1 Discrete group actions

Discrete groups and discrete group actions

- A discrete group is a group with a discrete topology. (Usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group.
- We have many notions of a group action $\Gamma \times X \to X$:
  - The action is effective, is free
  - The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with compact open topology.
  - The action has *discrete orbits* if every x has a neighborhood U so that the orbit points in U is finite.
  - The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
  - The action is *properly discontinuous* if for every compact subset K the set of γ such that $K \cap \gamma(K) \neq \emptyset$ is finite.
- discrete action < discrete orbit < wandering < properly discontinuous. This is a strict relation (Assuming X is a manifold.)
- The action is wandering and free and gives manifold quotient (possibly non-Hausdorff)
- The action of Γ is free and properly discontinuous if and only if X/Γ is a manifold quotient (Hausdorff) and $X \to X/\Gamma$ is a covering map.
- Γ a discrete subgroup of a Lie group G acting on X with compact stabilizer. Then Γ acts properly discontinuously on X.
- A complete (X, G) manifold is one isomorphic to X/Γ.
- Suppose X is simply-connected. Given a manifold M the set of complete (X, G)-structures on M up to (X, G)-isotopies are in one-to-one correspondence with the discrete representations of $\pi(M) \to G$ up to conjugations.

Examples

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. (free, properly discontinuous.)
- The modular group $PSL(2, \mathbb{Z})$ the group of Mobius transformations or isometries of hyperbolic plane given by $z \mapsto \frac{az+b}{cz+d}$ for integer a, b, c, d and $ad - bc = 1$. `http://en.wikipedia.org/wiki/Modular_group`. This is not a free action.

Convex polyhedrons

- A *convex subset* of H^n is a subset such that for any pair of points, the geodesic segment between them is in the subset.
- Using the Beltrami-Klein model, the open unit ball B, i.e., the hyperbolic space, is a subset of an affine patch $\mathbb{R}^n$. In $\mathbb{R}^n$, one can talk about convex hulls.
- Some facts about convex sets:
  - The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in H^n.
  - The interior C^o, the boundary ∂C are defined in $\hat{C}$.
  - The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.

Convex polytopes

- A side C is a nonempty maximal convex subset of ∂C.
- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in H^n.
- A polytope is a convex polyhedron with finitely many vertices and is the convex hull of its vertices inH^n.
- A polyhedron P in H^n is a generalized polytope if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices.

Examples of Convex polytopes

- A compact simplex: convex hull of $n + 1$ points in H^n.
- Start from the origin expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x \to sx$ for $s > 0$ and x is the coordinate vector of an affine patch using in fact any vector coordinates. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.
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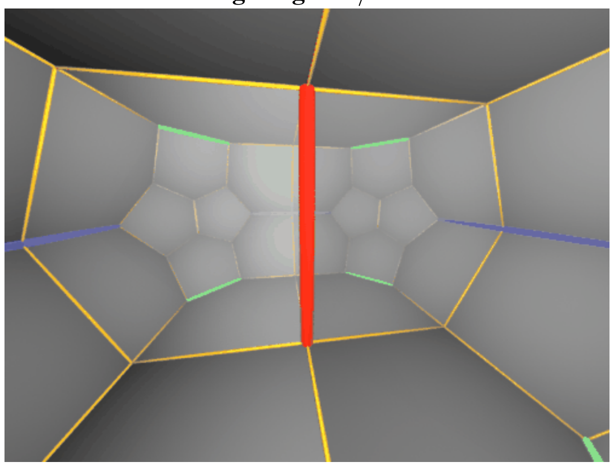
Regular dodecahedron with all edge angles $\pi/2$



Fundamental domain of discrete group action

- Let Γ be a group acting on X.
- A *fundamental domain* for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X.
- The fundamental domain is locally finite if the above closures are locally finite.
- The *Dirichlet domain* for $u \in X$ is the intersection of all $H_g(u) = \{x \in X|d(x,u) < d(x,gu)\}$. Under nice conditions, $D(u)$ is a convex fundamental polyhedron.
- The regular octahedron example of hyperbolic surface of genus 2 is an example of a Dirichlet domain with the origin as u.

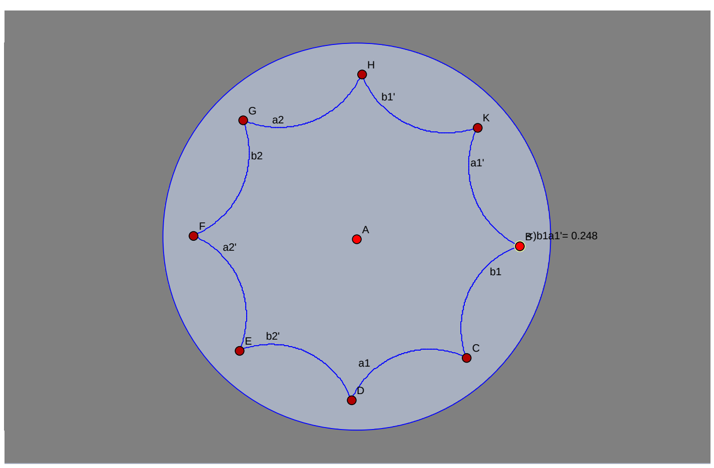
Tessellations

- A tessellation of X is a locally-finite collection of polyhedra covering X with mutually disjoint interiors.
- Convex fundamental polyhedron provides examples of exact tessellations.
- If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by $\Phi = \{g \in \Gamma|P \cap g(P)$ is a side of $P\}$.

Side pairings and Poincare fundamental polyhedron theorem

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}$.

- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.
- Cycle relations (This should be cyclic):
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.
- Then
  - There is an integer l such that $S_{i+l} = S_i$ for each i.
  - $\sum_{i=1}^{l} \theta(S_i', S_{i+1}) = 2\pi/k$.
  - $g_{S_1} g_{S_2} \ldots . g_{S_l}$ has order k.
- Example: the octahedron in the hyperbolic plane giving genus 2-surface.
- The period is the number of sides coming into a given side R of codimension two.



- $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C),$
- $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G),$
- $(b2, G), (b2', D), (a1, D), (a1', K), ...$
- Poincare fundamental polyhedron theorem is the converse. (See Kapovich P. 80–84):

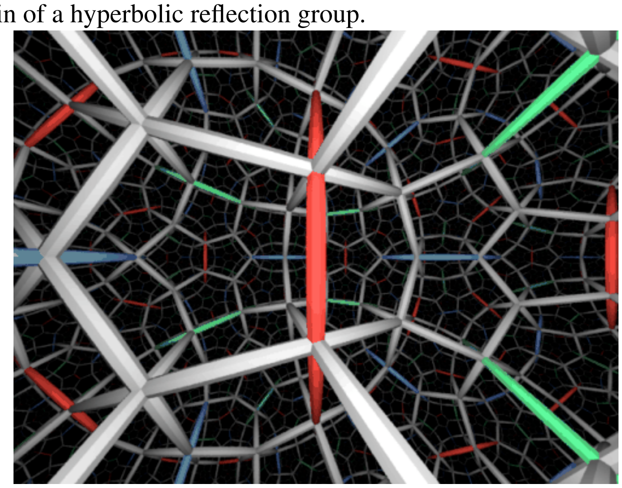
- Given a convex polyhedron P in X with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete group generated by the side-pairing isometries.
- If every k equals 1, then the result of the face identification is a manifold. Otherwise, we obtain orbifolds.
- The results are always complete.
- See Jeff Weeks `http://www.geometrygames.org/CurvedSpaces/index.html`

Reflection groups

- A discrete reflection group is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π.
- Then the side pairing such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.
- The reflection group has presentation $\{S_i : (S_i S_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$.
- These are examples of Coxeter groups. `http://en.wikipedia.org/wiki/Coxeter_group`

The dodecahedral reflection group

One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group.



Triangle groups

- Find a triangle in X with angles submultiples of π.
- We divide into three cases $\pi/a + \pi/b + \pi/c > 0, = 0, < 0$.
- We can always find ones for any integers a, b, c.
  - > 0 cases: $(2, 2, c), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ corresponding to dihedral group of order $4c$, a tetrahedral group, octahedral group, and dodecahedral group.
  - $= 0$ cases: $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$.
  - < 0 cases: Infinitely many hyperbolic tessellation groups.
- $(2, 4, 8)$-triangle group
- The ideal example `http://egl.math.umd.edu/software.html`

Higher-dimensional examples

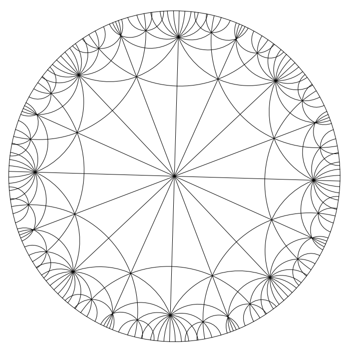
- To construct a 3-dimensional examples, obtain a Euclidean regular polytopes and expand it until we achieve that all angles are $\pi/3$. Regular octahedron with angles $\pi/2$. These are ideal polytope examples.
- Higher-dimensional examples were analyzed by Vinberg and so on. For example, there are no hyperbolic reflection group of compact type above dimension ≥ 30.

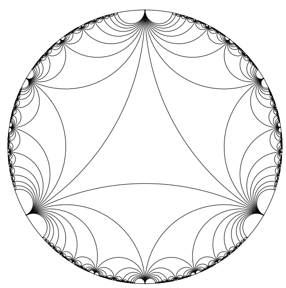
Crystallographic groups

- A crystallographic group is a discrete group of the rigid motions whose quotient space is compact.
- Bieberbach theorem:
  - A group is isomorphic to a crystallographic group if and only if it contains a subgroup of finite index that is free abelian of rank equal to the dimension.
  - The crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.

Crystallographic groups

- There are only finitely many crystallographic group for each dimension since once the abelian group action is determined, its symmetry group can only be finitely many.
- 17 wallpaper groups for dimension 2. `http://www.clarku.edu/~djoyce/wallpaper/` and see Kali by Weeks http://www.geometrygames.org/Kali/index.html.





- 230 space groups for dimension 3. Conway, Thurston, ... http://www.emis.de/journals/BAG/vol.42/no.2/b42h2con.pdf
- Further informations: http://www.ornl.gov/sci/ortep/topology.html