

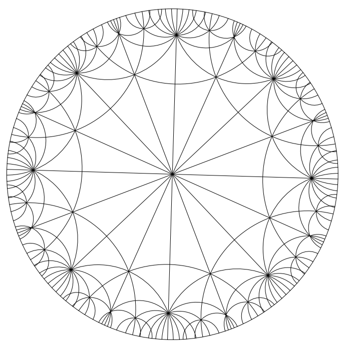


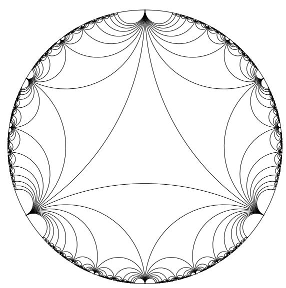
Triangle groups

- Find a triangle in X with angles submultiples of π.
- We divide into three cases $\pi/a + \pi/b + \pi/c > 0, = 0, < 0$.
- We can always find ones for any integers a, b, c.
  - > 0 cases: $(2, 2, c), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ corresponding to dihedral group of order $4c$, a tetrahedral group, octahedral group, and icosahedral group.
  - $= 0$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$.
  - < 0 cases: Infinitely many hyperbolic tessellation groups.
- $(2, 4, 8)$-triangle group
- The ideal example `http://egl.math.umd.edu/software.html`

Higher-dimensional examples

- To construct a 3-dimensional examples, obtain a Euclidean regular polytopes and expand it until we achieve that all angles are $\pi/3$. Regular octahedron with angles $\pi/2$. These are ideal polytope examples.
- Higher-dimensional examples were analyzed by Vinberg and so on. For example, there are no hyperbolic reflection group of compact type above dimension ≥ 30.





Crystallographic groups

- A crystallographic group is a discrete group of the rigid motions whose quotient space is compact.
- Bieberbach theorem:
  - A group is isomorphic to a crystallographic group if and only if it contains a subgroup of finite index that is free abelian of rank equal to the dimension.
  - The crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.

Crystallographic groups

- There are only finitely many crystallographic group for each dimension since once the abelian group action is determined, its symmetry group can only be finitely many.
- 17 wallpaper groups for dimension 2. `http://www.clarku.edu/~djoyce/wallpaper/` and see Kali by Weeks http://www.geometrygames.org/Kali/index.html.
- 230 space groups for dimension 3. Conway, Thurston, ... `http://www.emis.de/journals/BAG/vol.42/no.2/b42h2con.pdf`
- Further informations: `http://www.ornl.gov/sci/ortep/topology.html`